PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 33 (47), 1983, pp. 227–234

FINITENESS OF SPECTRA OF GRAPHS OBTAINED BY SOME OPERATIONS ON INFINITE GRAPHS

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Abstract. In this paper we consider some unary and binary operations on infinite graphs, and we investigate when the spectrum of the resulting graph is finite.

In particular, we consider the induced subgraphs of an infinite graph, relabeling of its vertices, the complementary graph, the union, Cartesian product, complete product and direct sum of two infinite graphs, the line graph .and the total graph of a graph.

For some of these operations we find that the spectrum of the graph so obtained is always infinite (direct sum, line and total graph). Among other things, we show that finiteness of the spectrum of an infinite graph does not change by any relabeling of its vertices.

1. Preliminaries concerning spectra

In [**4**], we began investigating the spectra of infinite graphs restricting ourselves to connected infinite simple graphs (undirected graphs without loops or multiple edges). But since in [**3**] M. Petrović considered infinite graphs without the restriction of connectedness, we shall formulate the needed spectral results for general (connected or disconnected) infinite graphs.

Throughout the paper, by a graph or an infinite graph, we always mean an infinite denumerable (connected or disconnected) simple graph. Its vertex set $V(G) = \{v_1, v_2, \dots\}$ is indexed by natural numbers, and we often identify it with the set N of natural numbers.

The adjacency matrix $A(G) = A = [a_{ij}]$ of a graph G is an infinite $N \times N$ matrix defined by

$$a_{ij} = \begin{cases} a^{i+j-2}, & \text{if } i, j \text{ are adjacent} \\ 0, & \text{otherwise} \end{cases}$$

where a is a fixed positive constant ($O < a < 1$).

Communicated 22. nov. 1980. on the First Yugoslav Seminar for Theory of graphs, held at University of Belgrade, Faculty of Electrical Eng.

AMS Subject Classification (1980): 05C50, Secondary 47A65.

Key words and phrases: Connected infinite graph, operation on infinite graphs, spectrum of a graph, finiteness of the spectrum.

Hence the whole graph G is labeled (or weighted), so that vertex $v_i = i$ $(i \in N)$ has the weight a^{i-1}.

The matrix $A = A(G)$ is a symmetric Hilbert-Schmidt matrix (or operator) in a separable Hilbert space, because its absolute norm

$$n(A)^2 = \sum_{i,j=1}^{\infty} |a_{ij}|^2$$

is finite.

The spectrum $\sigma(G)$ of a graph G is defined to be the spectrum a $\sigma(A) = \sigma(A(G))$ of this Hilbert-Schmidt operator $A(G)$. Then (see [**4**], or [**5**]) the following holds:

PROPOSITION 1. *The spectrum $\sigma(G)$ is always real, and it consists of the zero and of a finite or infinite sequence of non-zero eigenvalues $\lambda_1, \lambda_2, \dots$)$\lambda_1 > \lambda_2 \leq \dots$), whose multiplicities are finite.*

The sequence $\lambda_n \to 0$ (as $n \to \infty$), if it is infinite.

The spectral radius $r(G) = \|A\| \leq d = a\sqrt{2}/(1-a^2)\sqrt{1+a^2}$, so that the whole spectrum $\sigma(G)$ lies in the interval $[-d, d]$.

If G has a finite spectrum, i.e. $\sigma(G) = \{\lambda_1, \dots, \lambda_p; 0\}$ with exactly p (not necessarily distinct) non-zero eigenvalues $\lambda_1, \dots, \lambda_p$, then we say that G has a p-finite spectrum. In this case, the value $\lambda = 0$ is an eigenvalue and its multiplicity is infinite (codimension of the corresponding proper subspace is exactly p).

An infinite graph G has a finite spectrum if and only if it is *of finite type* (see [**4**] or [**5**]), i.e. $G = g(N_1, \dots, N_k)$, where g is its canonical image (a finite graph with k vertices, the quotient graph with respect to the following equivalence relation in the set $V(G) : x \sim y$ iff they have the same neighbors), and $N_1, \dots, N_k$ are the corresponding equivalence classes (consisting of totally non-adjacent vertices).

If G is a general (connected or disconnected) infinite graph, then we have ([**51**]).

PROPOSITION 2. (1°) *Any infinite graph G of finite type k has the p finite spectrum ($p = p(G) \leq k$).*

(2°) *Every graph G with p finite spectrum has a ,finite type k, where $k \leq 2^p - 1$.*

(3°) *If $G = g(N_1, \dots, N_k)$ is a graph of finite type, then its number of non-zero eigenvalues coincides with the number of non-zero eigenvalues of its canonical image g, i.e. $p(G) = p(g)$.*

PROPOSITION 3. *If the graph G is of the type k, $G = g(N_1, \dots, N_k)$, then its non-zero eigenvalues are determined by $\lambda = \alpha/a^2$, where α are the non-zero roots of the characteristic equation*

$$(*) \qquad \begin{vmatrix} -\alpha/A & b_{12} & \dots & b_{1k} \\ \dots & \dots & \dots & \dots \\ b_{k1} & b_{k2} & \dots & -\alpha/A_k \end{vmatrix} = 0$$

Here $B = [b_{ij}]$ *is the usual* $(0-1)$ *adjacency matrix of the canonical image* g *of* G *wtd* $A_i = \sum_{j \in N_i} a^{2j}$ $(i = 1, \dots, k)$.

We notice that *connected* infinite graphs are of our main interest, and remarks concerning spectra of general (connected or disconnected) graphs bad to be included only because some operations over connected infinite graphs lead to disconnected graphs.

2. Results

We consider separately some basic operations on infinite graphs, and investigate the finiteness of spectra of the graphs thus obtained.

(i) *Ralabeling of vertices*

Let G be any infinite graph with a fixed labeling of its vertices (vertex v_i has the weight a^{i-1}) and let $M = \{m_1, m_2, \dots\}$ (m_i – distinct) be an arbitrary not necessarily ordered, infinite subset of the set N.

If $A(G) = [a_{ij}]$ is the adjacency matrix of G with the constant a, and b is any other constant $(0 < b < 1)$, define the new adjacency matrix $\tilde{A}(G) = [\tilde{a}_{ij}]$ of G by

$$\tilde{a}_{ij} = \begin{cases} b^{m_i + m_j - 2}, & \text{if } i, j \text{ are adjacent} \\ 0, & \text{otherwise} \end{cases} .$$

Hence the vertex v_i now has the new weight $b^{m_i - 1}$ $(i \in N)$.

Then we say that the vertex set $V(G)$ has been relabeled (including the change of the constant a).

Theorem 1. *If a graph* G *has a* p *finite spectrum, then any relabeling of its vertex set does not change the number of its non-zero eigenvalues.*

Proof. One can see that all arguments from the three propositions above hold for the relabeled graph too, where

$$\tilde{d} = \left(\sum_{i,j=1}^{\infty} |\tilde{a}_{ij}|^2 \right)^{1/2}$$

in Proposition 1, and $A_i = \sum a^{2m_j}$ $(j \in N_i)$ in the last proposition (consult [**5**] for the details).

But, since the relabeling of a graph does not change the type of a graph, Proposition 2 (3°) completes the proof. □

Remark. This theorem implies that for any infinite graph, the property "to have a p-finite spectrum" does not depend on the constant a or on the way of labeling, so it is a pure spectral property of graphs.

In particular, any renumeration of weights of a graph with p-finite spectrum does not change the number $p = p(G)$.

(ii) *Induced subgraphs of a graph*

Consider next an arbitrary (connected or disconnected) infinite induced subgraph of a connected graph G. We assume that the subgraph G_0, considered has the induced weights from the graph G.

THEOREM 2. *If a graph G has a p finite spectrum, then any of its induced subgraphs G_0 has a y finite spectrum, where $0 \leq q \leq p$.*

Proof. Since the canonical image g_0 of the graph G_0 is an induced subgraph of the canonical image g of the graph G, Proposition 2 (3°) provides the proof.

It can be shown by examples, that both extreme cases $q = 0$ and $q = p$ may be obtained. □

(iii) *Complement of a graph*

If G is a connected infinite graph, then the complement $\overline{G}$ of G is an infinite graph with the same vertices as G, and two vertices adjacent in $\overline{G}$ if and only if they are, distinct and they are non-adjacent in G.

For the adjacency matrix $A(\overline{G})$ of the complementary graph $\overline{G}$ we take the complementary matrix of the matrix $A(G)$ (with the zero diagonal).

THEOREM 3. *If a connected infinite graph G has a finite type k, i.e. $G = g(N_1, \dots, N_k)$, then its complementary graph $\overline{G}$ is infinite (connected or disconnected) and has an infinite spectrum.*

Proof. If $G = g(N_1, \dots, N_k)$ then, obviously, at least one among the characteristic subsets $N_1, \dots, N_k$ must be infinite, so that the complementary graph $\overline{G}$ possesses at least one complete infinite induced subgraph. Hence the complementary graph $\overline{G}$ cannot be of a finite type, and consequently it always has an infinite spectrum. □

It can be easily seen that a connected infinite graph G has a complementary graph $\overline{G}$ of a finite type (that is with a finite spectrum) if and only if $G = h(M_1, \dots, M_i)$, where $M_1, \dots, M_i$ are complete induced subgraphs of G and h is a finite graph with l vertices. Then G has an infinite spectrum.

(iv) *Union of two graphs*

The union $G_1 \cup G_2$ of two infinite connected graphs G_1 and G_2 is the graph whose vertex set is the union of the *disjoint* sets $V(G_1)$, $V(G_2)$, and the set of edges is the union of the corresponding sets of edges in G_1 and G_2.

In $G_1 \cup G_2$ we always assume some fixed (but arbitrary) labeling of its vertices by the set N, and the constant a is assumed to be the same in all G_1, G_2, $G_1 \cup G_2$. So G_1, G_2 become subgraphs of $G_2 \cup G_2$ with induced labeling from $G_1 \cup G_2$ (these induced subgraphs are denoted by G_1^0 and G_2^0). The union $G_1 \cup G_2$ is always disconnected.

THEOREM 4. *The union $G = G_1 \cup G_2$ of two connected graphs G_1, G_2 has a finite spectrum if and only if G_1, G_2 have such spectra.*

The numbers $p = p(G)$, $p_i = p(G_i)$ $(i = 1, 2)$ *of the non-zero eigenvalues of* G, G_1, G_2 *satisfies the relation* $p = p_1 + p_2$.

Proof. If $A(G)$, $A(G_1^0)$, $A(G_2^0)$ are the adjacency matrices of graphs G, G_1, G_2, respectively, then

$$A(G) = A(G_1^0) \oplus A(G_2^0),$$

whence $\sigma(G) = \sigma(G_1^0) \cup \sigma(G_2^0)$.

Then Theorems 1 and 2 complete the proof. □

(v) *Complete product of two graphs*

The complete product $G_1 \nabla G_2$ of two connected infinite graphs G_1, G_2 is the graph obtained from $G_1 \cup G_2$ by joining every vertex from G_1 with every vertex from G_2. Here, again, $V(G_1) \cap V(G_2) = \emptyset$ is assumed.

We also suppose that in $G_1 \nabla G_2$ a fixed labeling of its vertices by the set of natural numbers is given. Then G_1^0, G_2^0 are the corresponding subgraphs of $G_1 \nabla G_2$ with the same vertices as G_1, G_2 and the induced labeling from $G_1 \nabla G_2$.

The complete product of two graphs is obviously always connected.

Theorem 5. *The complete product* $G_1 \nabla G_2$ *of two connected graphs* G_1, G_2 *is a graph of a finite type if and only if the graphs* G_1, G_2 *are such graphs.*

If G_i *has the type* $k_i\,(i = 1, 2)$, *then* $G_1 \nabla G_2$ *has type* $k = k_1 + k_2$, *and* p *non-zero eigenvalues, where*

$$(**) \qquad p_1 + p_2 \le p \le p_1 + p_1 + 2.$$

Proof. If the complete product $G_1 \nabla G_2$ of the graphs G_1, G_2 has a finite type, then the graphs G_1^0, G_2^0, which are induced subgraphs of $G_1 \nabla G_2$ must also have a finite type, and Theorem 1 implies that the graphs G_1, G_2 have finite type, too.

Conversely, let the graphs G_1, G_2 have finite types k_1, k_2, respectively, that is

$$G_1 = g(N_1', \dots, N_{k_1}'), \quad G_2 = g_2(N_1'', \dots, N_{k2}'').$$

Then, as easily seen, the characteristic subsets of $G_1 \nabla G_2$ are exactly the sets $N_1', \dots, N_{k_1}'$, $N_1'', \dots, N_{k_2}''$, so that $G_1 \nabla G_2$ has the type $k = k(G_1 \nabla G_2) = k_1 + k_2$.

If now B_i is the (0–1) adjacency matrix of the finite graph g_i $(i = 1, 2)$, then the adjacency matrix of the finite canonical image g of the graph $G_1 \nabla G_2$ has the form

$$B = \begin{bmatrix} B_1 & J \\ J' & B_2 \end{bmatrix},$$

where J is a $k_1 \times k_2$ matrix whose all of entries are equal to 1.

Since then $p(G) = \operatorname{rank}(B)$, we can obtain the estimates $(**)$ for $p(G)$. □

(vi) *The product of two graphs*

The product $G_1 \times G_2$ of two infinite graphs G_1, G_2 is the infinite graph whose vertices are the ordered pairs (x, y) of the vertices $x \in V(G_1)$, $y \in V(G_2)$, with two vertices (x_1, y_1), (x_2, y_2) adjacent if and only if x_1 is adjacent to x_2 in G_1, and y_1 is adjacent to y_2 in G_2.

Since vertices (x_0, y) $(y \in G_2)$ are not adjacent in $G_1 \times G_2$, the product of two graphs can be connected or disconnected.

THEOREM 6. *The product $G_1 \times G_2$ of two connected infinite graphs G_1, G_2 has a finite spectrum if and only $i_j G_1$ and G_2 do.*

If G_i has the type k_i $(i = 1, 2)$, then $G_1 \times G_2$ has the type $k = k_1 \cdot k_2$.

Proof. We assume that G_i has the finite type k_i $(i = 1, 2)$, that is

$$G_1 = g_1(N_1, \dots, N_{k_1}), \;\; G_2 = g_2(M_1, \dots, M_{k_2}),$$

and we prove that $k(G_1 \times G_2) = k_1 \cdot k_2$.

We will prove that the subsets $P_{ij} = N_i \times M_j$ $(i \le k_1; j \le k_2)$ are the characteristic subsets of the graph $G_1 \times G_2$.

Write $u \sim v$ if in a graph G the vertices u, v belong to the same characteristic subset of G (i.e. u, v are non-adjacent and have the same neighbors). Then for any $x, x_1, x_2 \in V(G_1)$, $y, y_1, y_2 \in V(G_2)$ the following holds:

1) $$x_1 \sim x_2,\; y_1 \sim y_2 \Rightarrow (x_1, y_1) \sim (x_2, y_2);$$

2) $$(x, y_1) \sim (x, y_2) \Leftrightarrow y_1 \sim y_2;$$

3) $$(x_1, y) \sim (x_2, y) \Leftrightarrow x_1 \sim x_2.$$

Next, let $(x_1, y_1) \sim (x_2, y_2)$ $(x_1 \ne x_2, y_1 \ne y_2)$. Then for every $x \in V(G_1)$, either x is adjacent to x_1 and x_2 or x is non-adjacent to both x_1, x_2, so that there is at least one $x \in V(G_1)$ adjacent to x_1, x_2, which implies that $y_1 \sim y_2$, and x_1 is non-adjacent to x_2 (because x_1 is non-adjacent to itself).

Similarly, we conclude that $x_1 \sim x_2$, whence we obtain

4) $$(x_1, y_1) \sim (x_2, y_2) \Leftrightarrow x_1 \sim x_2, y_1 \sim y_2 \;\; (\text{if } \; x_1 \ne x_2, y_1 \ne y_2).$$

Relations 1)–4) immediately imply that the subsets P_{ij} are the characteristic subsets of $G_1 \times G_2$, thus $k(G_1 \times G_2) = k_1 \cdot k_2$.

Further, assume that at least one of the graphs G_1, G_2 has an infinite spectrum. Then similarly one concludes that the products of the characteristic subsets in G_1, G_2 are characteristic subsets of $G_1 \times G_2$, hence $G_1 \times G_2$ must have an infinite type. $\square$

(vii) *The sum of graphs*

The sum $G_1 + G_2$ of two infinite graphs $G_1 G_2$ is the infinite graph whose vertices are the pairs (x, y) $(x \in V(G_1), y \in V(G_2))$, with two pairs $(x_1, y_1), (x_2, y_2)$ being adjacent if and only if $x_1 = x_2$ and y_1 is adjacent to y_2, or $y_1 = y_2$ and x_1 is adjacent to x_2.

Obviously, $G_1 + G_2$ is connected if G_1, G_2 are so.

For the sum of graphs we obtain a surprising results.

THEOREM 7. *If G_1, G_2 are connected infinite graphs, then the sum $G_1 + G_2$ is a graph with an infinite spectrum.*

Proof. We prove that $G_1 + G_2$ is always a graph of an infinite type, and moreover every of its characteristic subsets consists of one element only.

Let two distinct vertices $(x_1, y_1), (x_2, y_2)$ be equivalent in $G_1 + G_2$. Then if (x, y) is adjacent to (x_1, y_1), it must be adjacent to (x_2, y_2) too.

We prove that in this case $x_1 = x_2$, $y_1 = y_2$, contradiction. Indeed, let for instance $x_1 \neq x_2$. Then there is at least one $y_0 \in V(G_2)$ which is adjacent to $y_1 (y_0 \neq y_2)$, or an y_0 adjacent to $y_2 (y_0 \neq y_1)$.

In the first case, we would have that (x_1, y_0) is adjacent to (x_1, y_1), thus (x_1, y_0) is adjacent to (x_2, y_2), which (because $x_1 \neq x_2$) implies $y_0 = y_2$, a contradiction. Thus $x_1 = x_2$.

Similarly, $y_1 = y_2$, which is impossible.

Hence, in $G_1 + G_2$ the characteristic subsets must be singletons, so $G_1 + G_2$ always has an infinite type, and in view of connectedeness, an infinite spectrum. □

(viii) *The line graph of a graph*

If G is an arbitrary infinite graph, then $L(G)$, the line graph of G, is the graph whose vertices are the edges of G, with two vertices being adjacent in $L(G)$ iff the corresponding edges of G have exactly one vertex in common.

$L(G)$ is connected if G is so.

Next, if $x \in V(G)$ is an arbitrary vertex of G, let $d(x) \leq +\infty$ be its degree, i.e. the number of vertices adjacent to x in G.

THEOREM 8. *The line graph $L(G)$ of a connected infinite graph G has, for any labeling of its vertices, an infinite spectrum.*

Proof. Assume to the contrary, that $L(G)$ has the finite type m, i.e. $L(G) = g(\overline{N}_1, \ldots, \overline{N}_m)$. Then at least one among subsets $\overline{N}_i$ (say $\overline{N}_1$) must be infinite, which (because $m > 1$) implies that there is some $f \in \overline{N}_j$ adjacent to all edges $h \in \overline{N}_1$.

Hence, there is a vertex $x \in G$ whose degree $d(x) = +\infty$. Thus, there is a complete infinite induced subgraph in $L(G)$, which contradicts the assumption that $L(G)$ is of finite type. □

(ix) *The total graph of a graph*

The total graph $T(G)$ of an infinite graph G is the graph whose vertices are the vertices and the edges of the graph G, with two elements being adjacent in $T(G)$ iff they are adjacent or incident in G.

It is obviously connected, if G is so.

THEOREM 9. *The total graph $T(G)$ of a connected infinite graph G has, for any Labeling of its vertices, an infinite spectrum.*

Proof. Since the line graph $L(G)$ of G is obviously a (relabeled) induced subgraph of the total graph $T(G)$, Theorems 2 and 8 provide the proof. □

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(Received 07 01 1981)