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RANDOM LIFETIMES IN A TWO-COMPONENT SYSTEM

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Abstract. Partly generalizing a model of Marshall and Olkin (1967) we consider the "cumulative" and "ultimately fatal" effects of shocks on a two-component system, according to a scheme hypothetically representative of physical or biological phenomenons.

1. Introduction

Marshall and Olkin have considered the following fatal shock model: the components of a two-component system fail after receiving a shock which is always fatal. Three independent Poisson processes govern the occurrence of the shocks with parameters λ_1, λ_2 and λ_{12} respectively. Events in the first process are shocks only to the first component, events in the second process are shocks only to the second component and events in the third process are shocks to both components.

In this paper we consider two models A and B which generalize the above fatal shock model in the following ways :

A: the first component fails after receiving $r(\geq 1)$ shocks governed by the same Poisson process and analogously the second one after $s\,(\geq 1)$ shocks;

B: the first component fails after receiving $r(\geq 1)$ shocks cumulatively (not necessarily belonging to the same Poisson process) and the second on after $s(\geq 1)$ shocks cumulatively.

In both cases the occurrence of the shocks is governed again by the above three independent Poisson processes.

2. The distribution function for the model A

Let X and Y be the random lifetimes of the first and second component

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respectively; then, for $r \leq s$

$$\begin{aligned}\overline{F}(x,y) &= pr(X > x, Y > y) = \\ &= \exp\{-\lambda_1 x - \lambda_2 y - \lambda_{12}\max(x,y)\}\sum_{k=0}^{r-1}(\lambda_1^k x^k/k!)\sum_{h=0}^{s-1}(\lambda_2^h y^h/h!)\cdot \\ &\cdot\sum_{i=0}^{r-1}(\lambda_{12}^i x^i/i!)\sum_{j=0}^{s-1-i}[\lambda_{12}^j\{\max(x,y)-x\}^j/j!].\end{aligned} \tag{1}$$

In the case $r > s$, $\overline{F}(x,y)$ is obtainable from (1) if we put there r, λ_1, x instead of s, λ_2, y respectively and viceversa.

In particular, since the following relation holds

$$\sum_{i=0}^{r-1}(\lambda_{12}^i x^i/i!)\sum_{j=0}^{r-1-i}\{\lambda_{12}^j(y-x)^j/j!\} = \sum_{i=0}^{r-1}(\lambda_{12}y^i/i!), \quad 0 \leq x \leq y \tag{2}$$

it follows from (1) for $r = s$

$$\begin{aligned}\overline{F}(x,y) = \exp\{-\lambda_1 x - \lambda_2 y - \lambda_{12}\max(x,y)\}\sum_{k=0}^{r-1}(\lambda_1^k x^k/k!)\sum_{h=0}^{r-1}(\lambda_2^h y^h/h!)\cdot \\ \cdot\sum_{i=0}^{r-1}[\lambda_{12}^i\{\max(x,y)\}^i/i!].\end{aligned}$$

$\overline{F}(x,y)$ as given in (1), is an absolutely continuous distribution for $r \neq s$, while for $r = s$ it has both an absolutely continuous and a singular part.

Precisely, we have for $r = s$

$$\overline{F}(x,y) = \alpha\overline{F}_a(x,y) + (1-\alpha)\overline{F}_s(x,y), \tag{4}$$

where

$$\alpha = 1 - \sum_{k=0}^{r-1}\sum_{h=0}^{r-1}[\lambda_1^k\lambda_2^h\lambda_{12}^r(k+h+r-1)!/\{\lambda^{k+h+r}k!h!(r-1)!\}], \quad (\lambda = \lambda_1 + \lambda_2 + \lambda_{12})$$

and

$$\begin{aligned}\overline{F}_s(x,y) = \{1/(1-\alpha)\}\exp\{-\lambda\max(x,y)\}\sum_{k=0}^{r-1}\sum_{h=0}^{r-1}[\lambda_1^k\lambda_2^h\lambda_{12}^r(k+h+r-1)!/ \\ /\{\lambda^{k+h+r}k!h!(r-1)!\}]\sum_{i=0}^{r-1+k+h}[\lambda^i\{\max(x,y)^i/i!]\end{aligned}$$

is a singular distribution.

3. The distribution function for the model B

Let us introduce the following notation

$$p(n;\gamma,t)=\exp(-\gamma t)(\gamma t)^n/n! \tag{5}$$

We have now for $1\le r\le s$ and $0\le x\le y$

$$\begin{aligned}\overline{F}(x,y)&=\sum_{k=0}^{r-1}p(k;\lambda_1,x)\sum_{h=0}^{s-r+k}p(h;\lambda_2,y)\sum_{i=0}^{r-1-k}p(i;\lambda_{12},x)\sum_{j=0}^{s-1-h-i}p(j;\lambda_{12},y-x)+\\&+\sum_{k=0}^{r-2}p(k;\lambda_1,x)\sum_{h=s-r+k+1}^{s-1}p(h;\lambda_2,y)\sum_{i=0}^{s-1-h}p(i;\lambda_{12},x)\sum_{j=0}^{s-1-h-i}p(j;\lambda_{12},y-x)\end{aligned} \tag{6}$$

where the second sum on the right hand side is equal to zero for $r=1$.

To prove (6), note that for $1\le r\le s$ and $0\le x\le y$:

if $0\le k\le r-1$ and $r-k\le s-h$, that is $0\le h\le s-r+k$, $(r-1-k)$ events at most of the third process can occur in the time interval $(0,x)$ and $[(s-1-h)-\{\text{number of events in }(0,x)\}]$ events at most of this process can occur in the time interval (x,y);

if $0\le k\le r-2$ and $r-k>s-h$, that is $s-r+k+1\le h\le s-1$, $(s-1-h)$ events at most of the third process can occur in the time interval $(0,x)$ and $[(s-1-h)-\{\text{number of events in }(0,x)\}]$ events at most of this process can occur in the time interval (x,y).

Similarly, we have for $1\le r\le s$ and $0\le y\le x$

$$\begin{aligned}\overline{F}(x,y)&=\sum_{k=0}^{r-1}p(k;\lambda_1,x)\sum_{h=0}^{s-r+k}p(h;\lambda_2,y)\sum_{i=0}^{r-1-k}p(i;\lambda_{12},y)\sum_{j=0}^{r-1-k-i}p(j;\lambda_{12},x-y)+\\&+\sum_{k=0}^{r-2}p(k;\lambda_1,x)\sum_{h=s-r+k+1}^{s-1}p(h;\lambda_2,y)\sum_{i=0}^{s-1-h}p(i;\lambda_{12},y)\sum_{j=0}^{r-1-k-i}p(j;\lambda_{12},x-y),\end{aligned} \tag{7}$$

where the second sum on the right hand side is equal to zero for $r=1$.

Consequently, by combining (6) and (7) and making use of the formulas (2), (5) we have for $1\le r\le s$

$$\begin{aligned}\overline{F}(x,y)=\exp\{-\lambda_1x-\lambda_2y-\lambda_{12}\max(x,y)\}&\Bigg\{\sum_{k=0}^{r-1}(\lambda_1^kx^k/k!)\sum_{h=0}^{s-r+k}(\lambda_2^hy^h/h!)\cdot\\&\cdot\sum_{i=0}^{r-2}(\lambda_{12}^ix^i/i!)\sum_{j=0}^{s-1-h-i}[\lambda_{12}^j\{\max(x,y)-x\}^j/j!]+\\&+\sum_{k=0}^{r-2}(\lambda_1^kx^k/k!)\sum_{h=s-r+k+1}^{s-1}(\lambda_2^hy^h/h!)\sum_{i=0}^{s-1-h}(\lambda_{12}^iy^i/i!)\cdot\\&\cdot\sum_{j=0}^{r-1-k-i}[\lambda_{12}\{\max(x,y)-y\}^j/j!]\Bigg\},\end{aligned} \tag{8}$$

where the second sum on the right hand side is equal to zero for $r = 1$, in which case for $s = 1$, $\overline{F}(x, y)$ becomes the bivariate exponential distribution of Marshall and Olkin.

In the case $1 \leq s \leq r$ the distribution $\overline{F}(x, y)$ is easily obtainable from (8) by symmetry.

For $r = s$ the distribution (8) has both an absolutely continuous and a singular part; that is, the decomposition (4) holds with

$$\alpha = 1 - \sum_{k=0}^{r-1} [\lambda_1^k \lambda_2^k \lambda_{12}^{r-k} (r + k - 1)!/\{\lambda^{r+k} k! k! (r - k - 1)!\}]$$

and

$$\overline{F}_s(x, y) = \{1/(1 - \alpha)\} \exp\{-\lambda \max(x, y)\} \sum_{k=0}^{r-1} [\lambda_1^k \lambda_2^k \lambda_{12}^{r-k} (r + k - 1)!/\{\lambda^{r+k} k! k! \cdot \\ \cdot (r - k - 1)!\}] \sum_{i=0}^{r-1-k} [\lambda^i \{\max(x, y)\}^i / i!].$$

The moment generating function exists for both distributions (1) and (7), but we not report here the tedious calculation of it.

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