PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 33 (47), 1983, pp. 197–202

ANOTHER NOTE ON CAUCHY-REGULAR FUNCTIONS

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In [**11**] and [**12**], R. F. Snipes points out that there is a class of mappings strictly between the classes of continuous and of uniformly continuous mappings with very interesting properties and important applications. In 1977 we independently found essentially the same results. It is the purpose of this note to present some complements to [**12**].

Motivated by the properties of Cauchy-regular functions stated in sections 1 and 3 of [**12**], we call a mapping f from a uniform space $(X, \mathfrak{U})$ into a uniform space $(Y, \mathfrak{B})$ *Cauchy-regular* or a *Cauchy morphism* if it preserves Cauchy filterbases. A mapping f is called a *Cauchy isomorphism* if it is bijective and if both f and f^{-1} are Cauchy morphisms.

1. Cauchy morphisms and Cauchy nets

We are interested in three constructions connecting filterbases and nets:

(i) If $(x_\delta)_{\delta\in D}$, sometimes abbreviated by (x_δ), is a net in the set X, i.e., $x_\delta \in X$ for every $\delta \in D$, then the sets $B_\delta := \{(x_\alpha; \alpha \in D, \delta \leq \alpha\}$ form the so-called *corresponding filterbase* of (x_δ).

(ii) Let ,$\mathfrak{L}$ be a filterbase on the set X and $D := \{(x, B); x \in B \in \mathfrak{L}\}$. Together with the relation $\leq$ defined by

$$(x_1, B_1), (x_2, B_2) \in D;\ \ (x_1, B_1) \leq (x_2, B_2) :\Leftrightarrow B_1 \supset B_2,$$

$(D, \leq)$ becomes a directed set. The net $(y_\delta)_{\delta\in D}$ with $y_\delta = x$ for $\delta = (x, B) \in D$ is said to be the *canonical net* of $\mathfrak{L}$. It turns out that $\mathfrak{L}$ is the corresponding filterbase of its canonical net (cf. [**7**, p. 83, Problem L, (f), (ii)] or [**14**, p. 41, Example 6]. In general, $\leq$ is not antisymmetric. For a modified procedure ensuring antisymmetry cf. [**4**, p. 171, 172].

AMS Subject Classification (1980): 54E15
Key words and phrases: Uniform space.

(iii) If $\mathfrak{L}$ is a filterbase on the set X and, for every $B \in \mathfrak{L}$, an element x_B is chosen arbitrarily from B, then the net $(x_B)_{B\in\mathfrak{L}}$ with the directed domain $(\mathfrak{L}, \supset)$ is called a *net associated with* $\mathfrak{L}$.

For uniform spaces $(X, \mathfrak{U})$ Cauchy nets make sense ([**7**, p. 190] or [**14**, p. 217, Problem 14]), and we get

(1) *The net (x_δ) in X is Cauchy if and only if its corresponding filterbase is Cauchy,*

and from (1) and (ii) furthermore

(2) *The filterbase $\mathfrak{L}$ on X is Cauchy if and only if its canonical net is Cauchy.*

It seems now quite natural to ask whether Cauchy filterbases could be characterized also in terms of their associated nets. (For similar statements concerning convergent filterbases and nets or ultra-filterbases and universal nets see [**13**, p. 221–222], or [**14**, p. 41, Example 5], or [**1**].)

Lemma 1. *Let X be a non-empty set, $(Y, \mathfrak{B})$ a uniform space, $f: X \to Y$, and $\mathfrak{L}$ a filterbase on X with no smallest member. If, for every net $(x_B)_{B\in\mathfrak{L}}$ associated with $\mathfrak{L}$, $(fx_B)_{B\in\mathfrak{L}}$ is a Cauchy net Y, then $f(\mathfrak{L})$ is a Cauchy filterbase on Y.*

Proof. Assume that $f(\mathfrak{L})$ is not a Cauchy filterbase. Then there exists a connector $V_0 \in \mathfrak{B}$ such that $f(B) \times f(B) \not\subset V_0$ for every $B \in \mathfrak{L}$, i.e., such that there exist $z_B, z'_B \in f(B)$ with the property

$$(z_b, z'_B) \notin V_0 \text{ for every } B \in \mathfrak{L}. \tag{3}$$

Let V_1 be a connector in $\mathfrak{B}$ satisfying

$$V_1 \circ V_1 \subset V_0. \tag{4}$$

Since $\mathfrak{L}$ has no smallest member, every B in $\mathfrak{L}$ has at least two proper successors in $\mathfrak{L}$ with respect to the relation $\supset$. By [**13**, p. 2l7, Theorem 1] there are cofinal subsets $\mathfrak{L}_1, \mathfrak{L}_2$ of $\mathfrak{L}$ satisfying $\mathfrak{L}_1 \cap \mathfrak{L}_2 = \emptyset$, $\mathfrak{L}_1 \cup \mathfrak{L}_2 = \mathfrak{L}$. For each B in $\mathfrak{L}$ we define y_B by

$$y_B := z_B \text{ if } B \in \mathfrak{L}_1, \quad y_B := z'_B \text{ if } B \in \mathfrak{L}_2. \tag{5}$$

If $x_B \in B$ such that $f(x_B) = z_B$ for every $B \in \mathfrak{L}$?, then $(z_B)_{B\in\mathfrak{L}}$ is a Cauchy net by hypothesis, hence there exists a $B_0 \in \mathfrak{L}$ such that

$$B_1, B_2 \in \mathfrak{L};\ B_0 \supset B_1, B_0 \supset B_2 \text{ imply } (z_{B_1}, z_{B_2}) \in V_1. \tag{6}$$

If $w_B \in B$ such that $f(w_B) = y_B$ for every $B \in \mathfrak{L}_1$?, then $(y_B)_{B\in\mathfrak{L}}$ is a Cauchy net for the same reason, hence there exists a $C \in \mathfrak{L}$ such that

$$B_1, B_2 \in \mathfrak{L};\ C \supset B_1, C \supset B_2 \text{ imply } (y_{B_1}, y_{B_2}) \in V_1. \tag{7}$$

Since $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are cofinal in $\mathfrak{L}$ there exist a $B_1 \in \mathfrak{L}_1$ and a $B_2 \in \mathfrak{L}_2$ such that $B_0 \cap C \supset B_1$, $B_0 \cap C \supset B_2$. From (6), (7) we get $(z_{B_2}, z_{B_1}) \in V_1$, $(y_{B_1}, y_{B_2}) \in V_1$ i.e., by virtue of (5) $(z_{B_1}, z_{B_2}) \in V_1$, i.e., by (4) $(z_{B_2}, z'_{B_2}) \in V_1 \circ V_1 \subset V_0$, contradicting (3).

COROLLARY 2. *Let $\mathfrak{L}$ be a filterbase on the uniform space $(X, \mathfrak{U})$. Then:* a) *If $\mathfrak{L}$ is Cauchy, so is any net associated with $\mathfrak{L}$.* b) *If $\mathfrak{L}$ has a smallest member, the converse of* a) *is false.* c) *If $\mathfrak{L}$ has no smallest member, then $\mathfrak{L}$ is a Cauchy filterbase if and only if every net associated with $\mathfrak{L}$ is a Cauchy net.*

Proof. a) is well known (cf. [**14**, p. 217, Problem 15]). b) For the usual uniformity on R, the filterbase $\mathfrak{L}$ consisting of the interval $[-1,1]$ only is not Cauchy. On the other hand, any net the domain of which has a last element is trivially eventually constant, hence Cauchy. c) The "only if" part follows from a), and the "if" part from Lemma 1 when we put $X = Y$, $\mathfrak{U} = \mathfrak{B}$, and $f = \mathrm{id}_x$.

THEOREM 3. *A mapping f from a uniform space $(X, \mathfrak{U})$ into a uniform space $(Y, \mathfrak{B})$ is a Cauchy morphism if and only if it preserves Cauchy nets.*

Proof. 1) Let f be a Cauchy morphism and (x_δ) a Cauchy net in X. By (1), the corresponding filterbase $\mathfrak{L}$ of (x_δ) is Cauchy, hence $f(\mathfrak{L})$ is Cauchy. Since $f(\mathfrak{L})$ is the corresponding filterbase of the net $(f(x_\delta))$, the latter is Cauchy by (1). Thus f preserves Cauchy nets. – 2) Let f be Cauchy net preserving and $\mathfrak{L}$ a Cauchy filterbase on X. We may express this information about in two ways, namely (a) by its associated nets, or (b) by its canonical net, and, accordingly, we get two different proofs which we now sketch.

(a) Case 1: $\mathfrak{L}$ have no smallest member. Let $(x_B)_{B \in \mathfrak{L}}$ be a net associated with $\mathfrak{L}$. By Corollary 2a), (x_B) is a Cauchy net in X, hence $(f(x_B))$ is a Cauchy net in Y. By virtue of Lemma 1, $f(\mathfrak{L})$ is a Cauchy filterbase on Y. So f is a Cauchy morphism. – Case 2: $\mathfrak{L}$ possess a smallest member B_0. If $f(\mathfrak{L})$ were not Cauchy, there would exist $V_0 \in \mathfrak{B}$ and $x_1, x_2 \in B_0$ such that $(f(x_1), f(x_2)) \notin V_0$. For $x_n := x_1$ (n odd), $x_n := x_2$ (n even) $(x_n)_{n \in N}$ becomes a Cauchy net since $\mathfrak{L}$ is Cauchy. But for every odd n, $(f(x_n), f(x_{n+1})) \notin V_0$, hence $(f(x_n))$ is not Cauchy, contradicting the hypothesis.

(b) By (2), the canonical net $(y_\delta)_{\delta \in D}$ of $\mathfrak{L}$ is Cauchy, therefore $(f(y_\delta))$ is Cauchy. We show now that the canonical net $(w_\gamma)_{\gamma \in C}$ of $f(\mathfrak{L})$ is Cauchy. (If (w_γ) were a subnet of $(f(y_\delta))$, the proof would be already complete; but in general (w_γ) is not a subnet of $(f(y_\delta))$, thus some more effort is necessary). For any $V \in \mathfrak{B}$ there exists a $\delta_0 := (x_0, B_0) \in D$ such that $\delta_1, \delta_2 D$; $\delta_0 \le \delta_1$, $\delta_0 \le \delta_2$, imply $(f(y_{\delta_1}), f(y_{\delta_2})) \in V$. Let $\gamma_0 := (f(x_0), f(B_0))$. We have that $x_0 \in B_0 \in \mathfrak{L}$ implies $f(x_0) \in f(B_0) \in f(\mathfrak{L})$, i.e. $\gamma_0 \in C := \{(z, f(B)); z \in f(B) \in f(\mathfrak{L})\}$. Let $\gamma_1, \gamma_2 \in C$ be such that $\gamma_0 \le \gamma_1, \gamma_0 \le \gamma_2$ say $\gamma_1 = (z_1, f(B_1))$ and $\gamma_2 = (z_2, f(B_2))$. It follows that $z_1 \in f(B_1) \subset f(B_0)$, $z_2 \in f(B_2) \subset f(B_0)$, and there exist $x_1, x_2 \in B_0$ with the property $f(x_1) = z_1$, $f(x_2) = z_2$. For $\delta_1 := (x_1, B_0)$ and $\delta_2 := (x_2, B_0)$ we obtain $\delta_1, \delta_2 \in D$; $\delta_0 \le \delta_1$, $\delta_0 \le \delta_2$, i.e., $(f(y_{\delta_1}), f(y_{\delta_2})) \in V$, and by (ii) we get $(w_{\gamma_1}, w_{\gamma_2}) = (z_1, z_2) = (f(x_1), f(x_2)) = (f(y_{\delta_1}), f(y_{\delta_2})) \in V$. Hence (w_γ) is a Cauchy net, and by (2) we have that $f(\mathfrak{L})$ is a Cauchy filterbase; thus f is a Cauchy morphism.

2. Miscellaneous Remarks

Remark 1. There is a notion of boundedness of subsets of uniform spaces for which we have

$$\text{(8)} \qquad \textit{compact} \not\Leftarrow\Rightarrow \textit{precompact} \not\Leftarrow\Rightarrow \textit{bounded}$$

[**14**, p. 219, Problems 203, 205, 206]. Uniformly continuous mappings preserve bounded sets, continuous mappings preserve compact sets, and Cauchy morphisms preserve precompact sets; the latter follows from [**12**, p. 19, Theorem 3] and [**14**, p. 217, Problem 9], or via a filter-theoretic characterization of precompactness [**14**, p. 215, Theorem 11.3.6; p. 216, Problem 5]. Thus we have a satisfactory corresponence between (8) and

$$\textit{uniformly continuous} \not\Leftarrow\Rightarrow \textit{Cauchy morphism} \not\Leftarrow\Rightarrow \textit{continuous.}$$

Remark 2. *Completeness is preserved under Cauchy isomorphisms.* The steps of a simple proof are: $\mathfrak{L}'$ Cauchy filterbase on Y, $f^{-1}(\mathfrak{L}')$ Cauchy on X, $f^{-1}(\mathfrak{L}')$ convergent in X, $ff^{-1}(\mathfrak{L}')$ convergent in Y. On the other hand, bijective mappings $f : X \to Y$ with f uniformly continuous and f^{-1} continuous need not preserve completeness as the example $f : R \to]-1,1[$, $f(x) := x/(1+|x|)$ $(x \in R)$ shows (R and $]-1,1[$ equipped with the usual uniformities).

Remark 3. By $CFB(\mathfrak{U})$, and $CN(\mathfrak{U})$ we denote the set of all Cauchy filterbases on the uniform space $(X,\mathfrak{U})$ and the set of all Cauchy nets in the uniform space $(X,\mathfrak{U})$, respectively. If $\mathfrak{U}_1$ and $\mathfrak{U}_2$ are uniformities on X, Theorem 3 applied for $f = \mathrm{id}_X$ yields

$$CFB(\mathfrak{U}_1) \subset CFB(\mathfrak{U}_2) \Leftrightarrow CN(\mathfrak{U}_1) \subset CN(\mathfrak{U}_2).$$
$$CFB(\mathfrak{U}_1) = CFB(\mathfrak{U}_2) \Leftrightarrow CN(\mathfrak{U}_1) = CN(\mathfrak{U}_2).$$

In the latter case, $\mathfrak{U}_1$ and $\mathfrak{U}_2$ are called Cauchy equivalent [**14**, p. 217, Problem 17]. For this and further aspects of the comparison of uniformities cf. also [**5**, p. 53, Problem 3], [**8**, p. 168], [**10**, p. 103, Beispiel 1].

Remark 4. [**12**, p. 21, Theorem 5]. The fact that a Cauchy morphism $f : D \to Y$ has an extension $\overline{f} : \overline{D} \to \overline{Y}$ which is also a Cauchy morphism holds also for non-Hausdorff spaces $(Y,\mathfrak{B})$, but $\overline{f}$ is no longer uniquely determined. On the other hand, simple examples show that completeness of Y and denseness of D in $\overline{D}$ are essential for the extension theorem.

3. Cauchy morphisms in connection with topological groups

The left and right uniformities of the topological group $(X,\cdot)$ are denoted by $\mathfrak{M}_l$ and $\mathfrak{M}_r$ respectively.

Remark 5. The group operation $(x_1,x_2) \to x_1 \cdot x_2$ of any topological group $(X,\cdot)$ is a Cauchy morphism from $(X,\mathfrak{M}_l) \times (X,\mathfrak{M}_l)$ into $(X,\mathfrak{M}_l)$ and also from $(X,\mathfrak{M}_r) \times (X,\mathfrak{M}_r)$ into $(X,\mathfrak{M}_r)$ (cf. [**2**, §3, Proposition 6] or [**14**, p. 256, Lemma 12.2.4]). It is uniformly continuous if and only if $\mathfrak{M}_l = \mathfrak{M}_r$ [**2**, §3, Exercise 3].

Therefore the possible difficulties in the completion problem of topological groups are never caused by the group operation.

Remark 6. For uniform continuity of the inversion mapping $x \to x^{-1}$ with respect to different uniformities cf. [**14**, p. 258, Problem 105]. It is an $\mathfrak{M}_l$–$\mathfrak{M}_l$-Cauchy morphism ($\mathfrak{M}_r$–$\mathfrak{M}_r$-Cauchy morphism) if and only if $\mathfrak{M}_l$, and $\mathfrak{M}_r$ are Cauchy equivalent [**14**, p. 258, Problem 2]. Therefore this last condition is crucial for the existence of a group completion for $(X, \mathfrak{M}_l)$ and for $(X, \mathfrak{M}_r)$ [**2**, §3, Théorème 1]. For a famous example where the condition is violated see [**14**, p. 255, Example 3].

Remark 7. Every continuous homomorphism from a topological group $(X, \cdot)$ into a topological group $(Y, \cdot)$ is uniformly continuous with respect to the left (right) uniformities [**14**, p. 252, Theorem 12.2.2], hence a Cauchy morphism. But $\mathrm{id}_x : (X, \mathfrak{M}_l) \to (X, \mathfrak{M}_r)$ is a Cauchy morphism if and only $CFB(\mathfrak{M}_l) \subset CFB(\mathfrak{M}_r)$.

Remark 8. If $(X+1, \cdot)$, $(X_2, \cdot)$, $(Y, \cdot)$ are groups, a mapping $f : X_1 \times X_2 \to Y$ is called a *bimorphism* if $f(x_1 \cdot x_1', x_2) = f(x_1, x_2) \cdot f(x_1', x_2)$ and $f(x_1, x_2 \cdot x_2') = f(x_1 x_2) \cdot f(x_1, x_2')$ hold for all $x_1 x_1' \in X_1$; $x_2, x_2' \in X_2$. It is easily seen that any two elements of $f(X_1 \times X_2)$ must then commute: Note that $f(X_1 \times X_2)$ need not be a subgroup of Y [**9**, p. 194]. Nevertheless, for the sake of simplicity and because we still cover most important examples, we assume from now on that Y is commutative, and we write $+$ instead of $\cdot$ in Y. Accordingly, a bimorphism is a *biadditive mapping.*

Theorem 4. *If $(X_1, \cdot)$, $(X_2, \cdot)$, $(Y, +)$ are topological groups, Y commutative, with right uniformities $\mathfrak{M}_1$, $\mathfrak{M}_2$, $\mathfrak{N}$, respectively, then every continuous biadditive mapping $f : X_1 \times X_2 \to Y$ is a Cauchy morphism with respect to $\mathfrak{M}_1$, $\mathfrak{M}_2$, $\mathfrak{N}$.*

For commutative and separated X_1 and X_2 see [**2**, §6 Théorème 1]. That proof may be adapted to our more general situation.

Remark 9. Theorem 4 and Remark 4 open the way to the completions of topological rings, topological modules, and inner product spaces, in general in the absence of uniform continuity (cf. [**2**, §6, Nos. 5 and 6] and [**12**, p. 24/25]). Remark 5 and Theorem 4 also have obvious consequences for combining Cauchy morphisms with Cauchy morphisms by algebraic operations (cf. [**12**, end of section 1, and p. 23, Proposition 8]).

Corollary 5. *If, in the situation of Theorem 4, A_1 and A_2 are precompact subsets of X_1 and X_2, respectively, then the restriction of f to $A_1 \times A_2$ is uniformly continuous* (for a special case cf. [**3**, §1, N0. 4, Remarque 2]).

This follows from $A_1 \times A_2$ precompact [**14**, p. 227, Problem 122], Theorem 4, the restriction property of Cauchy morphisms, and [**12**, p. 19, Theorem 3].

Corollary 6. *Let $(X, \cdot)$, $(Y, +)$ be topological groups with right uniformities $\mathfrak{M}$, $\mathfrak{N}$, respectively, and let $(Y, +)$ be commutative. Let $\varphi : Y \to Y$, defined by $\varphi(y) = 2y$ $(y \in Y)$, be bijective and φ^{-1} continuous. If $q : X \to Y$ is continuous* *and satisfies the functional equations*

$$\text{(Q)} \qquad q(x_1 \cdot x_2) + q(x_1 \cdot x_2^{-1}) = 2q(x_1) + 2q(x_2) \quad (x_1, x_2 \in X),$$

$$\text{(C)} \qquad q(x_1 \cdot x_2 \cdot x_3) = q(x_2 \cdot x_1 \cdot x_3) \quad (x_1, x_2, x_3 \in X),$$

then g is a Cauchy morphism.

Proof. By [bf 6, p. 193, Theorem 3], the mapping $f : X \times X \to Y$ defined by

$$\text{(P)} \qquad f(x_1, x_2) = \varphi^{-1}[q(x_1 \cdot x_2) - q(x_1) - q(x_2)] \quad (x_1, x_2 \in X)$$

is biadditive. From the hypotheses we conclude that f is continuous, hence a Cauchy motphism by Theorem 4. From (Q) and the hypothesis on φ we get $q(x^2) = 4q(x)$, i.e.

$$\text{(D)} \qquad f(x, x) = q(x) \ \text{ for every } \ x \in X.$$

Since Cauchy morphisms behave nicely under composition and the formation of mappings into product spaces [**12**, p. 23 Proposition 8], q is a Cauchy morphism.

Remark 10. Condition (C) is trivially satisfied in the case of a commutative group $(X, \cdot)$. (C) is also necessary for f and q to be connected by the formulae (P) and (D) (cf. [**6**]). A solution of (Q) is called a quadratic functional, and now Corollary 6 and Remark 4 provide an extension theorem for continuous quadratic functionals.

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(Received 29 10 1981)