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1-FACTORIZATION OF THE COMPOSITION OF REGULAR GRAPHS

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Abstract. 1-factorability of the composition of graphs is studied. The followings sufficient conditions are proved: $G[H]$ is 1-factorable if G and H are regular and at least one of the following holds: (i) Graphs G and H both contain a 1-factor, (ii) G is 1-factorable (iii) H is 1-factorable. It is also shown that the tensor product $G \otimes H$ is 1-factorable, if at least one of two graphs is 1-factorable. This result in turn implies that the strong tensor product $G \otimes' H$ is 1-factorable, if G is 1-factorable.

1.0 Introduction. The source of inspiration for this paper is rightfully Kotzig's [**3**]. His simple sufficient conditions for the cartesian product of graphs to be 1-factorable naturally raise the question of when other well known products of graphs are 1-factorable.

In this paper we give analogous results for the composition of graphs and partial results for the tensor and strong tensor products, which extend those announced in [**5**].

We will leave the basic definitions of graph theory to any standard textbook, for example Harary's Graph Theory [**2**], and will limit ourselves to defining only lesser known terms and those which may cause confusion.

2.0 Definitions. If u and v are adjacent vertices of a graph, then we write $u \sim v$ and denote with uv the edge joining them.

For a graph G, let $V(G)$ denote the vertex set of G and $E(G)$ denote its edge set.

The composition, also known as the lexicographical product, of graphs G and H is defined as the graph $G[H]$ with the vertex set $V(G[H]) = V(G) \times V(H)$ and the edge set $E(G[H]) = \{(u,v)(u',v')$: either $(u = u'$ and $v \sim v')$ or $u \sim u'\}$.

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The tensor product of graphs G and H is defined as the graph $G \otimes H$ with vertex set $V(G) \times V(H)$ and the edge set

$$E(G \otimes H) = \{(u,v)(u',v') : u \sim u' \text{ and } v \sim v'\}$$

If by $\deg(v)$ we denote the degree of a vertex v, then for (u,v) a vertex in $G \otimes H$ we have $\deg((u,v) = \deg(u) \cdot \deg(v)$. Thus if G and H are regular, so is $G \otimes H$.

The graph $G\{m\}$ is defined as $G \otimes K_m$, where K_m is the complete graph on m vertices.

If G and H have the same vertex set $V = V(G) = V(H)$, and disjoint edge sets, $E(G) \cap E(H) = \emptyset$, then the sum $G \otimes H$ is the graph having the vertex set $V(G \oplus E(H) = V$ and the edge set $E(G \oplus H) = E(G) \cup E(H)$.

Several authors have defined $G(m)$ as $G[mK_1]$ [**1**,**4**]. Mohar and Pisanski studied 1-factorability of $G(m)$ in [**4**]. Here we only note that $G(m)$ and $G\{m\}$ are connected by the relation $G(m) = G\{m\} \oplus mG$.

The graph $G[H]$ can be expressed as the sum of the standard cartesian product $G \times H$ and the graph $G\{|V(H)|\}$:

$$G[H] = (G \times H) \oplus G\{|V(H)|\}.$$

If G is the sum of a series of graphs:

$$G = F_1 C \oplus F_2 \oplus \cdots \oplus F_k,$$

we can readily verify the following results:

$$G = F_1\{m\} \oplus F_2\{m\} \oplus \cdots \oplus F_k\{m\},$$
$$G \otimes H = (F_1 \otimes H) \oplus (F_2 \otimes H) \oplus \cdots \oplus (F_k \otimes H).$$

If each graph F_i is d-factorable, it is also clear that G is d-factorable, as it can be written as the sum of all the d-factors of the F_i.

The strong tensor product $G \otimes' H$ is defined on the vertex set $V(G) \times V(H)$ as

$$G \otimes' H = (G \otimes H) \oplus (G \times \{v_1\} \cup G \times \{v_2\} \cup \cdots \cup G \times \{v_m\}),$$
$$\text{where } V(H) = \{v_1, v_2, \ldots, v_m\}.$$

3.0 Known results. We first restate in our own words Kotzig's result for the cartesian product of regular graphs.

3.1 THEOREM (Kotzig 1979, [**3**]): *If G and H are two regular graphs for which at least one of the following conditions holds:*

(i) *Both G and H contain 1-factor,*

(ii) *G is 1-factorable,*

(iii) *H is 1-factorable,*
then the cartesian product $G \times H$ *1-factorable.*

Kotzig also showed that these conditions, though being sufficient are not necessary. In particular he showed that for any cubic graph G and any cycle of length n, n greater than three, the cartesian product $G \times C_n$ is 1-factorable [**3**, Theorem 7]. We shall in turn use this to show that our conditions for 1-factorability of $G[H]$ are also not necessary.

Finally we shall require the 1-factorability of K_{2n}, the complete graph on an even number of vertices, and König's well known theorem that a regular bipartite graph is 1-factorable. These theorems can be found in Harar 's book [**2** Theorems 9.1, 9.2].

4.0 Lemma and main theorems. We fist of all state a lemma concerning the graph $G(2m)$.

4.1 Lemma. *If graph G is regular, then* $G\{2m\}$ *is 1-factorable.*

Proof. In section 3 we mentioned that K_{2m} is 1-factorable. Using this result, let graphs $F_1, F_2, \dots, F_{2m-1}$ be 1-factors of K_{2m}, which together make up a 1-factorisation: $K_{2m} = F_1 \oplus F_2 \oplus \cdots \oplus F_{2m-1}$.

Now we have

$$G\{2m\} = G \otimes K_{2m} = (G \otimes F_1) \oplus (G \otimes F_2) \oplus \cdots \oplus (G \otimes F_{2m-1})$$

and since $F_i = mK_2$ $(1 < i < 2m)$, it follows that the tensor product $G \times F$ can be written as $G \otimes F_i = m(G \otimes K_2) = mG\{2\}$.

But the graph $G\{2\}$ is bipartite, since it has vertices on two levels $G \times \{1\}$ and $G \times \{2\}$, and edges pass only between these two disjoint sets. It is also reregular since G is regular, and thus it is 1-factorable (cf. Section 3).

Since the tensor product $G \otimes F_i$ is $mG\{2\}$, it is also 1-factorable for each i. This in turn means that the sum

$$(G \otimes F_1) \oplus (G \otimes F_2) \oplus \cdots \oplus (G \otimes F_{2n-1}) = G\{2m\}$$

is 1-factorable.

The main theorem follows readily:

4.2 Theorem. *If G and H are two regular graphs for which at least one of the following holds:*

(i) *both graphs G and H contain 1-factor,*

(ii) *G is 1-factorable,*

(iii) *H is 1 factorable,*
then the composition $G[H]$ *of G and H is 1-factorable.*

Proof. We use the identity $G[H] = G \times H \oplus G\{|V(H)|\}$.

By Theorem 3.1 $G \times H$ is 1-factorable in cases (i), (ii) and (iii).

In cases (i) and (iii) H has at least one 1-factor and thus the number of vertices $V(H)$ is even. This means that $G\{|V(H)|\}$ is 1-factorable by Lemma 4.1 and thus $G[H]$ is 1-factorable.

There remains only case (ii) when G is 1-factorable. Let $G = F_1 \oplus F_2 \oplus \cdots \oplus F_k$ be a 1-factorisation of G. Thus F_i is nK_2, where G has $2n$ vertices. Now let H have m vertices, giving

$$G\{|V(H)|\} = G\{m\} = F_1\{m\} \oplus F_2\{m\} \oplus \cdots \oplus F_k\{m\}.$$

Considering the structure of the F_i, we have $F_i\{m\} = n(K_2\{m\})$.

The graph $K_2\{m\}$ is a regular bipartite graph of degree $m-1$ and so is 1-factorable. This means that $F_i\{m\}$ is 1-factorable and so in turn are $G\{|V(H)|\}$ and $G[H]$. The theorem is proved.

That the conditions of this theorem are not necessary is demonstrated by the following theorem.

4.3 Theorem. *Let G be a cubic graph and n greater than three. Then $C_n[G]$ is 1-factorable.*

Proof. $C_n \times G = G \times C_n$ is 1-factorable by Kotzig's theorem [**3**, Theorem 7] concerning the cartesian product of cubic graphs and cycles of length greater than three and $C_n\{|V(G)|\}$ is 1-factorable by Lemma 4.1 since G has an even number of vertices.

If n is odd and G has no 1-factor, neither graph has a 1-factor and the conditions of Theorem 4.2 are certainly not satisfied. This counter-example is by no means unique. Other such graphs are for instance $G(2m)$ for graph G cubic or regular of even degree but not 1-factorable [**4**].

Let us now consider the tensor products.

4.4 Theorem. *If G and H are regular graphs at least one of which is 1-factorable, then the tensor product $G \otimes H$ is 1-factorable.*

Proof. Since the tensor product is commutative, we can without loss of generality take G to be 1-factorable. Let

$$G = F_1 oplus F_2 \oplus \cdots \oplus F_k$$

be a 1-factorisation of G. Thus $F_i = nK_2$, where G has $2n$ vertices. This gives

$$G \otimes H = (F_1 \otimes H) \oplus (F_2 \otimes H) \oplus \cdots \oplus (F_k \otimes H)$$

and $F_2 \otimes H = n(K_2 \otimes H) = nH\{2\}$. By Lemma 4.1 $H\{2\}$ is 1-factorable, this means that $F_i \otimes H$ is 1-factorable and so also is $G \otimes H$.

For the strong tensor product we derive the following results.

4.5 Theorem. *If G is a 1-factorable and H a regular graph, then the strong tensor product $G \oplus' H$ is 1-factorable.*

Proof. We can write the strong tensor product $G \otimes' H$ as $G \otimes' H = (G \otimes H) \oplus nG$, where n is the number of vertices of graph H. By Theorem 4.4, the tensor product $G \otimes H$ is 1-factorable, if G is 1-factorable. That nG is 1-factorable is also immediate. Thus $G \otimes' H$ is 1-factorable.

5.0 Concluding remarks. A question of interest is whether there exist simple necessary conditions for the different products of regular graphs to be 1-factorable. These are, however, likely to be difficult to find inasmuch as it is harder to disprove 1-factorability than to construct 1-factorisations for various classes of regular graphs.

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