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ON CONGRUENCY OF OPERATORS

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Introduction. In [**1**] the idea of a congruence relation has been discussed with the main stress on obtaining some analogy between the similarity and the congruency of operators. This motivates us to make the same type of study by proving several results analogous to those proved in [**3**], [**4**], [**5**].

Notations and Terminology. Throughout the note, a bounded linear transformation on a complex Hilbert space H will be called an operator. Notations $\sigma(T)$ and $CLW(T)$ will be used to denote the spectrum and the closure of the numerical range $W(T)$ of an operator T. Let D be the class of all operators T for which either $0 \notin W(T)$ or $\sigma(T) \cap \sigma(-T) = \emptyset$. The operators A and B are called congruent operators if there exists a non-singular operator X for which $A = X^*BX$. According to [**1**], an operator is defined to be conormal if its square is a unitary operator.

Results. In [**3**], we have shown that if T_1, is a left inverse of T for which there exists an operator X such that $T^* = S^{-1}T_1S$, where 0 is not in $CLW(T)$, then T is similar to an isometry operator. In an attempt to establish an analogue of this, we have

Theorem 1. *If $T^* = XT_1X$ with $0 \notin CLW(X)$, then T is congruent to a hyponormal operator.*

The proof of the theorem requires the following simple lemma.

Lemma. *A left invertible operator with a left inverse T_1, is hyponormal if T_1^*T is isometry.*

Proof. Let $V = T_1^*T$. Then $T^*V = T^*$ or $T^ast = V^*T$. Clearly then, for all x in H, $\|T^*x\| \leq \|Tx\|$. This completes the proof.

Proof of Theorem 1. By hypothesis, $T = X^*T_1^*X$ and so $T_1' = X^{*-1}T^*X^{*-1}$ is a left inverse of T. Also $T^*T_1' = X^*T_1T^*X^{*-1}$. Thus $T_1T^* = X^{*-1}T^*T_1'X^*$

or $(TT_1^*)^* = X^{*-1}(T^*T_1')X^*$. As T^*T_1' is a left inverse of TT_1^*, we infer by [**3**] that TT_1^*, is similar to isometry. Since $(T_1')^*T = X^{-1}TT_1'X$, $S^{-1}(T_1')^*TS = V$, where V is isometry and S is a similarity transformation. The last equation can be rewritten as $(S^{-1}(T_1')^*S^{-1})(S^*TS) = V$ or $(S^{-1}T_1'S^{*-1})^*(S^*TS) = V$. Since $S^{-1}T_1'S^{*-1}$ is a left inverse of S^*TS, we conclude by the Lemma that S^*TS is a hyponormal operator.

In the preceding theorem if it is assumed that X is also selfadjoint, then we get a stronger result as given below.

THEOREM 2. *Let T be a Ieft invertible operator with a left inverse T_1. If there exists a selfadjoint operator X such that $T^* = XT_1X$ pnd 0 is not in $CLW(X)$, then T is congruent to isometry.*

Proof. Since X is selfadjoint with $0 \notin CLW(X)$ X is positive invertible. By hypothesis, $T^*X^{-1}T = X$. Therefore if we let B to be $X^{-1/2}TX^{-1/2}$, then

$$\begin{aligned} B^*B &= (X^{-1/2}T^*X^{-1/2})(X^{-1/2}TX^{-1/2}) \\ &= X^{-1/2}T^*X^{-1}TX^{-1/2} \\ &= X^{-1/2}XX^{-1/2} \\ &= I. \end{aligned}$$

This shows that $T = X^{1/2}BX^{1/2}$, where B is isometry.

The next result is a converse to Theorem 2.

THEOREM 3. *If T is congruent to isometry, then there exists a left inverse T_1 of T and a selfadjoint operator X with $0 \notin CLW(X)$ such that $T^* = XT_1X$.*

Proof. By hypothesis, $T = S^*VS$, where V is an isometry operator and S is a non-singular operator. If we take $T_1 = S^{-1}V^*S^{*-1}$ and $X = S^*S$, then $T^* = XT_1X$ and $0 \notin CLW(X)$.

The following result is analogous to a result proved in [**5**].

THEOREM 4. *If T is a non-singular operator for which there exists an operator X such that $T^* = X^*T^{-1}X$ and 0 is not in $CLW(X)$, then T is congruent to a unitary operator.*

Proof. Since $T^* = X^*T^{-1}X$, $T = X^*T^{*-1}X$, and hence $(T^{*-1}T)^* = T^*T^{-1} = (X^*T^{-1}X)(X^{-1}T^*X^{*-1}) = X^*T^{-1}T^{*-1} = X^*(T^{*-1}T)^{-1}X^{*-1}$. Using [**5**] this implies that $T^{*-1}T$ is similar and hence congruent to a unitary operator (see [**1**, Remark 1]).

COROLLARY 1. *If T^2 is normal and $T^* = X^*T^{-1}X$ with $0 \notin CLW(X)$, then T is normal.*

Proof. From Theorem 4, it follows that T is congruent to a unitary operator. Therefore by [**1**, Theorem 5], the desired conclusion follows.

Remark 1. The following question naturally arises out of the above corollary: If T^2 is assumed to be normal in Theorem 1, then can it be concluded that T is also hyponormal? The answer is not known to us.

2. The converse part of Theorem 4 can be established easily with a slight modification in the proof of Theorem 2.

In [**4**] we have asserted that for an operator X in D if $XT^{-1} = T^*X$ and $XT^* = T^{-1}X$, then T is a unitary operator. The corresponding result for a congruence relation is given in

THEOREM 5. *Let X be an invertible operator in D. If $T^{-1} = X^*T^*X$ and $T^* = X^*T^{-1}X$, then T is a unitary operator.*

Proof. The condition $T^* = X^*T^{-1}X$ implies that $T^{*-1} = X^{-1}TX^{*-1}$. This along with the condition $T^{-1} = X^*T^*X$ yields $(TT^*)^{-1} = T^{*-1}T^{-1} = (X^{-1}TY^{*-1})(X^*T^*X) = X^{-1}(TT^*)X$. Then by [**4**, Theorem 2], we get $(TT^*)^2 = I$ or $TT^* = I$. This completes the proof.

COROLLARY 2. *Let X be an operator in D. If for a conormal operator T, $T^{-1} = X^*T^*X$, then T is a unitary operator.*

Proof. The desired conclusion follows from [**1**, Theorem 5] and Theorem 5.

Next we show that in Theorem 5 the condition $X \in D$ can be replaced by the condition that the unitary part of X belongs to D, without altering the conclusion.

THEOREM 6. *Lel X be an invertible operator with polar decomposition UP, where U is a unitary operator in D and P is a positive operator. If $T^{-1} = X^*T^*X$ and $T^* = X^*T^{-1}X$, then T is a unitary operator.*

Proof. The condition $T^{-1} = X^*T^*X$ implies

$$T^{-1} = PU^*T^*UP \tag{1}$$

Also the second condition implies $T^* = PU^*T^{-1}UP$ or $T = PU^*T^{*-1}UP$ and hence

$$T^{-1} = P^{-1}U^*T^*UP^{-1} \tag{2}$$

From (1) and (2), we get $P^{-1}ZP^{-1} = PZP$, where $T = U^*T^*U$ or $ZP^{-2} = P^2Z$. Thus $Zf(P^{-2}) = f(P^2)Z$ for all polynomials f. Consequently, $ZP^{-1} = PZ$. Now by (1), $T^{-1} = PZP = ZP^{-1}P = Z = U^*TU$. Since U is in D, the desired conclusion follows [**4**, Corollary 2].

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