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AN ALTERNATIVE THEOREM FOR CONTINUOUS RELATIONS AND ITS APPLICATIONS

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0. Introduction

In this paper, improving [**10**, Lemma 3.5] of M. S. Stanojević, we prove the following alternative theorem: If S is a continuous relation from a connected space X into a space Y and V is a subset of Y such that at least one of the following conditions is fulfilled: (i) V is both open and closed, (ii) S is open-valued and V is closed, (iii) S^{-1} is open-valued and V is open, (iv) both S and S^{-1} are open-valued; then either $S(x) \subset V$ for all $x \in X$, or $S(x) \setminus V \neq \emptyset$ for all $x \in X$.

This theorem has a lot of interesting applications. In particular, it can be used to obtain not only the improvement [**10**, Theorem 3.6] of Ponomarev's [**6**, Theorem 4.1] (which was later rediscovered by R. E. Smithson [**8**, Corollary 1.7]), but also to improve [**2**, Proposition 1], and to obtain [**2**, Proposition 2] of S. P. Franklin.

The hypothesis that a relation or/and its inverse is open-valued occurs very rarely in the literature. (See [**2**]). We think this absence is very surprising, despite the fact that such relations, under some natural additional conditions, are 'almost constant'. (See Corollaries 2.5 and 2.8, and Theorem 3.10). Namely, each open cover of a topological space may be viewed as an open-valued relation. For instance, a topological space X is compact if and only if each open-valued relation S from a set onto X has a restriction with finite domain which is still onto X. Note that all the concepts defined in terms of open covers (such as paracompactness, for instance) have similar reformulations. This suggests a very broad program for the investigation of open-valued relations.

Terminology and notation in this paper will mainly follow Kelley [**4**] and Smithson [**9**]. ([**4**] is our main reference concerning relations and topological spaces,

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[**9**] is the best brief introduction to the topological theory of relations which are usually called multifunctions in this context. Note that it would be useful to combine the algebraic and topological theories of relations). Moreover, our terminology also owes much to the paper [**12**] of G. T. Whyburn.

1. Simple properties of open-valued relations

Definition 1.1. A relation S from a set X into a topological space Y is said to be open (closed)-valued if $S(x)$ is open (closed) in Y for all $x \in X$.

Remark 1.2. This terminology is motivated by that of Whyburn [**12**]. Franklin [**3, 4**], Smithson [**9**] and Stanojević [**10**] would call such relations image-open (closed), point open (closed) and Y-open (closed), respectively. (Note that the latter term is more precise, but less descriptive than the former ones.)

Proposition 1.3. *Let S be a relation from a topological space X into a topological space Y. If S is open-valued then $S(A)$ is open in Y for each $A \subset X$. In particular if S^{-1} is open-valued, then S is lower semi-continuous.*

Proof. If S is open-valued and $A \subset X$, then from $S(A) = \bigcup_{x \in A} S(X)$, it is clear that $S(A)$ is open in X for each $B \subset Y$, which is a stronger property than the lower semi-continuity of S.

Proposition 1.4. *Let S be an open-valued, upper semi-continuous relation from a topological space X into a topological space Y. Then $S^{-1}(B)$ is closed in X for each $B \subset Y$. In particular, S^{-1} is closed-valued.*

Proof. Let $B \subset Y$. Then a straightforward computation shows that $X \setminus S^{-1}(B) = \{x \in X : S(x) \subset Y \setminus B\}$. Thus, if $x_0 \in X \setminus S^{-1}(B)$, then $S(x_0) \subset Y \setminus B$. Moreover, since S is upper semicontinuous at x_0 and $S(x_0)$ is a neighborhood of itself in Y, there exists a neighborhood U of x_0 in X such that $S(U) \subset S(x_0)$. Thus, we have $U \subset X \setminus S^{-1}(B)$. Consequently, $X \setminus S^{-1}(B)$ is open in X.

Remarc 1.5. If S is a relation from a topological space X into a topological space Y such that S is open in $X \times Y$, then both S and S^{-1} are open-valued.

Namely, if $y \in S(x)$, i.e., $x \in S^{-1}(y)$, then by definition of the product topology, there exist neighborhoods U and V of x and y in X and Y, respectively such that $(x, y) \in U \times V \subset S$. Hence, it follows that $y \in V \subset S(x)$ and $x \in U \subset S^{-1}(y)$. Consequently, $S(x)$ and $S^{-}(y)$ are open in Y and X, respectively.

Example 1.6. Consider $[0, 1]$ equipped with its usual topology, and define the relation S from $[0, 1]$ into itself such $S(0) = [0, 1]$ and $S(x) = [0, 1] \setminus \{x\}$ if $0 < x \leq 1$. Then both S and S^{-1} are open-valued, but S is not open $[0, 1]^2$.

Note that $S^{-1} = S$ and $S \setminus S^0 = \{(0, 0)\}$. (Moreover, it is noteworthy that S is lower semi-continuous but S is not upper semi-continuous. Note that S is upper semi-continuous only at the point 0.)

Problem 1.7. In the light of Remark 1.5 and Example 1.6, it is natural to ask when the open-valuedness of a relation or/and its inverse implies that it is open in the product space?

2. An alternative theorem and its applications

The following alternative theorem improves [**10**, Lemma 3.5] of M. S. Stanojević, and can be used to obtain not only the improvement [**10**, Theorem 3.6] of Ponomarev's [**6**, Theorem 4.1], but also to improve [**2**, Proposition 1], and to obtain [**2**, Proposition 2] of S. P. Franklin, and also to give a partial answer to Problem 1.7.

THEOREM 2.1. *Let S be a continuous relation from a connected space X into an arbitrary topological space Y and let V be a subset of Y such that at least one of the following conditions is fulfilled:*

(i) *V is both open and closed in Y;*

(ii) *S is open-valued and V is closed in Y;*

(iii) *S^{-1} is open-valued and V is open in Y;*

(iv) *both S and S^{-1} are open-valued.*

Then either $S(x) \subset V$ for all $x \in X$, or $S(x) \setminus V \neq \emptyset$ for all $x \in X$.

Proof. Since $\{x \in X : S(x) \subset V\} = X \setminus S^{-1}(Y \setminus V)$ and X is connected, we have to show only that in each of these cases $S^{-1}(Y \setminus V)$ is both open and closed in X. However, this is quite obvious by the definitions of semicontinuities and Propositions 1.3 and 1.4.

COROLLARY 2.2. *Let S be a continuous relation from a connected space X onto an arbitrary topological space Y such that $S(x_0)$ is connected in Y for some $x_0 \in X$. Then Y is also connected.*

Proof. Suppose that V is both open and closed in Y. Since $S(x_0)$ is connected in Y, either $S(x_0) \subset V$ or $S(x_0) \subset Y \setminus V$. Thus, by Theorem 2.1, either $S(X) \subset V$ or $S(X) \subset Y \setminus V$. Hence, since $S(X) = Y$, either $V = Y$ or $V = \emptyset$.

COROLLARY 2.3. *Let S be a continuous relation from a connected space X onto an arbitrary topologicul space Y such that $S(x_0)$ lies in a component C of Y for some $x_0 \in X$, and suppose that either S is open-valued or C is open. Then Y is also connected.*

Proof. It is well-known that C is always closed in Y. Thus, by Theorem 2.1, $Y = S(X) \subset C$, and hence $Y = C$.

COROLLARY 2.4. *Let S be an almost single-valued* [**10**, *Definition* 3.1]*, upper semi-continuous relation from a connected space X onto a T_0-space Y such that S^{-1} is open-valued. Then Y is either empty or a singleton.*

Proof. Suppose on the contrary that there exist $y_1, y_2 \in Y$ such that $y_1 \neq y_2$. Since Y is T_0, either $y_1 \notin \overline{\{y_2\}}$ or $y_2 \notin \overline{\{y_1\}}$. We may assume that $y_1 \notin \overline{\{y_2\}}$ Then, there exists an open neighborhood V of y_1 such that $y_2 \notin V$. Since S is almost single-valued, there exists an $x_1 \in X$ such that $S(x_1) \subset V$. Thus, by Theorem 2.1, $Y = S(X) \subset V$, which is a contradiction since $y_2 \notin V$.

Corollary 2.5. *Let S be an open-valued, continuous relation from a topological space X into a topological space Y such that either S is closed-valued or S^{-1} is open-valued. Then S is constant on each component of X.*

Proof. Since the restrictions of S to the components of X have the same properties, we may assume that X is connected. Then, by taking $V = S(x_0)$ for any $x_0 \in X$, from Theorem. 2.1, we get $S(X) \subset S(x_0)$, and hence $S(x_0) = S(X)$. Consequently, S is constant.

Corollary 2.6. *Let X, Y and S be as in Corollary* 2.5, *and suppose that the components of X are open. Then S is open in $X \times Y$.*

Proof. Let (X_i) be the family of all components of X, and $Y_i = S(X_i)$. Then, by Corollary 2.5, $S = \bigcup_i X_i \times Y_i$, whence it is clear that S is open in $X \times Y$.

Remark 2.7. Note that if a topological space is locally connected, or the family of its components is locally finite, then each of its components is open.

Corollary 2.8. *Let S be an upper semi-continuous relation from a topological space X into a T_1-space Y such that S^{-1} is open-valued. Then S is constant on each component of X.*

Proof. We may again suppose that X is connected. Let x_0X, and $\mathcal{V}$ be the family of all open neighborhoods of $S(x_0)$ in Y. Then, by Theorem 2.1, $S(X) \subset V$ for all $V \in \mathcal{V}$, Moreover, since Y is T_1, $\bigcap \mathcal{V} = S(x_0)$. Thus, we have $S(X) \subset S(x_0)$, and hence $S(x_0) = S(X)$.

Corollary 2.9. *Let S be an open-valued, continuous relation from a topological space X into a topological space Y. Then the relation T from X into Y defined by $T(x) = \overline{S(x)}$ is constant on each component of X.*

Proof. Suppose that X is connected. Then, by taking $V = T(x_0)$ for any $x_0 \in X$, from Theorem 2.1, we get $S(X) \subset T(x_0)$, and hence $T(x_0) = \overline{S(X)}$.

Corollary 2.10. *Let S be an upper semi-continuous relation from a topological space X into a regular space Y such that S^{-1} is open-valued. Then the relation T from X into Y defined by $T(x) = \overline{S(x)}$ is constant on each component of X.*

Proof. Suppose that X is connected. Let $x_0 \in X$ and $\mathcal{V}$ be the family of all open neighborhoods of $T(x_0)$ in Y. Then, by Theorem 2.1, $S(X) \subset V$ for all $V \in \mathcal{V}$. Moreover, since Y is regular, $\bigcap \mathcal{V} = T(x_0)$. Thus, we have $S(X) \subset T(x_0)$, and hence $T(x_0) = \overline{S(X)}$.

Remark 2.11. If S is an upper semi-continuous relation from a topological space X into a regular space Y, then by [**3**, Theorem (1)], we have $\overline{S}(x) = \overline{S(x)}$ for all $x \in X$.

3. Applications to nonmingled-valued relations

Definition 3.1. A relation S is said to be nonmingled-valued if $S(x_1) \cap S(x_2) \neq \emptyset$ implies that $S(x_1) = S(x_2)$.

Remark 3.2. Here, we again followed Whyburn's terminology [**12**]. Berge [**1**] and Smithson [**8, 9**] call such relations semi-single-valued. (The term 'almost single-valued', due to Ponomarev [**7**, p. 534], should also be replaced by a better one.)

Remark 3.3. Nonmingled-valued relations occur frequently in algebra, and also in analysis, since all equivalences, certain additive, and all linear relations [**5, 11**], are nonmingled-valued.

They are closely related to equivalence relations as shown by the following characterization which will be used frequently in a subsequent paper.

Proposition 3.4. *Let S be a relation from a nonempty set X into a nonempty set Y. Then S is nonmingled-valued if and only if there exist an eguivalence relation R on $S(X)$ and a function f from X into Y such that $S = R \circ f$.*

Proof. To prove the less trivial part, suppose that S is nonmingled-valued. Then the set $\{S(x)\}_{x \in X}$ forms a partition of $S(X)$ which determines an equivalence relation R on $S(X)$ such that $R(y) = S(x)$ if $y \in S(x)$. Moreover, by the axiom of choice, there exists a selection f for S (i.e., a function f from X into Y such that $f(x) \in S(x)$ for all $x \in X$). Thus, by the definition or R, we have $S(x) = R(f(x))$ for all $x \in X$.

The next characterization is certainly due to G. T. Whyburn [**12**]. A particular case of its 'only if part' was formerly observed by S. McLane [**5**]. (See also [**11**].)

Proposition 3.5. *A relation S is nonmingled-valued if and only if $S \circ S^{-1} \circ S = S$.*

Proof. We clearly have

$$S(S^{-1}(S(x_0))) = \bigcup \{S(x) : S(x) \cap S(x_0) \neq \emptyset\}$$

for any x_0, whence the assertion is quite obvious.

Remark 3.6. The above equality shows also that we have $S \subset S \circ S^{-1} \circ S$ for any relation S.

Corollary 3.7. *A relation S is nonmingled valued if and only if its inverse S^{-1} is nonmingled-valued.*

Proof. This follows immediately from Proposition 3.5, since we have $(S \circ S^{-1} \circ S)^{-1} = S^{-1} \circ S \circ S^{-1}$ for any relation S.

Theorem 3.8. *Let S be a nonmingled valued relation from a topological space X into a topological space Y such that either S is open-valued and lower semi-continuous or S^{-1} is open-valued. Then S is continuous.*

Proof. Let $x_0 \in X$ and V be a neighborhood of $S(x_0)$ in Y. Then $U = S^{-1}(S(x_0))$ is a neighborhood of x_0 in X. Moreover, by Proposition 3.5, we have $S(U) = S(S^{-1}(S(x_0))) = S(x_0) \subset V$, and thus S is is upper semi-continuous at x_0.

COROLLARY 3.9. *Let S be an open-valued and nonmingled-valued relation from a topological space X into a topological space Y. Then S^{-1} is continuous.*

Proof. Since, by Corollary 3.7, S^{-1} is also nonmingled-valued and $(S^{-1}) = S$ is open-valued, Theorem 3.8 implies that S^{-1} is continuous.

The next two theorems are derived directly from Corollaries 2.5, 2.8 and 2.10 using Theorem 3.8. The first one is also followed by a corollary which is again quite obvious by Corollary 3.7.

THEOREM 3.10. *Let S be nonmingled-valued relation from a topological space X into a topological space Y such that either both S and S^{-1} open-valued or merely S^{-1} is open-valued but Y is T_1. Then S is constant on each component of X.*

COROLLARY 3.11. *Let S be an open-valued and nonmingled-valued relation from a topological space X onto an topological space Y such that either S^{-1} is open-valued or X is T_1. Then S^{-1} is constant on each component of X.*

THEOREM 3.12. *Let S be nonmingled-valued relation from topological space X into a regular space Y such that S^{-1} is open-valued. Then the relation T from X into Y defined by $T(x) = \overline{S(x)}$ is constant on each component of X.*

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