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EDGE-COLORING OF A FAMILY OF REGULAR GRAPHS

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Abstract. Let $G(m)$ denote the composition graph $G[mK_1]$. An obvious necessary condition for $G(m)$ to be 1-factorable is that G is regular and mp is even, where p is the number of vertices of G. It is conjectured that this is also a sufficient condition. For regular G it is proved that $G(m)$ is 1-factorable if at least one of the following conditions is satisfied: (a) G is 1-factorable, (b) G is of even degree and m is even, (c) m is divisible by 4, (d) G has a 1-factor and m is even, (e) G is cubic and m is even. The results are used to solve some other problems.

1. Introduction. The necessary background and terminology of graph theory can be found in [**1, 5**], in particular in the paper by Fiorini and Wilson [**4**].

A factor F of G is a subgraph of G with the vertex set $V(F) = V(G)$, and a *factization* $G = F_1 \oplus F_2 \oplus \cdots \oplus F_k$ of G is a family of factors $F_1, F2, \ldots, F_k$ whose edge sets $E(F_1), E(F_2), \ldots, E(F_k)$ partition the edge set $E(G)$. A d-regular factor of G is said to be a *d-factor*. If there exists a factorization of G into d-factors then it is called a *d-factorization* and G is said to be *d factorable*.

Vizing's well-known theorem [**12**] which states that every graph G with maximum degree $\Delta(G)$ can be edge-colored with at most $\Delta(G)+1$ colors, gives rise to the famous *Classification problem*. Obviously there is no edge-coloring of G using less than $\Delta(G)$ colors. A graph G is said to be of *class* 1 if it can be edge-colored with $\Delta(G)$ colors, otherwise it is of *class* 2. A regular graph is of class 1 if and only if it is 1-factorable. The classification problem is extremely difficult even for regular graphs.

Various partial results in this field were obtained in several directions by different authors [**3, 6, 8, 9**]. A good survey is given by Fiorini and Wilson [**4**]. Recently Kotzig [**7**] investigated 1-factorization of Cartesian products of regular graphs. The aim of this paper is to generalize the results of Laskar and Hare [**8**], and Parker [**9**], and to prepare ground for 1-factorization of other products of regular graphs [**11**].

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By $G[H]$ we denote the composition of graphs, also known as the lexicographic product. In the particular case when H is the null graph mK_1 on m vertices, we give the composition $G[mK_1]$ a special symbol: $G(m)$. This notation almost agrees with that of Bouchet [**2**] except that he uses m as a subscript. In our notation the regular multipartite graph $K_{n(m)}$ can be written as $K_n(m)$.

If G has p vertices and q edges, then $G(m)$ has mp vertices and m^2q edges. Each vertex of degree d in G gives rise to m vertices of degree md in $G(m)$. This means that $G(m)$ is k-regular, if and only if $k = dm$ and G is d-regular. An obvious necessary condition for $G(m)$ to be 1-factorable is that G is regular and mp is even, where p is the number of vertices of G. The authors have no counter-example to the converse statement. This paper is devoted to the study of the following conjecture.

1.1. CONJECTURE. *Let G be a graph on p vertices and $m > 1$. Then $G(m)$ is 1-factorable if and only if G is regular and pm is even.*

2. Main results. The Conjecture 1.1 is supported by the following theorems:

2.1. THEOREM. *If G is 1-factorable, then $G(m)$ is 1-factorable.*

2.2. THEOREM. *If G is regular of even degree, then $G(2m)$ is 1-factorable.*

2.3. THEOREM. *For any regular graph G, the graph $G(4m)$ is 1-factorable.*

2.4. THEOREM. *If G is regular and has a 1-factor, then $G(2m)$ is 1-factorable.*

2.5. THEOREM. *If G is a cubic graph, then $G(2m)$ is 1-factorable.*

The proofs of these theorems are given in Sections 3 and 4. Here are three simple corollaries:

2.6. COROLLARY. *if G is a regular bipartite graph, then $G(m)$ if 1-factorable for arbitrary m.*

Proof. Apply König's well-known theorem [**5**, Theorem 9.2] and our Theorem 2.1!

2.7. COROLLARY. (Laskar and Hare, [**8**]) *The complete n-partite graph $Kn(m)$ each of whose parts has exactly m vertices is 1-factorable if and only if mn is even.*

Proof. If mn is odd then $K_n(m)$ obviously has no 1-factor. For even n the graph K_n is 1-factorable, [**5**, Theorem 8.1], hence Theorem 2.1 applies. If, however, n is odd and m is even, Theorem 2.2 applies.

2.8. COROLLARY. (Parker, [**9**]) *The generalized cycle $C_n(m)$ is 1-factorable if and only if mn is even.*

Proof. Substitute C for K in the proof of the Corollary 2.7 and omit the reference.

Some other partial results are given in the subsequent sections.

3. Proofs. In this section we introduce some lemmas necessary to prove all theorems stated in Section 2 except Theorem 2.5 which is proved in the next section.

3.1. Lemma. *Let $F_1 \oplus F_2 \oplus \cdots \oplus F_k$ be a factorization of G. Then $F_1(m) \oplus F_2(m) \oplus \cdots \oplus F_k(m)$ is a factorization of $G(m)$.*

Proof. Trivial!

Proof of Theorem 2.1. If G is 1-factorable then let $G = F_1 \oplus F_2 \oplus \cdots \oplus F_d$ be one of its 1-factorizations. $F_i(m)$ is a m-regular bipartite graph, for $i = 1, 2, \ldots, d$. By König's theorem [**5**, Theorem 9.2] it is 1-factorable. By Lemma 3.1 the graph $G(m)$ can be factored into factors $F_i(m)$, which turned out to be 1-factorable, therefore $G(m)$ itself is 1-factorable.

The easy proofs of the following two lemmas are omitted.

3.2. Lemma. *If $G_1, G_2, \ldots, G_k$ are components of graph G:*

$$G = G_1 \cup G_2 \cup \cdots \cup G_k$$

then

$$G(m) = G_1(m) \cup G_2(m) \cup \cdots \cup G_k(m)$$

3.3. Lemma. $G(km) = G(k)(m)$.

The key to Theorem 2.2 is the following lemma:

3.4. Lemma. *$C_n(2)$ is 1-factorable.*

Proof. Each vertex of graph $C_n(2)$ is labeled by an ordered pair (u, k), $u \in \{0, 1, \ldots, n-1\}$ and $k \in \{0, 1\}$. It is assumed that two consecutive vertices on a cycle are labeled by consecutive integers modulo n. Obviously all the edges joining $(i, 0)$ to $((i+1) \mod n, 1)$, for $i = 0, 1, \ldots, n-1$ constitute a 1-factor of $C_n(2)$. Removing this 1-factor we obtain a n-gonal prism. Each prism is edge-3-colorable: choose an arbitrary 3-coloring for both base cycles and use the missing color at each vertex for its lateral edge. This proves the lemma.

Proof of Theorem 2.2. Since by Lemma 3.3 we have $G(2m) = G(2)(m)$, by Theorem 2.1 its is enough to prove that $G(2)$ is 1-factorable. As G is regular of even degree, by Petersen's theorem [**10**, p. 200] it is 2-factorable. Using Lemmas 3.1 and 3.2 it is sufficient to prove that $C_n(2)$ is 1-factorable. This is established by Lemma 3.4.

Proof of Theorem 2.3. By Lemma 3.3 we have $G(4m) = G(2)(2m)$. Since $G(2)$ is regular of even degree, Theorem 2.2 applies.

Proof of Theorem 2.4. If the degree of G is even, this theorem is just a special case of Theorem 2.2. Otherwise remove the 1-factor and apply the same theorem.

4. Cubic graphs. In this section Theorem 2.5 is proved by induction on the number of bridges. The induction basis is established using Theorem 2.4 and

Petersen's theorem concerning the existence of 1-factors in cubic graphs [**10**, p. 218]. Lemma 4.3 is crucial in the proof of Theorem 2.5. It guarantees the existence of a special edge-coloring of $G(2)$, for an arbitrary cubic (multi) graph G.

For a graph G, an edge-coloring of $G(2)$ is called simple if it satisfies the following condition: for every edge e in G, either $e(2)$ is colored with only two colors or the edge e lies on a unique cycle C such that both $C(2)$ and $e(2)$ are edge-4-colored. In connection with this definition the following corollary to Lemma 3.4 is of interest:

4.1. Corollary. *For every cycle C there exists an edge-4-coloring of $C(2)$ such that for every edge e of C four colors are used to color the edges of $e(2)$.*

Proof. The construction of the proof of Lemma 3.4 gives the desired edge-4-coloring of $C(2)$.

This corollary and the following lemma are used in the proof of Lemma 4.3.

4.2. Lemma. *For every path P there exists an edge-4-coloring of $P(2)$ such that for every edge e of P two colors are used to color the edges of $e(2)$.*

Proof. Trivial.

4.3. Lemma. *For every cubic graph G there exists a simple edge-6-coloring of $G(2)$.*

Proof. Without loss of generality assume that C is connected. The lemma is proved using induction on the number of bridges in G. By Petersen's theorem [**1O**, p. 218] every cubic (multi)graph G with at most one bridge is a sum of a 1-factor and a 2-factor: $G = F_1 \oplus F_2$. By Theorem 2.1 the factor $F_1(2)$ of $G(2)$ is edge-2-colorable and by Corollary 4.1 the factor $F_2(2)$ is edge-4-colorable in such a way that these colorings give rise to a simple edge-6-coloring of $G(2)$. This takes care of the induction basis.

Let G be an arbitrary cubic graph with k bridges ($k \geq 2$). By the induction hypothesis, for all cubic graphs with less than k bridges there exists a simple edge-6-coloring. Two possibilities arise.

Case 1. In G there are 3 bridges incident with the same vertex v, as depicted in Figure 1a. We construct the three cubic graphs H_1, H_2 and H_3 of Figue 1b. In each of these three graphs the number of bridges is at least two less than in G. By the induction hypothesis we can get simple edge-6-colorings of $H_1(2)$, $H_2(2)$ and $H_3(2)$. The bridges e_1, e_2, and e_3 do not belong to any cycle, therefore $e_1(2)$, $e_2(2)$, and $e_3(2)$ are each colored with only two colors. By appropriate permutation of the colors we can construct such colorings that $e_1(2)$ is colored with colors 1 and 2, $e_2(2)$ with colors 3 and 4, and $e_3(2)$ with colors 5 and 6. Combining the three parts colored in this way, we get a simple edge-6-coloring of $G(2)$.

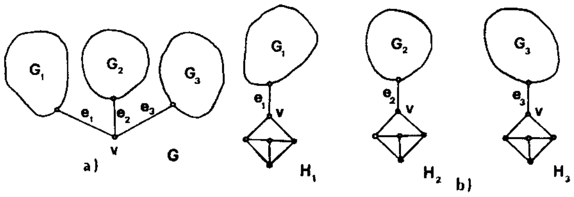

Fig 1

Case 2. No two bridges have a common vertex. Then we have a situation shown in Figure 2a, where u and v are distinct vertices and G_1 and G_3 are blocks. The same arguments as in Case 1 apply for the two graphs of Figure 2b. Graph $H_1(2)$ [or $H_3(2)$] admits and edge-6-coloring and the two colors used for $e_1(2)$ [or $e_3(2)$] can be chosen in advance. Graph L depicted in Figure 2c is obtained from G_2 by adding a new edge e, incident with u and v. By the induction hypothesis we can get a simple edge-6-coloring of $L(2)$. Cutting the edge e in L we obtain M, as depicted in Figure 2d. The simple edge-6-coloring of $L(2)$ induces an edge-6-coloring of $M(2)$ in which $f_1(2)$ and $f_2(2)$ are colored the same way as $e(2)$. Again we have two cases:

Case 2.1. If $e(2)$ is edge-2-colored in $L(2)$, the induced coloring of $M(2)$ is simple.

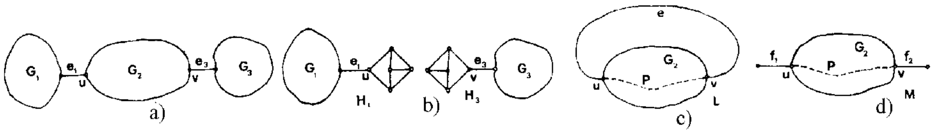

Fig 2.

Case 2.2. Otherwise there is a unique cycle C in L such that e lies on C and $C(2)$ is edge-4-colored in $L(2)$. Let $P = C - e$. Obviously P is a path. Starting with the induced edge-6-coloring of $M(2)$ we replace the coloring of $P(2)$ with the coloring of Lemma 4.2 and we use the two unused colors at u to color $f_1(2)$ and the two unused colors at v to color $f_2(2)$. This new coloring of $M(2)$ can be readily seen to be a correct simple edge-6-coloring.

Combining the simple edge-6-coloring of $M(2)$ [which was obtained in both sub-cases] with the suitably adjusted simple edge-6-colorings of $H_1(2)$ and $H_3(2)$ gives us the required simple edge-6-coloring of $G(2)$. The lemma is proved.

Proof of Theorem 2.5. By Lemma 3.3 and Theorem 2.1 it suffices to prove that $G(2)$ is 1-factorable. This follows from Lemma 4.3.

Lemma 4.3. guarantees that for any cubic graph G there exists a simple edge-6-coloring of $G(2)$. This in turn implies that for any bridge e of G the edges of $e(2)$ are colored with only two colors. The following proposition generalizes this observation to arbitrary edge-6-colorings of $G(2)$.

4.5. Proposition. *Let e be a bridge of a cubic graph G. For an arbitrary edge-6-coloring of $G(2)$, the edges of $e(2)$ are colored with exactly two colors.*

Proof. As $e(2)$ is isomorphic to the complete bipartite graph $K_{2,2}$ at least two colors are needed to color its edges. It is a bit harder to see that no more than two colors are used in an edge-6-coloring. Let u and v be the end-points of the bridge e, i.e. $e = uv$. The edges of $e(2)$ are $e_{i,j} = (u,i)(v,j)$, $i, j \in \{0,1\}$. Take any edge-6-coloring of $G(2)$ and any edge $e_{i,j}$ of $e(2)$. Let c be the color of $e_{i,j}$. If we show that $e_{1-i,1-j}$ is also colored by c, we will establish the lemma. Each color in the given edge-coloring determines a 1-factor in $G(2)$. In particular the color c determines a 1-factor in $G(2)$ and also in the two-vertex-deleted graph $H = G(2) - (u,i) - (v,j)$. Since $e_{1-i,1-j}$ constitutes a bridge joining two odd components in H, it lies on every 1-factor in H, in particular in the 1-factor determined by color c.

5. Concluding remarks. In order to apply our theorems to more general compositions of graph we need the following simple proposition:

5.1. Proposition. *If H is a 1-factorable graph on m vertices and $G(m)$ is 1-factorable, then the composition $G[H]$ is 1-factorable t 1-factorable too.*

Proof. Since $G[H]$ can be factored into $G(m)$ and pH, where p is the number of vertices of G, the Proposition follows readily.

This proposition, combined with the other results of this paper, gives an interesting corollary.

5.2. Corollary. *If H is 1-factorable, on m vertices then $G[H]$ is 1-factorable if at least one of the following is true:*

(a) *G is regular of even degree,*

(b) *m is divisible by 4,*

(c) *G is regular and has a 1-factor,*

(d) *G is cubic.*

A generalization of Proposition 5.1 and of Corollary 5.2 is obtained using different methods in a forthcoming paper [**11**]. However the methods do not give a generalization of the other results of this paper.

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Note added in proof. M. Truszczyński found a counterexample to our Conjecture 1.1.

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