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A FIXED POINT THEOREM IN A REFLEXIVE BANACH SPACE

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In [**1**] the following theorem is proved:

Theorem A *Let B a reflexive Banach space, K a nonempty bounded closed and convex subset of B and $T : K \to K$ a mapping satisfying the following conditions:*

$$\begin{gathered}\|Tx - Ty\| \leq \max\{\|x - Tx\|, \|y - Ty\|, (\|x - ty\| + \|y - Tx\|)/3, \\ (\|x - y\| + \|x - Tx\| + \|yTy\|)/3\}, \quad x, y \in K\end{gathered}$$

and

$$\sup_{z \in D} \|z - Tz\| \leq \delta(D)/2,$$

where D is any nonempty closed convex subset of K which is mapped into itself by T and $\delta(D) = \sup\limits_{x,y \in D} \|x - y\|$ the diameter of D. Then T has a unique fixed point in K.

In the present note we shall prove a theorem which is certain generalization of Theorem A, and its proof is simpler than that of Theorem A. Namely, we have the following:

Theorem 1. *Let B a reflexive Banach space, K a nonempty bounded closed and convex subset of B and $T : K \to K$ a mapping satisfying the following conditions:*

$$\begin{gathered}\|Tx - Ty\| \leq \max\{\|x - Tx\|, \|y - Ty\|, a\|x - Ty\| + b\|y - Tx\|, \\ (\|x - y\| + \|x - Tx\| + \|y - Ty\|)/3 \\ x, y \in K, \quad a \geq 0, b \geq 0, a + b < 1\end{gathered} \tag{1}$$

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$$\sup_{z\in D}\|z-Tz\|\le r\delta(D),\ \ 0\le r=r(D)<1, \tag{2}$$

where D and $\delta(D)$ have the same meaning as in Theorem A. Then T has a unique fixed point in K.

Proof. Let $\mathcal{F}$ denote the family of all nonempty bounded closed convex subsets of K, which are mapped by T into itself. $\mathcal{F}$ is nonempty since $K\in\mathcal{F}$. If $\|F_\alpha\}$ is any nonincreasing sequence in $\mathcal{F}$, then by the well known result [**2**] of Smulian, $F=\bigcap_\alpha F_\alpha$ is in $\{$. Now, by Zorn's lemma it follows that $\mathcal{F}$ has a minimal element. If C is such a minimal element of $\mathcal{F}$, we shall prove that C contains only one point, i.e. that T has a fixed point in K. Supposing that C contains more than one element we obtain

$$\sup_{x,y\in C}\|x-y\|=\delta(C)>0 \tag{3}$$

Since $T(C)\subseteq C$, for any $x,y\in C$ we have, by (1), (2) and (3)

$$\|Tx-Ty\|\le\max\{r\delta(C),(a+b)\delta(C),(\delta(C)+2r\delta(C))/3\}$$

Putting $\overline{r}=\max\{a+b,(1+2r)/3\}<1$ we have

$$\|Tx-Ty\|\le\{r\delta(C),(\overline{r}<1)\ \text{ for each }\ x,y\in C. \tag{4}$$

If by $\operatorname{co}D$ we denote the convex hull of D, and by $\overline{\operatorname{co}}D$ the closed convex hull of D we have

$$\overline{\operatorname{co}}T(C)\subseteq\overline{\operatorname{co}}C=\overline{C}=C \tag{5}$$

because $C\in\mathcal{F}$, $T(C)\subseteq C$ and C is closed and convex. Therefore

$$T(\overline{\operatorname{co}}T(C))\subseteq T(C)\subseteq\operatorname{co}T(C)\subseteq\overline{\operatorname{co}}T(C). \tag{6}$$

Since C is a minimal element of $\mathcal{F}$, by (5) and (6) we have $\overline{\operatorname{co}}T(C)=C$. Let $\overline{x},\overline{y}\in\operatorname{co}T(C)$. Then we can write

$$\overline{x}=\sum_{i=1}^n a_iTx_i,\ a_i\ge 0\,(i=1,\dots,n),\quad \sum_{i=1}^n a_i=1,\ x_i\in C$$
$$\overline{y}=\sum_{j=1}^m b_jTy_j,\ b_j\ge 0\,(j=1,\dots,m),\quad \sum_{j=1}^n b_j=1,\ y_j\in C$$

Now, by (4), and $\sum_{i,j}a_ib_j=1$

$$\|\overline{x}-\overline{y}\|=\left\|\sum_{i=1}^n a_iTx_i-\sum_{j=1}^m b_iTy_j\right\|=\left\|\sum_{i,j}a_ib_jTx_i-\sum_{i,j}a_ib_jTy_j\right\|=$$
$$=\left\|\sum_{i,j}a_ib_j(Tx_i-Ty_j)\right\|\le\sum_{i,j}a_ib_j\overline{r}\delta(C)=\overline{r}\delta(C)$$

Hence $\|\overline{x} - \overline{y}\| \leq \overline{r}\delta(C)$, for every $\overline{x}, \overline{y} \in \operatorname{co} T(C)$ and therefore

$$\sup_{\overline{x}, \overline{y} \in \operatorname{co} T(C)} \|\overline{x} - \overline{y}\| \leq \overline{r}\delta(C) \tag{7}$$

Now, $\delta(C) = \sup\limits_{x,y \in C} \|x - y\|$ and $\overline{\operatorname{co}} T(C) = C$ implies

$$\delta(C) = \sup_{\overline{x}, \overline{y} \in \operatorname{co} T(C)} \|x - y\|, \text{ and by (7) we obtain } \delta \leq \overline{r}\delta.$$

where $\overline{r} < 1$ and $\delta > 0$. This contradiction proves that C contains only one point, i.e. that T has a fixed point in K. Now we shall complete the proof demonstrating that the fixed point of T is unique. Let x_0 and y_0 be two fixed points of T.

Then by (1) we have

$$\begin{aligned} \|x_0 - y_0\| = \|Tx_0 - Ty_0\| \leq \max\{ & \|x_0 - Tx_0\|, \|y_0 - Ty_0\|, \\ & a\|x_0 - Ty_0\| + b\|y_0 - Tx_0\|, \\ & (\|x_0 - y_0\| + \|x_0 - Tx_0\| + \|y_0 - Ty_0\|)/3\} \\ = \max\{ & 0, 0, (a+b)\|x_0 - y_0\|, \|x_0 - y_0\|/3\} \end{aligned}$$

From this inequality immediately follows $x_0 = y_0$.

REFERENCES

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