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# SOME APPROACHES TO TOPOLOGICAL SPACES[1]

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**Summary.** There are several approaches to the concept of topological spaces which are based respectively on limit, neighborhood, deviation, coincidence degree, transfer of organization, topos, etc. From the historical point of view we stress in particular Newton's approach by limits.

**0.** An abstract space in any set $S$ and any $S$-un $f$ of its subsets: one speaks of the ordered pair or of the 2-nd $(S, f)$, where $f : PS \to PS$.[2] The fundamental fact here is that for every $X \subset S$ one has $fX \subset S$; in particular for the void $v$ one has $fv \subset S$; and one also has $fS \subset S$.

**1.** There are trivial cases of spaces. For example take the mapping $f$ to be constant or "linear" in the sense that $fX = X \cup A$ or $fX = -X \cup A$, where $A$ is a given member of $PS$ and $-X$ is the complement $CX := S - X := -X$.

**2.** Another very well known case is that of closures: isotone $S$-uns such that $\overline{X} \supset X$, $\overline{v} = v$ and $\overline{\overline{X}} = X$.

**3.** Historically, the derivation $D : PS \to PS$ of sets played a great role. It was defined by the condition that

3.1 $$a \in DX \Leftrightarrow x \in D(X \setminus \{a\}) \text{ and } X \setminus \{a\} \neq v.$$

These two approaches are linked by the equality

3.2 $$\overline{X} = X \cup DX.$$

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[2]If $S$ is any set, then a $S$-un is any mapping $f$ of $S$; for any number $k$, a $k$-un is any mapping of a set of power $k$; 0-un = the empty sequence; 1-un = any number, point, ...; 2-un = ordered pair; 3-un = ordered triplet, $k$-un = ordered $k$-tuple.

**4.** We have some other notions in a space $(S, ^-)$: for any set $X \subset S$ we have the interior $\int X$ of $X$, the exterior $\operatorname{ext} X$, the boundary or the border $F_r X$ of $X$ defined respectively by:

4.1 $$\int X = C\overline{C}X \qquad \text{(3 factors)}$$

4.2 $$\operatorname{ext} X = \int (CX) = C\overline{X} \qquad \text{(2 factors)}$$

4.3 $$F_r X = C(\int X \cup \operatorname{ext} X).$$

One has several relations between these operators; e.g.

4.4 $$F_r X = F_r CX$$

4.5 $$F_r(X \cup Y) = F_r X \cap \overline{CY} \cup \overline{CX} \cap F_r Y$$

4.6 $$F_r(X \cap Y) = F_r X \cap \overline{Y} \cup \overline{X} \cap F_r Y.$$

The proof of the relation 4.4 – 4.6 is obvious; e.g.

$$(4.6)_1 := \overline{X \cap Y} \cap \overline{C(X \cap Y)} = \overline{X} \cap Y \cap (\overline{CX} \cup \overline{CY}) = F_r X \cap \overline{Y} \cup \overline{X} \cap F_r Y := (4.6)_2.$$

4.7. We have also the operator $\overleftarrow{\operatorname{ext}} X := \overline{CX} = C(\int CX)$; this operator $\overleftarrow{\operatorname{ext}}$ could be taken as primitive because $\overline{X} = \overleftarrow{\operatorname{ext}}\, CX$.

4.8. The permutation $C \leftrightarrow ^-$ in any of the preceding operators $f$ yields an operator, say $p - f$; one has $p - \operatorname{ext} X := \overline{CX} := \overleftarrow{\operatorname{ext}} = C \int X$, i.e. $\int X = Cp - \operatorname{ext} X$, $p - \int X := \overline{C\overline{X}} = \overline{\operatorname{ext} X}$.

**5.** A great number of phenomena could give rise to self-mappings of power sets.

5.1. *Example.* If $R$ is any binary relation on $E$, i.e. if $R \subset E \times E$, then for any $x \in E$ one has the set $R[x] := \{y \therefore y \in E, R(x,y)\}$; for every subset $A \subset E$ let $R[A] := \bigcap_{x \in A} R[x]$. The set $R[A]$ is called the polar of $A$ with respect to $R$. So we have a definite mapping

5.2 $$R : A \in PE \to R[A] \in PE.$$

This mapping is decreasing; moreover, if $R$ is symmetric, then its square is extensive, i.e., it satisfies $f^2 X \supset X$ for every $X \subset E$.

5.2.1. One can prove the converse also: if $f : PE \to PE$ is decreasing and if $ff$ is extensive, then the relation

5.2.2 $$xRy \Longleftrightarrow y \in fa$$

is a symmetrical relation in $E$ (v. J. Schmidt [**10**], §8).

5.3. Let $R$ be any binary symmetrical relation in $E$; for any $M \subset E$ let $HM := \{x \therefore x \in E \& MR'x\}$ i.e. $\neg\forall_{m\in M}(m,x) \in S$; one has again a self-mapping of $PE$.

5.4 *Ordered sets, ordered spaces.*

5.4.0 In connexion with a given (quasi-) ordered set $(E, \leq)$ one has several kinds of ordered spaces $(E, f)$ with the same set $E$ and tied in a certain way with $\leq$.

5.4.1. THEOREM. *If a topological $T_2$-space $(E, {}^-)$ is equipped with an order $(E, \leq)$, such that every middle interval of $(E, \leq)$ is a nonlacunary chain and the given topology is equivalent to the open interval topology of $(E, \leq)$, then the half-cones $[p, \cdot)$, $(\cdot, p]$ are closed sets. Either of the conditions: $T_2$ and non lacunarity are indispensable.*

*Proof.* Let us assume that, on the contrary, there exists a point $p \in E$ such that the corresponding right cone $P := [p, \cdot)$ is not closed, i.e., there exists a point $x \in E$ such that $x \in \overline{P} \setminus P$; thus $x < p$ or $x \| p$.

5.4.1.1. Case $x < p$. Every neighborhood $V(x)$ contains at least one point of $P$; since the topology is equivalent to the open interval topology we may assume in particular that $V(x)$ is of the form $(b, p)_<$ and that there would be a point $q \in P$ in the same interval i.e., $b < q < p$-absurdity, because $p \leq P$; thus $p \leq q$.

Case $x \| p$. Since $x \in \overline{P} \setminus P$, one has $x \in P' := DP$.

5.4.1.2. Subcase: $x$ is not isolated in $P$ from left, i.e., the set $P(\cdot, x)$ has no last member: for every $a \in E$ such that $a < x$ there exists a point $b \in P$ such that $a < b < x$; now, $b < x$ with $b \in P$, i.e., $p \leq b$ would imply $p < x$, contradicting $x \| p$.

5.4.1.3. Subcase: $x$ is isolated from left in $(P, \leq)$. We claim that $\nu \neq E(x, \cdot) \subset E(p, \cdot)$. Now, $E(x, \cdot) \neq \nu$ because $x$ is a right limit point of $P$. Therefore, if $x < z \in E$, there is a $y \in P$ such that $x < y < z$ and consequently $z \in P$. Again, by our supposition, the intervals $E(x, z) := B$, $E(p, z) := C$ are chains; $B$ is a proper terminal part of $C$; so we have the corresponding cut $C \setminus B | B$; since $B$ has no first member and since, by our supposition, $C$ is without any gap, $C \setminus B$ has a last member, say $x'$. One has $x' \in P$, $x$ non $\in P$; so the distinct points $x$ and $x'$ would be both right limit points of a same chain $B$; therefore any $V(x)$ would meet any $V(x')$, which contradicts the assumed $T_2$-property of the space. This proves that $(E, \leq)[a, \cdot)$ is closed. Q.E.D.

5.4.1.4. Applying the last property to the dual order set $(E, \geq)$ one obtains the statement that the left half-cone $(E, \leq)(\cdot, p]$ is also closed. Q.E.D.

The following examples of spaces $R_0$ and $R_{00}$ show that non-lacunarity and $T_2$ are indispensable in the wording of 5.4.1.

5.4.1.5. $R_0$ is obtained from $R$ (reals) on replacing the ordinary order relation $\leq$ by the following reorganizations of $(R, \leq) : R(\cdot, 0) \| 0 < R(0, \cdot)$; here the chain $(\cdot, 0) \cup (0, \cdot)$ presents a gap because the remaining member 0 is $\| b$ for every negative real $b$; again $0 \in Cl(b, \cdot)$ because in particular $0 \in Cl(0, \cdot)$ but $0 \notin R_0(b, \cdot)$.

5.4.1.6. The space $R_{00}$ is obtained from $(R, \leq)$ by adjoining a new object $0'$ such that $(\cdot, 0] \| 0' < (0, \cdot)$: in the space $R_{00}$ the distinct points 0 and $0'$ are not separated:

$V(0) \cap V(0') \neq \nu$ holds identically; if $b < 0$, then again the halfcome $R_{00}[b, \cdot)$ is not closed because it does not contain its limit point $0'$.

5.4.1.7. Corollary. *Let $(E, \leq)$ be a pseudotree (ramified set), i.e. such that for every $x \in E$ the set $(E, \leq)(\cdot, x) := \{y \therefore y \leq x, y \in E\}$ is a chain; if the subchains of $(E, \leq)$ are without gap and if the ordered space $(E, \leq)$ is $T_2$ and definable by means of open intervals, then for every $x \in E$ the half-cones $(E, \leq)(\cdot, x)$, $(E, \leq)[x, \cdot) := \{y \therefore x \leq y \in E\}$ are closed.*

5.4.2. *A variant of real numbers.* Let $S \subset R$; let every $s \in S$ be replaced biuniquely by an object $s^+$ in such a way that $R(\cdot, s) < s^+ \| R(s, \cdot)$ and $\{s^+ \therefore s \in S\} := S^+$ is an antichain. So one obtains an ordered set $(R(S), \leq)$ which is dual to a pseudotree; for every $s \in S$ we have a maximal chain $R(\cdot, s) \cup R(s, \cdot)$ presenting a gap $(\cdot, s) \| (s, \cdot)$.

5.4.2.1. A corresponding construction $(E(S), \leq)$ for any $(E, \leq)$ and any $S \subset E$ is obvious. It repays to study such ordered spaces.

5.4.3. We considered also the following cone condition CC for ordered spaces $(E, \leq)$:

CC For every point $p \in E$ one has

$$(E, \leq)(\cdot, p) \cap Cl(p, \cdot) \cup ClE(\cdot, p) \cap E(p, \cdot) = \nu.$$

Of course, interval topologies on $(E, \leq)$ satisfy CC.

**6. Graphs and topologies.** Any quasi-ordered set is a particular kind of a graph. For any graph $(G, R)$ (thus $R \subset G \times G$) one may consider several topological spaces $(G, -)$ in particular under the condition that $R$ is a closed set in the space $(G, -) \times (G, -)$.

**7. Distance. Abstract distance.** The notion of distance $d(x, y)$ is of fundamental importance in Mathematics and in Applications, and goes back to Thales; M. Fréchet allowed the arguments $x$ and $y$ to run through any abstract set E. Đ. Kurepa [**5,6,7**] allowed $d$ to run through any structure $M$; the transfer of elementary geometrical considerations to this general situation of $M$-spaces, yields by force, uniform spaces and any topological space; in other words, abstract distance spaces coincide with topological spaces.

**8. Limit considerations.** They are basic for topology; the notion of limit evolved slowly and is tied with Zeno's paradoxes, exhaustion, Arch'medes considerations, etc. Let us quote the following text.

8.1. *A quotation from Newton.* "Ultimae rationes illae quibuscum quantitates evanescunt, revera non sunt rationes quantitatum ultimarum, sed limites ad quos quantitatum sine limite decrescentium rationes semper appropinquant, et quas propius assequi possunt quam pro data quavis differentia, nunquam vero transgredi, neque prius attingere quam quantitates diminuuntur in infinitum." ("Ultimate ratios in which quantities vanish, are not, strictly speaking, ratios of ultimate quantities, but limits to which the ratios of these quantities, decreasing without limit, approach, and which, though they can come nearer than any given difference whatever, they can neither pass over not attain before the quantities have diminished indefinitely" Newton, Philosophiae Naturalis Principia, End of Section I, London 1687, p. $36^{10-16}$).

8.2. This quotation is very important because the idea that was expressed in it ts an important starting point for topological spaces. The idea of a limes (limit) was introduced intuitively and since then was tremendously extended by Fréchet, Moore,*dots*; Filter theory with applications is a chapter of involved convergence considerations.

8.3. Probably, such a most general convergence approach is the best topological approach because the approaching constitutive parts are more and more complete approximations of the feature involved.

**9.** There are other approaches to topological spaces (for example, using topoi).

**10. Problem.** For each approach we can consider problem of its scope. In particular it is of interest to examine whether a given approach to topological spaces is universal in the sense of yielding every topological space.

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