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TWO RESULTS ON ASSOCIATIVITY OF COMPOSITE OPERATIONS IN GROUPS

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Introduction

Every group word $w(x,y)$ in variables x, y and elements of a given group G determines a binary operation on G. Considerable attention has been given to investigating when such an operation brings a new group structure on G; see [**1**], [**2**] and [**3**]. In some cases the problem when $w(x,y)$ is only associative is solved too; see [**4**] and [**5**]. The two theorems we are going to prove in this article are related to this problem of associativity.

The theorem of Hanna Neumann ([**4**]) states that all associative operations $w(x,y)$ in the case of a free G are of one of the following forms:

$$a, x, y, xay, yax,$$

where a is an arbitrary element of G. In the first part of this article we generalize this result. Theorem 1 shows that operations of the forms listed above are the only possible (except trivial cases) when we require $w(x,y)$ to satisfy not the associativity law, but any consequence of it (any weakened associativity law).

In the second part of the article we determine all associative operations $w(x,y)$ in the case of G free nilpotent of class two.

Part one

Terms and trees. We begin with a few comments on terms and trees of a special kind which stand in 1–1 correspondence with terms. The terms will be built of variables $x, y, z, x_1, x_2, \ldots$ and a binary operation symbol $\circ$. The tree $T = T(t)$ corresponding to the term t takes shape in a well-known way, following the inductive definition of terms. If t is of the simplest form $t = x_i$, then $T(t)$ is a single vertex labeled with x_i. If $t = t_1 \circ t_2$ and $T_i = T(t_i)$ $(i = 1, 2)$, then $T(t) = T_1 \circ T_2$, the tree

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obtained from the union of trees T_1 and T_2 by adding a new vertex v (base vertex of $T_1 \circ T_2$) and two edges vv_1 and vv_2, first labeled with λ, second with ρ. Here v_1 and v_2 are base vertices of T_1 and $T-2$. See fig. 1

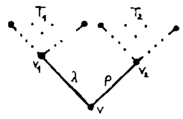

fig. 1

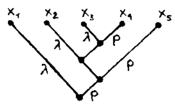

fig. 2

So, all vertices of degree 1 (they will be called peaks) of our term-trees are labeled with variables occurring in corresponding term. Edges are also labeled, but with λ and ρ – these letters symbolize "left" and "right" when drawing trees. For example, the labeled tree corresponding to the term $x_1 \circ ((x_2 \circ (x_3 \circ x_4)) \circ x_5)$ is shown on fig. 2.

All terms appearing in the sequel have a feature that no variable occurs more than once in them. Thus the peaks of trees we consider are labeled with different variables. The word obtained by reading successive labels of passed edges on the path from the base vertex of T to the peak (labeled with) x_i will be called *trace of x_i in the tree T* and denoted by $\mathrm{tr}_T(x_i)$, or simply $\mathrm{tr}(x_i)$. For example, in fig. 2 $\mathrm{tr}(x) = \rho\lambda\rho\rho = \rho\lambda\rho^2$. The following is almost obvious.

LEMMA 1. *If in the peaks off the trees T_1 and T_2 the same variables $x_1, \dots, x_o$ stand as labels and if $\mathrm{tr}_{T_1}(x_i) = \mathrm{tr}_{T_2}(x_i)$ for every i, $1 \le i \le n$, then $T_1 = T_2$.*

Suppose the tree T and the term t correspond in the way described above. Then for every subterm to of t the correspoding tree T_0 is a subtree of T. Especially, if $x_{i_1}, \dots, x_{i_r}$ are labels of some peaks of T, then $T[x_{i_1}, \dots, x_{i_r}]$ will denote the subtree of T corresponding to the minimal subterm of t which contains all variables $x_{i_1}, \dots, x_{i_r}$.

Suppose that terms t_1 and t_2 are obtained by placing brackets in the word $x_1 \circ x_2 \circ \cdots \circ x_n$. (Equivalent formulation: in both $T(t_1)$ and $T(t_2)$ we have $\mathrm{tr}(x_1) \prec \mathrm{tr}(x_2) \prec \cdots \prec \mathrm{tr}(x_n)$, where $\preceq$ is the lexicographical ordering on the set of words over the alphabet $\{\lambda, \rho\}$, induced by $\lambda \preceq \rho$). Any equality of the form $t_1 = t_2$ will be called the *weakened associativity law* for $\circ$. Trivial laws are those in which t_1 and t_2 are literally equal, that is, those for which $T(t_1) = T(t_2)$ holds.

Types. Let F be a free group, $X = \{x_1, x_2, \dots\}$ and let

$$u = a_0 x_{i_1}^{\varepsilon_1} a_1 x_{i_2}^{\varepsilon_2} a_2 \dots x_{i_r}^{\varepsilon_r} a_r \qquad (a_\nu \in F,\ \varepsilon \in Z)$$

be the reduced form of the element u of the free product $F * \langle X \rangle$. The type of the element u is the word $\tau(u) = \xi_1 \dots \xi_s$ obtained from the word $x_{i_1} \dots x_{i_r}$ by amalgamating successive equal letters in it. For example, $\tau(x_1^{-1} x_3^2 a x_3^2 b) = x_1 x_3$. By definition the type $\xi_1 \dots \xi_r$, (that is, any word over X) is *realized* in the element $u \in F * \langle X \rangle$ iff $\xi_1 \dots \xi_r$ is a subword of $\tau(u)$. As for the realization of types, we shall need only the following simple

LEMMA 2. *Let* $w(x_1,x_2) = a_0 x_{i_1}^{e_1} a_1 \ldots x_{i_r}^{\varepsilon_r} a_r$ $(a_\nu \in F, \varepsilon_\nu \in Z, i_\nu \in \{1,2\})$ *be the reduced form of the element* $w(x_1,x_2) \in F * (x_1,x_2)$ *and let* u_1,u_2 *be elements of* $F * \langle X \rangle$ *such that no* $x_i \in X$ *occurs in the reduced forms of both* u_1 *and* u_2. *Denote by* $w_1 = w(u_1,u_2)$ *the element of* $F * \langle X \rangle$ *obtained by substituting* u_1 *and* u_2 *for* x_1 *and* x_2 *in* $w(x_1,x_2)$. *Then all types* $\tau(u_{i_\nu}^{\varepsilon_\nu})$ $(\nu = 1,\ldots,r)$ *are realized in* w_1.

Operations defined by group words. If in the word $w(x_1,\ldots,x_N) \in F*\langle X \rangle$ we let $x_1,\ldots,x_n$ be variables taking values in F, then $w(x_1,\ldots,x_n)$ defines an n-ary operation on F. The following easily provable fact should be noticed: in the case of the free group F the operation $w(x_1,\ldots,x_n)$ takes the constant value 1 if and only if $w(x_1,\ldots,x_n)$ as an element of the group $F*\langle X \rangle$ is equal to the identity element of that group. This fact is implicitly used whenever we do not distinguish between equality of operations and equality of corresponding group words (see, for example, the first part of the proof of Theorem 1).

Further, whenever we refer to types of terms or realizations of types in terms we think of interpreted terms, that is, those in which the operation symbol $\circ$ is replaced by an operation of the form $w(x,y)$. The interpretation will be always clear from the context. For example, if $w(x,y) = xyx^{-1}$, then the term $(x \circ y) \circ z$ after the interpretation of $\circ$ by $w(x,y)$ becomes $xyx^{-1}\ zxy^{-1}x^{-1}$ and is of the type $xyxzxyx$.

THEOREM 1. *Let* $t_1(x_1,\ldots,x_n) = t_2(x_1,\ldots,x_n)$ *be a non-trivial weakened associativity law and let* $w(x,y)$ *be a group word in x, y and the elements of a free group F, in whose reduced form both x and y occur. If the operation* $x \circ y = w(x,y)$ *satisfies the law* $t_1 = t_2$, *then* $w(x,y) = xay$ *or* $w(x,y) = yax$, *for some* $a \in F$.

Proof. We shall limit our considerations to the case when in the word $w(x,y)$ the symbol x occurs before the symbol y, that is, when the type $\tau(w(x,y))$ begins with x. If it is not so, we can take the operation $w'(x,y) = w(y,x)$; $\tau(w')$ begins with x and $w'(x,y)$ satisfies the law $t_1' = t_2'$, where t_1' and t_2' are "mirror images" of t_1 and t_2. From the conclusion that $w'(x,y)$ is of the form xay or yax it immediately folows that $w(x,y)$ is also of one of these forms.

First step. Assumption of non-congruence of subterms of t_1 and t_2. Let $T_1 = T(t_1)$ and $T_2 = T_2(t_2)$. We are going to prove that in Theorem 1 the terms t_1 and t_2 may be assumed to satisfy the following condition:

$$\text{(C)} \qquad T_1[x_i,\ldots,x_j] \neq T_2[x_i,\ldots,x_j]$$

for every i,j $(1 \leq i < j \leq n)$.

LEMMA 3. *Let* $u(z) \in F * \langle z \rangle$ *and* $v(y_1,\ldots,y_m) \in F * \langle y_1,\ldots,y_m \rangle$. *If* $u(v(y_1,\ldots,y_m)) = 1$ *in* $F * \langle y_1,\ldots,y_m \rangle$, *then* $u(z) = 1$ *in* $F * \langle z \rangle$.

Proof of Lemma 3. Let $v(y_1,\ldots,y_m) = av'(y_1,\ldots,y_m)b$, where $a,b \in F$ and the extreme symbols of v' are some of the y_i. Let also $u(z) = u(azb)$, and suppose that $u(z) \neq 1$ in $F * \langle z \rangle$. Then $u'(z) \neq 1$ too. If in the reduced form $u'(z) = a_0 z^{\varepsilon_1} a_1 z^{\varepsilon_2} a_2 \ldots z^{\varepsilon_r} a_r$ we put $v'(y_1,\ldots,y_m)$ instead of z, the symbols $a_0,\ldots,a_r$

will remain uncancelled. Thus, $u'(v'(y_1,\ldots,y_m))\neq 1$, But $u'(v')=u'(a^{-1}vb^{-1})=u(v)$, and so we get $u(v(y_1,\ldots,y_m))\neq 1$, q.e.d.

Suppose now the condition (C) does not hold for terms t_1 and t_2, that is, for some i,j, $T_1[x_i,\ldots,x_j]=T_2[x_i,\ldots,x_j]$; let $t(x_i,\ldots,x_j)$ be the term corresponding to these subtrees. From the equality $t_1t_2^{-1}=1$ by means of Lemma 3 we obtain $\overline{t}_1\overline{t}_2^{-1}=1$ where $\overline{t}_1$ and $\overline{t}_2$ are terms in $x_1,\ldots,x_{i-1},z,x_{j+1},\ldots,x_n$, obtained by putting z instead of $t(x_i,\ldots,x_j)$. The number of variables in the law $\overline{t}_1=\overline{t}_2$ is less than in $t_1=t_2$ and so we can arrive, repeating the process above several times, to a law satisfied by the operation $x\circ y=w(x,y)$ for which the condition (C) holds. Thus, nothing is lost if we assume that this condition holds already for the law $t_1=t_2$.

Second step. The type of $w(x,y)$ is xy. Suppose the contrary: the type xyx is realized in $w(x,y)$. Clearly, there exists an index k $(1\leq k<n)$ such that $T_2[x_k,x_{k+1}]=T(x_k,x_{k+1})$. Let T' and T'' be subtrees of T_1 such that $T'\circ T''=T_1[x_k,x_{k+1}]$. In the peaks of T' are the variables $x_{k+1},\ldots,x_k$ $(i\leq k)$ and in the peaks of T'' are the variables $x_{k+1},\ldots,x_j$ $(j\geq k+1)$.

Now we are going to prove $T'=x_k$. Assuming the contrary we see that in T' there exists a subtree $T'''\circ x_k$, where the variables in the peaks of T''' are $x_{i+r},\ldots,x_{k-1}$ $(r\geq 0)$. If t' and t''' are the terms which correspond to the trees T' and T''', then $t'=w(t''',x_k)$. Since the type xyx is realized in $w(x,y)$, Lemma 2 implies that a certain type $\xi x_k\eta$ is realized in t', where ξ and η are variables occurring in t''' (some of $x_{i+r},\ldots,x_{k-1}$). Using the observation that if $\xi x_k\eta$ is realized in t, then in any power of $t\xi x_k\eta$ or $\eta x_k\xi$ is realized, several applications of Lemma 2 lead us to the conclusion that in t_1 one of the types $\xi x_k\eta$ is also realized, where ξ and η are variables from t'''. Now let us look at the situation in t_2; $w(x_k,x_{k+1})$ is a subterm of it. In the type of $w(x_k,x_{k+1})$ to every occurrence of the letter x_k at least one neighbor is a letter x_{k+1} and the same can be said for the term t_2 (again we refer to Lemma 2). So in t_2 no type $\xi x_k\eta$ can be realized for $\xi,\eta\neq x_{k+1}$. By the contradiction just obtained we have proved $T'=x_k$.

If the type yxy were realized in $w(x,y)$, in the same way as above, we could deduce $T''=x_{k+1}$ and then $T_1[x_k,x_{k+1}]=T(x_k\circ x_{k+1})=T_2[x_k,x_{k+1}]$ – contrary to the condition (C). Thus we are left with the only possibility that $w(x,y)$ is of type xyx. Let $t_1[x_k,x_{k+1}]$, $t_2[x_k,x_{k+1}]$ and t'' be the terms corresponding to the trees $T_1[x_k,x_{k+}]$, $T_2[x_k,x_{k+1}]$ and T''. We have $\tau(t_1[x_k,x_{k+1}])=T\tau(w(x_k,t''))=x_k\omega x_k$ and $\tau(t_2[x_k,x_{k+1}])=x_kx_{k+1}x_k$, where the letters occurring in ω are just the variables from t'', namely $x_{k+1},\ldots,xj$ $(j\geq k+2)$. Arguing in a manner similar to that of the preceding paragraph we can deduce that to every occurrence of x_{k+1} in the type of t_2 both adjacent symbols are x_k and that in the type of t_1 this is not the case. This contradiction breaks the assumption we have started with, so $\tau(w(x,y))=xy$.

Third step. $w(x,y)=bx^{\pm1}ay^{\pm1}c$ for some $a,b,c\in F$. According to what we have already proved in the previous step we have now

$$w(x,y)=a_0x^{\alpha_1}a_1x^{\alpha_2}\ldots a_{p-1}x^{\alpha_p}a_py^{\beta_1},a_{p+1}y^{\beta_2}\ldots a_{p+q-1}y^{\beta_q}a_{p+q}, \tag{1}$$

where $p, q \geq 1$, $\alpha_i, \beta_j \in Z$ and a_i are elements of F, all non-trivial except perhaps a_0, a_p and a_{p+q}.

Let us define the functions $\lambda_j(x_i)$ and $\rho_j(x_i)$ ($j \in \{l, 2\}, 1 \leq i \leq n$) as the numbers of occurrences of letters λ and ρ in the words $\mathrm{tr}_j(x_i) = \mathrm{tr}\, T_j(x_i)$. From the fact that $w(x, y)$ is of the form (1), by induction on the complexity of terms, with a simple analysis of possible cancellations that appear during the process of building terms, it follows that the number of syllables of the form x_i^ε in the reduced form of the term t_j is equal to

$$p^{\lambda_j(x_i)} q^{\rho_j(x_i)}.$$

Since $t_1 = t_2$

$$p^{\lambda_1(x_i)} q^{\rho_1(x_i)} = p^{\lambda_2(x_i)} q^{\rho_2(x_i)} \tag{2}$$

must hold for every i, $1 \leq i \leq n$.

In order to prove $p = 1$ assume the contrary: $p > 1$. Now if $\mathrm{tr}_j(x_i) = \sigma_j \lambda \rho^{\varepsilon_j}$ ($\varepsilon_j \geq 0$, σ_j - words over the alphabet $\{\lambda, \rho\}$), then $\mathrm{tr}_j(x_{i+1}) = \sigma_j \rho \lambda^{\zeta_j}$ for some $\zeta_j \geq 0$. Making use of this fact together with (2) we can proceed by induction and prove $\mathrm{tr}_1(x_i) = \mathrm{tr}_2(x_i)$ for every i, $1 \leq i \leq n$. By means of Lemma 1 these equalities imply $T_1 = T_2$, which cannot hold because of the assumed fulfilment of condition (C).

Thus we have proved $p = 1$ and the proof of $q = 1$ goes in the same manner. So we have $w(x, y) = bx^\alpha a y^\beta c$, for some $a, b, c \in T$ and $\alpha, \beta \in Z \setminus \{0\}$.

The sum of exponents of x_i in the term t_j is equal to

$$\alpha^{\lambda_j(x_i)} \beta^{\rho_j(x_i)}.$$

Assuming $|\alpha| > 1$, in the same way as in the proof above we can arrive to equalities $\mathrm{tr}_1(x_i) = \mathrm{tr}_2(x_i)$ for every i and hence to the absurd conclusion $T_1 = T_2$. By this the assumption $|\alpha| > 1$ fails. So we must have $|\alpha| = 1$, and quite analogously $|\beta| = 1$.

It remains only to discuss four cases which arise when α and β take the values ± 1.

Fourth step. Case 1: $w(x, y) = bxayc$. If $w(x, y)$ is of this form, then

$$t_1 = t_2 = a_0 x_1 a_1 x_2 a_2 \ldots x_n a_n,$$

for some $a_0, \ldots, a_n \in F$.

Let i and j be the smallest numbers such that $x_i \circ x_{i+1}$ is a subterm of t_1 and $x_j \circ x_{j+1}$ a subterm of t_2. Let us suppose $i < j$; $i = j$ does not hold because of the condition (C). The tree $T_2[x_i, x_{i+1}]$ is now of the form $x_i \circ T'$, where in the peaks of T' stand as labels the variables $x_{i+1}, \ldots, x_{i+r}$, $r \geq 2$. The term t' which correspond to the tree T' begins with bx_{x+1} and it follows that the term $t_2[x_i, x_{i+1}]$ begins with $bx_i abx_{x+1}$. On the other hand $t_1[x_i, x_{i+1}] = b_{x_i} a x_{i+1} c$. It follows that in t_1 between x_i and x_{i+1} stands a, and in t_2 between these variables stands ab. From this we derive $b = 1$ and the proof for $c = 1$ is quite analogous. Thus, $w(x,y) = xay$ and such an operation clearly satisfies all weakened associativity laws.

Case 2: $w(x,y) = bxay^{-1}c$. We shall prove only that the operation $w(x,y) = xy^{-1}$ does not satisfy any non-trivial weakened associativity law, the same result for all operations of the form $bxay^{-1}c$ being implied by this one. The following observation makes this implication clear: if $w(x,y)$ satisfies some law $t_1 = t_2$, then $w_0(x,y)$ satisfies the same law too, where w_0 is obtained by identifying all elements of F in w with 1.

Let $w(x,y) = xy^{-1}$ and let i, j and T' be as in Case 1. Since $t_1[x_i, x_{i+1}] = x_i x_{i+1}$, it follows that in the term t_1 the variables x_i and x_{i+1} also stand in adjacent places. On the other hand, $t_2[x_i, x_{i+1}] = x_i {t'}^{-1}$, where t' begins with x_{i+1} and does not end with that symbol. So in $t_2[x_i, x_{i+1}]$, and therefore in t_2 too, the variables x_i and x_{i+1} do not occur in adjacent places. By this contradiction we are done with Case 2.

Case 3: $w(x,y) = bx^{-1}ayc$. Analogous with Case 2.

Case 4: $w(x,y) = bx^{-1}ay^{-1}c$. Again we are going to prove that the operations we are dealing with do not satisfy non-trivial laws. With the same explanation as in Case 2 it suffices to consider only the operation $w(x,y) = x^{-1}y^1$. To reach the absurd more simply let us suppose that between nontrivial trivial weakened associativity laws satisfied by $x^{-1}y^{-1}t_1 = t_2$ is the one which involves the minimal number of variables.

The initial symbol of both t_1 and t_2 is that $x_k^{\pm 1}$ for which $\operatorname{tr}(x_k)$ in the trees T_1 and T_2 are of the form $\lambda_\rho \lambda_\rho \ldots \lambda_\rho$ or $\lambda_\rho \lambda_\rho \ldots \lambda_\rho \lambda$ (words beginning with λ,

in which any two adjacent letters are different). Taking into account the exponents with which the variables occur in t_1 and t_2 we see that the initial symbol in these terms is x_k or x_k^{-1} according to what the last letter in the corresponding trace of x_k is: ρ of λ. Thus, the last letters in $\operatorname{tr}_1(x_k)$ and $\operatorname{tr}_2(x_k)$ should be the same, say ρ, the other case being quite similar to consider.

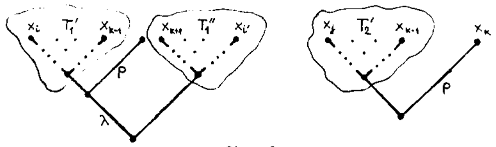

fig. 3

Let T_1' and T_2' be the subtrees of T_1 and T_2 such that $T_1' \circ x_k$ and $T_2' \circ x_k$ are also subtrees of T_1 and T_2. The variables in the peaks of T_1' are $x_j, \ldots, x_{k-1}$ $(x_i \leq k-1)$ and those in the peaks of T_2' are $x_j, \ldots, x_{k-1}$ $(j \leq k-1)$. Now t_1 begins with x_k and the next $k-i$ symbols in it are $x_i^{\pm 1}, \ldots, x_{k-1}^{\pm 1}$, in some order. The last but one letter of $\operatorname{tr}_1(x_k)$ being λ, there exists a subtree T_1'' of T_1 such that $(T_1' \circ x_k) \circ T_1''$ is a subtree of T_1. The variables in the peaks of T'' are $x_{k+1}, \ldots, x_{i'}$, $i' \geq k+1$ (see fig 3). Then the $(k-i+2)$th symbol in t_1 is one of $x_{k+1}^{\pm 1}, \ldots, x_{i'}^{\pm 1}$. If $i > j$ could hold, then since $k-i+2 \leq k-j+1$, the $(k-i+2)$th symbol in t_2 would be one of $x_j^{\pm 1}, \dots, x_{k-1}^{\pm 1}$, contrary to the conclusion of the. preceding sentence. In the same way the possibility $i < j$ fails, so we have $i = j$.

The initial segments of $k - i + 1 = k - j + 1$ symbols in t_1 and t_2 are $x_k t_1'$ and $x_k t_2'$, where t_1' and t_2' are the terms corresponding to the trees T_1' and T_2'. So $t_1' = t_2'$ must hold and the number of variables occurring in this law is less than in the law $t_1 = t_2 t_1' = t_2'$ is also non-trivial, because $T_1' \neq T_2'$ as a consequence of the condition (C) assumed to hold for $t_1 = t_2$. Thus we fell in contradictions with the assumption of minimality of the number of variables in $t_1 = t_2$. Hence the operation $x^{-1}y^{-1}$ does not satisfy any non-trivial weakened associativity law.

This finishes the proof of Theorem 1.

Remark on the case when $w(x,y)$ does not depend on both variables. To complete the discussion it remains to consider which weakened associativity laws are satisfied by the operations of the forms $x \circ y = w(x)$ and $x \circ y = w(y)$; the operations $x \circ y = \text{const.}$ clearly satisfy all the laws.

Assume that $x \circ y = w(x)$ satisfies the law $t_1 = t_2$ and let m_1 and m_2 be the numbers such that $\operatorname{tr}_i(x_1) = \lambda^{m_i}$ $(i = 1, 2)$. Then $t_i = w^{m_i}(x_1)$, where w^m denotes the operation w iterated n times. If $m_1 = m_2$, then clearly all operations $x \circ y = w(x)$ satisfy the law $t_1 = t_2$. We have still to analyze the possibility $m_1 \neq m_2$; for the sake of definiteness let us suppose $m_1 < m_2$. From $t_1 = t_2$ we infer $w^{m_2 - m_1}(w^{m_1}(x)) = w^{m_1}(x)$ and hence, by means of Lemma 3, $w^{m_2-m_1}(w^{m_1}(z)) = z$ for every $z \in F$. It is easy to prove that this equality holds only if $w(z) = z$ or $w(z) = z^{-1}$. The operation $x \circ y = x$ satisfies all weakens associativity laws, while $x \circ y = x^{-1}$ satisfies those for which the numbers m_1 and m_2 are both odd or even.

The final conclusion on satisfying weakened associativity laws by composite operations in free groups is the following:

- — the operations $x \circ y = a$, x, y, xay, yax $(a \in F)$ satisfy all the laws;
- — the operations $x \circ y = x^{-1}$, y^{-1} satisfy those laws for which the lengths of the traces $\operatorname{tr}_1(x_1)$ and $\operatorname{tr}_2(x_1)$, or $\operatorname{tr}_1(x_2)$ and $\operatorname{tr}_2(x_2)$, are at the same time odd or even;
- — the operations $x \circ y = w(x)$, $w(y)$ satisfy the laws for which $\operatorname{tr}_1(x_1) = \operatorname{tr}_2(x_1)$, or $\operatorname{tr}_1(x_n) = \operatorname{tr}_2(x_n)$ holds.

Part two

THEOREM 2. *Let G be a free nilpotent group of nilpotency class 2. If the operation $x \circ y = w(x,y)$ is associative, then $w(x,y)$ is of one of the following forms:*

$$a[(a,x)(x,y)(y,a)]^k, \quad x, y, xay[(x,a)(x,y)(a,y)]^k,$$

where (ξ, η) denotes the commutator $\xi^{-1}\eta^{-1}\xi\eta$ and $a \in F$ and $k \in Z$ are arbitrary.

Proof. Let $G = F/F_3$, where F is a free group and $F = F_1, F_2, F_3, \dots$ its lower central series. Let $w(x,y) \in F' = F * \langle x, y \rangle$. Our goal is to find all $w(x,y)$ such that

$$(1) \qquad w(w(x,y),z) = w(x,w(y,z)) \mod F_3$$

holds for every $x,y,z \in F$.

From $w(w(x,y),z) \equiv w(x,w(y,z)) \mod F_2$ after simple considerations it follows that

$$w(x,y) \equiv \overline{w}(x,y) \mod F_2,$$

where $\overline{w}(x,y)$ is of one of the following forms:

$$a, x, y, xay (a \in F)$$

(These are in fact all accosiative operations on the free Abelian group F/F_2). Now we have

$$w(x,y) \equiv \overline{w}(x,y)\psi(x,y) \mod F_3,$$

where $\overline{w}$ is of one of the above forms and $\psi(x,y)$ is an element of F_2'. We have to find all $\psi(x,y)$ for each of the above possibilities, provided $w(x,y)$ is associative modulo F_3.

Case 1: $\overline{w}(x,y) = a$. From the relations (1) and

$$\begin{aligned} w(w(x,y),z) &\equiv a\psi(a\psi(x,y),z) \mod F_3, \\ w(x,w(y,z)) &\equiv a\psi(x,a\psi(y,z)) \mod F_3 \end{aligned}$$

and the simple observation that

$$\psi(x\xi,y) \equiv \psi(x,y\xi) = \psi(x,y) \mod F_3$$

holds for all $\xi \in F_2$, $x,y \in F$, it follows that

$$\psi(a,z) \equiv \psi(x,a) \mod F_3 \tag{2}$$

holds for all $x,z \in F$. As an element of $F_2'(x,y)$ is of the form

$$\psi'(x,y) \equiv (x,b)(y,c)(x,y)^k \mod F_3,$$

where $b,c \in F$ and $k \in Z$. Now the relation (2) becomes

$$(a,b)(z,c)(a,z)^k \equiv (x,b)(a,c)(x,a)^k \mod F_3.$$

Putting here $x = z = 1$ we obtain $(a,b) \equiv (a,c)$ and

$$(z,c)(a,z)^k \equiv (x,b)(x,a)^k \mod F_3,$$

that is,

$$(z, ca^{-k}) \equiv (x, ba^k) \mod F_3.$$

The last relation holds only for $c = a^k$ and $b = a^{-k}$ so we get

$$\begin{aligned} \psi(x,y) &\equiv (x,a^{-k})(y,a)^k(x,y)^k \\ &\equiv [(a,x)(x,y)(y,a)]^k \mod F_3. \end{aligned}$$

It could be immediately checked that for any $a \in F$ and $k \in Z$

$$w(x,y) = a[(a,x)(x,y)(y,a)]^k$$

satisfies the associativity law (1).

Case 2: $\overline{w}(x,y) = x$ or $\overline{w}(x,y) = y$. For $\overline{w}(x,y) = x$ we have

$$w(w(x,y),z) \equiv x\psi(x,y)\psi(x\psi(x,y),z) \mod F_3,$$
$$w(x,w(y,z)) \equiv x\psi(x,y\psi(y,z)) \mod F_3,$$

which together with (1) implies $\psi(x,z) \equiv 1 \mod F_3$, so $w(x,y) = x \mod F_3$. The other possibility $\overline{w} = y$ gives the unique solution $w(x,y) \equiv y \mod F_3$.

Case 3: $\overline{w}(x,y) = xay$. Now we have

$$w(w(x,y),z) \equiv xay\psi(x,y)az\psi(xay\psi(x,y),z) \mod F_3,$$
$$w(x,w(y,z)) \equiv xayaz\psi(y,z)\psi(x,yaz\psi(y,z)) \mod F_3.$$

Putting this in (1), after considerations similar to those in Case 1 we obtain

$$\psi(x,a^{-1}) \equiv \psi(a^{-1},z) \mod F_3,$$

for every $x,z \in F$. Just as in Case 1, hence we get

$$\psi(x,y) \equiv [(x,a)(x,y)(a,y)]^k \mod F_3,$$

for some $k \in Z$. Again we immediately check every $w(x,y)$ of the form

$$w(x,y) = xay[(x,a)(x,y)(a,y)]^k \qquad (a \in F, k \in Z)$$

satisfies the relation (1).

This finishes the proof of Theorem 2. It remains only to note that

$$yax = xay[(x,a)(x,y)(a,y)]^{-1}$$

and that all operations of the form $xay[(x,a)(x,y)(a,y)]^k$ satisfy the group postulates – the identity element being a^{-1} and $a^{-1}x^{-1}a^{-1}$ the inverse of x.

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