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ON N-DIMENSIONAL IDEMPOTENT MATRICES

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An $n \times n$ matrix $A = (a_{ij})$ will be called a 2-dimensional matrix of order n. We shall study 2 m-dimensional idempotent matrices of order n with respect to an associative matrix product.

1. Introduction. We shall denote the set of all $n \times n$ matrices over a field F by $M_{2,n}(F)$ and the set of all $n \times n \times \cdots \times n = n^m$ matrices over F by $M_{m,n}(F)$. Any matrix $A = (a_{ij\ldots k})$ in $M_{m,n}(F)$ will be called an m-dimensional matrix of order n. For a determinant of an m-dimensional matrix, we refer [**2**, **3**, **6** and **7**]. Let $A = (a_{i_1 i_2 \ldots i_{2m}})$ and $B = (b_{j_1 j_2 \ldots j_{2m}})$ be members of $M_{2m,n}(F)$. We define a matrix product $AB = C = (c_{k_1 k_2 \ldots k_{2m}})$ as follows:

$$c_{k_1 k_2 \ldots k_{2m}} = \sum_{t_1=1}^{n} \sum_{t_2=1}^{n} \cdots \sum_{t_m=1}^{n} a_{k_1 k_2 \ldots k_m t_1 t_2 \ldots t_m} b_{t_1 t_2 \ldots t_m k_{m+1} \ldots k_{2m}}.$$

This matrix product is associative (see [**3**]) and with respect to this matrix product $AB = C$, $M_{2m,n}(F)$ forms a semigroup (and a ring). A is an idempotent if $AA = A$. We shall count the number of idempotents in the semigroup $M_{2m,n}(F)$, where F is a finite field, and we shall classify the idempotents.

2. The number of idempotents. Let S be a semigroup and let $a, b \in S$. We define aLb (aRb) to mean that a and b generate the same principal left (right) ideal of S. If aLb, we say that a and b are L-equivalent. By L_a we mean that the set of all elements of S which are L-equivalent to a. The join of the equivalence relations L and R is denoted by D. If X is a subset of the semigroup S, then we define $E(X) = \{x \in X : xx = x\}$. We need the following lemma.

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LEMMA 1 [**5**, Lemma 5]. *Let D, be the D-class of rank r in the semigroup $M_{2m,n}(F)$. Let $|F| = p$. Define $[p^k] = (p^k - 1)(p^k - p)\dots(p^k - p^{k-1}) = \prod_{i=1}^{k-1}(p^k - p^i)$. Then $|E(D_r)| = [p^n]/[p^r][p^{n-r}$. Let $t(r) = [p^n]/[p^r][p^{n-r}]$.*

THEOREM A. *The number of all idempotent matrices in $M_{2m,n}(F)$ is equal to t:*

$$t = \sum_{r=0}^{n^m} t(r).$$

Proof. If $A = (a_{ij\dots k}) \in M_{2m,n}(F)$ we identify A as $A' = (a'_{ij}) \in M_{2,n^m}(F)$. Then applying Lemma 1, we obtain the desired result.

Let S be a semigroup and let $a \in S$. We define $V(a) = \{x \in S : axa = a$ and $xax = x\}$. We need the following lemma to prove Theorem B.

LEMMA 2 [**4**, Theorem 1]. *If $A \in M_{2,n}(F)$, then the cardinal number of the inverse set $V(A)$ is equal to $|F|^{2r(n-r)}$, where r is the rank of the matrix A.*

Let r be an integer such that $0 \leq r \leq n^m$. We have the following. (We assume that $S = M_{2m,n}(F)$ and $|F| = p$).

THEOREM B. *If $A \in D_r$, then $|V(A)|$ is given by t, where $t = p^{2r(n^m - r)}$.*

Proof. In the semigroup $M_{2m,n}(F)$ there are $n^m + 1$ D-classes D_r of rank r. (See the proof of Theorem A). Applying Lemma 2 and replacing n by n^m in $|F|^{2r(n-r)}$, we obtain the desired result. (Note that $|F| = p$).

3. Classification of idempotents. We define $V_r(n) = \{(i_1, i_2, \dots, i_r) :$ i_j are positive integers such that $1 \leq i_j \leq n\}$. Let $A = (a_{ij,\dots k}) \in M_{2m,n}(F)$. For any entry $a_{ij\dots k}$ of A, there exists $\pi \in V_{2m}(n)$ such that $(ij\dots k) = \pi$; we write $a_{ij\dots k} = a_\pi$. For an element $\pi = (\pi_1, \pi_2, \dots, \pi_m) \in V_m(n)$, we write $\pi\pi$ to mean that $\pi\pi = (\pi_1, \pi_2, \dots, \pi_m, \pi_1, \pi_2, \dots, \pi_m) \in V_{2m}(n)$. We define a matrix $E_\pi = (a_{ij\dots k})$ as follows: $a_\pi = 1$ and $a_\mu = 0$ for all $\mu \in V_2m(n)$ such that $\pi \neq \mu$. We can see that $E_{\lambda\lambda}$ is an idempotent ($\lambda \in V_m(n)$) and we may call $E_{\lambda\lambda}$ a primitive idempotent. Define $I = \sum_{\lambda \in V_M(n)} E_{\lambda\lambda}$. Then we can see that $IA = AI = A$ for all $A \in M_{2m,n}(F)$. We denote the zero matrix by 0. Then for $A = (a_{ij\dots k})$ we have $A = \sum_{\pi \in V_{2m}(n)} a_\pi E\pi$ $(a\pi \in F)$. Define $(A)_\pi = a_\pi$ as the π-entry of A, and define $D(A) = \{\lambda \in V_m(n) : (A)_{\lambda\lambda} \neq 0\}$.

Types of idempotents. Let A be an idempotent. A is called an idempotent of type I if either $a_{\lambda\lambda} = 1$ and $a_{\lambda\mu} = 0$ $(\lambda \neq \mu)$ for all $\lambda \in D(A)$ or if $a_{\lambda\lambda} = 1$ and $a_{\mu\lambda} = 0$ $(\mu \neq \lambda)$ for all $\lambda \in D(A)$. A is called an idempotent of type II if A is not an idempotent of type I and if $a_{\lambda_i\lambda_i} = 1$.

An idempotent A which is neither of type I nor type II will be called an idempotent of type III. We assume the zero matrix 0 is an idempotent of type I.

We consider the idempotents of type I. Let $F_k = \{\lambda_1, \lambda_2, \dots, \lambda_k\}$ be a non-empty subset of $V_m(n)$. Let U_i be a subset of $V_m(n) \setminus F_k = \{x \in V_m(n) : x \notin F_k\}$, $(i = 1, 2, \dots, k)$. Then:

$$A = \left(E_{\lambda_1\lambda_1} + \sum_{\pi \in U_1} x_{\pi\lambda_1} E_{\pi\lambda_1}\right) + \left(E_{\lambda_2\lambda_2} + \sum_{\pi \in U_2} x_{\pi\lambda_2} E_{\pi\lambda_2}\right) + \cdots + \\ + \left(E_{\lambda_k\lambda_k} + \sum_{\pi \in U_k} x_{\pi\lambda_k} E_{\pi\lambda_k}\right)$$

and

$$B = \left(E_{\lambda_1\lambda_1} + \sum_{\pi \in U_1} x_{\lambda_1\pi} E_{\lambda_1\pi}\right) + \left(E_{\lambda_2\lambda_2} + \sum_{\pi \in U_2} x_{\lambda_2\pi} E_{\lambda_2\pi}\right) + \cdots + \\ + \left(E_{\lambda_k\lambda_k} + \sum_{\pi \in U_k} x_{\lambda_k\pi} E_{\lambda_k\pi}\right)$$

where $x_\mu \in F(\mu \in V_{2m}(n))$. Now we can state the following theorem.

THEOREM C. *Every idempotent of type I is either of the form A or the form B. The number of all idrmpotents of type I in $M_{m,n}(F)$ is given by t:*

$$t = 2\sum_{k=0}^{n^m} \binom{n^m}{k} p^{k(n^m-k)} - 2^{n^m}, \quad (p = |F|).$$

Proof. Let U, V, U_i and V_i be subsets of the set $V_m(n) \setminus F_k$, where $F_k = \{\lambda_1, \lambda_2, \dots, \lambda_k\} \subset V_m(n)$. Let $C = E_{\lambda_1\lambda_1} + E_{\lambda_2\lambda_2} + \cdots + E_{\lambda_k\lambda_k}$,

$$D = \sum_{\pi_i \in U_i}^{k} x_{\lambda_i\pi_i} E_{\lambda_i\pi_i}, \qquad E = \sum_{\mu_u \in V_i}^{k} x_{\mu_i\lambda_i} E_{\mu_i\lambda_i}$$

and $G = \sum_{\substack{\mu \in U \\ \nu \in V}} x^{\mu\nu} E_{\mu\nu}$. Assume that $X = C + D + G$ and $Y = C + E + G$ are idempotents of type I. The following is the product table for C, D, E and G.

	C	D	E	G
C	C	D	O	O
D	O	O	D'	D'
E	E	G'	O	O
G	O	O	E'	G'

In the table, $DE = D'$ means that DE takes the form D but $D' \neq D$. Similarly for E' and G'.

Then from $XX = X$ and $YY = Y$ we have that $X = C + D = B$ and $Y = C + E = A$. We now consider the number t of all idempotents of type I. We note that $|V_m(n)| = n^m$. The number of the ordered sets F_k is equal to $\binom{n^m}{k}$; the number of all possible terms $\sum_{\pi \in V_i} x_{\pi\lambda_i} E_{\pi\lambda_i}$ $(i = 1, 2, \ldots, k)$

$$\left(\sum_{\pi \in U_i} x_{\lambda_i \pi_i} E_{\lambda_i \pi_i} \text{ in } B \right) \text{ in } A \text{ is equal to } p^{k(n^m - k)}.$$

In the expression of t, the factor 2 appears because of the two forms A and B and, because we counted the number of terms $E_{\lambda_1\lambda_1} + E_{\lambda_2\lambda_2} + \cdots + E_{\lambda_k\lambda_k}$ twice in the first term for t, we must subtract 2^{n^m}.

Remark. For 2-dimensional matrices, analogous results of Theorem A and Theorem B are respectively Lemma 5 [**5**] and Theorem 1 [**4**]. For Theorem C, we do not have any reference, but we find that t in Theorem C is correct for $M_{2,3}(Z/(2))$, where Z is the set of all integers and $|Z/(2)| = 2$. For $M_{2,3}(Z/(2))$, $t = 44$ from our D-class table of the semigroup.

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