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REPRODUCTIVITY OF SOME EQUATIONS OF ANALYSIS, II

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This note is a direct continuation of (1).

4. Equations for linear functionals

4.1. Introduction. Let V be a vector space over a field S and let $A_1, \dots, A_n$ be linear functionals mapping V into S. In this part we shall consider various equations in $x \in V$ of the form

$$b + \sum_{k=1}^{n} a_k A_k x = 0 \tag{4.1.1}$$

and

$$x = b + \sum_{k=1}^{n} a_k A_k x \tag{4.1.2}$$

where b, $a_1, \dots, a_n \in V$ are given. By analogy with integral equations, equations of the form (4.1.1) will be called equations of the first kind, and equations of the form (4.1.2) will be called equations of the second kind. They are homogeneous if $b = 0$.

4.2. Equations of the first kind. 4.2.1. Let V be a vector space over and let $A : V \to S$ be a linear functional on V. Consider the equation in x:

$$Ax = 0. \tag{4.2.1}$$

Suppose that there exists $x_0 \in V$ such that $Ax_0 \neq 0$; otherwise (4.2.1) holds for all $x \in V$. Then, since A is linear, we have $x_0 \neq 0$, and the equation (4.2.1) is equivalent to the equation

$$x = x + \lambda x_0 Ax \qquad (\lambda \in S; \lambda \neq 0), \tag{4.2.2}$$

i. e. to the equation

$$x = Fx,$$

where

$$Fx = x + \lambda x_0 Ax.$$

The condition for reproductivity $F^2 = F$ becomes

$$(1 + \lambda Ax_0)Ax = 0.$$

Hence, for $\lambda = -1/Ax_0$, the equation (4.2.2) is reproductive, and its general solution, and so the general solution of (4.2.1) is given by

$$x = t - \frac{At}{Ax_0}x_0, \tag{4.2.3}$$

where $t \in V$ is arbitrary.

Example 1. If $g \not\equiv 0$ is a given continuous function on $[a, b]$, then there exists a function $h \in C[a, b]$ such that $\int g(x)h(x)dx \neq 0$, and the general solution of the equation in f:

$$\int_a^b g(x)h(x)dx = 0,$$

is given by

$$f(x) = T(x) - Bigg(\int_a^b g(x)T(x)dx\Bigg)\Bigg(\int_a^b g(x)h(x)dx\Bigg)^{-1} h(x),$$

where $T \in C[a, b]$ is arbitrary.

So, for example, the general solutions of the equations

$$\int_a^b f(x)dx = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} f(x)\sin x dx = 0$$

are

$$f(x) = T(x) - \frac{1}{b-a}\int_a^b T(x)dx, \quad f(x) = T(x) - \frac{x}{2\pi}\int_{-\pi}^{\pi} T(x)\sin x dx,$$

respectively, where in both cases T is arbitrary; in the first case $T \in C[a, b]$, and in the second $T \in C[-\pi, \pi]$.

Example 2. The form of the general solution (4.2.3) of the equation (4.2.1) shows that there exist different formulas for the general solution; the element x_0 is any element of V such that $Ax_0 \neq 0$, and by varying x_0 we obtain different general solutions of (4.2.1) Naturally, all the obtained formulas are equivalent.

Nevertheless, in applications those formulas may lead to different conclusions. The following example is an illustrations of that fact.

The well known Rolle's theorem states that if $f \in C[a,b]$, f is differentiable in (a,b) and if $f(a) = f(b)$, then there exists $c \in (a,b)$ such that $f'(c) = 0$. In order to remove the third supposition $f(a) = f(b)$, we solve the equation

$$f(a) - f(b) = 0 \qquad (a \neq b), \tag{4.2.4}$$

which is an equation of the form (4.2.1). Since for the function $f(x) = x$ we have $f(a) - f(b) = a - b \neq 0$, the general solution of (4.2.5) is

$$f(x) = T(x) - \frac{T(a) - T(b)}{a - b} x. \tag{4.2.5}$$

Applying Rolle's theorem to the function f, defined by (4.2.5), we obtain the Lagrange mean-value theorem: If $T \in C[a,b]$, and if T is differentiable in (a,b), then there exists $c \in (a,b)$ such that

$$\frac{T(a) - T(b)}{a - b} = T'(c).$$

However, the general solution of (4.2.4) can also be written in the form

$$f(x) = T(x) - \frac{T(a) - T(b)}{S(a) - S(b)} S(x), \tag{4.2.6}$$

where $S(a) \neq S(b)$. Applying Rolle's theorem to the function f defined by (4.2.6) we obtain the Cauchy mean-value theorem: If $T, S \in C[a,b]$, if they are differentiable in (a,b), and if $S(a) \neq S(b)$, then there exists $c \in (a,b)$ such that

$$T'(c) = \frac{T(a) - T(b)}{S(a) - S(b)} S'(c), \tag{4.2.7}$$

The additional hypothesis $S'(x) \neq 0$ for $x \in (a,b)$, which implies $S(a) - S(b) \neq 0$, enables us to write (4.2.7) in the familiar form

$$\frac{T(a) - T(b)}{S(a) - S(b)} = \frac{T'(c)}{S'(c)}.$$

Example 3. Suppose that $f \in C[a,b]$. The general solution of the equation in f:

$$\frac{1}{b-a} \int_a^b f(x)dx = f\left(\frac{a+b}{2}\right)$$

is given by

$$f(x) = T(x) - \frac{12}{(b-a)^2} \left(\frac{1}{b-a} \int_a^b T(x)dx - T\left(\frac{a+b}{2}\right) \right) x^2.$$

Example 4. Let V be the set of all real functions defined on $[a, b]$, such that for a fixed $x_0 \in (a, b)$ the limit $\lim\limits_{x \to x_0} f(x)$ exists. Then $f \in V$ is continuous at x_0 if and only if

$$f(x) = T(x) + (T(xa) - \lim_{x \to x_0} T(x)) \operatorname{sgn}(x - x_0)^2,$$

where $T \in V$ is arbitrary.

4.2.2. We now turn to the nonhomogeneous equation

$$Ax = \alpha \tag{4.2.8}$$

where $A : V \to S$ is a linear functional, $\alpha \in S$ is given, and $\alpha \neq 0$. Suppose that $x_0 \in V$ is such that $Ax_0 \neq 0$. Then the general solution of (4.2.8) is

$$x = \frac{\alpha}{Ax_0} x_0 + t - \frac{At}{Ax_0} x_0,$$

where $t \in V$ is arbitrary.

Notice that the equation (4.2.8) is possible if and only if there exists $x_0 \in V$ such that $Ax_0 \neq 0$. Hence, if the equation (4.2.8) is possible for a fixed $\alpha \in S$, it is possible for all $\alpha \in S$.

Example 5. Let c be the set of all convergent real sequences. The general solution of, the equation

$$\lim x_n = \alpha$$

is given by

$$x_n = \frac{\alpha}{\beta} x_n^0 + t_n - \frac{\lim t_n}{\lim x_n^0} x_n^0, \tag{4.2.9}$$

where $(t_n) \in c$ is arbitrary, and $\lim x_n^0 = \beta$. In particular we may take $(x_n^0) = (\beta)$, and (4.219) takes the simpler form

$$x_n = \alpha + t_n - \lim t_n.$$

Example 6. Let F be the set of all complex analytic functions for which $z = a$ is a regular point or an isolated singularity. The general solution of the equation in $f \in F$:

$$\operatorname*{Res}_{z=a} f(z) = \alpha$$

is

$$f(z) = \frac{\alpha}{z - a} + T(z) - \frac{1}{z - a} \operatorname*{Res}_{z=a} T(z),$$

where $T \in F$ is arbitrary,

Example 7. Let V be unitary vector space over S. The general solution of the equation in x:

$$(x, a) = \alpha$$

where $a \in V$ and $\alpha \in S$ are fixed, is given by

$$x = \frac{\alpha a}{(a,a)} + t - \frac{(t,a)}{(a,a)}a,$$

where $t \in V$ is arbitrary.

4.2.3. Let $A_1, \ldots, A_n$ be linear functionals on a vector space V over S and consider the equation in x:

$$\sum_{k=1}^{n} a_k A_k x = 0, \tag{4.2.10}$$

where $a_1, \ldots, a_n \in V$ are given. Since those vectors can be taken to be linearly independent, the equation (4.2.10) splits into the system

$$A_1 x = 0 \wedge \cdots \wedge, A_n x = 0, \tag{4.2.11}$$

which consists of n equations of the form (4.2.1).

Example 8. The integral equation

$$\int_{-1}^{1} (5tu^3 + 4t^2 u) x(u) du = 0 \qquad (x \in C[-1,1]) \tag{4.2.12}$$

can be replaced by the system

$$\int_{-1}^{1} u^3 x(u) du = 0 \wedge \int_{-1}^{1} u x(u) du = 0. \tag{4.2.13}$$

The general solution of the first equation of this system is

$$x(t) = S(t) - \frac{5}{2} t \int_{-1}^{1} t^3 S(t) dt \qquad (S \in C[-1,1]) \tag{4.2.14}$$

and substituting (4.2.14) into the second equation of the system (4.2.13) we obtain the equation for S:

$$\int_{-1}^{1} \left(u - \frac{5}{3} u^3 \right) S(u) du = 0,$$

with the general solution

$$S(t) = T(t) + \frac{105}{8} t^3 \int_{-1}^{1} \left(u - \frac{5}{3} u^3 \right) T(u) du, \tag{4.2.15}$$

where $T \in C[-1,1]$ is arbitrary.

Combining (4.2.14) and (4.2.15) we obtain the general solution of the equation (4.2.9):

$$x(t) = T(t) + \frac{l5t}{8}(7t^2 - 5)\int_{-1}^{1} tT(t)dt + \frac{35t}{8}t(3 - 5t^2)\int_{-1}^{1} t^3T(t)dt \tag{4.2.16}$$

where $T \in C[-1, 1]$ is arbitrary.

Notice that (4.2.16) is not only the general solution of (4.2.12), but of any equation of the form

$$\int_{-1}^{1} (A(t)u^3 + B(t)u)x(u)du = 0,$$

where A and B are linearly independent functions.

Remark. The nonhomogeneous equation

$$\sum_{k=1}^{n} a_k A_k x = b \quad (\neq 0)$$

can be treated in a similar manner. Indeed, from the equation itself follows that b must be of the form $\sum\limits_{k=1}^{n} \alpha_k a_k$; otherwise the equation has no solutions. Hence, it can be reduced to

$$\sum_{k=1}^{n} a_k(A_k x - \alpha_k) = 0,$$

and the last equation splits into the system

$$A_k x = \alpha_k \qquad (k = 1, \dots, n).$$

Example 9. The equation

$$\int_{-1}^{1} (5tu^3 + 4t^2u)x(u)du = 16t^2$$

splits into the system

$$\int_{-1}^{1} u^3x(u)du = 0 \wedge \int_{-1}^{1} ux(u)du = 4,$$

and its general solution is easily obtained. It is:

$$x(t) = T(t) - \frac{105}{2}t^3 + \frac{105}{8}t^3 \int_{-1}^{1} \left(u - \frac{5}{3}u^3\right) T(u)du,$$

where $T \in C[-1,1]$ is arbitrary.

Again, the obtained solution is also the general solution of the equation

$$\int_{-1}^{1} (A(t)u^3 + B(t)u)x(u)du = 4B(t),$$

where A and B are linearly independent functions.

4.2.4. Let $A_k : V \to S$ $(k = 1, \ldots, n)$ be linear functionals on V, and suppose that there exist $a_k \in V$ $(k = 1, \ldots, n)$ such that $A_j a_j = \delta_{ij}$ $(i, j = 1, \ldots, n)$ where δ_{ij} is the Kronecker delta.

Then the general solution of the system

$$A_k x = 0 \qquad (k = 1, \ldots, n)$$

is

$$x = t - \sum_{k=1}^{n} (A_k t) a_k, \tag{4.2.17}$$

where $t \in V$ is arbitrary.

Hence, if $A_1, \ldots, A_n$ is a complete set of linear functionals (i. e. $A_k x = 0$ for $k = 1, \ldots, n$ implies $x = 0$), from (4.2.17) follows the representation

$$t = \sum_{k=1}^{n} (A_k t) a_k \tag{4.2.18}$$

for arbitrary $t \in V$. Moreover, the condition $A_i a_j = \delta_{ij}$ implies that the vectors a_k are linearly independent, which means that the representation (4.2.18) is unique.

Example 10. Let P_n be the set of all real polynomials with degree $\leq n$. Then if $P \in P_n$, the functionals $A_1, \ldots, A_{n+1}$ defined by

$$A_k P = P(xk) \qquad (k = 1, \ldots, n+1)$$

where $x_1, \ldots, x_{n+1}$ are distinct real numbers, form a complete set. Moreover, for the polynomials $a_k \in P_n$ $(k = 1, \ldots, n+1)$ defined by

$$a_k(x) = \frac{(x - x_1) \ldots (x - x_{k-1})(x - x_{k+1}) \ldots (x - x_{n+1})}{(x_k - x_1) \ldots (x_k - x_{k-1}(x_k - x_{k+1}) \ldots (x_k - x_{n+1})} (k = 1, \ldots, n+1)$$

we have $A_i a_j = \delta_{ij}$. Hence, if $P \in P_n$ we obtain the Lagrange interpolation formula

$$P(x) = \sum_{k=1}^{n+1} \frac{(x - x_1) \ldots (x - x_{k-1})(x - x_{k+1}) \ldots (x - x_{n+1})}{(x_k - x_1) \ldots (x_k - x_{k-1})(x_k - x_{k+1}) \ldots (x_k - x_{n+1})} P(x_k).$$

Again, the functionals F_k defined by $F_kP = P^{(k)}(a)$ for $k = 0, 1, \ldots, n$, also form a complete set and the corresponding polynomials a_k such that $F_i a_j = \delta_{ij}$ are defined by $a_k(x) = (x-a)^k/k!$. This implies the Taylor expansion

$$P(x) = \sum_{k=0}^{n} \frac{(x-a)^k}{k!} P^{(k)}(a).$$

Example 11. Let V be an n-dimensional unitary vector space with an orthonormal basis $(e_1, \ldots, e_n)$. Then the system of linear functionals $A_1, \ldots, A_n$, defined by

$$A_k x = (x, e_k) \qquad (k = 1, \ldots, n)$$

is a complete system. Since $A_i e_j = \delta_{ij}$, we obtain the familiar representation

$$x = \sum_{k=1}^{n} (x, e_k) e_k.$$

Remark. Similar conclusions can be obtained in the case when $A_1, A_2, \ldots$ is a countable set of linear functionals, but in this case it is necessary to examine the convergence of the series $\sum_{k=1}^{\infty} (A_k t) a_k$. As special cases we mention the Taylor expansion for analytic functions, the Fourier expansion for elements of a Hilbert space, etc.

Remark. It is interesting to note that the representation (4.2.18) is itself reproductive; namely, if we denote the right hand side of (4.2.18) by Ft, then $F^2 = F$.

4.2.5. At the end of this section we mention one more possibility, of a rather formal nature. Suppose that V is the space of all functions which map a vector space U into the scalar field S, and that $A : V \to V$ is a linear operator. The equation in $f \in V$:

$$Af = 0 \tag{4.2.19}$$

is a functional equation. However, for a fixed $a \in U$ the expression $(Af)(a)$ defines a linear functional on V, and the equation

$$(Af)(a) = 0 \tag{4.2.20}$$

can be solved by the method exposed here. Suppose that S_a is the set of all solutions of the equation (4.2.20). Then if S denotes the set of all solutions of the equation (4.2.19), we have, formally,

$$S = \bigcap_{a \in U} S_a.$$

Example 12. If S is the set of all solutions of the equation

$$f(x+1) = f(x) \qquad (f : R \to R) \tag{4.2.21}$$

then, since the general solution of the equation

$$f(a+1)=f(a) \qquad (a\in R \text{ is fixed})$$

is

$$f(x)=T(x)-(T(a+1)-T(a))x \qquad (T:R\to R \text{ arbitrary})$$

we have

$$S=\bigcap_{a\in R}\{f\,|\,f(x)=T(x)-(T(a+1)-T(a))x;\ T\in R^R \text{ arbitrary}\}.$$

The set S defined by (4.2.22) gives one more formal expression for the general solution of (4.2.21).

4.3. Equations of the second kind. Let V be a vector space over S and let $A: V\to S$ be a linear functional on V. Consider the equation in x:

$$x=b+aAx, \tag{4.3.1}$$

where $a,b\in V$ are given. From (4.3.1) follows

$$(1-Aa)Ax=Ab. \tag{4.3.2}$$

We distinguish between the following cases:

(i) $Aa\neq 1$. Then from (4.3.2) follows

$$Ax=\frac{Ab}{1-Aa},$$

which, substituted into (4.3.1), gives the unique solution of that equation:

$$x=b+\frac{Ab}{1-Aa}a.$$

(ii) $Aa=1$, $Ab\neq 0$. Then the equation (4.3.1) has no solutions.

(iii) $Aa=1$, $Ab=0$. Then it is easily verified that the equation (4.3.1) is reproductive, and hence its general solution is

$$x=b+aAt, \tag{4.3.3}$$

where $t\in V$ is arbitrary.

Remark. In the special case $b=0$, i. e. in the case of the homogeneous equation

$$x=aAx \tag{4.3.4}$$

we have the following possibilities:

(i) $Aa\neq 1$. The trivial solution $x=0$ is the only solution of the equation (4.3.4).

(ii) $Aa = 1$. The equation (4.3.4) is reproductive, and its general solution is

$$x = aAt, \tag{4.3.5}$$

where $t \in V$ is arbitrary.

Remark. Unless $Ax = 0$ for all $x \in V$, A maps V onto S, and hence in the solutions (4.3.3) and (4.3.5) At can be replaced by α where $\alpha \in S$ is arbitrary.

In a similar manner we may treat the equation in x:

$$x = b + \sum_{k=1}^{n} a_k A_k x, \tag{4.3.6}$$

where $b, a_1, \ldots, a_n \in V$ are given, and $A_k : V \to S$ $(k = 1, \ldots, n)$ are linear functionals on V. We again suppose that $a_1, \ldots, a_n$ are linearly independent.

Now, from (4.3.6) follows

$$A_k x = A_k b + \sum_{i=1}^{n} (A_k a_i)(A_i x) \qquad (k = 1, \ldots, n) \tag{4.3.7}$$

and this is a linear system in $A_1 x, \ldots, A_n x$. Let $A = \|A_i a_j\|_{n\times n}$, $B = \|A_k b\|_{n\times 1}$. If $\det(I - A) \neq 0$, the system (4.3.7) has a unique solution, $(\alpha_1, \ldots, \alpha_n)$ say, and the equation (4.3.6) has the unique solution $x = b + \sum_{k=1}^{n} \alpha_k a_k$.

If $\det(I - A) = O \wedge \operatorname{rank}(I - A) < \operatorname{rank}\|I - A|B\|$, the system (4.3.7) has no solutions, implying that the equation (4.3.6) has no solutions.

If $\det(I - A) = 0 \wedge \operatorname{rank}(I - A) = \operatorname{rank}\|I - A|B\|$, then some of the $A_k x's$ can be expressed as linear combinations of others. When this is done, the equation (4.3.6) becomes

$$x = b + \sum_{k=1}^{m} b_k A_{i_k} x \qquad (i_k \in \{1, \ldots, n\}),$$

where $m < n$, and it is easily verified that it is a reproductive equacion with the general solution

$$x = b + \sum_{k=1}^{m} b_k A_{i_k} t, \qquad (t \in V \text{ arbitrary})$$

or equivalently,

$$x = b + \sum_{k=1}^{m} \alpha_k b_k \qquad (\alpha_k \in S \text{ arbitrary}).$$

Remark. Notice that the equation (4.3.6) implies that its solution x, if it exists, must be of the form

$$x = b + \sum_{k=1}^{n} \alpha_k a_k.$$

The scalars α_k are obtained by substituting (4.3.8) into (4.3.6) and equating coefficients. Though this is a simpler method, we have emphasized here the method which uses the notion of reproductivity.

Example 13. Let $V = C[a,b]$, and let $A_k x = \int_a^b b_k(u)x(u)du$. The equation (4.3.6) becomes

$$x(t) = b(t) + \int_a^b \left(\sum_{k=1}^n a_k(t) b_k(u) \right) x(u)du,$$

and we see that the complete theory of Fredholm integral equations of the second kind with degenerate kernel is a consequence of the above result. Concrete examples of such equations need not be given here.

Example 14. Suppose that f is an integrable function on $[0,1]$ and consider the equation in f:

$$f(x) = x(x-1)\int_0^1 f(x)dx + xf(1) - (x-1)f(0).$$

This is a homogeneous equation and it has the trivial solution $f(x) = 0$. We look for nontrivial solutions.

From (4.3.9) follows

$$\int_0^1 f(x)dx = -\frac{1}{6}\int_0^1 f(x)dx + \frac{1}{2}f(1) + \frac{1}{2}f(0); \quad f(1) = f(1); \quad f(0) = f(0).$$

Hence,

$$\int_0^1 f(x)dx = \frac{3}{7}(f(1) + f(0)), \tag{4.3.10}$$

and substituting (4.3.10) into (4.3.9), we obtain the reproductive equation

$$f(x) = \frac{1}{7}(3x^2 + 4x)f(1) + \frac{1}{7}(3x^2 - 10x + 7)f(0),$$

with the general solution

$$f(x) = (3x^2 + 4x)T(1) + (3x^2 - 10x + 7)T(0),$$

where T is an arbitrary function, integrable on $[0,1]$ or equivalently,

$$f(x) = (3a + 3b)x^2 + (4a - 10b)x + 7b,$$

where $a(= T(1))$ and $b(= T(0))$ are arbitrary real numbers.

By a direct verification we see that (4.3.11) satisfies (4.3.9) which means that (4.3.1 I) is the general solution of the given equation (4.3.9).

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