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LEVEL SETS OF POLYNOMIALS IN SEVERAL* REAL VARIABLES

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By generalizing the concept of homogeneous polynomial and by adapting Cauchy's technique for obtaining bounds on the zeros of polynomials in one complex variable, the level surfaces of a real polynomial in E^n are studied with respect to their intersection with certain curves, including all lines, passing through the origin. In addition, it is shown that the equipotential surface of any axisymmetric harmonic polynomial in E^3 is unbounded if and only if it is asymptotic to a finite union of cones each of which is parallel to a cone having the origin as its vertex.

This paper extends results obtained by M. Marden and P. A. McCoy in 1976.

THEOREM 1. *Let H be a real-valued polynomial in n variables, $\sigma \in E^n$, α. a real number and q an n-tuple of positive integers. Then*

$$L_\alpha(H) \cap (0,\infty)_{\sigma,q} \subseteq (c_{1,q}(\sigma,q), c_2(\sigma,q))_\sigma,$$

where $L_\alpha(H)$ denotes the level set $\{\sigma \in E^n : H(\sigma) = \alpha\}$,$(a,b)_{\sigma,q}$ denotes the image of the open interval (a,b), $0 \le a < b \le \infty$, under the mapping $\mapsto \langle r^{1/q_i}\sigma_i\rangle$ and tlle endpoints c_j are defined in section 3.

THEOREM 2. *The equipotential surface of any axisymmetric harmonic polynomial in E^3*

$$\sum_{k=0}^{n} a_k r^k P_k(\cos\theta), \quad a_n \neq 0,$$

is unbounded if and only if it is asymptotic to a finite union $\bigcup c_i$ of cones each of which is parallel to a cone having the origin as its vertex. Moreover, each c_i has the origin as its vertex if and only if $a_{n-1} = 0$.

1. Basis notation. Let E^n denote n dimensional Euclidean space. The letters a and τ denote points of E^n. Alternatively denote a point σ of E^n by using

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vector notation $\langle \sigma_i \rangle$, where σ_i denotes the i-th component of σ. The scalar product of two points σ and τ of E^n will be denoted by $\sigma \cdot \tau$ and the origin of E^n by $\vec{0}$.

For any set S, let S^n denote the n-fold Cartesian product of the set S with itself. Let Z_+ denote the set of positive integers. For z in Z_+ let $\vec{z} = (z, z, \ldots, z) \in (Z_+)^n$.

2. Generalized homogeneous polynomials. Marden and McCoy's extension of Cauchy's technique involves writing polynomials in several variables as sums of homogeneous polynomials and then using the special properties of homogeneous polynomials to obtain estimates of the locations of the zeros of the general polynomial. This method does not distinguish between the two polynomials $h_1(\sigma) = \sigma_1^2 + \sigma_1^4 + \sigma_2^4$ and $h_2(\sigma) = \sigma_1^2 + \sigma_2^2 + \sigma_1^4$, both being viewed as a sum of homogeneous polynomials of degree 2 and 4. In the following paragraph the concept of homogeneous polynomial is generalized in such a way that if the methods of Marden and McCoy are applied using this more general concept then polynomials such as h_1 and h_2 are distinguished resulting in better estimates.

Since a polynomial $\sum_p \alpha_p \sigma_1^{p_1} \ldots \sigma_n^{p_n}$ in n variables is homogeneous if and only if there exists a positive integer d such that each index p satisfies the equation $p_1 + p_2 + \cdots + p_n = d$, the concept of homogeneity may be generalized by defining the polynomial $\sum_p \alpha_p \sigma_1^{p_1} \ldots \sigma_n^{p_n}$ to be q-homogeneous if and only if there exists a point q of $(Z_+)^n$ and a positive rational number I such that each index p satisfies the equation $p_1/q_1 + \cdots + p_n/q_n = I$, Call I the index of q-homogeneity of the polynomial. It is straightforward to show that a polynomial h is q-homogeneous of index I if and only if

$$h\left(\left\langle r^{1/q_i} \sigma_i \right\rangle\right) = r^I h(\sigma) \tag{1}$$

for all $r > 0$ and all σ. Returning to the examples of the preceding paragraph we see that h_1 may be viewed as the sum of $\sigma_1^2 + \sigma_2^4$ and σ_1^4, q-homogeneous polynomials of indices 1 and 2 respectively, for $q = (2, 4)$, whereas h_2 may be viewed as the sum of $\sigma_1^2 + \sigma_2^2$ and σ_1^4, q-homogeneous polynomials of indices 1 and 2 respectively, for $q = (2, 2)$. Further discussion of this generalization of the concept of homogeneous polynomial can be found in [**1**].

3. Definitions of $c_j(\sigma, q)$. Throughout this section let H be a given real-valued polynomial in n variables, α be a given real number and $q \in (Z_+)^n$. It is straightforward to show that $H(\sigma) - \alpha$ may be written in exactly one way as a sum $\sum_{j=1}^{k} h_j(\sigma)$ of polynomials where each polynomial h_j is q-homogeneous and where $I_j(q)$, the index of q-homogeneity of h_j, is a strictly increasing function of j. Writing the rational numbers numbers $I_1, I_2, \ldots, I_k$ in reduced form, let $l = l(q)$ denote the least rational common multiple of their denominators. For $i = 1$, k let

$M_i(\sigma, q) = \max_{j \neq i} |h_j/h_i|(\sigma)$. Define $c_j(\sigma, q)$ by

$$\begin{aligned} c_1(\sigma, q) &= (1 + M_1(\sigma, q))^{-l} \text{ if } h_1(\sigma) \neq 0, \\ &= 0 \qquad \text{otherwise} \end{aligned}$$

and

$$\begin{aligned} c_2(\sigma, q) &= (1 + M_k(\sigma, q))^{-l} \text{ if } h_k(\sigma) \neq 0, \\ &= \infty \qquad \text{otherwise} \end{aligned}$$

4. Corollaries and remarks. The following corollary is given in order to relate Theorem 1 to the corresponding work of Marden and McCoy.

Corollary 1. *Let α be a real number, H be a real-valued polynomial written as a sum $\sum_{j \geq v} h_j$ of homogeneus polynomials, where the degree of h_j is j. If $|\sigma| = 1$ and $h_v(\sigma) \neq 0$, then*

$$L_\alpha(H) \cap (0, \infty)_{\sigma, \vec{v}} \subseteq (c_1(\sigma, \vec{v}), c_2(\sigma, \vec{v}))_{\sigma, \vec{v}}. \tag{2}$$

Corollary 1, a specialization of Theorem 1 of this paper, is a strengthening of Theorem 1 of [**2**]. In fact, the latter theorem is equivalent to the result obtained by weakening the conclusion of Corollary 1 to read

$$L_\alpha(H) \cap (0, \infty)_{\sigma, \vec{v}} \subseteq (c_1(\sigma, \vec{v})^{v/l}, c_2(\sigma, \vec{v})^{v/l})_{\sigma, \vec{v}}. \tag{3}$$

That this is a weaker conclusion follows from the three inequalities $O < c_1 < 1$, $1 < c_2$, $l \leq v$. That l can be strictly less that v is illustrated by the example $H(\sigma) = 4\sigma_1^2 - \sigma_1^4 - \sigma_2^4$, $\alpha = 0$, $\sigma = (1/\sqrt{2}, 1/\sqrt{2})$ for which $l = 1$, $v = 2$, and $c_2(\sigma, \vec{v}) = c_1(\sigma, \vec{v})^{-1} = 5$.

Before stating Corollary 2, which is given because it can give surprisingly sharper estimates than Theorem 1, some additional notation is necessary. For $p \in (Z_+)^n$, let $\top_p$ denote the curve having the equation $\sum_{i=1}^{n} \tau_i^p i = 1$. Since the family of sets $\{(0, \infty)_{\tau, p}\}_{\tau \in \top_p}$ partitions $E^n \setminus \{\vec{0}\}$ it is immediate that to each point $\sigma \in E^n \setminus \{\vec{0}\}$ there corresponds exactly one point $\tau \in \top_p$ such that $\sigma \in (0, \infty)_{\tau, p}$. Denote this point τ by τ_σ.

Corollary 2. *Let H be a real-valued polynomial in n variables, α a real number, $\sigma \in E^n \setminus \{\vec{0}\}$ and $p \in (Z_+)^n$. Then*

$$\sigma \not\subset (c_1(\tau_\sigma, p), c_2(\tau_\sigma, p))_{\tau_\sigma, p} \Rightarrow \sigma \notin L_\alpha(H).$$

Again considering the preceding example, with $q = \vec{v}$ and $p = (2, 4)$, one can deduce from Corollary 2 that every zero of H on the ray $(0, \infty)_{\sigma, q}$ lies between the origin and the point (m,m) where $m^2 = 2(28+\sqrt{704}/(27+\sqrt{704})$. In actuality the only such zero is the point $(\sqrt{2},\sqrt{2})$ so that in terms of distance from the origin the estimate obtained from Corollary 2 involves a relative error of less than 2.1% in comparison with relative errors of over 253% and 1265% incurred by the use of Theorem 1 of this paper and [**2**] respeetively.

Theorem 2 of this paper contradicts Theorem 2 of [**2**] and corrects Theorem 6 of [**2**]. Adopting the notation and terminology of [**2**], the error in the proof of these theorems is the assumption that if $r_m \to \infty$ and $\theta_m \to \overline{\theta}$, where $\theta = \overline{\theta}$ is a null cone of the highest order term of H, then (r_m, θ_m) is asymptotic to a subset of the cone $\theta = \overline{\theta}$. This is false even if H is an axisymmetric harmonic polynomial in E^3 and if for some α and all m the point (r_m, θ_m) belongs to $L_\alpha(H)$. To see this, consider the example $H(r,\theta) = r\cos\theta - r_2(3\cos^2\theta - 1)/2$, for which $A_1(\theta) = \cos\theta$ and $2A_2(\theta) = -3\cos^2\theta + 1$. Clearly $L_0(H)$ is unbounded since if $\{\theta_m\}$ is chosen so that $\theta_m \neq \overline{\theta}$, for all m, $\theta_m \to \overline{\theta}$, as $m \to \infty$, where $\overline{\theta}$ is a zero of $A_2(\theta)$, then $r_m(\theta) \equiv -A_1(\theta_m)/A_2(\theta_m) \to \infty$, as $m \to \infty$ and $(\theta_m, r_m) \in L_0(H)$ for all m. Let d_m denote the distance between the cone $\theta = \overline{\theta}$ and the point (r_m, θ_m). Then $\lim\limits_{m\to\infty} dm = \lim\limits_{m\to\infty} |r_m \sin(\overline{\theta} - \theta_m)| = \lim\limits_{m\to\infty} 2|\cos\theta_m| \sin(\overline{\theta} - \theta_m)/(3\cos^2\theta_m - 1)| = 1/\sqrt{6}$, so that the example under discussion is a counterexample to Theorems 2 and 6 of [**2**].

5. Proof the Theorem 1. Write $H(\sigma) - \alpha$ in the form $\sum\limits_{j=1}^{k} h_j(\sigma)$ where h_j is a q-homogeneous polynomial of index I_j, I_j being a rational number in reduced form, and where I_j is a strictly increasing function of j. Recall that l denotes the least common multiple of the denominators of $I_1, I_2, \dots, I_k$. To prove Theorem 1 it suffices to show for $\sigma \in E^n$, $q \in (Z_+)^n$ that

$$\left|H\left(\left\langle r^{1/q_1}\sigma_i\right\rangle\right) - \alpha\right| > 0 \text{ if } c_2(\sigma,q) \leq r < \infty \tag{4}$$

and that

$$\left|H\left(\left\langle r^{1/q_i}\sigma_i\right\rangle\right) - \alpha\right| > 0 \text{ if } 0 < r \leq c_1(\sigma,q) \tag{5}$$

In proving (4) it can be assumed that $r > 1$ since $c_2(\sigma,q) > 1$. It can also be assumed that $h_k(\sigma) \neq 0$ since otherwise $c_2(\sigma,q) = \infty$ in which case (4) is trivial. But for $h_k(\sigma) \neq 0$, $r > 1$

$$\begin{aligned}
&\left|H\left(\left\langle r^{1/q_i}\sigma_i\right\rangle\right) - \alpha\right| \geq |h_k(\sigma)| r^{I_k} - \sum_{j=1}^{k-1} |h_j(\sigma)| r^{I_j} \geq \\
&\geq |h_k(\sigma)| r^{I_k} \left(1 - M_k(\sigma,q) \sum_{j=l(I_k-I_{k-1})}^{l(I_k-I_1)} (r^{1/l})^{-j}\right) = \\
&= |h_k(\sigma)| r^{I_k} \left(1 - M_k(\sigma,q)(r^{1/l - I_k + I_{k-1}} - r^{-I_k+I_1})\right)(r^{1/l} - 1)^{-1} > \\
&> |h_k(\sigma)| r^{I_k} \left(1 - M_k(\sigma,q)\right)(r^{1/l} - 1)^{-1},
\end{aligned}$$

from which (4) follows.

In proving (5) it can be assumed that $r < 1$ since $c_1(\sigma, q) < 1$. It can also be assumed that $h_1(\sigma) \neq 0$ since otherwise $c_1(\sigma, q) = 0$ in which case (5) is trivial. But for $h_1(\sigma) \neq 0$, $0 < r < 1$,

$$\begin{aligned}
&\left| H\left(\left\langle r^{1/q_i}\sigma_i\right\rangle\right) - \alpha\right| \le r^{I_1}|h_1(\sigma)| - \sum_{j=1}^{k} r^{I_j}|h_j(\sigma)| \ge \\
&\ge r^{I_1}|h_1(\sigma)|\left(1 - M_1(\sigma,q)\sum_{j=l(I_2-I_1)}^{l(I_k-I_1)} (r^{1/l})^j\right) = \\
&= r^{I_1}|h_1(\sigma)|\left((r^{1/l})^{-1} - 1 - M_k(\sigma,q)\left((r^{1/l})^{l(I_2-I_1)-1} - (r^{1/l})^{l(l_m-I_1)}\right)\right) \\
&\qquad \left((r^{1/l})^{-1} - 1\right) > \\
&r^{I_1}|h_1(\sigma)|\left((r^{1/l})^{-1} - 1 - M_k(\sigma,q)\right)\left((r^{1/l})^{-1} - 1\right)^{-1},
\end{aligned}$$

from which (5) and hence Theorem 1 follows.

6. Proof of Theorem 2. By an equipotential surface of an axisymmetric harmonic polynomial in E^3 is meant a level surface of a real harmonic polynomial of degree n,

$$H(r,\theta) = \sum_{k=0}^{n} a_k r^k P_k(\cos\theta), \;\; a_n \neq 0,$$

$P_k(\cos\theta)$ being the Legendre polynomial of degree k in $\cos\theta = \sigma r^{-1}$. To prove Theorem 2 it suffices to prove that

given (r_m, θ_m) in $L_\alpha(H)$, $m = 1, 2, \ldots$, such that $\theta_m \to \overline{\theta}$ and $r_m \to \infty$, as $m \to \infty$, there exists a number d such that the distance d_m, between (r_m, θ_m) and $\theta = \overline{\theta}$ approaches d as $m \to \infty$ and such that $d = 0$ if and only if $a_{n-1} = 0$ (6)

and that

given (r_m, θ_m) in $L_\alpha(H)$, $m = 1, 2, \ldots$ such that $r_m \to \infty$, as $m \to \infty$, there exists a finite partition $\{Z_i\}$ of the positive integers such that $\{\theta_m(\mod 2\pi)\}_{m\in Z_i}$ is a Cauchy sequence for each i. (7)

The following result will be used in the proofs of (6) and (7).

If $(r_m, \theta_m) \in L_\alpha(H)$ and $r_m \to \infty$, as $m \to \infty$, then $P_n(\cos\theta_m,) \to 0$ as $m \to \infty$. (8)

Letting $\sigma_m = (1, \cos\theta_m)$ and $q = (1,1)$ it follows from Theorem 1 that $(r_m, \cos\theta_m) \in (c_1(\sigma_m, q), c_2(\sigma_m, q))_{\sigma_m \cdot q}$. Thus, $c_2(\sigma_m, q)$ and hence $\max\limits_{j\neq n} |a_j P_j / a_n P_n|(\cos\theta_m)$ tend to infinity as $m \to \infty$, which implies (8). From (8) the proof of (7) is straightforward, using that $P_n(\cos\theta)$ is continuous and has a finite number of zeros in $[0, 2\pi]$, a compact set.

Tu prove (6) note that $d_m = r_m|\sin(\alpha - \theta_m)|$ and

$$a_n r_m = \left(a_0 r_m^{-n+1} P_0(\cos\theta_m) + a_1 r^{-n+2} P_1(\cos\theta_m) + \right. \\ \left. + \cdots + a_{n-1} P_{n-1}(\cos\theta_m)\right) P_n(\cos\theta_m)$$

so that

$$\lim_{m\to\infty} d_m = \left|a_{n-1}P_{n-1}(\cos\overline{\theta})/a_n\right| \lim_{m\to\infty} \left|\sin(\overline{\theta} - \theta_m)/P_n(\cos\theta_m)\right|.$$

From (6) and (8) it follows that both the numerator and denominator of the ratio on the right approach zero as $n \to \infty$. Applying L'Hospital's rule yields

$$\lim_{m\to\infty} d_m = \left|a_{n-1}P_{n-1}(\cos\overline{\theta})/a_n P_n'(\cos\overline{\theta})\sin\theta\right|,$$

which proves (6) since $P_n(\cos\overline{\theta}) = 0$ implies the following three facts: $\cos\overline{\theta} \neq \pm 1$ (see [**3**, p. 178]), $P_{n-1}(\cos\overline{\theta}) \neq 0$ and $P_n'(\cos\overline{\theta}) = 0$. The second fact follows the recursion formula

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0 \quad ([\mathbf{3}, \text{ p. } 178])$$

and the fact that $P_0(x) = 1$ for all x. The third fact follows from the second fact and the recursion formula

$$(x^2 - 1)P_n'(x) = nxP_n(x) - nP_{n-1}(x) \quad ([\mathbf{3}, \text{ p. } 179]).$$

REFERENCES

[1] C. H. Heiberg, *Extreme for functions of several variables*, Amer. Math. Monthly **86** (1979), 767–770.

[2] M. Marden, P. A. McCoy, *Level sets of polynomials in n real variables*, Pacific J. Math. **66** (1976), 491–498.

[3] G. Sansone, *Orthogonal Functions*, Interscience, New York, 1959.

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