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A GENERALIZATION OF A THEOREM OF A. D. OTTO

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Abstract. In this paper we prove that if G is a finite p-group of class c with G/G' of exponent p^r and L_i/L_{i+} is cyclic of order p^r for $i = 1, 2, \ldots, c-1$, where L_i, $i = 0, 1, \ldots, c$ is the lower central series of G, then the order of G divides the order of the group $A(G)$ of automorphisms of G.

Introduction and notation. Let G be a finite p-group of class c. Let $G = L_0 \supset L_1 \supset \cdots \supset L_c = 1$, $1 = Z_0 \subset Z_1 \subset \cdots \subset Z_c = G$ be the lower and the upper central series of G respectively, where $L_1 = G' = [G, G]$ and $Z_1 = Z = Z(G)$. If G has no non-trivial abelian direct factor, then G is called a PN-group. A. D. Otto in [**1**] proved that if G is a PN-group with $|L_i/L_{i+1}| = p$ for all $i = 1, 2, \ldots, c-1$ and $\exp(G/G') = p$, then the order of G divides the order of the group of automorphisms of G. We generalize this result by showing that if G is any finite p-group with L_i/L_{i+1} cyclic of order p' for all $i = 1, 2, \ldots, c-1$ and $\exp(G/G') = p'$, then $|G|$ divides $A(G)$. We also show that the same result holds if Z_i/Z_{i-1} is cyclic of order p', $i = 1, 2, \ldots, c-1$, and $L_j = Z_{c-j}$ for some j, $1 \leq j \leq c-1$. Throughout this paper, G is a finite non-abelian p-group, $|G|$ is the order of G, $C(p^x)$ is the cyclic group of order p^x, $A(G)$, $I(G)$, $A_c(G)$ are the groups of automorphisms, inner automorphisms, central automorphisms of G.

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We begin with

Lemma 1. *Let G be a PN-group. If $\exp(G/G') \leq |Z|$, then $|A_c(G)| \geq |C/G'|$.*

Proof. Let $|G/G'| = p^m$ and $G/G' = C(p^{m_1}) \times C(p^{m_2}) \times \cdots \times C(p^{m_t})$, where $m_1 \geq m_2 \geq \cdots \geq m_t \geq 1$ and $\sum_{j=1}^{t} m_j = m$. Similarly let $|Z| = p^k$ and $Z = C(p^{k_1}) \times C(p^{k_2}) \times \cdots \times C(p^{k_s})$ with $k_1 \geq k_2 \geq \cdots \geq k_s \geq 1$ and $\sum_{i=1}^{s} k_i = k$. If

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a_x is the number of times p^x appears in the invariants of G/G', then $\sum_{x\geq 1}^{m} xax = m$. Since G is a PN-group $|A_c(G)| = |\mathrm{Hom}\,(G,Z)| = |\mathrm{Hom}\,(G/G',Z)|$ [**2**]. So we have $|A_c(G)| = |\mathrm{Hom}\,(G/G',Z)| = |\mathrm{Hom}\prod_{j=1}^{t}(C(p^{m_j}),\prod_{i=1}^{s}C(p^{k_i}))|$. Hence $|A_c(G)| = \prod_{j,i}^{t,s}|\mathrm{Hom}\,(C(p^{mj}),C(p^{k_i}))\prod_{j,i}^{t,s}p^{\min(m_j,k_i)} = p^A$ for some A. Summing powers over $m_j = 1,2,\dots,m_1$ for $k_i = k_s,\dots,k_1$ we have

$$\begin{aligned} A &= \left(\sum_{x\geq 1}^{k_s} xa_x + k_s\sum_{x>k_s}^{m_1} a_x\right) + \cdots + \left(\sum_{x\geq 1}^{k_1} xa_x + k_1\sum_{x>k_1}^{m_1} a_x\right) = \\ &= \sum_{i=2}^{s}\left(\sum_{x\geq 1}^{k_i} xa_x + k_i\sum_{x>k_i}^{k_1} a_x\right) + \sum_{i=1}^{s}\left(k_i\sum_{x>k_1}^{m_1} a_x\right) + \sum_{x\geq 1}^{k_1} xa_x = \\ &= \sum_{i=2}^{s}\theta_i + k\sum_{x>k_1}^{m_1} a_x + \sum_{x\geq 1}^{k_1} xa_x, \quad \text{where } \theta_i = \sum_{x\geq 1}^{k_i} xa_x + k_i\sum_{x>k_i}^{k_1} a_x. \end{aligned}$$

Since $k \geq m_1$ we have $k\sum_{x>k_1}^{m_1} a_x \geq \sum_{x>k_1}^{m_1} xa_x$ and so $A \geq k\sum_{i=2}^{s}\theta_i + \sum_{x>k_1}^{m_1} xa_x = \sum_{i=2}^{s}\theta_i + m$.

LEMMA 2. [**4**]. *Let G be a finite non-abelian p-group. Let $G = L_0 \supset L_1 \supset \cdots \supset L_c = 1$, $1 = Z_0 \subset Z_1 \subset \cdots \subset Z_c = G$ be the lower and the upper central series of G. If L_i/L_{i+1} is cyclic of order p^r for all $i = 1,2,\dots,c-1$. then $L_i \cap Z_{c-i-1} = L_{i+1}$, $i = 1,2,\dots,c-1$.*

LEMMA 3. [**3**]. *If G is a finite non-abelian group and Z_i/Z_{i-1} is cyclic of order p^r for all $i = 1,2,\dots,c-1$, then $[G,Z_{i+1}] = Z_i$ for $i = 1,2,\dots,c-1$.*

THEOREM 1. *Let G be a finite group of order p^n and class c. If L_i/L_{i+1} is cyclic of order p^r for all $i = 1,2,\dots,c-1$ and $\exp(G/G') = p^r$, then $|G|$ divides, $|A(G)|$.*

Proof. Consider the following:

A: G is a PN-group. Since $L_i \subseteq Z_{c-i}$ and $L_i \not\subset Z_{c-i-1}$ we have $(Z_{c-i}/L_i) \supseteq (L_iZ_{c-i-1}/L_i) \simeq Z_{c-i-1}/L_i \cap Z_{c-i-1} = Z_{c-i-1}/L_{i+1}$ (by Lemma 2). Hence

$$|Z_{c-i}/Z_{c-i-1}| \geq |L_i/L_{i+1}| = p^r$$

for all $i = 1,2,\dots,c-1$. But $|G/Z_{c-1}| = p^{2r}$ [**4**] and so $|G/Z_2| \geq p^{(c-1)r}$ which gives $|Z_2| \leq p^{n-(c-1)r}$. If $|Z| = p^k$ then $|I(G)| = |G/Z| = p^{n-k}$ and $|Z_2/Z| \leq p^{n-(c-1)r-k}$. Since $L_{c-1} \subseteq Z$ and $|L_{c-1}| = p^r$ we have $|Z| \geq p^r = \exp(G/G')$

and so by Lemma 1, $|A_c(G)| \geq |G/G'|$. But $|L_i/L_{i+1}| = p^r$ for $i = 1, 2, \ldots, c-1$ which implies that $|L_{c-i}| = p^{ir}$ and so $|L_1| = p^{(c-1)r}$. Therefore we have $|G/G'| = |G/L_1| = p^{n-(c-1)r}$, and so $|A_c(G)| \geq p^{n-(c-1)r}$. Since $A_c(G)$ centralizes $I(G)$ in $A(G)$ we have $|I(G) \cap A_c(G)| = |Z(I(G))| = |Z(G/Z)| = |Z_2/Z| \leq p^{n-(c-1)r-k}$. Hence

$$\begin{aligned} |A(G)|_p \geq |I(G)A_c(G)| = |I(G)| \cdot |A_c(G)|/|I(G) \cap A_c(G)| \geq \\ \geq p^{n-k} p^{n-(c-1)r} / p^{n-(c-1)r-k} = p^n. \end{aligned}$$

B: $G = H \times K$, where H is abeilian of order p^e and K is a PN-group. By [**1**], $|A(G)|_p \geq p^e |A(K)|_p$. Since $G = H \times K$, $|G'| = |K'|$, and by induction $L_i(G)| = |L_i(K)|$ for all $i = 1, 2, \ldots, c$. Hence $L_i(K)/L_{i+1}(K)$ is cyclic of order p^r for $i = 1, 2, \ldots, c-1$. Moreover $G/G' = H \times K/K'$ and so $\exp(K/K') \leq p^r$. But $\exp(L_i(K)/L_{i+1}(K)) = p^r$ and so $\exp(K/K') = p^r$. Therefore by A, $|A(K)|_p \geq |K|$ and so $|A(G)|_p \geq p^e$. $|K| = |G|$.

Corollary. *Let G be a PN-group. If $|L_i/L_{i+1}| = p$, $i = 1, 2, \ldots, c-1$, and $\exp(G/G') \leq |Z|$, then $|G|$ divides $|A(G)|$.*

Theorem 2. *Let G be a finte p-group of order p^n and class c. If Z_{i+1}/Z_i is cyclic of order p^r for all $i = 0, 1, \ldots, c-2$, and $Z_{c-j} = L_j$ for some j, $1 \leq j \leq c-1$, then $|G|$ divides $|A(G)|$.*

Proof. By Lemma 3, $L_{j+1} = [L_j, G] = [Z_{c-j}, G] = Z_{c-j-1}$ and so $p^r = \exp(Z_{c-j}/Z_{c-j-1}) = \exp(L_j/L_{j+1}) \leq \exp(L_{j-1}/L_j) \leq |L_{j-1}/L_j| \leq |Z_{c-j-1}/Z_{c-j}| = p^r$. Hence $|L_{j-1}| = |Z_{c-j+1}|$ and since $L_{j-1} \subseteq Z_{c-j+1}$ we have $L_{j-1} = Z_{c-j+1}$. Therefore $L_j = Z_{cj}$ for all $j = 1, 2, \ldots, c$, and so L_j/L_{j+1} is cyclic of order p^r for all j. By [**4**], $G/Z_{c-1} = p^{2r}$ and so $|G/L_1| = |G/G'| = p^{2r}$. Let $p^{m_1} \geq p^{m_2} \geq \cdots \geq p^{m_t}$ by the invariants of G/G', if $m_2 < r$, then $\exp(L_1/L_2) \geq p^{m_2} < p^r$, which is a contradiction. Hence $m_2 \geq r$ and so $m_1 = m_2 = r$ and $\exp(G/G') = p^r$. The result follows from Theorem 1.

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