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SUBSPACES OF RECURRENT FINSLER SPACES

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1. Introduction. In the Finsler space F_n the metric function is $L(x, \dot{x})$. Let us define the m vector fields $B_a^\alpha(x, \dot{x})$ and $n-m$ vector fields $N_k^\alpha(x, \dot{x})$

$$\alpha, \beta, \gamma, \delta, \varepsilon, \varkappa, \dots = 1, 2, \dots, n$$
$$a, b, c, d, e, f, \ \dots = 1, 2, \dots, m$$
$$k, l, m, n, p, q, \dots = m+1, \dots, n$$

in such a way that these vector fields are linearly independent at each $(x, \dot{x})$ and satisfy the relations

$$g_{\alpha\beta} = B_a^\alpha N_k^\beta = 0 \tag{1.1}$$

for each $a = 1, 2, \dots, m \quad k = m+1, \dots n$. Let us define

$$g_{ab} = g_{\alpha\beta} B_{ab}^{\alpha\beta} \tag{1.2}$$

$$g_{kl} = g_{\alpha\beta} N_k^\alpha N_l^\beta \tag{1.3}$$

$$B_\beta^b = g^{ab} g_{\alpha\beta} B_a^\alpha \tag{1.4}$$

$$N_\alpha^k = g^{km} g_{\alpha\beta} N_m^\beta \tag{1.5}$$

$g_{\alpha\beta}$, B_a^α and N_k^α have zero degree of homogenity in $\dot{x}$, g^{ab} and g^{km} are inverse matrices of g_{ab} and g_{km}, respectively. From (1.3) and (1.5) we have

$$N_\alpha^k N_p^\alpha = g^{kl} g_{\alpha\beta} N_l^\beta N_p^\alpha = g^{kl} g_{lp} = \delta_p^k. \tag{1.6}$$

As usually

$$\delta_\beta^\alpha = B_a^\alpha B_\beta^a + N_k^\alpha N_\beta^k \tag{1.7}$$

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Let us denote the absolute differential which corresponds to the motion from $(x, \dot{x})$ to $(x + dx, \dot{x} + d\dot{x})$ by D. Then we may write

$$dx^\alpha = B_a^\alpha du^a + N_k^\alpha dv^k, \qquad (1.8)$$
$$\dot{x}^\alpha = B_a^\alpha \dot{u}^a + N_k^\alpha N_k^\alpha \dot{v}^k.$$

If $\xi^\alpha(x, \dot{x})$ is a vector field in F_n, then $\xi^\alpha = B_a^\alpha + N_k^\alpha \xi^k$. The induced differentials $\overline{D}\xi^a$, $\overline{D}\xi^k$ are defined by $B_\alpha^a D\xi^\alpha = \overline{D}\xi^a$, $N_\alpha^k D\xi^\alpha = \overline{D}\xi^k$ and

$$D\xi^\alpha = B_a^\alpha \overline{D}\xi^a + N_k^\alpha \overline{D}\xi^k \qquad (1.9)$$

We shall use the notation $l^\alpha = L^{-1}(x, \dot{x})\dot{x}^\alpha = L^{-1}(B_a^\alpha \dot{u}^a + N_k^\alpha \dot{v}^k) = B_a^\alpha + N_k^\alpha l^k$ where $l^a = L^{-1}\dot{u}^a$, $l^k = L^{-1}\dot{v}^k$.

From (1.8) we have

$$Dl^\alpha = B_a^\alpha \overline{D}l^a + N_k^\alpha \overline{D}l^k \qquad (1.10)$$

2. The connection coefficients for the recurrent Finsler space. We shall suppose that the metric tensor is determined by

$$g_{\alpha\beta}(x, \dot{x}) = \dot{\partial}_\alpha \dot{\partial}_\beta L^2(x, \dot{x})/2 \qquad (2.1)$$

and that space F_n is recurrent, i.e.

$$Dg_{\alpha\beta} = K(x, \dot{x}) g_{\alpha\beta} \qquad (2.2)$$

As

$$Dg_{\alpha\beta} = g_{\alpha\beta|\gamma} dx^\gamma + g_{\alpha\beta}|_\gamma Dl^\gamma$$

(2.2) is valid if

$$g_{\alpha\beta|\gamma} = \lambda_\gamma(x, \dot{x}) g_{\alpha\beta} \qquad (2.3)$$
$$g_{\alpha\beta}|_\gamma = \mu_\gamma(x, \dot{x}) g_{\alpha\beta} \qquad (2.4)$$
$$K(x, \dot{x}) = \lambda_\gamma(x, \dot{x}) dx^\gamma + \mu_\gamma(x, \dot{x}) Dl^\gamma. \qquad (2.5)$$

The absolute differential of $g_{\alpha\beta}$ is

$$Dg_{\alpha\beta} - dg_{\alpha\beta} - (\Gamma^{*\delta}_{\alpha\gamma} + \Gamma^{*\delta}_{\beta\gamma} dx^\gamma) - (A^\delta_{\alpha\gamma} g_{\delta\beta} + A^\delta_{\beta\gamma} g_{\alpha\delta}) Dl^\gamma. \qquad (2.6)$$

We shall determine connection coefficients under condition (2.2) and

$$\Gamma^{*\delta}_{\alpha\gamma} = \Gamma^{*\delta}_{\gamma\alpha} \qquad (2.7)$$
$$A^\delta_{\alpha\gamma} = A^\delta_{\gamma\alpha} \qquad (2.8)$$

which have a well known geometric interpretation. Substituting

$$dg_{\alpha\beta} = \partial_\gamma g_{\alpha\beta} dx^\gamma + \dot{\partial}_\gamma g_{\alpha\beta} d\dot{x}^\gamma = \qquad (2.9)$$
$$= \partial_\gamma g_{\alpha\beta} dx^\gamma + \dot{\partial}_\gamma g_{\alpha\beta}(L_D l^\gamma + l^\gamma DL - \Gamma^{*\gamma}_\delta dx^\delta - LA^\gamma_{0\delta} Dl^\delta)$$

into (2.6) and using (2.3), (2.4) we get

$$g_{\alpha\beta|\delta} = \partial_\gamma g_{\alpha\beta} - \dot{\partial}_\delta g_{\alpha\beta}\Gamma^{*\delta}_{\gamma} - \Gamma^{*\delta}_{\alpha\gamma} g_{\delta\beta} - \Gamma^{*\delta}_{\beta\gamma} g_{\alpha\delta} = \lambda_\gamma g_{\alpha\beta} \tag{2.10}$$

$$g_{\alpha\beta|\gamma} = L\dot{\partial}_\delta g_{\alpha\beta}(\delta^\delta_\gamma - A^\delta_{0\gamma}) - A^\delta_{\alpha\gamma} g_{\delta\beta} - A^\delta_{\beta\gamma} g_{\alpha\delta} = \mu_\gamma g_{\alpha\beta}. \tag{2.11}$$

Using (2.7) and (2.10) from the expression for $g_{\alpha\beta|\gamma} + g_{\beta\gamma|\alpha} - g_{\gamma\alpha|\beta}$ we obtain

$$\begin{aligned} 2\Gamma^*_{\alpha\beta\gamma} = (\partial_\gamma g_{\alpha\beta} + \partial_\alpha g_{\beta\gamma} - \partial_\beta g_{\alpha\gamma}) - -(\dot{\partial}_\delta g_{\alpha\beta}\Gamma^{*\delta}_{\gamma} + \\ + \dot{\partial}_\delta g_{\beta\gamma}\Gamma^{*\delta}_{\alpha} - \dot{\partial}_\delta g_{\alpha\gamma}\Gamma^{*\delta}_{\beta}) - (\lambda_\gamma g_{\alpha\beta} + \lambda_\alpha g_{\beta\gamma} - \lambda_\beta g_{\alpha\gamma}) \end{aligned} \tag{2.12}$$

In the same way using (2.11) and (2.8) we have

$$\begin{aligned} 2A_{\alpha\beta\gamma} = L(\dot{\partial}_\alpha g_{\beta\gamma} + \dot{\partial}_\gamma g_{\alpha\beta} - \dot{\partial}_\beta g_{\alpha\gamma}) - L(\dot{\partial}_\delta g_{\alpha\beta} A^\delta_{0\alpha} + \\ + \dot{\partial}_\delta g_{\alpha\beta} A^\delta_{0\gamma} - \dot{\partial}_\delta g_{\alpha\gamma} A^\delta_{0\beta}) - (\mu_\alpha g_{\beta\gamma} + \mu_\gamma g_{\alpha\beta} - \mu_\beta g_{\alpha\gamma}) \end{aligned} \tag{2.13}$$

On the right-hand side of (2.12) are the undetermined quantities

$$\Gamma^{*\delta}_{\alpha} = L\Gamma^{*\delta}_{0\alpha}.$$

Multiplying (2.12) by l^α we get

$$2\Gamma^*_{0\beta\gamma} = (\partial_\gamma g_{\alpha\beta} + \partial_\alpha g_{\beta\gamma} - \partial_\beta g_{\alpha\gamma})l^\alpha - L\dot{\partial}_\delta g_{\beta\gamma}\Gamma^{*\delta}_{00} - (\lambda_\gamma l_\beta + \lambda_0 g_{\beta\gamma} - \lambda_\beta l_\gamma) \tag{2.14}$$

Multiplying (2.14) by l^γ we obtain

$$2\Gamma^*_{0\beta 0} = (\partial_\gamma g_{\alpha\beta} + \partial_\alpha g_{\beta\gamma} - \partial_\beta g_{\alpha\gamma})l^\alpha l^\gamma - (2\lambda_0 l_\beta + \lambda_\beta) \tag{2.15}$$

and now $\Gamma^*_{\alpha\beta\gamma}$ is determined by (2.12), (2.14) and (2.15).

In the same way from (2.13) we obtain

$$2A_{0\beta\gamma}^{=} - L\dot{\partial}_\delta g_{\beta\gamma} A^\delta_{00} - (\mu_\alpha l^\alpha g_{\beta\gamma} - \mu_\gamma l^\beta - \mu_\beta l_\gamma) \tag{2.16}$$

$$2A_{0\beta 0} = -(2\mu_0 l_\beta - \mu_\beta). \tag{2.17}$$

3. The induced connection for the subspace of the recurrent space. Using the method of O. Varga we define DB^α_a and DN^α_k as vectors in F_n which may be decomposed at each lineelement $(x, \dot{x})$ in the direction of the vectors B^α_a and N^α_k

$$DB^\alpha_a = \overline{w}^d_a(d)B^\alpha_d + \overline{w}^m_a(d)N^\alpha_m \tag{3.1}$$

where

$$\overline{w}^d_a(a) = \overline{\Gamma}^{*d}_{ab}du^b + \overline{\Gamma}^{*d}_{ak}dv^k + \overline{A}^d_{ab}\overline{D}l^b + \overline{A}^d_{ak}\overline{D}l^k, \tag{3.2}$$

$$\overline{w}^m_a(a) = \overline{\Gamma}^{*m}_{ab}du^b + \overline{\Gamma}^{*m}_{ak}dv^k + \overline{A}^m_{ab}\overline{D}l^b + \overline{A}^m_{ak}\overline{D}l^k, \tag{3.3}$$

and

$$DN_k^\alpha = \overline{w}_k^d(d)B_d^\alpha + \overline{w}_k^m(d)N_m^\alpha, \tag{3.4}$$

where

$$\overline{w}_k^d(d) = \overline{\Gamma}_{kb}^{*d}du^b + \overline{\Gamma}_{kn}^{*d}dv^n + \overline{A}_{kb}^d\overline{D}l^b + \overline{A}_{kn}^d\overline{D}l^n \tag{3.5}$$

$$\overline{w}_k^m(d) = \overline{\Gamma}_{kb}^{*m}du^b + \overline{\Gamma}_{kn}^{*m}dv^n + \overline{A}_{kb}^m\overline{D}l^b + \overline{A}_{kn}^m\overline{D}l^n. \tag{3.6}$$

From (1.1) and (2.2) we have

$$D(g_{\alpha\beta}B_a^\alpha N_k^\beta) = g_{\alpha\beta}(DB_a^\alpha)N_k^beta + g_{\alpha\beta}B_a^\alpha DN_k^\beta = 0$$

from which

$$\begin{aligned} g_{mk}(\overline{\Gamma}_{ab}^{*m}du^b + \overline{\Gamma}_{an}^{*m}dv^n + \overline{A}_{ab}^m\overline{D}l^b + \overline{A}_{an}^m\overline{D}l^n) = \\ - g_{ad}(\overline{\Gamma}_{kb}^{*d}du^b + \overline{\Gamma}_{kn}^{*d}dv^n + \overline{A}_{kb}^d\overline{D}l^b + \overline{A}_{kn}^d\overline{D}l^n) \end{aligned}$$

follows.

From the relation above we obtain

$$\begin{aligned} &a)\quad \overline{\Gamma}^*_{akb} = -\overline{\Gamma}^*_{kab} \\ &b)\quad \overline{\Gamma}^*_{akn} = -\overline{\Gamma}^*_{kan} \\ &c)\quad \overline{A}_{akb} = -\overline{A}_{kab} \\ &d)\quad \overline{A}_{akn} = -\overline{A}_{kan} \end{aligned} \tag{3.7}$$

i.e.

$$\overline{w}_{ak} = -\overline{w}_{ka} \quad (g_{km}\overline{w}_b^m = -g_{ab}\overline{w}_k^b). \tag{3.8}$$

After some calculation using (1.8) and (1.16) we obtain

$$\begin{aligned} DB_a^\alpha = (B_{a|\beta}^\alpha du^b + B_a^\alpha|_\beta \overline{D}l^b)B_b^\beta + \\ + (B_{a|\beta}^\alpha dv^k + B_a^\alpha|_\beta \overline{D}l^k)N_k^\beta \end{aligned} \tag{3.9}$$

where

$$B_{a|\beta}^\alpha = \partial_\beta B_a^\alpha - \dot{\partial}_\gamma B_a^k \Gamma_\beta^{*\gamma} + \Gamma_{\gamma\beta}^{*\alpha}B_a^\gamma \tag{3.10}$$

$$B_a^\alpha|_\beta = L\dot{\partial}_\delta B_a^\alpha(\delta_\beta^\delta - A_{0\beta}^\delta) + A_{0\beta}^\alpha B_a^\delta \tag{3.11}$$

Comparing (3.9) and (3.1) using (3.2) and (3.3) we obtain

$$\begin{aligned} &\text{a)}\quad \overline{\Gamma}^*_{acb} = g_{\alpha\delta}B_c^\delta B_b^\beta B_{a|\beta}^\alpha \\ &\text{b)}\quad \overline{\Gamma}^*_{ack} = g_{\alpha\delta}B_c^\delta N_k^\beta B_{a|\beta}^\alpha \\ &\text{c)}\quad \overline{A}_{acb} = g_{\alpha\delta}B_c^\delta B_b^\beta B_a^\alpha|_\beta \\ &\text{d)}\quad \overline{A}_{ack} = g_{\alpha\delta}B_c^\delta N_k^\beta B_a^\alpha|_\beta \\ &\text{e)}\quad \overline{\Gamma}^*_{anb} = g_{\alpha\delta}N_n^\delta B_b^\beta B_{a|\beta}^\alpha \\ &\text{f)}\quad \overline{\Gamma}^*_{ank} = g_{\alpha\delta}N_n^\delta N_k^\beta B_{a|\beta}^\alpha \\ &\text{g)}\quad \overline{A}_{anb} = g_{\alpha\delta}N_n^\delta B_b^\beta B_a^\alpha|_\beta \\ &\text{h)}\quad \overline{A}_{ank} = g_{\alpha\delta}N_n^\delta N_k^\beta B_a^\alpha|_\beta \end{aligned} \tag{3.12}$$

On the other hand we have for the vector field N_k^α

$$DN_k^alpha = (N^\alpha_{k|\beta}du^b + N^\alpha_k|_\beta\overline{D}l^b)B^\beta_b + (N^\alpha_k|_\beta dv^m + N^\alpha_k|_\beta \overline{D}l^m)N^\beta_m, \tag{3.13}$$

where

$$N^\alpha_{k|\beta} = \partial_\beta N^\alpha_k - \dot{\partial}_\gamma N^\alpha_k \Gamma^{*\gamma}_{\beta} + \Gamma^{*\alpha}_{\gamma\beta}N^\gamma_k \tag{3.14}$$

$$N^\alpha_k|_\beta = L\dot{\partial}_\delta N^\alpha_k(\delta^\delta_\beta - A^\delta_{0\beta}) + A^\alpha_{\delta\beta}N^\delta_k \tag{3.15}$$

multiplying (3.4) and (3.13) first with $g_{\alpha\delta}B^\delta_c$ and then with $g_{\alpha\delta}N^\delta_n$, and comparing the coefficients of du^b, dv^l and $\overline{D}l^l$ in these relations we get

$$\begin{array}{llll} \text{a)} & \overline{\Gamma}^*_{kcb} = g_{\alpha\delta}B^\delta_c B^\beta_b N^\alpha_{k|\beta} & \text{e)} & \overline{\Gamma}^*_{knb} = g_{\alpha\delta}N^\delta_n B^\beta_b N^\alpha_{k|\beta} \\ \text{b)} & \overline{\Gamma}^*_{kcl} = g_{\alpha\delta}B^\delta_c N^\beta_l N^\alpha_{k|\beta} & \text{f)} & \overline{\Gamma}^*_{knl} = g_{\alpha\delta}N^\delta_n N^\beta_l N^\alpha_{k|\beta} \\ \text{c)} & \overline{A}_{kcb} = g_{\alpha\delta}B^\delta_c B^\beta_b N^\alpha_k|_\beta & \text{g)} & \overline{A}_{knb} = g_{\alpha\delta}N^\delta_n B^\beta_b N^\alpha_k|_\beta \\ \text{d)} & \overline{A}_{kcl} = g_{\alpha\delta}B^\delta_c N^\beta_l N^\alpha_k|_\beta & \text{h)} & \overline{A}_{knl} = g_{\alpha\delta}N^\delta_n N^\beta_l N^\alpha_k|_\beta \end{array} \tag{3.16}$$

4. Recurrent subspaces of recurrent Finsler spaces. From (1.1), (1.2) and (1.3) it follows that

$$g_{\alpha\beta} = g_{ab}B^{ab}_{\alpha\beta} + g_{nk}N^n_\alpha N^k_\beta \tag{4.1}$$

From (4.1) we obtain

$$\begin{aligned} Dg_{\alpha\beta} = {} & dg_{ab}B^{ab}_{\alpha\beta} + g_{ab}DB^a_\alpha B^b_\beta + g_{ab}B^a_\alpha DB^b_\beta + \\ & + dg_{nk}N^n_\alpha N^k_\beta + g_{nk}DN^n_\alpha N^k_\beta + g_{nk}N^n_\alpha DN^k_\beta \end{aligned} \tag{4.2}$$

Using (1.7) we have

$$0 = DB^b_\beta B^\gamma_b + B^b_\beta DB^\gamma_b + DN^n_\beta N^\gamma_n + N^n_\beta DN^\gamma_n. \tag{4.3}$$

On the other hand

$$g_{ab}B^b_\beta = g_{\alpha\varkappa}B^{\alpha\chi}_{ab}B^b_\beta = g_{\alpha\varkappa}B^\alpha_a(\delta^\chi_\beta - N^x_n N^n_\beta) = g_{\alpha\beta}B^\alpha_a.$$

$$g_{kn}N^k_\beta = g_{\alpha\varkappa}N^\alpha_k N^\varkappa_n N^k_\beta = g_{\alpha\varkappa}N^\varkappa_n(\delta^\alpha_\beta - B^\alpha_a B^a_\beta) = g_{\beta\varkappa}N^\varkappa_n. \tag{4.5}$$

Using (4.3), (4.4) and (4.5) we have

$$\begin{aligned} g_{ab}DB^a_\alpha B^b_\beta + g_{nk}DN^n_\alpha N^k_\beta &= g_{\varkappa\beta}B^\varkappa_a DB^a_\alpha + g_{\varkappa\beta}N^\varkappa_n DN^n_\alpha = \\ &= g_{\varkappa\beta}(B^a_\alpha DB^\varkappa_a + N^n_\alpha DN^\varkappa_n) \end{aligned} \tag{4.6}$$

$$\begin{aligned} g_{ab}B^a_\alpha DB^b_\beta + g_{nk}N^n_\alpha DN^k_\beta &= g_{\alpha\varkappa}B^\varkappa_b DB^b_\beta + g_{\alpha\varkappa}N^\varkappa_k DN^k_\beta = \\ &= g_{\alpha\varkappa}(B^b_\beta DB^\varkappa_b + N^k_\beta DN^\varkappa_k) \end{aligned} \tag{4.7}$$

Substituting (4.6) and (4.7) into (4.2) and using (3.1) and (3.4) we get

$$\begin{aligned} Dg_{\alpha\beta} = {} & B^{ab}_{\alpha\beta}dg_{ab} - g_{\beta\varkappa}B^a_\alpha(B^\varkappa_a\overline{w}^d_a + N^\varkappa_k\overline{w}^k_a) - g_{\beta\varkappa}N^n_\alpha(B^\varkappa_d\overline{w}^d_n + N^\varkappa_k\overline{w}^k_n) + \\ & + N^n_\alpha N^k_\beta dg_{nk} - g_{\alpha\varkappa}B^b_\beta(B^\varkappa_d\overline{w}^d_b + N^\varkappa_k\overline{w}^k_\beta) - g_{\alpha\varkappa}N^k_\beta(B^\varkappa_d\overline{w}^d_k + N^\varkappa_n\overline{w}^k_n). \end{aligned} \tag{4.8}$$

Using the relations

$$\text{(4.9)}\qquad \begin{aligned} &\text{a)}\quad \dot{d}u^a = L\overline{D}l^a - \overline{\Gamma}^{*a}_c du^c - \overline{\Gamma}^{*a}_k dv^k + l^a dL \\ &\text{b)}\quad \dot{d}v^k = L\overline{D}l^k - \overline{\Gamma}^{*k}_c du^c - \overline{\Gamma}^{*k}_l dv^l + l^k dL \end{aligned}$$

we obtain

$$\text{(4.10)}\qquad \begin{aligned} dg_{ab} = \partial_c g_{ab} du^c + \partial_k g_{ab} dv^k + \dot{\partial}_c g_{ab}(L\overline{D}l^c - \overline{\Gamma}^{*c}_d du^d - \overline{\Gamma}^{*c}_k dv^k + l^c dL) + \\ + \dot{\partial}_k g_{ab}(L\overline{D}l^c - \overline{\Gamma}^{*k}_d du^d - \overline{\Gamma}^{*k}_t dv^t + l^k dL) \end{aligned}$$

(4.10) is valid if g_{ab} is replaced by g_{nk}. Using again (4.4) and (4.5) some terms in (4.8) become zero, i.e.

$$\text{(4.11)}\qquad \begin{aligned} &- g_{\beta\varkappa} B^a_\alpha N^\varkappa_k \overline{w}^k_a - g_{\alpha\varkappa} N^k_\beta B^\varkappa_d \overline{w}^d_k = \\ &= -g_{tk} B^a_\alpha N^t_\beta \overline{w}^k_a - g_{da} N^k_\beta B^a_\alpha \overline{w}^d_k = B^a_\alpha N^t_\beta(-\overline{w}_{at} - \overline{w}_{ta}) = 0 \end{aligned}$$

because of (3.8). Similarly we get

$$\text{(4.12)}\qquad -g_{\beta\varkappa} N^n_\alpha B^\varkappa_d \overline{w}^d_n - g_{\alpha\varkappa} B^b_\beta N^\varkappa_k \overline{w}^k_b = -N^n_\alpha B^b_\beta(-\overline{w}_{nb} - \overline{w}_{bn}) = 0.$$

AS $g_{ab} = g_{\alpha\beta} B^\alpha_a B^\beta_b$ is homogeneous of degree zero in $\dot{x}$ so $\dot{\partial}_\varepsilon g_{ab} \dot{x}^\varepsilon = 0$ or

$$\text{(4.13)}\qquad \dot{\partial}_\varepsilon g_{ab}(B^\varepsilon_c \dot{u}^c + N^\varepsilon_k \dot{v}^k) = \dot{\partial}_c g_{ab} \dot{u}^c + \dot{\partial}_k g_{ab} \dot{v}^k = 0$$

Substituting (4.10), (4.11) (4.12) and (4.13) into (4.8) using (4.4), (4.5) we obtain

$$\text{(4.14)}\qquad \begin{aligned} &Dg_{\alpha\beta} = B^{ab}_{\alpha\beta}(g_{ab\top c} du^c + g_{ab\top k} dv^k + g_{ab}\top_c \overline{D}l^c + g_{ab}\top_k \overline{D}l^k) + \\ &N^{nt}_{\alpha\beta}(g_{nt\top c} du^c + g_{nt\top k} dv^k + g_{nt}\top_c \overline{D}l^c + g_{nt}\top_k \overline{D}l^k), \end{aligned}$$

where

$$\text{(4.15)}\qquad \begin{aligned} &\text{a)}\ g_{ab\top x} = \partial_x g_{ab} - \dot{\partial}_d g_{ab} \overline{\Gamma}^{*d}_x - \dot{\partial}_m g_{ab} \overline{\Gamma}^{*m}_x - g_{db} \overline{\Gamma}^{*d}_{ax} - g_{ad} \overline{\Gamma}^{*d}_{ax}, \\ &\quad x = c \ \text{ or } \ x = k, \\ &\text{b)}\ g_{mt\top x} = \partial_x g_{nt} - \dot{\partial}_d g_{ab} \overline{\Gamma}^{*d}_x - \dot{\partial}_m g_{nt} \overline{\Gamma}^{*m}_x - g_{mt} \overline{\Gamma}^{*m}_{nx} - g_{nm} \overline{\Gamma}^{*m}_{tx}, \\ &\quad x = c \ \text{ or } \ x = k, \\ &\text{c)}\ g_a\top_{bx} = L\dot{\partial}_x g_{ab} - g_{db} A^d_{ax} - g_{ad} A^d_{bx}, \ \ x = c \ \text{ or } \ x = k, \\ &\text{d)}\ g_{nt}\top_x = L\dot{\partial}_x g_{nt} - g_{mt} A^m_{nx} - g_{nm} A^m_{tx}, \ \ x = c \ \text{ or } \ x. = k, \end{aligned}$$

Using the relation

$$\text{(4.16)}\qquad Dg_{\alpha\beta} = B^\delta_c g_{\alpha\beta|\delta} du^c + N^\delta_k g_{\alpha\beta|\delta} dv^k + B^\delta_c g_{\alpha\beta}|_\delta D l^c + N^\delta_k g_{\alpha\beta}|_\delta \overline{D}l^k$$

from (4.14) we obtain

$$\text{(4.17)}\qquad B^\delta_c g_{\alpha\beta\top\delta} = B^{ab}_{\alpha\beta} g_{ab\top c} + N^{nt}_{\alpha\beta} g_{nt\top c}$$

$$\text{(4.18)}\qquad N^\delta_k g_{\alpha\beta|\delta} = B^{ab}_{\alpha\beta} g_{ab\top k} + N^{nt}_{\alpha\beta} g_{nt\top k}.$$

Multiplying (4.17) by B^c_γ and (4.18) by N^k_γ and adding these relations we get

$$g_{\alpha\beta|\gamma} = B^{abc}_{\alpha\beta\gamma} g_{ab\top c} + B^{ab}_{\alpha\beta} N^k_\gamma g_{ab\top k} + N^{nt}_{\alpha\beta} B^c_\gamma g_{nt\top c} + N^{ntk}_{\alpha\beta\gamma} g_{nt|k}. \tag{4.19}$$

In the same way we obtain

$$B^\delta_c g_{\alpha\beta}|_\delta = B^{ab}_{\alpha\beta} g_{ab} \top_c + N^{nt}_{\alpha\beta} g_{nt} \top_c \tag{4.20}$$

$$N^\delta_k g_{\alpha\beta}|_\delta = B^{ab}_{\alpha\beta} g_{ab} \top_k + N^{nt}_{\alpha\beta} g_{nt} \top_k \tag{4.21}$$

$$g_{\alpha\beta}|_\gamma = B^{abc}_{\alpha\beta\gamma} g_{ab} \top_c + B^{ab}_{\alpha\beta} N^k_\gamma g_{ab} \top_k + N^{nt}_{\alpha\beta} B^c_\gamma g_{nt} \top_c + N^{ntk}_{\alpha\beta\gamma} g_{nt} \top_k. \tag{4.22}$$

From (4.19) we have

Lemma 4.1. *If*

$$g_{ab|c} = \lambda_c g_{ab}, \quad g_{ab|k} = \lambda_k g_{ab}, \quad g_{nt|c} = \lambda_c g_{nt}, \quad g_{ab|k} = \lambda_k g_{nt} \tag{4.23}$$

then $g_{\alpha\beta|\gamma} = \lambda_\gamma g_{\alpha\beta}$ *where,*

$$\lambda_\gamma = B^c_\gamma \lambda_c + N^k_\gamma \lambda_k. \tag{4.24}$$

and λ_c, λ_k, λ_γ *are covariant vectors.*

Proof. Substituting (4.23) and (4.24) into (4.19) we have

$$g_{\alpha\beta|\gamma} = B^{ab}_{\alpha\beta}(B^c_\gamma \lambda_c g_{ab} + N^k_\gamma \lambda_k g_{ab}) + N^{nt}_{\alpha\beta}(B^c_\gamma \lambda_c g_{nt} + N^k_\gamma \lambda_k g_{nt} =$$
$$\lambda_\gamma B^{ab}_{\alpha\beta} g_{ab} + \lambda_\gamma N^{nt}_{\alpha\beta} g_{nt} = \lambda_\gamma g_{\alpha\beta}.$$

In the same way from (4.22) we have

Lemma 4.2. *If*

$$g_{ab|c} = \mu_c g_{ab}, \quad g_{ab|k} = \mu_k g_{ab}, \quad g_{nt|c} = \mu_c g_{nt}, \quad g_{ab|k} = \mu_k g_{nt} \tag{4.25}$$

then $g_{\alpha\beta|\gamma} = \mu_\gamma g_{\alpha\beta}$ *where,*

$$\mu_\gamma = B^c_\gamma \mu_c + N^k_\gamma \mu_k. \tag{4.26}$$

From (4.19) we have

Lemma 4.3. *If* $g_{\alpha\beta|\gamma} = \lambda_\gamma g_{\alpha\beta}$ *then* $g_{ab|c} = \lambda_c g_{ab}$, $g_{nt|c} = \lambda_c g_{nt}$, $g_{nt|k} = \lambda_k g_{nt}$ *where:*

$$\lambda_c = B^\gamma_c \lambda_\gamma, \quad \lambda_k = N^\gamma_k \lambda_\gamma \tag{4.27}$$

Proof. Multiplying (4.19) by $B^\gamma_d\ B^\beta_e\ B^\alpha_f$ and using the relations:

$$N^k_\alpha B^\alpha_c = 0 \quad B^\alpha_a B^c_\alpha = \delta^c_a \quad N^\alpha_k N^n_\alpha = \delta^n_k \tag{4.28}$$

we get

$$g_{\alpha\beta|\gamma} B^{\gamma\beta\alpha}_{def} = \delta^a_f \delta^b_e \delta^c_d g_{ab|c} \Rightarrow \lambda_\gamma B^\gamma_d g_{\alpha\beta} B^{\alpha\beta}_{fe} = \lambda_\gamma B^\gamma_d g_{je} = g_{fe|d}.$$

Multiplying (4.19) by $B^\alpha_f\ B^\beta_e\ N^\gamma_n$ and using (4.28) we have

$$g_{\alpha\beta|\gamma} B^\alpha_f B^\beta_e N_n \gamma = \delta^{abk}_{fen} g_{ab|k} \Rightarrow \lambda_\gamma g_{\alpha\beta} B^\alpha_f B^\beta_e N^\gamma_n = \lambda_\gamma N^\gamma_n g_{fe} = g_{fe|n}.$$

Multiplying (4.19) by $N^{\alpha\beta}_{kl}\ B^\gamma_d$ and using (4.28) we have

$$g_{\alpha\beta|\gamma} N^{\alpha\beta}_{kl} B^\gamma_d = \delta^n_k \delta^t_l \delta^c_d g_{nt|c} \Rightarrow \lambda_\gamma g_{\alpha\beta} N B^{\alpha\beta}_{kl} B^\gamma_d = \lambda_\gamma B^\gamma_d g_{kl} = g_{kl|d}.$$

Multiplying (4.19) by $N^{\alpha\beta\gamma}_{plm}$ and using (4.28) we obtain

$$g_{\alpha\beta|\gamma} N^{\alpha\beta\gamma}_{plm} = \delta^{ntk}_{plm} g_{nt|k} \Rightarrow \lambda_\gamma g_{\alpha\beta} Nplm^{\alpha\beta\gamma} = \lambda_\gamma N^\gamma_m g_{pl} = g_{pl|d}.$$

LEMMA 4.4 *If* ${}_{\alpha\beta}|_\gamma = \mu_\gamma g_{\alpha\beta}$, *then* $g_{ab}|_c = \mu_c g_{ab}$, $g_{ab}|_k = \mu_k g_{ab}$, $g_{nt}|_c = \mu_c g_{nt}$, $g_{nt}|_k = \mu_k g_{nt}$, *where* $\mu_c = B^\gamma_c \mu_\lambda$, $\mu_k = N^\lambda_k \mu_\lambda$.

The proof is similar to that of Lemma 4.3 if we start from (4.22).

From lemma 4.1 and 4.3 we have

THEOREM 4.1. *The necessary and sufficient conditions for* $g_{\alpha\beta|\gamma} = \lambda_\gamma g_{\alpha\beta}$ *are* (4.23) *and* (4.24)

From lemmas 4.2 and 4.4 we have

THEOREM 4.2. *The necessary and sufficient conditions for* $g_{\alpha\beta|\gamma} = \mu_\gamma g_{\alpha\beta}$ *are* (4.25) *and* (4.26)

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