PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 33 (47), 1983, pp. 29–33

DISCUSSING GRAPH THEORY WITH A COMPUTER II, THEOREMS SUGGESTED BY THE COMPUTER

Dragoš M. Cvetković

Abstract. The formulation and proof of several theorems in graph theory as well as in other areas of mathematics have been found after testing some special cases on a computer. Some results of this type are described in this paper. Special attention is paid to the interactive programming system "Graph" (described in [**7**], the first part of this paper) in which a lot of graph theoretical algorithms have been implemented and whose purpose is, among other things, to enable quick formulation, checking, or disproving of conjectures in graph theory.

The computation of extensive tables of cubic graphs [**1**], which also included other information about these graphs, gave several ideas for subsequent investigations. One such idea was that of ordering graphs lexicographically according to their eingenvalues; this is a natural way of ordering graphs. (This topic is also discussed in [**5**]). Looking at these tables one of the first observations was the fact that cubic graphs with bridges were never bipartite. This led to the following theorem which could serve as a nice exercise in teaching of graph theory.

THEOREM 1. *Let G be a regular graph: of odd degree $r\,(r \geq 3)$ having a bridge. Then G is not bipartite.*

Proof I. According to well-known theorems, any 1-factor of a regular graph of odd degree contains each bridge of the graph and, on the other hand, any regular bipartite graph has a factorization into 1-factors.

This contradiction proves the theorem.

Proof II. Without lose of generality assume that G is connected. After removing a bridge from G we get two connected components. Suppose G is bipartite and consider one of the mentioned components G_1. G_1 must be bipartite; assume that it has n_1, n_2 vertices in its respective parts. Vertex degrees of G_1 are r except for one which is $r-1$. The number of edges of G_1, counted in two ways, gives the relation $n_1 r = n_2 r - 1$, i.e. $(n_2 - n_1)r = 1$, which is impossible since $r > 1$.

This completes the proof.

AMS Subject Classification (1980): Primary 05C99, Secondary 68E10.

Another theorem obtained in just the same may is the following one:

THEOREM 2. *Let G be a cubic graph having exactly one bridge. Then the order of the automorphism group is a power of two.*

The proof is left to the reader.

Other examples of theorems suggested by a computer can be found in [**3**]. It is well known that a connected regular graph is strongly regular if and only if it has exactly 3 distinct eigenvalues. For a graph operation called NEPS (noncomplete extended p-sum of graphs), the spectrum of the resulting graph can be expressed in terms of spectra of graphs on which the operation is performed (see, for example, [**6**], p. 69). An experiment on a computer has shown that in some examples under some conditions the NEPS of complete graphs has just 3 distinct eigenvalues. This led to some conjectures which were proved and gave some constructions of strongly regular graphs and symmetric block designs.

On the other hand, in paper [**2**] the brute force of the computer was used to complete the proof of a theorem.

Inspired, among other things, by these examples and similar experience of other researchers, the author of this paper has undertaken the implementation of an interactive programming system, called "Graph", for the classification and extension of the knowledge in the field of graph theory. The basic ideas of the system "Graph" are described in (4). The system contains a subsystem for solving problems on particular graphs which basically is meant for quick checking, posing and disproving conjectures in graph theory. This part of the system is described with some detail in [**7**]. We shall repeat here a few basic facts about it.

The work with "Graph" consists of a dialogue in (a formalized subset of) the natural English language. Graphs and other objects enter into the system under names given by the user. After typing the command "SET G." the user can draw a graph on the screen by a light pen and the system will memorize this graph under the name G. The picture of any graph G in the system can be obtained on the screen by the command "DRAW G." Using again the light pen the user can modify the picture after typing "MODIFY G.". A graph can also be defined by typing its edges; for example, by the command "SET G= $(4,5)$ 1 2, 2 3, 3 4, 4 1, 1 3." a graph G is defined which has 4 vertices and 5 edges, the edges given by its endpoints. The user can give orders to the system to perform several tasks with graphs defined in the system, as the following commands illustrate: "TYPE G.", "FORGETG.", "CHECK WHETHER G IS PLANAR." "DETERMINE RADIUS OF G.", "FORM H PRODUCT OF G1 AND G2.", "CREATE G A RANDOM GRAPH ON 15 POINTS.", etc.

The system "Graph" is written in FORTRAN IV and it runs now on a PDP-11/34 computer. With slight modifications it can be installed at any computer, at least in principle.

We shall describe some of our first experience in working with the system "Graph".

1. The graphs $C_4 \cup K$, and $K_{1,4}$ are known to be the smallest pair of cospectral non-isomorphic graphs[1]. The curiosity of the author of this paper was aroused by the fact that components of the product of P_3 with itself (see Fig. 1) are just the graphs C_4 and $K_{1,4}$. As it is well-known, the product of any two connected bipartite graphs has exactly two components. It was natural to ask whether these two components always give rise to a pair of cospectral non-isomorphic graphs as they do in the above case. The facilities of the system "Graph" enable one to check conjecture on examples very easily. Positive results in a few examples convinced the author that the conjecture is true and we have now the following theorem (after a definition).

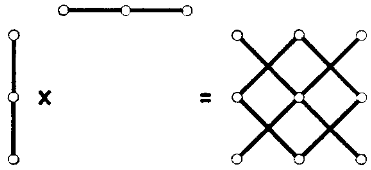

Fig. 1

Definition. Two graphs are said to be almost cospectral if their non-zero eigenvalues (and their multiplicities) coincide.

THEOREM 3. *If G and H are connected bipartite graphs then the graph $G \times H$ has exactly two components which are almost cospectral.*

Proof. By appropriate labeling of vertices, the adjacency matrices of G and H can be represented in the form

$$\begin{Vmatrix} O & B^T \\ B & O \end{Vmatrix}, \quad \begin{Vmatrix} O & C^T \\ C & O \end{Vmatrix},$$

where B and C are $(0,1)$ matrices. Vertices of $G \times H$ can be ordered so that the adjacency matrix of $G \times H$ takes the form

$$\begin{Vmatrix} O & O & O & B^T \otimes C^T \\ O & O & B^T \otimes C & O \\ O & B \otimes C^T & O & O \\ B \otimes C & O & O & O \end{Vmatrix}.$$

It remains to prove that matrices

$$(1) \qquad \begin{Vmatrix} O & B^T \otimes C^T \\ B \otimes C & O \end{Vmatrix}, \quad \begin{Vmatrix} O & B^T \otimes C \\ B \otimes C^T & O \end{Vmatrix}$$

which represent adjacency matrices of components of $G \otimes H$, have the same nonzero eigenvalues. The squares of the matrices (1) are:

$$(2) \qquad \begin{Vmatrix} B^TB \otimes C^TC & O \\ O & BB^T \otimes CC^T \end{Vmatrix}, \quad \begin{Vmatrix} B^TB \otimes CC^T & O \\ O & BB^T \otimes C^TC \end{Vmatrix}.$$

[1]Although it is very likely that many persons discovered this pair of graphs, it seems that the author's thesis (Graphs and their spectram Univ. Beograd, Publ. Elektrotehn. Fak., Ser. Mat. Fiz. No 354 — No 356 (1971), 1–50) was the first published paper mentioning these graphs

Since matrices C^TC and CC^T have the same nonzero eigenvalues, the same holds for matrices (2), and hence the theorem is true.

If the components arising from the theorem have the same number of vertices, then they are cospectral. To derive a necessary and sufficient condition for that, suppose that matrices B and C have orders $m_1 \times n_1$ and $m_2 \times n_2$ respectively. Then the matrices (1) are of orders $m_1m_2 + n_1n_2$ and $m_1n_2 + m_2n_1$ respectively. The difference of these numbers

$$m_1m_2 + n_1n_2 - (m_1n_2 + m_2n_1) - (m_1 - n_1)(m_2 - n_2)$$

is equal to 0 if and only if one of the matrices B, C is a square matrix. So we can formulate the following corollary;

COROLLARY. *Components of $G \times H$ from Theorem* 3 *are cospectral if and only if one of the bipartite graphs G, H have the same number of vertices in both parts.*

It turned out that Theorem 3 has already been discovered by C. D. Godsil and B. McKay [**8**]. Now we have the following conjecture whose verification would give a generalization of Theorem 3.

CONJECTURE. *If the NEPS of bipartite graphs is disconnected* (see [**6**], p. 204) *its components are almost cospectral.*

2. The ten self-complementary graphs on 8 vertices have been put into the system and several their properties (e.g. vertex degrees, spectrum etc.) have been examined. It turned out that all 10 graphs had an even number of triangles. This gives the rise to the following theorem.

THEOREM 4. *A self-complementary graphs G on $n = 4k$ $(k \in N)$ vertices contains an even number of triangles.*

Proof. Let $d(x)$ be the degree of the vertex x and let X be the vertex set of G. Since t, the number of triangles of G, is equal to the number of sets of three mutually non-adjacent vertices, we have

$$2t + \frac{1}{2}\sum_{x \in X} d(x)(n - 1 - d(x)) = \binom{n}{3}.$$

Since $n = 4k$ implies $\binom{n}{3} = 4s$ $(s \in N)$, we get

$$t = 2s - \frac{1}{4}\sum_{x \in X} d(x)(n - 1 - d(x)).$$

As it is well-known, any permutation π of X which takes G into $\overline{G}$ consists of cycles whose lengths are divisible by 4. The product $d(x)(n - 1 - d(x))$ is even and does not depend on x if x runs over vertices forming a cycle of π. Therefore the sum $\sum_{x \in X} d(x)(n - 1 - d(x))$ is divisible by 8 and the theorem follows from (3).

Remark. Theorem 4 does not hold for self complementary graphs on $4k+1$ vertices since there is a self complementary graphs on 5 vertices that has just one triangle.

3. According to Mathematical Reviews 80 k: 05079 the following conjecture has been posed in [**9**]:

For $m \geq 3$ and $n \geq 0$, among unicyclic graphs with $m+n$ vertices and cycle length m the largest eigenvalue of the graph $C_m - P_n$ (C_m with P_n attached at a vertex) is strictly smallest.

Using system "Graph" the conjecture has been disproved very quickly. An ad hoc counterexample is provided by $C_{19} - P_2$ with largest eigenvalue 2.0945 and C_{19} with two pendant lines attached to two vertices of C_{19} which are at distance 5 with largest eigenvalue 2.0880.

Several other investigations involving system "Graph" are in progress.

Aknowledgement. Support has been given by the Mathematical institute, Beograd.

REFERENCES

[1] F. C. Bussemaker, S. Čobeljić, D. M. Cvetković, J. J. Seidel, *Computer investigation of cubic graphs*, Technological University Eindhoven, T. H. Report. 76–WSK–O1; *Cubic graphs on ≤ 14 vertices*, J. Comb. Theory (B) **23** (1977), 234–235.

[2] D. M. Cvetković, *The number of antichains of finite power sets*, Publ. Inst. Math. (Beograd) (N.S.) **13** (**27**) (1972), 5–9.

[3] D.M. Cvetković, *A note on constructions of graphs by means of their spectra*, Publ. Inst. Math. (Beograd) (N.S) **27** (**41**) (1980), 27–30.

[4] D. Cvetković, *A project for using computers in further development of graph theory*, The theory and application of graphs, Proc. of the 4-th Internat. Conf. on the Theory and Appl. of Graphs Kalamazoo, 1980, Ed, G. Chastrand, Y. Alavi, D. L. Goldsmith, L. Lesniak-Foster John Wiley & Sons, New York–Chichester–Brisbane–Toronto–Singapore 1981, 285–296.

[5] D. M. Cvetković, *Some possible directions in further investigations of graph spectra*, Algebraic Methods in Graph Theory, Vol. 1, ed. L. Lovász, V. T. Sós, North-Holland, Amsterdam–Oxford–New York 1981, 47–67.

[6] D. Cvetković, M. Doob, H. Sachs, *Spectra of graphs. Theory and application*, Deutscher Verlag der Wissenschaften-Academic Press, Berlin–New York, 1980.

[7] D. M. Cvetković, L. L. Kraus, S. K. Simić, *Discussing graph theory with a computer I, Implementation of graph theoretic algorithms*, Univ. Beograd, Publ. Elektrotehn. Fak., Ser. Mat. Fiz. N° 716 – N° 734 (1981), 100–104.

[8] C. D. Godsil, B. McKay, *Products of graphs and their spectra, Combinatorial Mathematics IV*, Lecture Notes in Mathematics 560, ed. L. R. A. Casse, W. D. Wallis, Springer-Verlag, Berlin–Heldelberg–New York 1976, 61–72.

[9] Q. Li, K. Q. Feng, *On the largest eigenvalue of a graph* (Chinese), Acta Math. Appl. Sinica **2** (1979), 167–175.

Elektrotehnički fakultet
11000 Beograd
Jugoslavija

(Received 26 05 1982)