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AN ERROR ESTIMATE FOR GAUSS-JACOBI QUADRATURE FORMULA WITH THE HERMITE WEIGHT $w(x) = \exp(-x^2)$

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The purpose of this paper is to give an estimate of the error in approximating the integral $\int_{-\infty}^{\infty} f(x)\exp(-x^2)dx$ by the Gauss-Jacobi quadrature formula $Q_n(w; f)$, assuming that f is an entire function satisfying a certain growth condition which depends on the Hermite weight function $w(x) = \exp(-x^2)$.

1. Introduction. Let $d\alpha$ be a non-negative measure supported in the interval (a, b), $-\infty \le a < b \le \infty$. Let the support of $d\alpha$ contain infinitely many points and let

$$\int_a^b x^n d\alpha(x) < \infty, \quad for \;\; n = 0, 1, 2, \dots .$$

Then there exists a uniquely determined sequence of orthonormal polynomials $\{p_n(d\alpha; x)\}$ generated by this measure (see e. g. [**1**; Ch. I], [**5**; Ch. II]); they are determined by the properties

(a) $p_n(d\alpha; x) = \gamma_n x^n + \dots$ is a polynomial of degree n and $\gamma_n > 0$.

(b) $\displaystyle\int_a^b p_n(d\alpha)p_m(d\alpha)d\alpha = \begin{cases} 0; & \text{if } m \ne n \\ 1; & \text{if } m = n. \end{cases}$

It is well known that all zeros x_{kn} $(k = 1, 2, \dots, n)$ of $p_n(d\alpha; x)$ are real, simple and are contained in (a, b). We shall assume, as usual, that $x_{1n} > x_{2n} > \dots > x_{nn}$.

If, in addition, $d\alpha$ is an absolutely continuous measure, then $d\alpha(x) = \alpha'(x)dx$ and $\alpha'(x)$ is called a weight function. In this case, $\alpha'(x)$ will be denoted by $w(x)$ and $p_n(d\alpha)$ by $p_n(w)$.

If f is an arbitrary function defined in (a, b), the Gauss-Jacobi quadrature formula is defined by the interpolatory quadrature formula

$$Q_n(d\alpha; f) = \sum_{k=1}^{n} \lambda_n(d\alpha, x_{kn}) f(x_{kn})$$

and it has the property that for every polynomial π having degree $2n-1$, at most, we have

$$Q_n(d\alpha; \pi) = \int_a^b \pi d\alpha.$$

The coefficient $\lambda_n(d\alpha; x_{kn})$ in this formula for $Q_n(d\alpha)$ are called the Christoffel numbers and are the values of the function (see [**4**]) $\lambda_n^{-1}(d\alpha; x) = \sum_{\nu=0}^{n-1} p_\nu^2(d\alpha; x)$ at $x = x_{kn}$ $(k = 1, 2, \ldots, n)$.

The nodes x_{kn} are called the Gaussian abscise with respect to $d\alpha$.

2. Preliminary results. To prove our main result, we are going to use the following three lemmas. Lemmas 1 and 2 are due to G. Freud [**2**, **3**].

LEMMA 1. (See [**2**]). *Let $f(z)$ be an analytic function in a domain containing the Gaussian abscise x_{kn} $(k = 1, 2, \ldots, n)$ and $x_{j,n+1}$ $(j = 1, 2, \ldots, n+1)$. If $p_n(d\alpha; x) = \gamma_n x^n + \ldots$ is the orthonormal polynomial of degree n associated with the measure $d\alpha$, we have*

$$Q_{n+1}(d\alpha; f) - Q_n(d\alpha; f) = \frac{\gamma_{n+1}}{\gamma_n} \cdot \frac{1}{2\pi i} \cdot \oint_{C_n} \frac{f(z)dz}{p_n(d\alpha; z) p_{n+1}(d\alpha; z)}$$

where $C_n \subset \mathcal{D}$ is a simple closed curve containing the zeros of $p_n(d\alpha)$ and $p_{n+1}(d\alpha, z)$ in its interior. The error term of the quadrature formula is

$$\int_a^b f d\alpha - Q_n(d\alpha; f) = \sum_{\nu=n}^{\infty} \frac{\gamma_{\nu+1}}{\gamma_\nu} \cdot \frac{1}{2\pi i} \cdot \oint_{C_\nu} \frac{f(z)dz}{p_\nu(d\alpha; z) p_{\nu+1}(d\alpha; z)} \tag{2.1}$$

LEMMA 2. (See [**3**]). *For every even weight function $w(x)$, we have*

$$\max_{1 \le k \le n-1} \gamma_{k-1}/\gamma_k \le x_{1n} \le 2 \max_{1 \le k \le n-1} \gamma_{k-1}/\gamma_k \tag{2.2}$$

LEMMA 3. *Let $w(x)$ be an even weight function. Then we have*

$$\sum_{k=1}^{[n/2]^{1}} x_{kn}^2 = \sum_{k=1}^{n-1} \left(\frac{\gamma_{k-1}}{\gamma_k} \right)^2 \tag{2.3}$$

[1] Here, [ ] is the greatest integer function

Proof. From the fact that w is an even weight function, it follows that (see e.g. [**5**; §2.3(2)]) $p_n(w;x)$ is an even or an odd polynomial according as n is even or odd. Hence, we can write

$$p_n(w;x) = \gamma_n x^n - \beta_n x^{n-2} + \dots \tag{2.4}$$

Recalling the recursion formula for the orthogonal polynomials generated by an even weight function (see [**5**; §3.2(1)] or (**1**; §I.2]), we have

$$xp_n(w;x) = \gamma_n \gamma_{n+1} p_{n+1}(w;x) + \gamma_{n-1}\gamma_n^{-1} p_{n-1}(w;x) \tag{2.5}$$

Combining (2.4) and (2.5) and comparing the coefficients of x^{n-1} on both sides of (2.5) we get

$$-\beta_n = -\beta_{n+1}\gamma_n/\gamma_{n+1} + \gamma_{n-1}^2/\gamma_n$$

i. e.,

$$\beta_{n+1}/\gamma_{n+1} = \beta_n/\gamma_n + (\gamma_{n-1}/\gamma_n)^2$$

which implies that $\frac{\beta_n}{\gamma_n} = \sum_{k=1}^{n-1}\left(\frac{\gamma_{k-1}}{\gamma_k}\right)^2$.

Since it is easy to see that $\frac{\beta_n}{\gamma_n} = \sum_{k=1}^{[n/2]} x_{kn}^2$, the proof of the lemma is completed.

3. The main result.

THEOREM. *Let $f(z)$ be an entire function satisfying the condition*

$$\beta = \lim_{R\to\infty} \sup R^{-2} \max_{|z|=R}(\log|f(z)|) < \rho \tag{3.1}$$

where $\rho\,(\approx .70541786)$ is such that $\rho = (3\varepsilon_0+1)(1-\varepsilon_0)/8\varepsilon_0$ and ε_0 is the solution of $\frac{1-x}{4}\exp\left(\frac{1-x}{2x}\right) = 1$, Then we have

$$\lim_{n\to\infty} \sup \left| \int_{-\infty}^{\infty} f(x)w(x)dx - Q_n(w;f)\right|^{1/n} < 1$$

where $w(x) = \exp(-x^2)$,

Proof. Since $w(x) = \exp(-x^2)$, it is well known that the n-th orthonormal polynomial generated by this weight function is the n-th Hermite polynomial $h_n(x)$ and it is also well known (see. e. g. [**5**; §5.5]) that

$$\gamma_n^2 = 2^n/\sqrt{n}/n! \tag{3.2}$$

which easily implies

$$\gamma_{n-1}/\gamma_n = \sqrt{n/2} \tag{3.3}$$

Combining (3.3) and (2.2), we get

$$x_{1,n+1} \le \sqrt{2n} \tag{3.4}$$

Combining (3.3) and (2.3), we find that

$$\sum_{k=1}^{[n/2]} x_{kn}^2 = \frac{n(n-1)}{4} \quad n = 2, 3, 4, \ldots \tag{3.5}$$

To prove this theorem we are going to use (2.1). First we will find an inequality for $|h_n(z)|^{-1}$. Since w is an even weight function, it follows that (see [**5**; §2.3 (2)])

$$h_n(z) = \gamma_n z^{n-2[n/2]} \prod_{k=1}^{[n/2]} (z^2 - x_{kn}^2)$$

Hence,

$$\begin{aligned} |h_n(z)| &= \gamma_n |z|^{n-2[n/2]} \prod_{k=1}^{[n/2]} |z^2 - x_{kn}^2| = \gamma_n |z|^n \exp\left\{ \sum_{k=1}^{[n/2]} \log |1 - x_{kn}^2/z^2| \right\} \\ &\geq \gamma_n |z|^n \exp\left\{ \sum_{k=1}^{[n/2]} \log \left(1 - \frac{x_{kn}^2}{|z|^2}\right) \right\} \geq \gamma_n |z|^n \exp\left\{ -\sum_{k=1}^{[n/2]} \frac{x_{kn}^2}{(|z|^2 - x_{kn}^2)} \right\}, \end{aligned}$$

for every complex number z such that $x_{1n} < |z|$. Next we have

$$1/(|z|^2 - x_{kn}^2) \leq 1/(|z|^2 - x_{1n}^2)$$

and so

$$|h_n(z)| \geq \gamma_n |z|^n \exp\left\{ -\frac{1}{|z|^2 - x_{1n}^2} \sum_{k=1}^{[n/2]} x_{kn}^2 \right\}.$$

Using (3.5), we find that

$$|h_n(z)| \geq \gamma_n |z|^n \exp\{-n(n-1)/4(|z|^2 - x_{1n}^2)\}$$

Therefore

$$1/|h_n(z)| \leq \gamma_n^{-1} |z|^n \exp\{n(n-1)/4(|z|^2 - x_{1n}^2)\}.$$

And so, for $x_{1,n+1} < |z|$, it follows that

$$\begin{aligned} \frac{1}{|h_n(z) h_{n+1}(z)|} &\leq \frac{1}{\gamma_n \gamma_{n+1}} \cdot \frac{1}{|z|^{2n+1}} \exp\left\{ \frac{n(n-1)}{4(|z|^2 - x_{1n}^2)} + \frac{n(n+1)}{4(|z|^2 - x_{1,n+1}^2)} \right\} \\ &\leq \frac{1}{\gamma_n \gamma_{n+1}} \frac{1}{|z|^{2n+1}} \exp\left\{ \frac{n^2}{2(|z|^2 - x_{1,n+1}^2)} \right\}. \end{aligned}$$

Since $\beta = \lim_{R\to\infty} \sup R^{-2} \max_{|z|=R} (\log |f(z)|)$ for any $\sigma > 0$, we can find N_σ such that

$$|f(z)| \leq \exp\{(\beta + \sigma)|z|^2\}, \text{ for all } |z| \geq N_\sigma. \tag{3.6}$$

Denoting by I_n the expresion $\frac{\gamma_{n+1}}{\gamma_n} \cdot \frac{1}{2\pi i} \oint_{C_n} \frac{f(z)dz}{h_n(z)h_{n+1}(z)}$, taking the path of integration to be the circle $|z| = R$, where

$$R^2 \geq x_{1,n+1}^2/(1-\varepsilon), \quad (0 < \varepsilon 1), \tag{3.7}$$

we find, that for $|z| = R$,

$$\frac{1}{|h_n(z)h_{n+1}(z)|} \leq \frac{1}{\gamma_n \gamma_{n+1}} \cdot \frac{1}{R^{2n+1}} \exp\left\{\frac{n^2}{2(R^2 - x_{1,n+1}^2)}\right\}$$

$$\leq \frac{1}{\gamma_n \gamma_{n+1}} \cdot \frac{1}{R^{2n+1}} \exp\left\{\frac{n^2}{2(R^2-(1-\varepsilon)R^2)}\right\} \leq \frac{1}{\gamma_n \gamma_{n+1}} \cdot \frac{1}{R^{2n+1}} \exp\left\{\frac{n^2}{2(2\varepsilon R^2}\right\}.$$

Using the last inequality, (3.2) and (3.6), we conclude that for $R \geq N_\sigma$,

$$|I_n| \leq \sqrt{\pi} n! 2^{-n} R^{-2n} \max_{|z|=R} |f(z)| \cdot \exp\{n^2/2\varepsilon R^2\} \leq$$
$$\leq \sqrt{\pi} n! 2^{-n} R^{-2n} \exp\{(\beta+\sigma)R^2 + n^2/2\varepsilon R^2\}.$$

Next, R will be chosen so as to minimize the right hand side of this inequality, and at the same time, to satisfy (3.7).

Consider the function $h(R) = R^{-1n} \exp\{(\beta+\sigma)R^2 + n^2/2\varepsilon R^2\}$.

Differentiating $h(R)$ and setting $h'(R) = 0$, we get

$$2(\beta+\sigma)\varepsilon R^4 - 2n\varepsilon R^2 - n^2 = 0. \tag{3.8}$$

If we denote by R_n the positive solution of this equation, we find that

$$R_n^2 = \frac{1 + \{1 + 2(\beta+\sigma)/\varepsilon\}^{1/2}}{2(\beta+\sigma)} \cdot n$$

(We can easily check that $f(R)$ attains its minimum value at $R = R_n$). For $n \geq N$, for a suitable $N > 0$, we will have $R_n \geq N_\sigma$. Also, from (3.4), it follows that

$$R_n^2 \geq \frac{1 + \{1 + 2(\beta+\sigma)/\varepsilon\}^{1/2}}{4(\beta+\sigma)} \cdot x_{1,n+1}^2$$

and consequently, condition (3.8) will be satisfied if

$$4(\beta+\sigma)1 + \{1 + 2(\beta+\sigma)/\varepsilon\}^{1/2} = 1 - \varepsilon \tag{3.9}$$

Since R_n satisfies equation (3.8), we find that

$$(\beta+\sigma)R_n^2 = n + n^2/2\varepsilon R_n^2$$

and it follows that

$$|I_n| \leq \sqrt{\pi}\frac{n!}{2^n} \cdot \frac{1}{R_n^{2n}} \exp\left\{n + \frac{n^2}{\varepsilon R_n^2}\right\} \leq$$
$$\sqrt{\pi} \cdot \frac{n!}{2^n} \left\{\frac{2(\beta+\sigma)}{1 + (1 + 2(\beta+\sigma)/\varepsilon)^{1/2}}\right\}^n \cdot \exp\left\{\frac{2(\beta+\sigma)n}{\varepsilon(1 + (1 + 2(\beta+\sigma)/\varepsilon)^{1/2})}\right\}$$

Using (3.9), we find that

$$|I_n| \le \sqrt{\pi} n! 2^{-n} n^{-n} ((1-\varepsilon)/2)^n \exp\{n + (1-\varepsilon)n/2\varepsilon\}$$

Using the Stirling formula $n! \approx \sqrt{2\pi n} n^n e^{-n}$, we find that

$$|I_n| \le K\sqrt{n} \left\{ \frac{1-\varepsilon}{4} \exp\left(\frac{1-\varepsilon}{2\varepsilon}\right) \right\}^n$$

$K =$ constant and n is sufficiently large.

It is easy to see that $g(\varepsilon) = \dfrac{1-\varepsilon}{4} \exp\left(\dfrac{1-\varepsilon}{2\varepsilon}\right)$ is a decreasing function on $(0,1)$. Consequently, if ε_0 is the unique solution of $g(\varepsilon) = 1$, then for $\varepsilon_0 < \varepsilon < 1$ we have $0 < g(\varepsilon) < 1$.

Thus, $\sum_{k=1}^{\infty} |I_k|$ is a convergent series.

If $\Delta_n = \sum_{k=n}^{\infty} I_k$, we have

$$|\Delta_n| \le \sum_{k=n}^{\infty} |I_k| \le K \sum_{k=n}^{\infty} \sqrt{k} \left(\frac{1-\varepsilon}{4} \exp\left(\frac{1-\varepsilon}{2\varepsilon}\right) \right)^k.$$

Since $\displaystyle\sum_{k=n}^{\infty} kx^k \le \frac{(n+2)x^n}{(1-x)^2}$ for $0 < x < 1$, it follows that

$$|\Delta_n| \le K(n+3) \left\{ 1 - \frac{1-\varepsilon}{4} \exp\left(\frac{1-\varepsilon}{2\varepsilon}\right) \right\}^{-2} \left(\frac{1-\varepsilon}{4} \exp\left(\frac{1-\varepsilon}{2\varepsilon}\right) \right)^n$$

and so

$$\lim_{n\to\infty} \sup |\Delta_n|^{1/n} \le \frac{1-\varepsilon}{4} \exp\left(\frac{1-\varepsilon}{2\varepsilon}\right) < 1 \ \text{ for every } \ \varepsilon_0 < \varepsilon < 1.$$

Finally, it remains to justify the assumption (3.1).

From (3.9) and the choice of $\varepsilon_0 < \varepsilon < 1$, we see that

$$\frac{4(\beta+\sigma)}{1 + \{1 + 2(\beta+\sigma)/\varepsilon_0\}^{1/2}} < \frac{4(\beta+\sigma)}{1 + \{1 + 2(\beta+\sigma)/\varepsilon\}^{1/2}} = 1 - \varepsilon < 1 - \varepsilon_0$$

and therefore,

$$\frac{4(\beta+\sigma)}{1 + \{2(\beta+\sigma)/\varepsilon_0\}^{1/2}} < 1 - \varepsilon_0.$$

Solving this inequality, we find that

$$0 < \beta + \sigma < (3\varepsilon_0 + 1)(1 - \varepsilon_0)/8\varepsilon_0 = \rho \ \ (\approx .70541786)$$

Hence, we must have $0 < \beta < \rho$, which completes the proof of the theorem.

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