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A UNIFIED CLASS OF POLYNOMIALS*

Hukum Chand Agrawal**

Summary. In this paper we propose to study the polynomial set $\left\{f_n^{(\alpha)}\right\}(x)$ satisfying the functional relation

$$T(\Delta_\alpha)\left\{f_n^{(\alpha)}(x)\right\} = f_{n-1}^{(\alpha+1)}(x), \qquad n = 1, 2, 3, \ldots,$$

where $f(\alpha)_n(x)$ is the polynomial of degree n in x and T is the operator of infinite order defined by

$$T(\Delta_\alpha) = \sum_{k=0}^{\infty} h_k^{(\alpha)} \Delta_\alpha^{k+1}, \;\; h_0^{(\alpha)} \neq 0,$$

in which $\Delta_\alpha\{f(\alpha)\} = f(\alpha+1) - f(\alpha)$.

1. Introduction. In his recent communication the author [**1**] studied the polynomial set $\left\{(p_n^{(\alpha)}(x)\right\}$ satisfying the condition

$$\Delta_\alpha\left\{p_n^{(\alpha)}(x)\right\} = p_(n-1)^{(\alpha+1)}(x), \qquad n = 1, 2, 3, \ldots \tag{1.1}$$

A list of twelve polynomials is given which satisfy the above functional relation. In this paper we study another classification of polynomials which includes the class above as a particular case.

Consider the polynomial set $\left\{f_n^{(\alpha)}(x)\right\}$; $f_n^{(\alpha)}(x)$ are the polynomials of degree n in x, and the infinite operator

$$T(\Delta_\alpha) \equiv T = \sum_{k=0}^{\infty} h_k^{(\alpha)} \Delta_\alpha^{k+1}, \;\; h_0^{(\alpha)} \neq 0, \tag{1.2}$$

in which $\Delta_\alpha\{f(\alpha)\} = f(\alpha+1) - f(\alpha)$.

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We are concerned here with those polynomials $f_n^{(\alpha)}(x)$ which satisfy the condition

$$T\left\{f_n^{(\alpha)}(x)\right\} = f_{n-1}^{(\alpha+1)}(x), \qquad n = 1, 2, 3, \ldots \tag{1.3}$$

Obviously, for $h_0^{(\alpha)} = 1$ and $h_1^{(\alpha)} = h_2^{(\alpha)} = \cdots = 0$ the condition (1.3) reduces to (1.1).

2. Certain Fundamental Properties of T–Operators.

THEOREM 1. *If $f_n^{(\alpha)}(x)$ is a simple set of polynomials in α, then there exists a unique difference operator of the form*

$$T = \sum_{k=0}^{\infty} h_k^{(\alpha)} \Delta_\alpha^{k+1}, \;\; h^{(\alpha)} \neq 0 \tag{2.1}$$

where $h_k^{(\alpha)}$ is a polynomial of degree $\leq k$ in α, for which

$$T\left\{f_n^{(\alpha)}(x)\right\} = f_{n-1}^{(\alpha+1)}(x), \qquad n \geq 1, \tag{2.2}$$

Proof. From (2.1) and (2.2), we have

$$\sum_{k=0}^{n-1} h_k^{(\alpha)} \Delta_\alpha^{k+1} \left\{f_n^{(\alpha)}(x)\right\} = f_{n-1}^{(\alpha+1)}(x). \tag{2.3}$$

The above equation shows that $h_k^{(\alpha)}$ is uniquely defined and is of degree $\leq k$, because $f_{n-1}^{(\alpha+1)}(x)$ is of degree $n-1$ for each $n (n \neq 0)$.

One can easily show that

THEOREM 2. *A necessary and sufficient condition that the simple sets of polynomials $f_n^{(\alpha)}(x)$ and $m_n^{(\alpha)}(x)$ belong to the same operator T is that there exist polynomial coefficients $b_k(x)$ of degree $\leq k$ in x and independent of a and n, such that*

$$f_n^{(\alpha)}(x) = \sum_{k=0}^{n} b_k(x) m_{n-k}^{(\alpha)}(x), \quad b_0(x) \neq 0. \tag{2.4}$$

Definition. Let $f_n^{(\alpha)}(x)$ be the simple set of polynomials belonging to the operator T defined by (1.2). If the maximum degree of the coefficient $h_k^{(\alpha)}$ in α is m, we say that the set $f_n^{(\alpha)}(x)$ is of α-type m. If the degree of $h_k^{(\alpha)}$ is unbounded, we say that $f_N^{(\alpha)}(x)$ is of α-type ∞.

3. Some Properties of Sets of α-type Zero. According to the definition of α-type zero, any polynomial $f_n^{(\alpha)}(x)$ corresponding to the operator T is said to be of α-type zero, if

$$T = \sum_{k=0}^{\infty} h_k \Delta_\alpha^{k+1}, \quad h_0 \neq 0 \tag{3.1}$$

and

$$T\left\{f_n^{(\alpha)}(x)\right\} = f_{n-1}^{(\alpha+1)}(x), \tag{3.2}$$

where h_k are independent of α.

Let $A(t)$ (independent of α) be a formal power series obtainable from the symbolic correspondence

$$T(A(t)) = t(1 + A(t)), \tag{3.3}$$

where $T(A(t))$ stands for $\sum_{k=0}^{\infty} h_k(A(t))^{k+1}$, $(h_0 \neq 0)$,

Again, let

$$A(t) = \sum_{r=1}^{\infty} u_r t^r, \tag{3.4}$$

and denote

$$[A(t)]^k = \left[\sum_{r=1}^{\infty} u_r t^r\right]^k \quad \text{by} \quad \sum_{r=k}^{\infty} u_{rk} t^r.$$

Then, (3.3) on equating the coefficient of t^r on both sides, gives

$$u_r = \sum_{k=0}^{r} h_k u_{(r+1)(k+1)}, \quad r = 1, 2, \ldots; \tag{3.5}$$

with $h_0 u_{11} = 1$.

THEOREM 3. *A necessary and sufficient condition for $f_n^{(\alpha)}(x)$ to be of α-type zero corresponding to the operator T is that $f_n^{(\alpha)}(x)$ possesses a generating function of the type*

$$(1 + A(t))^{\alpha} Q(x, t) = \sum_{n=0}^{\infty} f_n^{(\alpha)}(x) t^n, \tag{3.6}$$

where[1] *$A(t)$ and $Q(x,t)$ are independent of α and $A(t)$ is given by* (3.3) *and* (3.4).

Proof. Transforming both sides of (3.6) by T, we obtain

$$\sum_{n=0}^{\infty} t^n T\left\{f_n^{(\alpha)}(x)\right\} = (1 + A(t))^{\alpha} T(A(t)) Q(x,t) =$$

$$= t(1 + A(t))^{\alpha+1} Q(x,t) = \sum_{n=0}^{\infty} f_n^{(\alpha+1)}(x) t^{n+1},$$

which gives $T\left\{f_n^{(\alpha)}(x)\right\} = f_{n-1}(\alpha+1)(x)$. Therefore, $f_n^{(\alpha)}(x)$ is of α-type zero.

Conversely, let $f_n^{(\alpha)}(x)$ be of α-type zero. Then from (3.2), we get

$$[(1-t)\Delta_{\alpha} - t] \sum_{n=0}^{\infty} t^n f_n^{(\alpha)}(x) = 0. \tag{3.7}$$

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[1] The generating function (3.6), in fact includes the generating functions given and studied by Appell [**2**], Sheffer [**8**], Brenke [**4**], Boas and Buck [**3**], and Rainville [**6**, §77].

Solving the above homogeneous-linear-difference equation, we get (3.6). Thus the theorem is proved.

COROLLARY 1. *A necessary and sufficient condition for $f_n^{(\alpha)}(x)$ to be of α-type zero and Sheffer A-type zero corresponding to the operator T and J*[2]*, respectively, is that $f_n^{(\alpha)}(x)$ possesses the generating function*

$$(1+A(t))^{\alpha}\exp\{xH(t)\}=\sum_{n=0}^{\infty}f_n^{(\alpha)}(x)t^n, \tag{3.8}$$

where $A(t)$ and $H(t)$ are independent of α and are given by (3.3) *and $J(H(t))=H(J(t))=t$, respectively.*

THEOREM 4. *Let $\left\{f_n^{(\alpha)}(x)\right\}$ be a set of α-type zero polynomials having the generating function*

$$(1+A(t))^{\alpha}Q(x,t)=\sum_{n-0}^{\infty}f_n^{(\alpha)}(x)t^n.$$

A necessary and sufficient condition for $f_n^{(\alpha)}(x)$ to satisfy the recurrence relation

$$nf_n^{(\alpha)}(x)=\sum_{r=0}^{n-1}(\alpha l_r+m_r(x))f_{n-r-1}^{(\alpha)}(x),\quad n\geq 1 \tag{3.9}$$

is that there exist constants l_k and polynomial coefficients $m_k(x)$ of degree $\leq k$ in x, independent of α and n, given by

$$A'(t)/(1+A(t))=\sum_{r=0}^{\infty}l_rt^r \tag{3.10}$$

and

$$Q'(x,t)/Q(x,t)=\sum_{r=0}^{\infty}m_r(x)t^r, \tag{3.11}$$

respectively. Prime denotes differentiation with respect to t.

Proof. Differentiating both sides of (3.6), with respect to t, we get

$$\begin{aligned}\sum_{n=0}^{\infty}nt^nf_n^{(\alpha)}(x)&=t[\alpha A'(t)/(1+A(t))+Q'(x,t)/Q(x,t))(1+A(t))^{\alpha}Q(x,t)\\&=\sum_{n=0}^{\infty}\sum_{r}^{n}(\alpha l_r+m_r(x))f_{n-r}^{(\alpha)}(x)t^{n+1}.\end{aligned}$$

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[2]Here, as well as in what follows, J is defined by

$$J(D)\equiv J=\sum_{k=0}^{\infty}c_kD^{k+1},\ \ c_0\neq 0,\ \ D\equiv d/dx,$$

where the $c_{k'}s$ are independent of α.

Equating the coefficients of t^n, we get (3.9). Thus, the sufficient part of the theorem is proved.

For, the necessary part, let

$$P = \sum_{n=0}^{\infty} f_n^{(\alpha)}(x)t^n, \quad L = \sum_{n=0}^{\infty} \beta_n t^n \text{ and } M = \sum_{n=0}^{\infty} \gamma_n t^n \tag{3.12}$$

where $r\beta_r = l_{r-1}1$, $\beta_0 = 0$, $r\gamma_r = m_{r-1}(x)$ and $\gamma_0 = 0$.

With these assumptions, (3.9) can be written as

$$\frac{dP}{dt} = \left[\alpha \frac{dL}{dt} + \frac{dM}{dt}\right] P,$$

which after some simplifications, gives

$$\alpha A'(t)/(1 + A(t)) + Q'(x,t)/Q(x,t) = \alpha \sum_{r=0}^{\infty} \beta_r r t^{r-1} + \sum_{r=1}^{\infty} \gamma_r r t^{r-1}.$$

Since $A(t)$ and $Q(x,t)$ are independent of α, comparing the coefficient of α, we obtain

$$A'(t)/(1 + A(t)) = \sum_{r=1}^{\infty} \beta_r r t^{r-1} = \sum_{r=1}^{\infty} l_{r-1} t^{r-1}$$

and

$$Q'(x,t)/Q(x,t) = \sum_{r=1}^{\infty} \gamma_r r t^{r-1} = \sum_{r=1}^{\infty} m_{r-1}(x) t^{r-1},$$

which are (3.10) and (3.11), respectively. Hence the theorem is proved.

Explicit form. The α-type zero polynomials satisfy the recurrence relation (3.9), viz.,

$$\begin{aligned} n f_n^{(\alpha)}(x) &= (\alpha l_0 + m_0(x)) f_{n-1}(\alpha)(x) + (\alpha l_1 + m_1(x)) f_{n-2}(\alpha)(x) + \dots \\ &\quad + (\alpha l_{n-1} + m_{n-1}(x)) f_0^{(\alpha)}(x). \end{aligned}$$

Eliminating $f_{n-1}^{(\alpha)}(x)$, $f_{n-2}^{(\alpha)}(x), \dots, f_0^{(\alpha)}(x)$, we get the following explicit form for $f_n^{(\alpha)}(x)$

$$f_n^{(\alpha)}(x) = \sum \frac{s_1^{r_1} s_2^{r_2} \dots s_n^{r_n}}{r_1! r_2! \dots r_n!}, \tag{3.13}$$

where $\alpha l_k + m_k(x) = (k+1)s_{k+1}$, for $k = 0, 1, \dots, n-1$; $f_0^{(\alpha)}(x) = 1$ and the summation is taken over all positive integral values of r_1, $r_2, \dots, r_n$ such that $r_1 + 2r_2 + \dots + nr_n = n$. (3.13) shows that $f_n^{(\alpha)}(x)$ is a polynomial of degree n in α.

THEOREM 5. *A necessary and sufficient condition for $f_n^{(\alpha)}(x)$ to be of α-type zero is that it satisfies a difference eguation of the form*

$$\Delta_\alpha \left\{ f_n^{(\alpha)}(x) \right\} = \sum_{k=1}^{\infty} u_k f_{n-k}^{(\alpha)}(x), \tag{3.14}$$

where u_k is independent of n, α and is given by (3.3) *and* (3.4).

Proof. Applying Δ_α to both sides of (3.6), we obtain

$$\sum_{n=0}^{\infty} t_u \Delta_\alpha \left\{ f_n^{(\alpha)}(x) \right\} = A(t)(1+A(t))^\alpha Q(x,t) = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} u_k f_n^{(\alpha)}(x) t^{n+k}.$$

Equating the coefficients of t^n, we get (3.14).

Conversely, (3.14) can be written as

$$\Delta_\alpha\{P\} = A(t)P, \tag{3.15}$$

in the notation of (3.12).

The solution of the difference equation (3.15) is

$$P = \sum_{n=0}^{\infty} f_n^{(\alpha)}(x) t^n = (1+A(t))^\alpha Q(x,t).$$

Thus the theorem is proved.

The difference equation (3.15) can be generalized as $T(P) = t(1+A(t))P$. Thus, $P = \sum_{n=0}^{\infty} f_n^{(\alpha)}(x)t^n$ is also the solution of the difference equation $T(P) = t(1+A(t))P$.

The following results can be proved easily.

Corollary 2. *If $f_n^{(\alpha)}(x)$ is of α-type zero and Sheffer A-type zero corresponding to the operator T and J, respectively, then so are the sets*

$$\{(\Delta_\alpha D) f_{n+1}^{(\alpha)}(x)\},\ \ \{(\Delta_\alpha^2 D^2) f_{n+2}^{(\alpha)}(x)\}, \ldots;$$

where $D \equiv d/dx$.

Corollary 3. *If $\left\{ f_n^{(\alpha)}(x) \right\}$ is α-type and Sheffer A-type zero, then*

$$f_n^{(\alpha+\beta)}(x+y) = \sum_{r=0}^{n} f_{n-r}^{(\alpha)}(x) f_r^{(\beta)}(y). \tag{3.16}$$

(3.16) can also be written as

$$n! f_n^{(\alpha+\beta)}(x+y) = \sum_{r=0}^{n} \binom{n}{r} ((n-r)! f_{n-r}^{(\alpha)}(x))(r! f_r^{(\beta)}(y)),$$

which shows that $\{n! f_n^{(\alpha)}(x)\}$ is a cross-sequence (for definition, see [**7**]).

The result (3.16) can be generalized as

$$f_n^{(\alpha_1+\cdots+\alpha_r)}(x_1+\cdots+x_r) = \sum_{m_1+\cdots+m_r=n} f_{m_1}^{(\alpha_r)}(x_1)\ldots f_{m_r}^{(\alpha_r)}(x_r). \tag{3.17}$$

THEOREM 6. *If* $\left\{f_n^{(\alpha)}(x)\right\}$ *is of* α*-type zero corresponding to the operator* Δ_α*, then so is* $\left\{f_n^{(\alpha+\beta_n)}(x)\right\}$ *corresponding to the operator* τ*, defined by*

$$\tau(z) = z(1+z)^{-\beta} \tag{3.18}$$

where $z = u(t)/(1-u(t))$ *and* $t = u(t)(1-u(t))^\beta$*.*

Proof. With the help of the following generating relation [**5**]

$$\frac{1-u(t)]^{1-\alpha}F(x,u(t))}{1-(1+\beta)u(t)} = \sum_{n=0}^{\infty} f_n^{(\alpha+\beta_n)}(x)t^n \tag{3.19}$$

where $t = u(t)(1-u(t))^\beta$, the theorem can be proved easily.

4. A Characterization for α-type Zero Polynomials. Let us consider the set of polynomials $\left\{\psi_n^{(\alpha)}(x,A,Q)\right\}$ defined by

$$\psi_n^{(\alpha)}(x,A,Q) = \left\{E_\alpha^{-1}(1+A(\nabla_\alpha))\right\}^\alpha Q(x,\nabla_\alpha)\frac{(\alpha)^n}{n!}, \tag{4.1}$$

where $A(t)$ and $Q(x,t)$ are formal power series in t independent of n.

(4.1), gives

$$\sum_{n=0}^{\infty} \psi_n^{(\alpha)}(x,A,Q)t^n = \left\{E_\alpha^{-1}(1+A(\nabla_\alpha))\right\}^\alpha Q(x,\nabla_\alpha)(1-t)^{-\alpha}. \tag{4.2}$$

Now the application of the formula $\Phi(\nabla_\alpha)\{a^\alpha\} = a^\alpha\Phi(1-a^{-1})$, with $\Phi(x) = \sum_{r=0}^{\infty} b_r x^r$, reduces (4.2) to the form

$$(1+A(t))^\alpha Q(x,t) = \sum_{n=0}^{\infty} \Psi_n^{(\alpha)}(x,A,Q)t^n. \tag{4.3}$$

Hence, we conclude that

THEOREM 7. *A necessary and sufficient condition for* $f_n^{(\alpha)}(x)$ *to be of* α*-type zero is that it is given by the operational formula*

$$f_n^{(\alpha)}(x) = \left\{E_\alpha^{-1}(1+A(\nabla_\alpha)\right\}^\alpha Q(x,\nabla_a)(\alpha)_n/n!, \tag{4.4}$$

and then the polynomial is defined by the generating function

$$(1+A(t))^\alpha(x,t) = \sum_{n=0}^{\infty} f_n^{(\alpha)}(x)t^n.$$

5. Algebraic Structure. Consider the set G_1 consisting of all α-type zero polynomials corresponding to the operator T as its elements, i. e.,

$$G_1 = \left\{f_n^{(\alpha)}(x) : \tau(f_n^{(\alpha)}(x)) = f_{n-1}^{(\alpha+1)}(x)\right\}, \tag{5.1}$$

where τ is fixed and given by (3.1). For the sake of brevity, we denote the elements of G_1 by $f_n^{(\alpha)}$, $g_n^{(\alpha)}, \ldots$.

THEOREM 8. *The set G_1 is an Abelian Group with respect to the operation $*$ defined by*

$$p_n^{(\alpha)} * q_n^{(\alpha)} = \sum_{k=0}^{\infty} p_{n-k}^{(0)} q_k^{(\alpha)}. \tag{5.2}$$

Before proving Theorem 8, we derive the following lemma:

LEMMA 1. *If $(1 + A(t))^\alpha = \sum_{n=0}^{\infty} I_n^{(\alpha)} t^n$, then* (5.3)

(i) $I_n^{(\alpha)} \in G_1$,

(ii) $$I_r^{(0)} = \begin{cases} 0 \ \textit{for} \ r \neq 0 \\ 1 \ \textit{for} \ r = 0, \end{cases} \tag{5.4}$$

(iii) *the explicit form of any element $f_n^{(\alpha)} \in G_1$ is*

$$\sum I_r^{(\alpha)} f_{n-r}^{(0)}, \tag{5.5}$$

(iv) $I_n^{(\alpha)}$ *is the identity element for the set $(G_1, *)$.*

Proof of the Lemma 1. By Theorem 3, it is evident that $I_n^{(\alpha)} \in G_1$. Putting $\alpha = 0$ in (5.3), we obtain

$$1 = \sum_{n=0}^{\infty} I_n^{(0)} t^n.$$

On comparing the coefficients of various powers of t, we get (5.4).

For every $f_n^{(\alpha)} \in G_1$, by Theorem 3, we have

$$(1 + A(t))^\alpha Q(x, t) = \sum_{n=0}^{\infty} f_n^z al(x) t^n,$$

in which the substitution $\alpha = 0$ gives

$$Q(x, t) = \sum_{n=0}^{\infty} f_n^{(0)}(x) t^n. \tag{5.6}$$

Now, putting the value of $(1 + A(t))^\alpha$, $Q(x, t)$ from (5.3) and (5.6), respectively, in (3.6), and equating the coefficients of t^n, we get the required result (5.5).

Since

$$f_n^{(\alpha)} * I_n^{(\alpha)} = \sum_{r=0}^{n} I_r^{(\alpha)} f_{n-r}^{(0)} = f_n^{(\alpha)}, \qquad \text{(by (5.5))},$$

and

$$I_n^{(\alpha)} * f_n^{(\alpha)} = \sum_{r=0}^{n} I_r^{(0)} f_{n-r}^{(\alpha)} = f_n^{(\alpha)}, \qquad \text{(by (5.4))};$$

$I_n^{(\alpha)}$ is the identity element for the set $(G_1, *)$. Thus the lemma is proved.

Proof of Theorem 8. With the help of the lemma above, the theorem can be proved easily.

THEOREM 9. *The mapping $\rho : G_1 \to G_1$ such that*

$$\rho(f^{(\alpha)}) = T(f_n^{(\alpha)}), \;\; \forall f_n^{(\alpha)} \in G_1 \tag{5.7}$$

is an isomorphism.

Proof. The proof is simple and hence omitted.

6. Polynomials of β-type m. Before defining β-type m polynomials consider the following example.

The classical Hermite polynomials are defined by means of the relation [**6**]

$$\exp(2xt - t^2) = \sum_{n=0}^{\infty} H_n(x) t^n / n!,$$

which gives

$$\Delta_x\{H_n(x)\} = 2H_{n-1}(x) + 4(x+1)H_{n-2}(x) + 8x(x+2)H_{n-3}(x) + \dots \tag{6.1}$$

The example above suggests the following extension of Theorem 5.

LEMMA 2. *For every polynomial $f_n^{(\alpha)}(x)$ there exist unique polynomial coefficients $u_k^{(\alpha)}$ of degree $\leq k$ in α and independent of n, such that*

$$\Delta_d\left\{f_n^{(\alpha)}(x)\right\} = u_1^{(\alpha)} f_{n-1}^{(\alpha)} + u_2 f_{n-2}^{(\alpha)}(x), + \dots + u_n^{(\alpha)} f_0^{(\alpha)}(x), \;\; (n \geq 1). \tag{6.2}$$

Definition. A set of polynomials $\left\{f_n^{(\alpha)}(x)\right\}$ is said to be of β-type m if in (6.2) the maximum degree of the coefficients $u_k^{(\alpha)}$ is m. If the degree of $u_k^{(\alpha)}$ is unbounded as $k \to \infty$ we say that the set $\left\{f_n^{(\alpha)}(x)\right\}$ is of β-type ∞.

From Theorem 5 and Lemma 2, we conclude

THEOREM 10. *The set of polynomials $\left\{f_n^{(\alpha)}(x)\right\}$ is of β-type zero if, and only if, it is of α-type zero.*

caps Theorem 11. *A necessary and sufficient condition for the polynomial $f_n^{(\alpha)}(x)$ to be of β-type m is that*

$$\sum_{n=0}^{\infty} t^n f_n^{(\alpha)} = C \exp\left[\Delta_\alpha^{-1} \log\left(1 + \sum_{r=1}^{\infty} u_r^{(\alpha)} t^r\right)\right], \tag{6.3}$$

where C is an arbitrary periodic function of period unity in α.

Proof. From (6.2), we have

$$\Delta_\alpha \left\{ \sum_{n=0}^{\infty} f_n^{(\alpha)}(t^n \right\} = \sum_{n=0}^{\infty} f_n^{(\alpha)}(x) t^n \sum_{r=1}^{\infty} u_r^{(\alpha)} t^r$$

or

$$\left[\Delta_\alpha - \sum_{r=1}^{\infty} u_r^{(\alpha)} t^r \right] \sum_{n=0}^{\infty} f_n^{(\alpha)}(x) t^n = 0. \tag{6.4}$$

(6.4) is a homogeneous-linear-difference equation of order one, whose solution is (6.3). The converse part can be proved easily by transforming both sides of (6.3) by Δ_α, and hence the proof is omitted. Thus, the theorem is proved.

7. Polynomials of γ-type m. In this section we define another class of polynomials which are said to be of γ-type m, based on Lemma 3 (an extension of Theorem 4) given below. We also give a generating function for γ-type m polynomials.

LEMMA 3. *For every polynomial $f_n^{(\alpha)}(x)$ there exist unique polynomial coefficients $\nu_k^{(\alpha)}(x)$ of degree $\le k$ in α and independent of n, such that*

$$n f_n^{(\alpha)}(x) = \nu_1^{(\alpha)}(x) f_{n-1}^{(\alpha)}(x) + \nu_2^{(\alpha)}(x) f_{n-2}^{(\alpha)}(x) + \cdots + \nu_n^{(\alpha)}(x) f_0^{(\alpha)}(x), \quad (n \ge 1). \tag{7.1}$$

Definition. A set of polynomials $\left\{ f_n^{(\alpha)}(x) \right\}$ is said to be of γ-type m if in (7.1) the maximum degree of the coefficients $\nu_k^{(\alpha)}(x)$ is $(m+1)$ in α.

From the definition above and Theorem 4, it is evident that every α-type zero polynomial is also of γ-type zero.

THEOREM 12. *A necessary and sufficient condition for the polynomials $f_n^{(\alpha)}(x)$ to be of γ-type zero is that*

$$\sum_{n=0}^{\infty} t^n f_n^{(\alpha)}(x) = K \exp \left(\sum_{r=0}^{\infty} \nu_{r+1}^{(\alpha)}(x) t^{r+1}/(r+1) \right), \tag{7.2}$$

where K is an arbitrary constant (independent of t).

Proof. The sufficient part of the theorem can be proved easily by differentiating both sides of (7.2) with respect to t.

For the converse part write (7.1) as

$$\frac{\delta}{\delta t} \sum_{n=0}^{\infty} f_n^{(\alpha)}(x) t^n = \sum_{n=0}^{\infty} t^{n-1} \sum_{r=1}^{n} \nu_r^{(\alpha)}(x) f_{n-r}^{(\alpha)}(x) = \sum_{n=0}^{\infty} f_n^{\infty}(x) t^n \sum_{r=0}^{\infty} \nu_{r+1}^{(\alpha)}(x) t^r,$$

or

$$\left[\frac{\delta}{\delta t} - \sum_{r=0}^{\infty} \nu_{r+1}^{(\alpha)}(x) t^r \right] \sum_{n=0}^{\infty} f_n^{(\alpha)}(x) t^n = 0. \tag{7.3}$$

On solving the difference equation (7.3), we get (7.2).

8. Generalized α-type Zero Polynomials. We conclude this paper by giving a generalization of α-type zero polynomials, introduced in § 3. We shall also give two characterizations for these polynomials.

Let us consider the following difference-operator of infinite order

$$T(\Delta_\alpha) \equiv T = \sum_{k=0}^{\infty} g_k \Delta_\alpha^{k+r}, \tag{8.1}$$

in which $g_0 \neq 0$, $g_k\,(k \geq 0)$ are independent of α and r is some fixed positive integer.

Definition. Any polynomial $G_n^{(\alpha)}(x)$ for which there exists an operator T of the form (8.1), such that

$$T\left\{G_n^{(\alpha)}(x)\right\} = G_{n-r}^{(\alpha+r)}(x), \quad (n = r, r+1, \dots) \tag{8.2}$$

where r is some fixed positive integer, we call a Generalized α-type zero polynomial. Obviously, for $r = 1$, (8.2) reduces to the condition required for $G_n^{(\alpha)}(x)$ to be of α-type zero.

THEOREM 13. *For any polynomial $G_n^{(\alpha)}(x)$ to be a Generalized α-type zero polynomial, the necessary and sufficient condition is that it satisfies a generating relation of the form*

$$\sum_{i=1}^{r} Q_i(x,t)(1 + B(\varepsilon_i t))^\alpha = \sum_{n=o}^{\infty} G_n^{(\alpha)}(x) t^n, \tag{8.3}$$

where, $B(t)$ is defined by the relation

$$T(B(t)) = t^r (1 + B(t))^r, \tag{8.4}$$

and $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r$ are the r roots of unity.

Proof. By operating both sides of (8.3), by T it can be shown easily with the help of (8.4) that $G_n^{(\alpha)}(x)$ satisfies the condition (8.2).

Conversely, let us write

$$\sum_{n=0}^{\infty} G_n^{(\alpha)}(x) t^n = G \tag{8.5}$$

Therefore, from (8.2) and (8.5), we have

$$[T - t^r E_\alpha^r]\, G = 0. \tag{8.6}$$

It is always possible to find out another difference-operator of the form

$$M(\Delta_\alpha) = \sum_{k=0}^{\infty} j_k \Delta_\alpha^{k+1}, \quad j_0 \neq 0 \tag{8.7}$$

such that

$$T(\Delta_\alpha) = (M(\Delta_\alpha))^r. \tag{8.8}$$

Hence from (8.6) and (8.8), we obtain

$$[(M(\Delta_\alpha))^r - (1+\Delta_\alpha)^r t^r]G = 0.$$

Consequently, if $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r$, are the r roots of unity, we have

$$[M(\Delta_\alpha) - (1+\Delta_\alpha)\varepsilon_i t]G = 0, \quad (i = 1, 2, \dots, r). \tag{8.9}$$

Solving the homogeneous-linear-difference equations above we get

$$G = \sum_{i=1}^{r} Q_i(x,t)(1 + B(\varepsilon_2 t))^\alpha$$

where $B(t)$ is given by

$$M(B(t)) = t(1 + B(t)). \tag{8.10}$$

Therefore, the theorem is proved.

Like Theorem 2, one can show that

Theorem 14. *If the set* $\left\{G_n^{(\alpha)}(x)\right\}$ *corresponds to the operator T, then a necessary and sufficient condition for the set* $\left\{K_n^{(\alpha)}(x)\right\}$ *to correspond also to the same operator T is that there exist polynomial coefficients* $d_k(x)$ *of degree* $\leq h$ *in* x *and independent of* n *and* α, *such that*

$$G_n^{(\alpha)}(x) = \sum_{i=0}^{n} d_i(x) K_n^{(\alpha)}(x), \quad d_0(x) \neq 0. \tag{8.11}$$

Finally, we give still another characterization for generalized α-type zero polynomials.

Theorem 15. *Let* $M(\Delta_\alpha)$ *be the operator of type* (8.7) *and* $u(\alpha)$ *a function of bounded variation on* $(0,\infty)$ *such that* $\int_0^\infty du(\alpha) \neq 0$. *Then* $G_n^{(\alpha)}(x)$ *is a Generalized* α*-type zero polynomial if, and only if,*

$$\int_0^\infty \{M(\Delta_\alpha)\}^k G_n^{(\alpha-n)}(x) du(\alpha) = c_{n,k}, \quad (k = 0, 1, 2, \dots) \tag{8.12}$$

where $c_{n,k}$ *are elements of an infinite triangular matrix, in which* $c + n + r, k + r = c_{n,k}$.

Before proving the theorem above we first prove the following lemma:

Lemma 4. (8.12) *is satisfied by one and only one* $G_n^{(\alpha-n)}(x)$ *for some given* $M(\Delta_\alpha)$ *and* $u(\alpha)$ *satisfying the conditions stated in the theorem above.*

Proof of Lemma 4. From Lemma 1 we know that the polynomials $I_n^{(\alpha)}$ defined by

$$(1+B(t))^\alpha = \sum_{n=0}^{\infty} I_n^{(\alpha)} t^n$$

are of α-type zero corresponding to the operator $M(\Delta_\alpha)$, where $M(B(t)) = t(1+B(t))$. We also have $I_0^{(\alpha)} = 1$.

Let $G_n^{(\alpha-n)}(x)$ satisfy (8.12); then we can write

$$G_n^{(\alpha-n)}(x) = \sum_{i=0}^{n} A(n,i,x) I_i^{(\alpha)}.$$

Therefore

$$\{M(\Delta_\alpha)\}^k G_n^{(\alpha-n)}(x) = \sum_{i=0}^{n-k} A(n,i+k,x) I_i^{(\alpha+k)}. \tag{8.13}$$

If we write

$$\int_0^\infty I_i^{(\alpha+k)} du(\alpha) = e_{i,k},$$

then from (8.12) and (8.13), we get

$$\sum_{i=0}^{n-k} A(n,i+k,x) e_{i,k} = c_{n,k}, \quad (k = 0,1,2,\ldots,n). \tag{8.14}$$

Since $A(n,i,x) = 0$, if $i > n$, it follows that the determinant of the system (8.14) is $\prod_{k=0}^{n} e_{0,k} \neq 0$, and since $e_{0,k} = \int_0^\infty du(\alpha) \neq 0$, we conclude that $A(n,i,x)$ $(i = 0,1,\ldots,n)$ are uniquely determined.

Proof of Theorem 15. Let $G_n^{(\alpha)}(x)$ be a Generalized α-type zero polynomial. Since $G_n^{(\alpha)}(x)$ is a polynomial of degree n in α, we have

$$[M(\Delta_\alpha)]^i G_n^{(\alpha-n)}(x) = 0, \quad \text{if } i > n$$

or

$$c_{n,i} = 0, \quad \text{if } i > n.$$

Again, since

$$M^k(\Delta_\alpha) G_n^{(\alpha-)}(x) = M^{k+r}(\Delta_\alpha) G_{n+r}^{(\alpha-n-r)}(x)$$

we get $c_{n,k} = c_{n+r,k+r}$.

Conversely, let $G_n^{(\alpha-n)}(x)$ satisfy (8.12). Then

$$\int_0^\infty M^{k+i}(\Delta_\alpha)G_{n+i}^{(\alpha-n-i)}(x)du(\alpha)=c_{n+i,k+i}=c_{n,k}.$$

The substitution

$$S_n^{(\alpha)}(x)=M^i(\Delta_\alpha)G_{n+i}^{(\alpha-n-i)}(x)$$

reduces (8.15) to

$$\int_0^\infty M^k(\Delta_k)S_n^{(\alpha)}(x)du(\alpha)=c_{n,k}.$$

But by Lemma 4, (8.12) is satisfied by the unique polynomial $G_n^{(\alpha-n)}$. Therefore $S_n^{(\alpha)}(x)=G_n^{(\alpha-n)}(x)$.

This completes the proof of the theorem.

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Department of Mathematics
Lucknow University
Lucknow, India

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