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EQUIVALENCE OF BASES IN NON-ARCHIMEDEAN BANACH SPACES

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1. Introduction

In [5, 6], it has been shown that a non-archimedean (n.a) Banach space of countable type has a basis. Very recently, the authors have obtained a necessary and sufficient condition for a general n.a. Banach space (countable or uncountable type) to have a basis and derived certain criterion for a basis to be orthogonal in [1] whereas a close connection between the existence of bases and projections has been studied in [2]. Once a n.a. Banach space is known to hawe a basis, it is natural to study its uniqueness. The notion of equivalence of bases serves a very useful tool for the study in this direction. In this paper we obtain a necessary and sufficient condition for two bases to be equivalent and study the stability properties of bases. Finally, we consider the idea of Block basis for the space.

2. Notation and Terminology

Let K be a n.a. non-trivial valued field which is complete under the metric of valuation of rank one. Throughout, by E we shall mean n.a. Banach space over the field K with n.a. norm $\|\ \|$. Let E' denote the dual of E. For general properties of n.a. Banach spaces we refer to [3, 5, 7].

Let $X \subset E \setminus \{0\}$ be any system of vectors. We may write

$$X = \{x_\lambda : \lambda \in \Lambda\},$$

where Λ is an index set of any cardinality. Let Σ be the set of all finite subsets of Λ directed by inclusion. A system X is said to be summable to x in E if $\lim_\sigma \Sigma_{\lambda\in\sigma} x_\lambda = x$, $\sigma \in \Sigma$ exists, the convergence being in the n.a. norm topology of E, where $\lim_\sigma y_\sigma$ denotes the limit of a net $\{y_\sigma : \sigma \in \Sigma\}$ in E. Further, a system X is a basis for E if to each x in E there exists a unique system $\{\alpha_\lambda : \lambda \in \Lambda\}$ of scalars such that the system $\{\alpha_\lambda x_\lambda : \lambda \in \Lambda\}$ is summable to x i.e.

$$x = \lim_\sigma \sum_{\lambda\in\sigma} \alpha_\lambda x_\lambda, \quad \sigma \in \Sigma. \tag{2.1}$$

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Clearly, with each basis $\{x_\lambda\}$ there is associated a unique family $\{f_\lambda\}$ of linear functionals on E such that $f_\lambda(x) = \alpha_\lambda$, where x is given by (2.1). Thus, without ambiguity, we may write a basis $\{x_\lambda\}$ as $\{x_\lambda, f_\lambda\}$ as and when we need so. If for a basis $\{x_\lambda, f_\lambda\}$ the family $\{f_\lambda\}$ is in E', then the basis is said to be Schauder.

A double system $\{x_\lambda, f_\lambda\}$, $x_\lambda \in E$, $f_\lambda \in E'$ is called a biorthogonal system if $f_\mu(x_\lambda) = \delta_{\lambda\mu}$, where $\delta_{\lambda\mu}$ denotes the Kronecker symbol.

3. A critarion for equivalent bases

DEFINITION 3.1. Two bases $\{x_\lambda\}$ and $\{y_\lambda\}$ for n.a. Banach spaces E and F over K are said to be equivalent if for a family $\{\alpha_\lambda\}$ in K,

$$\{\alpha_\lambda x_\lambda\} \text{ is summable } \Leftrightarrow \{\alpha_\lambda y_\lambda\} \text{ is summable.}$$

THEOREM 3.1. *Let E and F be n.a. Banach spaces over K. Then bases $\{x_\lambda\}$ and $\{y_\lambda\}$ for E and F, respectively, are equivalent if and only if there is a topological isomorphism T of E onto F such that $Tx_\lambda = y_\lambda$, $\lambda \in \Lambda$.*

The proof of this theorem requires the following lemma the proof of which is simple and is omitted.

LEMMA. *Let $X = \{x_\lambda, f_\lambda\}$ be a basis for E. Let*

$$\hat{E} = \{\{f_\lambda(x) : \lambda \in \Lambda\},\ x \in E\},$$

be a n.a. normed space over K whith the n.a. norm

$$\|\alpha\|_{\hat{E}} = \sup\{\|\alpha_\lambda x_\lambda\| : \lambda \in \Lambda\},$$

where $\alpha = \{\alpha_\lambda : \lambda \in \Lambda\} \in \hat{E}$. Then $\hat{E}$ is a n.a. Banach space over K which is topologically isomorphic to E, and

$$\left\| \sum_{\lambda\in\sigma} \alpha_\lambda x_\lambda \right\| \le \left\| \sum_{\lambda\in\sigma} \beta_\lambda x_\lambda \right\|, \quad (\sigma \in \Sigma),$$

for all $\{\alpha_\lambda\}$, $\{\beta_\lambda\}$ in K such that $|\alpha_\lambda| \le |\beta_\lambda|$ on Λ.

PROOF OF THE THEOREM. Suppose first that the bases $\{x_\lambda\}$ and $\{y_\lambda\}$ in E and F are equivalent. By lemma, E and F are, respectively, topologically isomorphic to $\hat{E}$ and $\hat{F}$ and, further, $\hat{E} = \hat{F}$. Let $\hat{I} : \hat{E} \to \hat{F}$ be the identity map. Let $\{\alpha^{(n)}\} \subset \hat{E}$ converge to α such that $\{\hat{I}\alpha^{(n)}\} \subset \hat{F}$ converges to β. Then

$$\|\alpha - \alpha^{(n)}\|_{\hat{E}} < \varepsilon,\ \ \|\beta - \hat{I}\alpha^{(n)}\|_{\hat{F}} < \varepsilon,$$

for all $n \ge n_0 = n_0(\varepsilon)$, $\varepsilon > 0$. Therefore, by the definitions of $\|\ \|_{\hat{E}}$ and $\|\ \|_{\hat{F}}$ we have

$$|\alpha_\lambda - \alpha_\lambda^{(n)}|\, \|x_\lambda\| < \varepsilon,\ \ |\beta_\lambda - \hat{I}\alpha^{(n)}|\, \|y_\lambda\| < \varepsilon,$$

for all $n \geq n_0$, where $\alpha = \{\alpha_\lambda\}$ etc. Thus

$$\begin{aligned}|\alpha_\lambda - \beta_\lambda| &\leq \max\{|\alpha_\lambda - \alpha_\lambda^{(n)}|,\ |\beta_\lambda - \hat{I}\alpha_\lambda^{(n)}|\} \\ &\leq \max\{\varepsilon\|x_\lambda\|^{-1},\ \varepsilon\|y_\lambda\|^{-1}\},\end{aligned}$$

for all $n \geq n_0$. This verifies that $\alpha = \beta$. Therefore, $\hat{I}$ is closed and so it is continuous.

Now, take $T = T_F^{-1}\hat{I}T_E$, where T_E and T_F are topological isomorphisms of E to $\hat{E}$ and F to $\hat{F}$, respectively. Clearly, T is a topological isomorphism such that $Tx_\lambda = y_\lambda$, $\lambda \in \Lambda$.

The proof of the converse part is simple and the details are omitted.

Corollary. *Every basis in n.a. Banach space is Schauder.*

4. Stability properties

Bases have certain stability properties: If we perturb each element of a basis by a sufficiently small amout we still get a basis. The perturbed basis is equivalent to the original one and inherits from it many of its nice properties. A few results in this direction are the following.

Theorem 4.1. *Let $\{x_\lambda, y_\lambda\}$ be a basis for a subspace of E and $\{y_\lambda : \lambda \in \Lambda\}$ be a system of vectors in E with*

$$\sup\{\|f_\lambda\|\ \|x_\lambda - y_\lambda\| : \lambda \in \Lambda\} < 1.$$

Then $\{y_\lambda\}$ is a basis system[2] *which is equivalent to $\{x_\lambda\}$.*

For consiseness the proof of this theorem is ommitted.

Theorem 4.2. *Let $\{x_\lambda, f_\lambda\}$ be a basic system in E. Assume also that there is a projection P from E onto $\overline{sp}\{x_\lambda\}$. Let $\{y_\lambda : \lambda \in \Lambda\}$ be a system of vectors in E such that*

$$\|P\| \lim_\sigma \left\{ \sum_{\lambda\in\sigma} \|f_\lambda\|\ \|x_\lambda - y_\lambda\| \right\} < 1. \tag{4.1}$$

Then $\{y_\lambda\}$ is a basic system and $\overline{sp}\{y_\lambda\}$ is complemented in E.

Proof. Define a map $T: E \to E$ by

$$Tx = x - Px + \lim_\sigma \sum_{\lambda\in\sigma} f_\lambda(Px) y_\lambda.$$

[2] $\{y_\lambda : \lambda \in \Lambda\}$ is said to be a basic system in E if $\{y_\lambda\}$ is a basic for $\overline{sp}\{y_\lambda\}$.

By the hypothesis T is a well defined one-to-one linear transformation. Also, we note that

$$\begin{aligned}\|(I-T)x\| &= \lim_{\sigma}\left\|\sum_{\lambda\in\sigma} f_\lambda(Px)(x_\lambda - y_\lambda)\right\| \\ &\leq \lim_{\sigma}\max\{\|f_\lambda\|\,\|P\|\,\|x\|\,\|x_\lambda - y_\lambda\|: \lambda\in\sigma\} \\ &\leq \{\lim_{\sigma}\sum_{\lambda\in\sigma}\|f_\lambda\|\,\|x_\lambda - y_\lambda\|\}\|P\|\,\|x\| \\ &\leq \|x\|,\end{aligned}$$

In view of (4.1). Therefore, $\|Tx\| \leq \|x\|$ and so T is continuous. Further, we observe that $T(\overline{sp}\{x_\lambda\}) = \overline{sp}\{y_\lambda\}$ and $Tx_\lambda = y_\lambda$, $\lambda \in \Lambda$. Therefore

$$T^{-1}\left(\lim_{\sigma}\sum_{\lambda\in\sigma}\alpha_\lambda y_\lambda\right) = \lim_{\sigma}\sum_{\lambda\in\sigma}\alpha_\lambda T^{-1}y_\lambda = \lim_{\sigma}\sum_{\lambda\in\sigma}\alpha_\lambda . x_\lambda. \tag{4.2}$$

Thus T maps $\overline{sp}\{x_\lambda\}$ onto $\overline{sp}\{y_\lambda\}$. The TPT^{-1} is the projection of E on $\overline{sp}\{y_\lambda\}$. Finally, $\{y_\lambda\}$ is a basis for $\overline{sp}\{y_\lambda\}$ follows from (4.2).

5. Block Basis

Definition 5.1. Let $\{x_\lambda: \lambda\in\Lambda\}$ be a basis for a n.a. Banach space E over the field K and $\{\sigma_a: a\in\Lambda\}$ be any collection of mutually disjoint sets in Σ. Then, if $\{\alpha_\lambda: \lambda\in\Lambda\}$ in K is such that $\{y_a\}$ in E, defined by

$$y_a = \sum_{\lambda\in\sigma}\alpha_\lambda x_\lambda,$$

never attiains zero on A, $\{y_a\}$ is said to be a block basis with respect to $\{x_\lambda\}$.

It is easy to check that $\{y_a\}$ is a basis for $\overline{sp}\{y_a\}$.

Theorem 5.1. *Let $\{x_\lambda, f_\lambda\}$ be a basis for E. Let $\{y_\lambda\}$ be a family in E such that $\{S_\sigma\}$, where $S_\sigma = \sum_{\lambda\in\sigma} y_\lambda$, is bounded away from zero and* $\lim_\sigma f_\lambda(S_\sigma) = 0$, *$\lambda\in\Lambda$. Then there exists a subfamily $\{y_{p_\lambda}\}$ of $\{y_\lambda\}$ which is a basis for $\overline{sp}\{y_{p_\lambda}\}$ and is equivalent to a block basis with respect to $\{x_\lambda\}$.*

Proof. Let $\{\sigma_a: a\in A\}$ be a collection of mutually disjoint sets in Σ. By the hypothesis, there exists $\varepsilon > 0$ and a family $\{\sigma_b: b\in B\}$ in Σ with $\sigma_b \subset \sigma_0$ such that $\|S_\sigma\| > \varepsilon$, $\forall\sigma\in\Sigma$ and

$$\left\|\sum_{\lambda\in\Lambda\sim\sigma_a} f_\lambda(S_{\sigma_b})x_\lambda\right\| < \frac{\varepsilon}{8M_1M_2},\quad \forall b\in B,$$

where $M_1 \geq 1$ will be fixed later on and $M_2 = 0(\sigma_b)$.

Consider a subfamily $\{y_{p\lambda}: p_\lambda \in \cup_{b\in B}\sigma_b \sim \sigma_0\}$ of the family $\{y_\lambda: \lambda \in \Lambda\}$. Then, for a y_{p_λ} in this subfamily, we note that

$$\begin{aligned}
\varepsilon \le \|y_{p_\lambda}\| &= \left\| \sum_{\mu\in\sigma_a} f_\mu(y_{p_\lambda})x_\mu + \sum_{\mu\in\Lambda\sim\sigma_a} f_\mu(y_{p_\lambda})x_\mu \right\| \\
&\le \max\left\{ \left\| \sum_{\mu\in\sigma_a} f_\mu(y_{p_\lambda})x_\mu \right\|, \left\| \sum_{\mu\in\Lambda\sim\sigma_a} f_\mu(S_{\sigma_b})x_\mu \right\|, \right. \\
&\qquad \left. \left\| \sum_{\mu\in\Lambda\sim\sigma_a} f_\mu\left(S_{\sigma_b\sim\{p_\lambda\}}\right) x_\mu \right\| \right\} \\
&\le \left\| \sum_{\mu\in\sigma_a} f_\mu(y_{p\lambda})x_\mu \right\| + \varepsilon/2,
\end{aligned}$$

which gives

$$\left\| \sum_{\mu\in\sigma_a} f_\mu(y_{p_\lambda})x_\mu \right\| > \varepsilon/2. \tag{5.1}$$

Also for $p_\lambda \in \sigma_b \sim \sigma_0$, we have

$$\left\| y_{p_\lambda} - \sum_{\mu\in\sigma_a} f_\mu(y_{p_\lambda})x_\mu \right\| < \frac{\varepsilon}{8M_1M_2},$$

which implies

$$\sum_{p_\lambda\in\sigma_b\sim\sigma_0} \left\| y_{p_\lambda} - \sum_{\mu\in\sigma_a} f_\mu(y_{p_\lambda})x_\mu \right\| < \frac{\varepsilon}{8M_1},\ \ \forall b \in B. \tag{5.2}$$

Define a family $\{z_{ap_\lambda}\}$ as

$$z_{ap_\lambda} = \sum_{M\in\sigma_a} f_\mu(y_{p_\lambda})x_\mu.$$

Clearly $\{z_{ap_\lambda}\}$ is a block basis with respect to $\{x_\lambda\}$, and hence is a basis for $\overline{sp}\{z_{ap_\lambda}\}$. Therefore, there exists a biorthogonal family $\{z^*_{ap_\lambda}\}$ in the dual of $\overline{sp}\{z_{ap_\lambda}\}$. Then ([5], p. 75)

$$\|z^*_{ap_\lambda}\| = \sup\left\{ \frac{\|z^*_{ap_\lambda}(z)\|}{\|z\|}: 0 \ne z \in \overline{sp}\{z_{ap_\lambda}\right\} < \frac{2M}{\varepsilon},$$

in view of [[1], Th. 1] and (5.1). Consequently, (5.2) verifies

$$\sup\{\|z^*_{ap_\lambda}\|\ \|y_{p_\lambda} - z_{ap_\lambda}\|\} < 1.$$

Hence, the result follows by Theorem 4.1.

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