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THE HILBERT COMPACTIFICATION OF THE UNIVERSAL MODULI SPACE OF SEMISTABLE VECTOR BUNDLES OVER SMOOTH CURVES

Alexander Schmitt

Abstract

We construct the Hilbert compactification of the universal moduli space of semistable vector bundles over smooth curves. The Hilbert compactification is the GIT quotient of some open part of an appropriate Hilbert scheme of curves in a Graßmannian. It has all the properties asked for by Teixidor.

Introduction

For every smooth curve C and integers χ and $r > 0$, one has the projective moduli space $\mathcal{U}(C;\chi,r)$ of semistable vector bundles $\mathcal{E}$ of rank r and Euler characteristic $\chi(\mathcal{E}) = \chi$. An automorphism σ of C acts on $\mathcal{U}(C;\chi,r)$ via $[\mathcal{E}] \longmapsto [\sigma^*\mathcal{E}]$. Let $\mathfrak{M}_g$ be the moduli space of smooth curves of genus g. It is possible to construct a universal moduli space $\mathfrak{U}(g;\chi,r) \longrightarrow \mathfrak{M}_g$, such that the fibre over $[C]$ is $\mathcal{U}(C;\chi,r)/\mathrm{Aut}(C)$. This leads to the problem of compactifying $\mathfrak{U}(g;\chi,r)$ over $\overline{\mathfrak{M}}_g$, the moduli space of stable curves of genus g. There are two natural approaches to this [**19**]: first, given a stable curve C of genus g, one can look at torsion-free sheaves $\mathcal{E}$ of uniform rank r on C with Euler characteristic $\chi(\mathcal{E}) = \chi$ which are semistable w.r.t. the canonical polarization. These objects form again a projective moduli space $\mathcal{U}(C;\chi,r)$. Pandharipande [**14**] has constructed a projective moduli space $\overline{\mathfrak{U}}(g;\chi,r) \longrightarrow \overline{\mathfrak{M}}_g$, such that the fibre over a stable curve $[C]$ is $\mathcal{U}(C;\chi,r)/\mathrm{Aut}(C)$. Second, for a stable curve C, instead of looking at torsion-free sheaves on C, one can look at vector bundles on semistable models of C. This viewpoint

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has advantages for certain degeneration arguments. As an approach to the above problem, it has been formalized by Gieseker [**6**] and further studied by Gieseker and Morrison [**7**], Nagaraj and Seshadri [**12**], Teixidor i Bigas [**19**], Kausz ([**8**],[**9**]), and the author [**16**]. It was also used by Caporaso [**2**] to solve the problem for $r = 1$. Without loss of generality, we may assume that, for every smooth curve of genus g and every semistable vector bundle $\mathcal{E}$ on C of rank r with $\chi(\mathcal{E}) = \chi$, $\mathcal{E}$ is globally generated, $H^1(\mathcal{E}) = 0$, and the evaluation map $\mathrm{ev}\colon H^0(\mathcal{E}) \otimes \mathcal{O}_C \longrightarrow \mathcal{E}$ gives rise to a closed embedding $C \hookrightarrow \mathrm{Gr}(H^0(\mathcal{E}), r)$ into the Graßmannian of r-dimensional quotients of $H^0(\mathcal{E})$. Thus, we fix a vector space V^χ of dimension χ, define $\mathfrak{G} := \mathrm{Gr}(V^\chi, r)$, and look at $\mathfrak{H}(g;\chi,r)$, the closure of the Hilbert scheme of smooth curves in $\mathfrak{G}$ with Hilbert polynomial $P(m) = d \cdot m + (1-g)$ in the whole Hilbert scheme. Here, $\chi = d + r(1-g)$ and $\mathfrak{G}$ is polarized by $\mathcal{O}_{\mathfrak{G}}(1)$, the determinant of the universal quotient bundle. Note that we have a natural action of $\mathrm{SL}(V^\chi)$ on $\mathfrak{H}(g;\chi,r)$. Our candidate for the Hilbert compactification is, therefore, $\mathfrak{HC}(g;\chi,r) := \mathfrak{H}(g;\chi,r)/\!\!/\,\mathrm{SL}(V^\chi)$. Before we can form the GIT-quotient, we have, however, to find appropriate linearized ample line bundles on $\mathfrak{H}(g;\chi,r)$. First, there are some obvious ones. For this, let $\mathfrak{C} \subset \mathfrak{G} \times \mathfrak{H}(g;\chi,r)$ be the universal curve. For every natural number m, we have, on $\mathfrak{G} \times \mathfrak{H}(g;\chi,r)$, the exact sequence

$$0 \longrightarrow \mathcal{I}_{\mathfrak{C}} \otimes \pi_{\mathfrak{G}}^*\mathcal{O}_{\mathfrak{G}}(m) \longrightarrow \pi_{\mathfrak{G}}^*\mathcal{O}_{\mathfrak{G}}(m) \longrightarrow \big(\pi_{\mathfrak{G}}^*\mathcal{O}_{\mathfrak{G}}(m)\big)_{|\mathfrak{C}} \longrightarrow 0.$$

For large m, $\pi_{\mathfrak{H}(g;\chi,r)*}$ of this sequence leads to a surjective homomorphism of vector bundles

$$\Psi_{\mathfrak{H}(g;\chi,r)}\colon S^m V^\chi \otimes \mathcal{O}_{\mathfrak{H}(g;\chi,r)} \longrightarrow \mathfrak{E}_m := \pi_{\mathfrak{H}(g;\chi,r)*}\Big(\big(\pi_{\mathfrak{G}}^*\mathcal{O}_{\mathfrak{G}}(m)\big)_{|\mathfrak{C}}\Big).$$

The rank of $\mathfrak{E}_m$ is $P(m)$, and $\wedge^{P(m)}\Psi_{\mathfrak{H}(g;\chi,r)}$ yields a closed immersion

$$\mathfrak{H}(g;\chi,r) \hookrightarrow \mathbb{P}\Big(\bigwedge^{P(m)} S^m V^\chi\Big)$$

which is equivariant w.r.t. the $\mathrm{SL}(V^\chi)$-actions. Let $\mathfrak{L}_m := \mathfrak{L}_m(g;\chi,r) := \det(\mathfrak{E}_m)$ be the pullback of $\mathcal{O}(1)$. Then, we can define $\mathfrak{HC}_m(g;\chi,r) := \mathfrak{H}/\!\!/_{\mathfrak{L}_m}\mathrm{SL}(V^\chi)$. This, space will be, however, only useful, if we can settle the following properties:

1) If $[C] \in \mathfrak{H}(g;\chi,r)$ is a smooth curve and $q_C\colon V^\chi \otimes \mathcal{O}_C \longrightarrow \mathcal{E}$ is the pullback of the universal quotient, then $[C]$ is (semi)stable w.r.t. the linearization in $\mathfrak{L}_m$, if and only if $\mathcal{E}$ is a (semi)stable vector bundle.

2) If $[C] \in \mathfrak{H}(g;\chi,r)$ is semistable w.r.t. the linearization in $\mathfrak{L}_m$ and $q_C\colon V^\chi \otimes \mathcal{O}_C \longrightarrow \mathcal{E}$ is the pullback of the universal quotient, then C is a semistable curve and $V^\chi \longrightarrow H^0(\mathcal{E})$ is an isomorphism.

Points (1) and (2) have been settled in the rank two case by Gieseker and Morrison [**7**], and (1) in general by the author [**16**]. Unfortunately, nothing is known about (2) in general, and, even if it were true, the computations of the correct notion of semistability would still be extremely difficult (cf. [**19**]). The way out is to adapt a strategy due to Nagaraj and Seshadri [**12**]. In our setting, it is described as follows: for every stable curve C, let $\mathcal{Q}(C;\chi,r)$ be the quot scheme parameterizing quotients $V^\chi \otimes \mathcal{O}_C \longrightarrow \mathcal{E}$ where $\mathcal{E}$ is a coherent sheaf of uniform rank r with Euler characteristic $\chi(\mathcal{E}) = \chi$. From Pandharipande's construction, we get a universal quot scheme $\overline{\mathfrak{Q}}(g;\chi,r) \longrightarrow \overline{\mathfrak{M}}_g$, such that the fibre over $[C]$ is just $\mathcal{Q}(C;\chi,r)/\mathrm{Aut}(C)$, and a natural $\mathrm{SL}(V^\chi)$-linearized ample line bundle $\mathfrak{N}$ on $\overline{\mathfrak{Q}}(g;\chi,r)$. Next, let $\mathfrak{H}^0(g;\chi,r) \subset \mathfrak{H}(g;\chi,r)$ be the open part corresponding to semistable curves with the following property: if $\pi\colon C \longrightarrow C'$ is the projection onto the stable model and if $q_C\colon V^\chi \otimes \mathcal{O}_C \longrightarrow \mathcal{E}$ is the pullback of the universal quotient on $\mathfrak{G}$, then $\pi_*(q_C)\colon V^\chi \otimes \mathcal{O}_{C'} \longrightarrow \pi_*(\mathcal{E})$ is surjective and $\pi_*(\mathcal{E})$ is a torsion-free sheaf which is semistable w.r.t. the canonical polarization. There is a natural map $\mathfrak{H}^0(g;\chi,r) \longrightarrow \overline{\mathfrak{Q}}\ (g;\chi,r)$, landing in the $\mathfrak{N}$-semistable locus. Let $\widehat{\mathfrak{H}}(g;\chi,r) \subset \mathfrak{H}(g;\chi,r) \times \overline{\mathfrak{Q}}(g;\chi,r)$ be the closure of the graph of the above morphism. This is an $\mathrm{SL}(V^\chi)$-invariant subscheme. For $a \gg 0$, the semistable points w.r.t. the linearization in

$$\mathfrak{L}_{a,m} := \mathfrak{L}_{a,m}(g;\chi,r) := \left(\pi^*_{\widehat{\mathfrak{H}}(g;\chi,r)}\mathfrak{L}_m(g;\chi,r) \otimes \pi^*_{\overline{\mathfrak{Q}}(g;\chi,r)}\mathfrak{N}^{\otimes a}\right)_{|\widehat{\mathfrak{H}}(g;\chi,r)}$$

will lie in the set of the preimages of the points in $\overline{\mathfrak{Q}}(g;\chi,r)$ which are semistable w.r.t. the linearization in $\mathfrak{N}$. Note that, for every $l \geq 0$, we can perform the same constructions w.r.t. g, r, and $\chi_l := \chi + l\cdot r\cdot(2g-2)$. The result of this note is:

Main Theorem. *There exist an l_0 and for every $l \geq l_0$ an $m(l)$, such that for all $l \geq l_0$ and $m \geq m(l)$ the following properties hold true:*

i) *All points in $\widehat{\mathfrak{H}}(g;\chi_l,r)$ which are semistable w.r.t. the linearization in $\mathfrak{L}_{a,m}(g;\chi_l,r)$ for all $a \gg 0$ lie in the graph of the morphism $\mathfrak{H}^0(g;\chi_l,r) \longrightarrow \overline{\mathfrak{Q}}(g;\chi_l,r)$.*
ii) *Let $[C]$ be contained in the graph of the morphism $\mathfrak{H}^0(g;\chi_l,r) \longrightarrow \overline{\mathfrak{Q}}(g;\chi_l,r)$, and let $q_C\colon V^{\chi_l} \otimes \mathcal{O}_C \longrightarrow \mathcal{E}$ be the pullback of the*

universal quotient. Then, $[C]$ *is (semi)stable w.r.t. the linearization in* $\mathfrak{L}_{a,m}(g;\chi_l,r)$ *for all* $a \gg 0$*, if and only if* $(C, \mathcal{E}\otimes\omega_C^{\otimes -l})$ *is H-(semi)stable in the sense of Definition* 2.2.10*.*

Remark.

i) If $\mathcal{E}$ is a vector bundle of rank r on the semistable curve C and if $\pi\colon C \longrightarrow C'$ is the map onto the stable model, then the condition that $\pi_*(\mathcal{E})$ be torsion-free is a precise condition on the restriction of $\mathcal{E}$ to any chain R of rational curves attached at only two points, say p_1 and p_2. Namely, there must not exist a nonzero section of $\mathcal{E}_{|R}$ vanishing in both p_1 and p_2. This implies, for example, that R has at most r components and that $\mathcal{E}_{|R}$ is *strictly standard*, i.e., for any component $R_i \cong \mathbb{P}_1$ of R, $\mathcal{E}_{|R_i} \cong \mathcal{O}_{R_i}^{\oplus\varepsilon_i}\oplus\mathcal{O}_{R_i}(1)^{\oplus(r-\varepsilon_i)}$ with $0 \leq \varepsilon_i < r$, $i = 1,\ldots,s$. For the detailed discussion, we refer the reader to [**12**].

ii) The condition to be semistable with respect to the linearization in $\mathfrak{L}_{a,m}(g;\chi_l,r)$ *for all* $a \gg 0$ is explained as follows: let G be a reductive algebraic group, $\rho_i\colon G \longrightarrow \mathrm{GL}(W_i)$, $i = 1,2$, two finite-dimensional representations, and $\widehat{\mathfrak{H}} \subset \mathbb{P}(W_1)\times\mathbb{P}(W_2)$ a G-invariant closed subscheme. Denote by $\mathfrak{L}_{a,m}$ the restriction of $\mathcal{O}_{\mathbb{P}(W_1)\times\mathbb{P}(W_2)}(m,a)$ to $\widehat{\mathfrak{H}}$ and by $\widehat{\mathfrak{H}}^{(\mathrm{s})\mathrm{s}}_{a,m}$ the set of points which are (semi)stable w.r.t. the linearization in $\mathfrak{L}_{a,m}$. Then, there is an ε_∞, such that

$$\pi_2^{-1}\big(\mathbb{P}(W_2)^{\mathrm{s}}\big) \quad\subset\quad \widehat{\mathfrak{H}}^{(\mathrm{s})\mathrm{s}}_{a',m'} = \widehat{\mathfrak{H}}^{(\mathrm{s})\mathrm{s}}_{a,m} \quad\subset\quad \pi_2^{-1}\big(\mathbb{P}(W_2)^{\mathrm{ss}}\big)$$

whenever both m'/a' and m/a are smaller than ε_∞. This follows from the corresponding assertion for $\mathbb{C}^*$-actions and the master space construction of Thaddeus ([**20**], [**13**]).

iii) The intrinsically defined concept of H-(semi)stability has the following properties:
- For smooth curves, it agrees with Mumford-(semi)stability.
- If $(C, \mathcal{E}\otimes\omega_C^{\otimes -l})$ is H-semistable and $\pi\colon C \longrightarrow C'$ is the morphism to the stable model, then $\pi_*(\mathcal{E}\otimes\omega_C^{\otimes -l})$ is semistable w.r.t. the canonical polarization.
- A pair $(C, \mathcal{E}\otimes\omega_C^{\otimes -l})$ with C a semistable curve and $\pi_*(\mathcal{E}\otimes\omega_C^{\otimes -l})$ a stable sheaf, $\pi\colon C \longrightarrow C'$ being the morphism to the stable model, is H-stable.

Therefore, $\mathfrak{HC}(g;\chi,r) := \widehat{\mathfrak{H}}(g;\chi_l,r)/\!\!/_{\mathfrak{L}_{a,m}(g;\chi_l,r)} \mathrm{SL}(V^{\chi_l})$ with $l \geq l_0$, $m \geq m(l)$, and $a \gg 0$ is a well-defined moduli space which compactifies $\mathfrak{U}(g;\chi,r)$ over $\overline{\mathfrak{M}}_g$ and, furthermore, maps to $\overline{\mathfrak{U}}(g;\chi,r)$.

We remark that the authors of [**12**] were well aware of the fact that their approach might be used in this generality. The formulation of an intrinsic semistability concept for vector bundles on semistable curves, however, seems to be new.

The final section of this paper is devoted to the study of the geometry of the Hilbert compactification and its map to the moduli space of stable curves.

Notation. Let C be a semistable curve. Then, $\mathcal{O}_C(1)$ will stand for the canonical sheaf ω_C, although this line bundle will be ample if and only if C is stable. Likewise, if $\mathcal{E}$ is a coherent sheaf on C, we write $P(\mathcal{E})$ for the polynomial $l \longmapsto \chi(\mathcal{E}(l))$. A *scheme* will be a scheme of finite type over the field of complex numbers.

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1. Preliminaries

1.1. Suitable linearizations. Let $\rho_i\colon G \longrightarrow \mathrm{GL}(W_i)$, $i = 1, 2$, two representations of a reductive group G. This yields an action of G on $\mathbb{P}(W_1) \times \mathbb{P}(W_2)$. Assume $\mathfrak{X}$ is a G-invariant subscheme, and let $\pi\colon \mathfrak{X} \longrightarrow \mathbb{P}(W_2)$ be the induced morphism. Finally, let $\mathfrak{L}_{n_1,n_2}$ be the G-linearized ample line bundle $\mathcal{O}_{\mathbb{P}(W_1)\times\mathbb{P}(W_2)}(n_1,n_2)_{|\mathfrak{X}}$. Define $\mathfrak{X}^{(\mathrm{s})\mathrm{s}}_{n_1,n_2}$ as the set of points in $\mathfrak{X}$ which are (semi)stable w.r.t. the linearization in $\mathfrak{L}_{n_1,n_2}$. Likewise, $\mathbb{P}(W_2)^{(\mathrm{s})\mathrm{s}}$ is defined. The following is well-known and easy to prove.

Proposition 1.1.1. *If n_2/n_1 is large enough, then*

$$\pi^{-1}\big(\mathbb{P}(W_2)^{\mathrm{s}}\big) \subset \mathfrak{X}^{\mathrm{s}}_{n_1,n_2} \subset \mathfrak{X}^{\mathrm{ss}}_{n_1,n_2} \subset \pi^{-1}\big(\mathbb{P}(W_2)^{\mathrm{ss}}\big).$$

Remark 1.1.2. Let n_2/n_1 be so large that the conclusion of Proposition 1.1.1 holds. Then, a point $x \in \mathfrak{X}$ will be (semi)stable w.r.t. the linearization in $\mathfrak{L}_{n_1,n_2}$, if and only if $\pi(x)$ is semistable and for every one-parameter subgroup $\lambda\colon \mathbb{C}^* \longrightarrow G$ with $\mu_{\mathcal{O}_{\mathbb{P}(W_2)}(1)}(\lambda,\pi(x)) = 0$, one has $\mu_{\mathcal{O}_{\mathbb{P}(W_1)}(1)}(\lambda,\pi'(x))$ $(\geq)$ 0, $\pi'\colon \mathfrak{X} \longrightarrow \mathbb{P}(W_1)$ being the induced morphism.

1.2. Pandharipande's moduli space; the universal quot scheme. Let C be a stable curve with irreducible components $C_1, ..., C_c$, and $\mathcal{F}$ a coherent sheaf on C. The tuple $\underline{r}(\mathcal{F}) := (\operatorname{rk}\mathcal{F}_{|C_1}, \ldots, \operatorname{rk}\mathcal{F}_{|C_c})$ is called the *multirank of* $\mathcal{F}$. We say that $\mathcal{F}$ has *uniform rank* r *on* C, if $\underline{r}(\mathcal{F}) = (r, \ldots, r)$. Finally, set $e_\iota := \deg \omega_{C|C_\iota}$, $\iota = 1, \ldots, c$. Then, the *total rank of* $\mathcal{F}$ is the quantity $\operatorname{trk}\mathcal{F} := \sum_{\iota=1}^{c} e_\iota \cdot \operatorname{rk}\mathcal{F}_{|C_\iota}$. Now, a coherent $\mathcal{O}_C$-module $\mathcal{E}$ is called (*semi*)*stable*, if it is torsion-free and for every subsheaf $0 \subsetneq \mathcal{F} \subsetneq \mathcal{E}$, the inequality

$$\frac{\chi(\mathcal{F})}{\operatorname{trk}\mathcal{F}}(\leq)\frac{\chi(\mathcal{E})}{\operatorname{trk}\mathcal{E}}$$

is satisfied. One also introduces the notions of *S-equivalence* and *polystability*. Pandharipande studies the functor $\underline{\mathrm{U}}(g;\chi,r)$ which associates to every scheme S the set of equivalence classes of pairs $(\mathcal{C}_S, \mathcal{E}_S)$, consisting of a flat family $\pi\colon \mathcal{C}_S \longrightarrow S$ of stable curves and an S-flat coherent sheaf $\mathcal{E}_S$ on $\mathcal{C}_S$ such that $\mathcal{E}_{S|\pi^{-1}(s)}$ is a semistable sheaf of uniform rank r and Euler characteristic χ on $\pi^{-1}(s)$ for all closed points $s \in S$. Here, $(\mathcal{C}_S, \mathcal{E}_S)$ and $(\mathcal{C}'_S, \mathcal{E}'_S)$ are considered *equivalent*, if there are an S-isomorphism $\psi\colon \mathcal{C}_S \longrightarrow \mathcal{C}'_S$ and a line bundle $\mathcal{L}_S$ on S such that $\mathcal{E}_S \cong \psi^*\mathcal{E}'_S \otimes \pi^*\mathcal{L}_S$. In [**14**], a coarse moduli space $\overline{\mathfrak{U}}(g;\chi,r)$ for the functor $\underline{\mathrm{U}}(g;\chi,r)$ is constructed.

We now briefly review the construction, because we need some of the details. If C is a stable curve, then $\mathcal{O}_C(10)$ defines a closed embedding $C \hookrightarrow \mathbb{P}_N$, $N := 10(2g-2)-g$. Let $\mathfrak{H}_g$ be the Hilbert scheme of curves in $\mathbb{P}_N$ with the respective Hilbert polynomial. There is a natural left action of $\mathrm{SL}(N+1)$ on $\mathfrak{H}_g$ together with a linearization in an ample line bundle $\mathfrak{L}_{\mathfrak{H}_g}$. As Gieseker [**5**] has shown, the GIT quotient $\mathfrak{H}_g /\!\!/_{\mathfrak{L}_{\mathfrak{H}_g}} \mathrm{SL}(N+1)$ yields $\overline{\mathfrak{M}}_g$.

There is a constant l_0, such that for every $l \geq l_0$, every stable curve C of genus g, and every (semi)stable sheaf $\mathcal{E}$ of uniform rank r and Euler characteristic χ on C, one has:

- $\mathcal{E}(10 \cdot l)$ is globally generated.
- $H^1(\mathcal{E}(10 \cdot l)) = 0$.
- After identification of $H^0(\mathcal{E}(10 \cdot l))$ with a previously fixed vector space $V^{\chi_{10\cdot l}}$ of dimension $\chi_{10\cdot l} = \chi + 10 \cdot l \cdot r \cdot (2g-2)$, the point $\mathrm{ev}\colon V^{\chi_{10\cdot l}} \otimes \mathcal{O}_C \longrightarrow \mathcal{E}(10 \cdot l)$ in the quot scheme $\mathcal{Q}(C;\chi_{10\cdot l},r)$ is (semi)stable.

Let $\mathcal{C}_g \hookrightarrow \mathfrak{H}_g \times \mathbb{P}_N$ be the universal curve. We then have a relative quot scheme $\rho\colon Q = Q(g;\chi_{10\cdot l},r) \longrightarrow \mathfrak{H}_g$, such that the fibre over $C \in \mathfrak{H}_g$

is the quot scheme $\mathcal{Q}(C;\chi_{10\cdot l},r)$ of quotients $\mathcal{E}$ of $V^{\chi_{10\cdot l}}\otimes\mathcal{O}_C$ with $\chi(\mathcal{E}\otimes_{\mathcal{O}_C}\mathcal{O}_{\mathbb{P}_N}(l))=\chi+10\cdot l\cdot r\cdot(2g-2)$ for all l. The natural action of $\mathrm{SL}(V^{\chi_{10\cdot l}})\times\mathrm{SL}(N+1)$ on Q is linearized in a suitable ρ-ample line bundle $\mathfrak{L}_Q$. For any $a>0$, define $\mathfrak{L}_a:=\mathfrak{L}_Q\otimes\rho^*\mathfrak{L}_{\mathfrak{H}_g}^{\otimes a}$. For large a, we have, by Proposition 1.1.1, $Q^{\mathrm{ss}}_{\mathfrak{L}_a}\subset\rho^{-1}(\mathfrak{H}^{\mathrm{ss}}_{g,\mathfrak{L}_{\mathfrak{H}_g}})$. Then, $\overline{\mathfrak{U}}(g;\chi,r):=Q/\!\!/_{\mathfrak{L}_a}\big(\mathrm{SL}(V^{\chi_{10\cdot l}})\times\mathrm{SL}(N+1)\big)$.

Now, we can form this GIT quotient in two steps [**13**]: first, we divide by the $\mathrm{SL}(N+1)$-action and then by the $\mathrm{SL}(V^{\chi_{10\cdot l}})$-action. As $\mathfrak{H}^{\mathrm{s}}_{g,\mathfrak{L}_{\mathfrak{H}_g}}=\mathfrak{H}^{\mathrm{ss}}_{g,\mathfrak{L}_{\mathfrak{H}_g}}$ (more precisely, this holds on an appropriate closed subscheme, see [**5**], Proposition 2.0.0), 1.1.1 shows that a point $(C,q_C)\in Q$ is $\mathrm{SL}(N+1)$-semistable w.r.t. the linearization in $\mathfrak{L}_{\mathfrak{H}_g}$, if and only if C is stable, i.e., there is no condition on the quotient q_C. Define $\overline{\mathfrak{Q}}(g;\chi_{10\cdot l},r):=Q/\!\!/_{\mathfrak{L}_a}\mathrm{SL}(N+1)$. Let $\underline{\mathrm{Q}}(g;\chi_{10\cdot l},r)$ be the functor which assigns to a scheme S the set of equivalence classes of pairs $(\mathcal{C}_S,q_S\colon V^{\chi_{10\cdot l}}\otimes\mathcal{O}_{\mathcal{C}_s}\longrightarrow\mathcal{E}_S)$ consisting of a flat family $\pi\colon\mathcal{C}_S\longrightarrow S$ of stable curves and a quotient q_S onto an S-flat sheaf $\mathcal{E}_S$, such that $\chi(\mathcal{E}_{S|\pi^{-1}(s)})=\chi_{10\cdot l}$ for every closed point $s\in S$. Two families $(\mathcal{C}_S,q_S)$ and $(\mathcal{C}'_S,q'_S)$ will be considered *equivalent*, if there are an S-isomorphism $\psi\colon\mathcal{C}_S\longrightarrow\mathcal{C}'_S$ and an isomorphism $\varphi_S\colon\mathcal{E}_S\longrightarrow\psi^*\mathcal{E}'_S$ with $\psi^*q'_S=\varphi_S\circ q_S$. The space $\overline{\mathfrak{Q}}(g;\chi_{10\cdot l},r)$ obviously is the coarse moduli scheme for the functor $\underline{\mathrm{Q}}(g;\chi_{10\cdot l},r)$. In particular, the fibre over $[C]\in\overline{\mathfrak{M}}_g$ identifies with $\mathcal{Q}(\overline{C};\chi_{10\cdot l},r)/\mathrm{Aut}(C)$. As explained in the first section of [**13**], there is an induced $\mathrm{SL}(V^{\chi_{10\cdot l}})$-action on $\overline{\mathfrak{Q}}(g;\chi_{10\cdot l},r)$ and some multiple of $\mathfrak{L}_a$ descends to an $\mathrm{SL}(V^{\chi_{10\cdot l}})$-linearized ample line bundle $\mathfrak{N}$. Moreover, the points in $\overline{\mathfrak{Q}}(g;\chi_{10\cdot l},r)$ which are $\mathrm{SL}(V^{\chi_{10\cdot l}})$-(semi)stable for the given linearization are just the images of the $(\mathrm{SL}(V^{\chi_{10\cdot l}})\times\mathrm{SL}(N+1))$-(semi)stable points. Let us note the following elementary fact:

Lemma 1.2.1. *Let C be a stable curve and $q_C\colon V^{\chi_{10\cdot l}}\otimes\mathcal{O}_C\longrightarrow\mathcal{E}$ a quotient with $\chi(\mathcal{E})=\chi_{10\cdot l}$. Let $h\in\mathfrak{H}_g$ be a point such that the fibre of the universal curve over h is isomorphic to C. Then, for any one-parameter subgroup $\lambda\colon\mathbb{C}^*\longrightarrow\mathrm{SL}(V^{\chi_{10\cdot l}})$,*

$$\mu_{\mathfrak{L}_a}\big(\lambda,(h,q_C)\big)\ (\geq)\ 0\quad\Longleftrightarrow\quad\mu_{\mathfrak{N}}\big(\lambda,[C,q_C]\big)\ (\geq)\ 0.$$

Proof. The quotient morphism $Q^{\mathrm{SL}(N+1)-\mathrm{ss}}_{\mathfrak{L}_a}\longrightarrow\overline{\mathfrak{Q}}(g;\chi_{10\cdot l},r)$ yields the $\mathrm{SL}(V^{\chi_{10\cdot l}})$-equivariant and finite morphism

$$Q_h:=\rho^{-1}(h)\longrightarrow\mathcal{Q}(C;\chi_{10\cdot l},r)/\mathrm{Aut}(C).$$

This immediately implies the assertion. q.e.d.

1.3. Auxiliary results from Nagaraj and Seshadri. We here recall some results of the paper [**12**]. Additional information may be found there. We also point out that [**12**] works with semistable curves the stable model of which is an irreducible curve with exactly one node. As one easily check, this assumption is not essential.

Proposition 1.3.1.

i) *Let R be a chain of projective lines, and $\mathcal{F}$ a globally generated torsion-free sheaf on R. Then, for any component $R_\iota \cong \mathbb{P}_1$, the restriction map $H^0(R,\mathcal{F}) \longrightarrow H^0(R_\iota, \mathcal{F}_{|R_\iota})$ is surjective. Moreover, $H^1(R,\mathcal{F}) = 0$.*

ii) *Let $\pi\colon C' \longrightarrow C$ be a morphism between semistable curves which contracts only some chains of projective lines. Suppose $\mathcal{E}$ is a vector bundle on C' the restriction of which to every projective line contracted by π has nonnegative degree. In that situation:*
   1) $\pi_*\mathcal{O}_{C'} = \mathcal{O}_C$.
   2) *$R^i\pi_*(\mathcal{E}) = 0$ for $i > 0$. In particular, $H^j(C',\mathcal{E}) = H^j(C,\pi_*(\mathcal{E}))$ for all j.*
   3) *Let R^j be a chain of projective lines which is contracted by π and attached at the points p_1 and p_2. Let $\widetilde{C}$ be the closure of $C' \setminus R^j$. If $H^1(\widetilde{C}, \mathcal{I}_{p_1,p_2}\mathcal{E}_{|\widetilde{C}}) = H^1(C', \mathcal{I}_{R^j}\mathcal{E}) = 0$, then the restriction map $H^0(C',\mathcal{E}) \longrightarrow H^0(R^j, \mathcal{E}_{|R^j})$ is surjective, so that, by* i), *the restriction map $H^0(C',\mathcal{E}) \longrightarrow H^0(R^j_\iota, \mathcal{E}_{|R^j_\iota})$ to any component R^j_ι of R^j is surjective, too.*

Proof. Part ii) is Proposition 3 in [**12**].

Ad i): The restriction of $\mathcal{F}$ to a component R_ι is of the form

$$\mathcal{O}_{\mathbb{P}_1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}_1}(a_{\operatorname{rk}\mathcal{F}_{|R_\iota}}) \oplus \text{Torsion}$$

with $a_i \geq 0$, $i = 1,\ldots,\operatorname{rk}\mathcal{F}_{|R_\iota}$. Therefore, $H^1(R_\iota, \mathcal{F}_{|R_\iota}(-1)) = 0$. By successively removing components which are attached at one point only, the result becomes an easy induction on the length of R. q.e.d.

Remark 1.3.2. In the situation of Proposition 1.3.1 ii), there are precise conditions for $\pi_*(\mathcal{E})$ to be torsion-free ([**12**], Proposition 5). In particular, any chain of rational curves contracted by π can have length at most $\operatorname{rk}\mathcal{E}$. This already bounds the family of semistable curves which might appear in our investigations.

Proposition 1.3.3. *Let C be a semistable curve containing the disjoint chains $R^1,\ldots,R^c$ of projective lines which are attached at two*

points only, define $\widetilde{C}_j$ *as the closure of* $C \setminus R^j$*, and let* p_1^j, p_2^j *be the points where* R^j *is attached,* $j = 1, \ldots, c$*. Also set* $R := \bigcup_{j=1}^{c} R^j$ *and define* $\widetilde{C}$ *as the closure of* $C \setminus R$*. Suppose* $\mathcal{E}$ *is a vector bundle on* C *which satisfies the following properties:*

1) *The restriction* $\mathcal{E}_{|R_i^j}$ *of* $\mathcal{E}$ *to any component of* R^j *has positive degree,* $j = 1, \ldots, c$*.*
2) $H^1\left(\widetilde{C}_j, \mathcal{I}_{p_1^j, p_2^j}\mathcal{E}_{|\widetilde{C}_j}\right) = 0$*,* $j = 1, \ldots, c$*.*
3) *The homomorphism*
$$H^0\left(\widetilde{C}_j, \mathcal{I}_{p_1^j, p_2^j}\mathcal{E}_{|\widetilde{C}_j}\right) \longrightarrow \left(\mathcal{I}_{p_1^j, p_2^j}\mathcal{E}_{|\widetilde{C}_j}\right) / \left(\mathcal{I}^2_{p_1^j, p_2^j}\mathcal{E}_{|\widetilde{C}_j}\right)$$
is surjective, $j = 1, \ldots, c$*.*
4) *For any point* $x \in \widetilde{C} \setminus \{p_1^j, p_2^j,\ j = 1, \ldots, c\}$*, the homomorphism*
$$H^0\big(C, \mathcal{I}_R\mathcal{E}\big) \longrightarrow \mathcal{E}_{|\widetilde{C}} / \left(\mathcal{I}_x^2\mathcal{E}_{|\widetilde{C}}\right)$$
is surjective.
5) *For any two points* $x_1 \neq x_2 \in \widetilde{C} \setminus \{p_1^j, p_2^j,\ j = 1, \ldots, c\}$*, the homomorphism*
$$H^0\big(C, \mathcal{I}_R\mathcal{E}\big) \longrightarrow \mathcal{E}_{|\{x_1\}} \oplus \mathcal{E}_{|\{x_2\}}$$
is surjective.

Then, the evaluation map $\mathrm{ev}\colon H^0(C, \mathcal{E}) \otimes \mathcal{O}_C \longrightarrow \mathcal{E}$ *yields a closed embedding*
$$C \hookrightarrow \mathrm{Gr}\big(H^0(C, \mathcal{E}), \mathrm{rk}\,\mathcal{E}\big).$$

Proof. This is proved like Proposition 4 in [**12**]. q.e.d.

Remark 1.3.4. Note that the conditions (2)–(5) will be satisfied for $\mathcal{E}(l)$, $l \gg 0$.

Proposition 1.3.5 ([**12**], Lemma 4)**.** *Suppose we are given a commutative diagram*
$$\begin{array}{ccc} Z & \xrightarrow{\pi} & W \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ T & = & T \end{array}$$
in which p *and* q *are projective,* p *is flat, and* $\pi_*\mathcal{O}_Z = \mathcal{O}_W$*. Assume that* $\mathcal{E}_Z$ *is a vector bundle on* Z*, such that, for every point* $t \in T$*, we*

have $R^i\pi_{t*}(\mathcal{E}_{Z|p^{-1}(t)}) = 0$ *for all* $i > 0$, $\pi_t\colon p^{-1}(t) \longrightarrow q^{-1}(t)$ *being the induced morphism. Then,*

$$\pi_*(\mathcal{E}_Z)_{|q^{-1}(t)} = \pi_{t*}\big(\mathcal{E}_{Z|p^{-1}(t)}\big), \quad \textit{for all closed points } t \in T.$$

Remark 1.3.6. Let $\mathcal{O}_W(1)$ be a q-ample line bundle. Assume

$$H^i\big(p^{-1}(t), (\mathcal{E}_Z \otimes \pi^*\mathcal{O}_W(n))_{|p^{-1}(t)}\big) = 0, \quad \text{for } n \gg 0,\ i > 0, \text{ and all } t \in T.$$

Then, $q_*\big((\pi_*\mathcal{E}_Z)(n)\big) = p_*\big(\mathcal{E}_Z \otimes \pi^*\mathcal{O}_W(n)\big)$ is locally free for all $n \gg 0$, whence $\pi_*\mathcal{E}_Z$ is T-flat.

1.4. Modules over Discrete Valuation Rings. Let $(R, v\colon R \longrightarrow \mathbb{Z})$ be a discrete valuation ring with uniformizing parameter t, i.e., $v(t) = 1$. A finitely generated module M over R will be called *almost torsion-free*, if its torsion submodule is annihilated by t. Likewise, a submodule M' of a free module M of finite rank is called *almost saturated*, if the module M/M' is almost torsion-free.

Proposition 1.4.1. *Let* $M = R^{\oplus r}$ *be a free module of rank* r *and*

$$(*) \qquad 0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_n \subsetneq M$$

a chain of almost saturated submodules. Then, there is a basis $e_1, \ldots, e_r$ *for* M*, such that* M_i *is generated by* $\rho_1^i e_1, \ldots, \rho^i_{\operatorname{rk} M_i} e_{\operatorname{rk} M_i}$ *for appropriate elements* $\rho_j^i \in R$ *with* $v(\rho_j^i) \in \{0, 1\}$, $j = 1, \ldots, \operatorname{rk} M_i$, $i = 1, \ldots, n$.

Note that we do not require $\operatorname{rk}(M_i) > \operatorname{rk}(M_{i-1})$, $i = 2, \ldots, n$.

Proof. We carry out an induction over r, the case $r = 1$ being clear. For the first submodule M_1, we may find a basis $e_1^*, \ldots, e_r^*$ with the asserted property [**18**], 10.5. Take $e_1 := e_1^*$. Then,

$$(**) \quad 0 \subsetneq \widetilde{M}_1 := M_1/(M_1 \cap \langle e_1\rangle) \subsetneq \cdots \subsetneq \widetilde{M}_n := M_n/(M_n \cap \langle e_1\rangle) \subsetneq \subsetneq \widetilde{M} := M/\langle e_1\rangle$$

is a filtration of the free module $M/\langle e_1\rangle$ of rank $r-1$. We claim that $\widetilde{M}_i$ is an almost saturated submodule of $\widetilde{M}$, $i = 1, \ldots, n$. If $e_1 \in M_i$, this is just the isomorphism theorem. Otherwise, $te_1 \in M_i$ and $M_i' := \widetilde{M}/\widetilde{M}_i$ is a quotient of $M_i'' := M/M_i = (M/\langle te_1\rangle)/\widetilde{M}_i$. As M_i' and M_i'' have the same rank, no element of the free part of M_i'' can map to the torsion of M_i', i.e., the torsion of M_i' is a quotient of the torsion of M_i'' which implies the claim.

Therefore, we can apply the induction hypothesis to $(**)$. Let $e_2', ..., e_r'$ be any lift of the appropriate basis $\bar{e}_2, \ldots, \bar{e}_r$ for $\widetilde{M}$. Suppose that we

have already found a basis of the form $e_1, \dots, e_{\operatorname{rk} M_i}, e'_{\operatorname{rk} M_i+1}, \dots, e'_r$, so that the assertion holds for $M_1, \dots, M_i$ and $e_2, \dots, e_{\operatorname{rk} M_i}$ also lift $\overline{e}_2, \dots, \overline{e}_{\operatorname{rk} M_i}$. Then, M_{i+1} is spanned w.r.t. that basis by vectors of the form

$$v_j = \begin{pmatrix} *_j \\ 0 \\ \vdots \\ 0 \\ q_j \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

with $q_j, *_j \in R$, $v(q_j) \in \{0, 1\}$, and q_j at the j-th place, $j = 1, \dots, \operatorname{rk} M_{i+1}$. If $v(q_1) \leq v(*_j)$, we may clearly set $*_j = 0$. If $v(q_1) > v(*_j)$, i.e., $v(q_1) = 1$ and $v(*_j) = 0$, and also $v(q_j) = 0$, we set $e_j := v_j$. In fact, this case can only occur, if $j > \operatorname{rk} M_i$ or in all the M_ι with $\iota \leq i$, $t \cdot e_j$ was the respective basis vector. But then, the result still holds for $M_1, \dots, M_i$ and the basis $e_1, \dots, e_{j-1}, v_j, e_{j+1}, \dots, e_{\operatorname{rk} M_i}, e'_{\operatorname{rk} M_i+1}, ..., e'_r$. Finally, the case $v(q_1) = 1$, $v(*_j) = 0$, and $v(q_j) = 1$ cannot occur. Indeed, in that case, the class of the vector $(0, \dots, t, \dots, 0)^T$, t at the j-th place, in M/M_{i+1} is nonzero, i.e., the class of $(0, \dots, 1, \dots, 0)^T$ in M/M_{i+1} is not annihilated by t, a contradiction. q.e.d.

2. Proof of the main theorem

2.1. Construction of the Hilbert compactification. The set of pairs $(\nu\colon \widetilde{C} \longrightarrow C, \nu^*\mathcal{E})$, ν being a partial normalization of the stable curve C and $\mathcal{E}$ a semistable sheaf of rank r with Euler characteristic χ is clearly bounded. Therefore, we may find an l_1, such that, for every $l \geq l_1$, the following assumptions are met:

Assumption 2.1.1. For every stable curve C, every partial normalization $\nu\colon \widetilde{C} \longrightarrow C$, resolving the nodes, say, $N_1, \dots, N_\kappa$, every semistable sheaf $\mathcal{E}$ on C, any two points $p_1, p_2 \in \widetilde{C}$ mapping to a node of C, and $Z := \nu^{-1}\big(\{N_1, \dots, N_\kappa\}\big)$, one has:

1) $H^1(\widetilde{C}, \mathcal{I}_{p_1,p_2}\nu^*\mathcal{E}(l)) = 0$.
2) The homomorphism

$$H^0\big(\widetilde{C}, \mathcal{I}_{p_1,p_2}\nu^*\mathcal{E}(l)\big) \longrightarrow \big(\mathcal{I}_{p_1,p_2}\nu^*\mathcal{E}(l)\big)/\big(\mathcal{I}^2_{p_1,p_2}\nu^*\mathcal{E}(l)\big)$$

is surjective.

3) For every $x \in \nu^{-1}\big(C \setminus \{N_1, \ldots, N_\kappa\}\big)$, the homomorphism
$$H^0\big(\widetilde{C}, \mathcal{I}_Z \nu^* \mathcal{E}(l)\big) \longrightarrow \nu^* \mathcal{E}(l) / \big(\mathcal{I}_x^2 \nu^* \mathcal{E}(l)\big)$$
is surjective.

4) For any two points $x_1 \neq x_2 \in \nu^{-1}(C \setminus \{N_1, \ldots, N_\kappa\})$, the homomorphism
$$H^0\big(\widetilde{C}, \mathcal{I}_Z \nu^* \mathcal{E}(l)\big) \longrightarrow \nu^* \mathcal{E}(l)_{|\{x_1\}} \oplus \nu^* \mathcal{E}(l)_{|\{x_2\}}$$
is surjective.

In the following, l is assumed to be at least l_1. Let $\mathfrak{H} := \mathfrak{H}(g; \chi_l, r)$ be as in the introduction, and let $\mathcal{C}_{\mathfrak{H}} \hookrightarrow \mathfrak{H} \times \mathfrak{G}$ be the universal curve. Let $\widetilde{\mathfrak{H}}$ be the open subset of points h for which $C_h := \mathcal{C}_{\mathfrak{H}} \times_{\mathfrak{H}} \{h\}$ is semistable. Let $q_{\widetilde{\mathfrak{H}}} \colon V^{\chi_l} \otimes \mathcal{O}_{\mathcal{C}_{\widetilde{\mathfrak{H}}}} \longrightarrow \mathcal{E}_{\widetilde{\mathfrak{H}}}$ be the pullback of the universal quotient. Then, there is a flat family $q \colon \mathcal{C}^*_{\widetilde{\mathfrak{H}}} \longrightarrow \widetilde{\mathfrak{H}}$ of stable curves together with an $\widetilde{\mathfrak{H}}$-morphism $\pi \colon \mathcal{C}_{\widetilde{\mathfrak{H}}} \longrightarrow \mathcal{C}^*_{\widetilde{\mathfrak{H}}}$, such that $\pi_* \mathcal{O}_{\mathcal{C}_{\widetilde{\mathfrak{H}}}} = \mathcal{O}_{\mathcal{C}^*_{\widetilde{\mathfrak{H}}}}$ and π is fibrewise the contraction onto the stable model. By 1.3.1 ii) (2), we are in the position to apply Proposition 1.3.5. Moreover, we see that the assumptions of Remark 1.3.6 are satisfied. We get the homomorphism
$$\pi_*(q_{\widetilde{\mathfrak{H}}}) \colon V^{\chi_l} \otimes \mathcal{O}_{\mathcal{C}^*_{\widetilde{\mathfrak{H}}}} \longrightarrow \pi_* \mathcal{E}_{\widetilde{\mathfrak{H}}}$$
of $\widetilde{\mathfrak{H}}$-flat sheaves. Let $\mathfrak{H}^0$ be the open set of points h for which:

- $\big(\pi_* \mathcal{E}_{\widetilde{\mathfrak{H}}}\big)_{|q^{-1}(h)}$ is a semistable sheaf.
- $\pi_*\big(q_{\widetilde{\mathfrak{H}}}\big)_{|q^{-1}(h)}$ is surjective.
- $H^0\big(\pi_*(q_{\widetilde{\mathfrak{H}}})_{|q^{-1}(h)}\big)$ is an isomorphism.

Set $\mathcal{C}^*_{\mathfrak{H}^0} := \mathcal{C}^*_{\widetilde{\mathfrak{H}}} \times_{\widetilde{\mathfrak{H}}} \mathfrak{H}^0$. Then, the quotient family
$$\pi_{\mathfrak{H}^0} := \pi_*(q_{\widetilde{\mathfrak{H}}})_{|\mathcal{C}^*_{\mathfrak{H}^0}} \colon V^{\chi_l} \otimes \mathcal{O}_{\mathcal{C}^*_{\mathfrak{H}^0}} \longrightarrow \mathcal{E}_{\mathfrak{H}^0} := (\pi_* \mathcal{E}_{\widetilde{\mathfrak{H}}})_{|\mathcal{C}^*_{\mathfrak{H}^0}}$$
defines a morphism $\mathfrak{H}^0 \longrightarrow \overline{\mathfrak{Q}}(g; \chi_l, r)$. Let $\mathfrak{X} \subset \mathfrak{H} \times \overline{\mathfrak{Q}}(g; \chi_l, r)$ be the closure of the graph of the above morphism. Let $\mathfrak{L}_{a,m}$ be as in the introduction with a so large, that the semistable points in $\mathfrak{X}$ lie in the preimage of the points in $\overline{\mathfrak{Q}}(g; \chi_l, r)$ which are semistable w.r.t. the linearization in $\mathfrak{N}$.

Theorem 2.1.2. *Let $x \in \mathfrak{X}$ be a point which is semistable w.r.t. the linearization in $\mathfrak{L}_{a,m}$, $a \gg 0$. Then, x is contained in the graph Γ of the morphism $\mathfrak{H}^0 \longrightarrow \overline{\mathfrak{Q}}(g; \chi_l, r)$.*

Proof. By construction of $\mathfrak{H}$ and $\mathfrak{X}$, the set of (semi)stable points $y \in \mathfrak{X}$ corresponding to smooth curves is dense. Let y represent the smooth curve C. By the result of [**16**], y is (semi)stable (w.r.t. the linearization in $\mathfrak{L}_{a,m}$ for all $a \gg 0$), if and only if $V^{\chi_l} \longrightarrow H^0(\mathcal{E}(l))$ is an isomorphism and $\mathcal{E}$ is a (semi)stable bundle of rank r with Euler characteristic χ, $\mathcal{E} := \mathcal{E}_{\mathfrak{H}^0|C}$. Hence, we may find a smooth curve K, a point $k \in K$, and a morphism $\kappa\colon K \longrightarrow \mathfrak{X}$ with $\kappa(k) = x$, such that $\kappa(K \setminus \{k\})$ is contained in the locus of pairs $(C, q_C\colon V^{\chi_l} \otimes \mathcal{O}_C \longrightarrow \mathcal{E}(l)) \in \Gamma$ for which C is a smooth curve and $\mathcal{E}$ is a stable vector bundle. We have an induced morphism $\overline{\kappa}\colon K \longrightarrow \overline{\mathfrak{Q}}(g; \chi_l, r)$ which lands, by assumption, in the semistable locus. Without loss of generality, we may assume that this morphism is induced by a family $(\mathcal{C}^*_K, q_K\colon V^{\chi_l} \otimes \mathcal{O}_{\mathcal{C}^*_K} \longrightarrow \mathcal{E}_{\mathcal{C}^*_K})$. The surface $\sigma\colon S := \mathcal{C}^*_K \longrightarrow K$ is smooth outside the nodes of $\sigma^{-1}(k)$ and has singularities of type A_n in these nodes. We may resolve these singularities in the usual way in order to get a flat family $\widetilde{\sigma}\colon \widetilde{S} = \widetilde{\mathcal{C}}_K \longrightarrow K$ of semistable curves with $\widetilde{S}$ smooth. Let $\widetilde{q}_K$ be the pullback of q_K to $\widetilde{S}$. Then, $\widetilde{q}_K$ defines a rational map $\widetilde{S} \longrightarrow \mathfrak{G}$ which is defined outside some nodes of $\widetilde{\sigma}^{-1}(k)$. By blowing up these nodes and points which are infinitely near to them, we get a new flat family $\sigma'\colon S' = \mathcal{C}'_K \longrightarrow K$ and $q'_K\colon V^{\chi_l} \otimes \mathcal{O}_{S'} \longrightarrow \mathcal{E}'_K$ where $\mathcal{E}'_K$ is locally free. Set $C'_k := {\sigma'}^{-1}(k)$. Let $\widetilde{C}$ be the closure of C'_k with all chains of rational curves attached at only two points removed. Then, $\nu\colon \widetilde{C} \longrightarrow C$ is a partial normalization of C. By construction, the morphisms $\widetilde{C} \longrightarrow \mathfrak{G}$ induced by $q'_{K|\widetilde{C}}$ and $\nu^*(q_{K|\sigma^{-1}(k)})$ agree, whence these quotients are equivalent. Now, our Assumptions 2.1.1 and Proposition 1.3.3 imply that the image of C'_k under the map $S' \longrightarrow \mathfrak{G}$ is a semistable curve C''_k, and $\pi\colon C'_k \longrightarrow C''_k$ just contracts all rational chains on which $\mathcal{E}'_K$ is trivial. Next, look at the commutative diagram

$$\begin{array}{ccccc}
\widetilde{C} & = & \widetilde{C} & \xrightarrow{\nu} & C \\
\| & & \subset\downarrow & & \| \\
\widetilde{C} & \xrightarrow{\subset} & C'_k & \xrightarrow{\nu'} & C \\
\| & & \pi\downarrow & & \| \\
\widetilde{C} & \xrightarrow{\subset} & C''_k & \xrightarrow{\nu''} & C.
\end{array}$$

The quotients $\pi_*\left(q'_{K|\sigma'^{-1}(k)}\right)_{|\widetilde{C}}$ and $\nu^*(q_{K|\sigma^{-1}(k)})$ are also equivalent. Thus, there is an isomorphism $\alpha\colon \left(\pi_*\mathcal{E}'_{K|\sigma'^{-1}(k)}\right)_{|\widetilde{C}} \longrightarrow \nu^*\big(\mathcal{E}_{K|\sigma^{-1}(k)}\big)$, making the following diagram commute:

$$\begin{array}{ccc}
V^{\chi_l}\otimes\mathcal{O}_{C''_k} & \xrightarrow{\pi_*\left(q'_{K|\sigma'^{-1}(k)}\right)} & \pi_*\mathcal{E}'_{K|\sigma'^{-1}(k)} \\
\text{restriction to } \widetilde{C}\big\downarrow & & \big\downarrow\text{restriction to } \widetilde{C} \\
V^{\chi_l}\otimes\mathcal{O}_{\widetilde{C}} & \xrightarrow{\pi_*\left(q'_{K|\sigma'^{-1}(k)}\right)_{|\widetilde{C}}} & \left(\pi_*\mathcal{E}'_{K|\sigma'^{-1}(k)}\right)_{|\widetilde{C}} \\
\| & & \big\downarrow\alpha \\
V^{\chi_l}\otimes\mathcal{O}_{\widetilde{C}} & \xrightarrow{\nu^*\left(q_{K|\sigma^{-1}(k)}\right)} & \nu^*\mathcal{E}_{K|\sigma^{-1}(k)}.
\end{array}$$

This latter diagram finally provides us, via projection onto C, with the following commutative diagram:

$$\begin{array}{ccc}
V^{\chi_l}\otimes\mathcal{O}_C & \xrightarrow{\nu''_*\pi_*\left(q'_{K|\sigma'^{-1}(k)}\right)} & \nu''_*\pi_*\mathcal{E}'_{K|\sigma'^{-1}(k)} \\
\big\downarrow & & \big\downarrow \\
V^{\chi_l}\otimes\nu_*\mathcal{O}_{\widetilde{C}} & \xrightarrow{\nu_*\nu^*\left(q_{K|\sigma^{-1}(k)}\right)} & \mathcal{E}_{K|\sigma^{-1}(k)}\otimes\nu_*\mathcal{O}_{\widetilde{C}}\,.
\end{array}$$

Furthermore, we have the commutative diagram

$$\begin{array}{ccc}
V^{\chi_l}\otimes\mathcal{O}_C & \xrightarrow{q_{K|\sigma^{-1}(k)}} & \mathcal{E}_{K|\sigma^{-1}(k)} \\
\big\downarrow & & \big\downarrow \\
V^{\chi_l}\otimes\nu_*\mathcal{O}_{\widetilde{C}} & \xrightarrow{\nu_*\nu^*\left(q_{K|\sigma^{-1}(k)}\right)} & \mathcal{E}_{K|\sigma^{-1}(k)}\otimes\nu_*\mathcal{O}_{\widetilde{C}}
\end{array}$$

in which the vertical arrows are injective. Therefore, the image of $\nu''_*\pi_*\mathcal{E}'_{K|\sigma'^{-1}(k)}$ in the sheaf $\mathcal{E}_{K|\sigma^{-1}(k)}\otimes\nu_*\mathcal{O}_{\widetilde{C}}$ is $\mathcal{E}_{K|\sigma^{-1}(k)}$. The kernel of the surjection

$$\nu''_*\pi_*\mathcal{E}'_{K|\sigma'^{-1}(k)} \longrightarrow \mathcal{E}_{K|\sigma^{-1}(k)}$$

must be zero, because both sheaves have the same Hilbert polynomial w.r.t. $\mathcal{O}_C(1)$.

To conclude, the quotient q'_K defines a K-morphism $\vartheta\colon \mathcal{C}'_K \longrightarrow K\times\mathfrak{G}$ and we have seen:

(a) The image is a flat family of curves $\sigma''\colon \mathcal{C}''_K \longrightarrow K$ with C'' as the fibre over k.
(b) The family $q''_K := \vartheta_*(q'_K)\colon V^{\chi_l}\otimes\mathcal{O}_{\mathcal{C}''_K} \longrightarrow \mathcal{E}_{\mathcal{C}''_K} := \vartheta_*\mathcal{E}_{\mathcal{C}'_K}$ is a flat quotient (by Proposition 1.3.5), such that $\sigma''_*(q''_K)$ is a quotient onto a family of semistable torsion-free coherent sheaves.

Therefore, the family $\mathcal{C}''_K$ defines a morphism $\kappa'\colon K \longrightarrow \Gamma \subset \mathfrak{X}$. Since we have not altered the original family outside the point k, κ' agrees with the original morphism κ outside k, and, thus, everywhere, because $\mathfrak{X}$ is separated. This shows $x = \kappa'(k) \in \Gamma$. q.e.d.

Remark 2.1.3. One could also use the arguments presented in the paper [**12**]. We have chosen the alternative way, also suggested in [**12**], because it reflects more of the moduli problem we are dealing with.

Therefore, the GIT-quotient $\mathfrak{H}^0/\!\!/_{\mathfrak{L}_{a,m}}\mathrm{SL}(V^{\chi_l})$ exists as a projective scheme over $\overline{\mathfrak{M}}_g$. It also comes with an $\overline{\mathfrak{M}}_g$-morphism to $\overline{\mathfrak{U}}(g;\chi,r)$. The hard task will be to give it a modular interpretation.

2.2. Analysis of semistability. Fix the data g, χ, and r, and set $\chi_l := \chi + l\cdot r\cdot(2g-2)$. Let C be a semistable curve of genus g. Denote by $\pi\colon C \longrightarrow C'$ the contraction onto the stable model of C. Let $\mathcal{E}$ be a vector bundle of uniform rank r on C with Euler characteristic $\chi(\mathcal{E}) = \chi$, such that $\mathcal{E}$ has positive degree on each rational component and $\pi_*(\mathcal{E})$ is torsion-free. From Proposition 1.3.3, we infer that, for sufficiently large l, $H^0(\mathcal{E}(l))\otimes\mathcal{O}_C \longrightarrow \mathcal{E}(l)$ will give rise to a closed embedding $C \hookrightarrow \mathfrak{G}(H^0(\mathcal{E}(l)),r)$ and $H^1(\mathcal{E}(l)) = \{0\}$. Identifying $H^0(\mathcal{E}(l))$ with some fixed vector space V^{χ_l} of dimension χ_l, we may ask whether $[C]\in\mathfrak{H}^0(g;\chi_l,r)$ is $\mathrm{SL}(V^{\chi_l})$-(semi)stable w.r.t. the linearization in $\mathfrak{L}_{a,m}$ for large a. We already know from Pandharipande's construction and Proposition 1.1.1 that:

(a) If $\pi_*(\mathcal{E})$ is stable w.r.t. the canonical polarization, then $[C]$ will be stable.
(b) If $[C]$ is semistable, then $\pi_*(\mathcal{E})$ is semistable w.r.t. the canonical polarization.

If $\pi_*(\mathcal{E})$ is properly semistable, then there will be additional conditions for $[C]$ to be (semi)stable. We will have to analyze those conditions. Abstractly, by Remark 1.1.2, they can be described as follows: suppose we are given $q_C\colon V^{\chi_l}\otimes\mathcal{O}_C \longrightarrow \mathcal{E}(l)$, such that $H^0(q_C)$ is an isomorphism

and $\pi_*(\mathcal{E})$ is semistable w.r.t. the canonical polarization. Then, $[C]$ will be $\mathrm{SL}(V^{\chi_l})$-(semi)stable, if and only if for every one-parameter subgroup $\lambda\colon \mathbb{C}^* \longrightarrow \mathrm{SL}(V^{\chi_l})$ with $\mu_{\mathfrak{N}}(\lambda, [\pi_*(q_C)]) = 0$, one has

(1) $\quad \mu_{\mathfrak{L}_m}\big(\lambda, \wedge^{P(m)}\Psi_{\mathfrak{H}(g;\chi_l,r)}\big)(\geq)0, \quad \Psi_{\mathfrak{H}(g;\chi_l,r)}$ as in the introduction.

Let us first recall when $\mu_{\mathfrak{N}}(\lambda, [\pi_*(q_C)]) = 0$ happens. For this, suppose λ is given with respect to the basis $v_1, \ldots, v_{\chi_l}$ by the weight vector

$$(2) \quad \underline{\gamma} = \sum_{i=1}^{\chi_l - 1} \alpha_i \big(\underbrace{i-\chi_l, \ldots, i-\chi_l}_{i\times}, \underbrace{i, \ldots, i}_{\chi_l \times}\big), \qquad \alpha_i \in \mathbb{Q}_{\geq 0},\ i = 1, \ldots, \chi_l - 1,$$

and let $i_1 < i_2 < \cdots < i_k$ be the indices with $\alpha_i > 0$. Then, we get a filtration $V^\bullet : \{0\} =: V_0 \subset V_1 \subset \cdots \subset V_k \subset V_{k+1} := V^{\chi_l}$ with $V_j := \langle\, v_1, \ldots, v_{i_j} \,\rangle$, $j = 1, \ldots, k$. Recall that $\mu_{\mathfrak{N}}(\lambda, [\pi_*(q_C)])$ depends only on the pair $(V^\bullet, \underline{\alpha})$, $\underline{\alpha} := (\alpha_{i_1}, \ldots, \alpha_{i_k})$ ([**11**], Section 2.2). Set

$$\widetilde{\mathcal{F}}_j := \pi_*\Big(q_C\big(V_j \otimes \mathcal{O}_C\big)(-l)\Big), \qquad j = 1, \ldots, k.$$

Now, from the construction of $\overline{\mathfrak{U}}(g; \chi_l, r)$ and Lemma 1.2.1, one knows that the equality $\mu_{\mathfrak{N}}(\lambda, [\pi_*(q_C)]) = 0$ will occur, if and only if:

- $\sum_{j=1}^{k} \alpha_{i_j}\big(P(\pi_*(\mathcal{E}))\,\mathrm{trk}(\widetilde{\mathcal{F}}_j) - P(\widetilde{\mathcal{F}}_j)\,\mathrm{trk}(\mathcal{E})\big) = 0$, i.e., each $\widetilde{\mathcal{F}}_j$ destabilizes $\mathcal{E}$.
- $H^0(\pi_*(q_C))(V_j) = H^0(\widetilde{\mathcal{F}}_j(l))$, $j = 1, \ldots, k$.

Recall that the family of pairs $(C, \mathcal{F})$ with C a stable curve and $\mathcal{F}$ a destabilizing subsheaf of a semistable torsion-free sheaf $\mathcal{E}$ of uniform rank r on C with $\chi(\mathcal{E}) = r$ is bounded. In view of Proposition 1.3.1, we can find an l_1, such that for all $l \geq l_1$, the following assumptions are verified:

Assumption 2.2.1. Let $[C] \in \mathfrak{H}^0(g; \chi_l, r)$ be semistable w.r.t. the linearization in $\mathfrak{L}_{a,m}(g; \chi_l, r)$ for $a \gg 0$. Denote by $\pi\colon C \longrightarrow C'$ the contraction onto the stable model, and by $q_C\colon V^{\chi_l} \otimes \mathcal{O}_C \longrightarrow \mathcal{E}$ the pullback of the universal quotient. Then, $\pi_*(\mathcal{E}(-l))$ is, by definition, semistable and $H^0(\pi_*(q_C))$ is an isomorphism. For every destabilizing subsheaf $\mathcal{F}' \subset \mathcal{E}(-l)$, define

$$\mathcal{F} := q_C\bigg(\Big(H^0\big(\pi_*(q_C)\big)^{-1}\big(H^0(\mathcal{F}'(l))\big)\Big) \otimes \mathcal{O}_C\bigg).$$

Note that $\pi_*(\mathcal{F}(-l)) = \mathcal{F}'$. Then, we assume:

1) For every irreducible component C_ι, the restriction homomorphism $H^0(C,\mathcal{F}) \longrightarrow H^0(C_\iota,\mathcal{F}_{|C_\iota})$ is surjective.
2) For every irreducible component C_ι which is not a rational curve attached at only two points and on which $\mathcal{F}$ has positive rank, and any two points $p_1 \neq p_2 \in C_\iota$, the evaluation map
$$H^0(C_\iota,\mathcal{F}_{|C_\iota}) \longrightarrow \mathcal{F}_{|C_\iota} \otimes_{\mathcal{O}_C} \mathcal{O}_{\{p_1,p_2\}}$$
is surjective.
3) For every irreducible component C_ι which is not a rational curve attached at only two points and on which $\mathcal{F}$ has positive rank and any point $p \in C_\iota$, the evaluation map
$$H^0(C_\iota,\mathcal{F}_{|C_\iota}) \longrightarrow \mathcal{F}_{|C_\iota} \otimes_{\mathcal{O}_C} \left(\mathfrak{m}_{p,C_\iota}/\mathfrak{m}^2_{p,C_\iota}\right)$$
is surjective.

Next, we fix a maximal filtration $\mathcal{F}_\bullet\colon 0 =: \mathcal{F}'_0 \subsetneq \mathcal{F}'_1 \subsetneq \mathcal{F}'_2 \subsetneq \cdots \subsetneq \mathcal{F}'_k \subsetneq \mathcal{F}'_{k+1} := \pi_*(\mathcal{E})$ of $\pi_*(\mathcal{E})$ by destabilizing subsheaves and a vector $\underline{\alpha} := (\alpha_1,\ldots,\alpha_k)$ of nonnegative rational numbers. Let us also fix some l and define $V_j := H^0(\mathcal{F}'_j(l))$ under the identification of V^{χ_l} with $H^0(\pi_*(\mathcal{E})(l))$ and $\mathcal{F}_j := q_C(V_j \otimes \mathcal{O}_C)$, $j = 1,\ldots,k$.

Remark 2.2.2. The $\mathcal{F}_j$ are saturated subsheaves of $\mathcal{E}$, $j = 1,\ldots,k$. For, if $\widetilde{\mathcal{F}}_j$ is the saturation of $\mathcal{F}_j$, then $\widetilde{\mathcal{F}}_j(l')$ will be globally generated for some l' large enough. This is clear for points which lie on a component on which ω_C is ample. Now, let R be the disjoint union of all maximal chains of rational curves attached at only two points, set $C^* := C \setminus R$, let $\overline{C}^*$ be the closure of C^*, and define $\underline{x} := \overline{C}^* \setminus C^*$. Then, $H^1\big(\overline{C}^*, \mathcal{I}_{\underline{x}}\widetilde{\mathcal{F}}_j(l')\big)$ is zero for $l' \gg 0$, i.e., $H^0\big(C,\widetilde{\mathcal{F}}_j(l')\big) \longrightarrow H^0\big(R,\widetilde{\mathcal{F}}_{j|R}(l')\big)$ is surjective. As $\widetilde{\mathcal{F}}_{j|R}(l')$ is globally generated, the claim is settled. Now, if $\widetilde{\mathcal{F}}_j$ strictly contains $\mathcal{F}_j$, then $h^0\big(C,\widetilde{\mathcal{F}}_j(l')\big) > h^0\big(C,\mathcal{F}_j(l')\big)$ for all $l' \gg 0$. This is, however, not possible, because the inclusion $\mathcal{F}'_j(l+l') = \pi_*\mathcal{F}_j(l') \subset \pi_*\widetilde{\mathcal{F}}_j(l')$ must be an equality as both sheaves have the same multi-rank and $\mathcal{F}'_j$ is saturated.

Note that, given a basis $v_1,\ldots,v_{\chi_l}$ of V^{χ_l} such that $\langle\, v_1,\ldots,v_{\dim V_j}\,\rangle = V_j$, $j = 1,\ldots,k$, the vector $\underline{\alpha}$ defines a one-parameter subgroup λ of $\mathrm{SL}(V^{\chi_l})$, by Formula (2). By [**11**], 2.2, again, the values $\mu_{\mathfrak{N}}(\lambda,[\pi_*(q_C)])$ and $\mu_{\mathfrak{L}_m}(\lambda,[\wedge^{P(m)}\Psi_{\mathfrak{H}(g;\chi_l,r)}])$ do not depend on the choice of such a basis. It will be our task to compute the value of $\mu_{\mathfrak{L}_m}(\lambda,[\wedge^{P(m)}\Psi_{\mathfrak{H}(g;\chi_l,r)}])$. In other words, we will have to find a basis for $H^0(C,\det(\mathcal{E}(l)^{\otimes m}))$ for

which the weight of the associated element in $\mathrm{Hom}(\wedge^{P(m)}S^mV,\mathbb{C})$ becomes minimal. Then, $\mu_{\mathfrak{L}_m}(\lambda,\wedge^{P(m)}\psi)$ will be minus that weight.

Case A). The irreducible components of C are smooth. Write $C=\bigcup_{\iota=1}^{c}C_\iota$ as the union of its irreducible components and set $\mathcal{L}_m:=\det(\mathcal{E}(l))^{\otimes m}$. For large m, the restriction map $H^0(C,\mathcal{L}_m)\longrightarrow H^0(C_\iota,\mathcal{L}_m^\iota)$, $\mathcal{L}_m^\iota:=\mathcal{L}_{m|C_\iota}$, will be surjective for $\iota=1,\ldots,c$. We may, therefore, compute first the weights of a basis for $H^0(C_\iota,\mathcal{L}_m^\iota)$, $\iota=1,\ldots,c$. Set $\mathcal{E}_\iota:=\mathcal{E}(l)_{|C_\iota}$ and $\mathcal{F}_j^\iota:=\mathrm{Im}(\mathcal{F}_j\longrightarrow\mathcal{E}_\iota)$, $j=1,\ldots,k$, $\iota=1,\ldots,c$. We also define $\widetilde{V}_j:=\langle\, v_{\dim V_{j-1}+1},\ldots,v_{\dim V_j}\,\rangle$, $j=2,\ldots,k+1$, and $\widetilde{V}_1:=V_1$. Then

$$\bigwedge^r V=\bigoplus_{(\rho_1,\ldots,\rho_{k+1}):\sum\rho_j=r}W_{\rho_1,\ldots,\rho_{k+1}},$$
$$W_{\rho_1,\ldots,\rho_{k+1}}:=\bigwedge^{\rho_1}\widetilde{V}_1\otimes\bigwedge^{\rho_2}\widetilde{V}_2\otimes\cdots\otimes\bigwedge^{\rho_{k+1}}\widetilde{V}_{k+1}.$$

The spaces $W_{\underline{\rho}}$ are weight spaces for λ for the weight

$$w_{\iota,\underline{\rho}}^{\underline{\alpha}}(l):=\rho_1\cdot\gamma_1(l)+\cdots+\rho_{k+1}\cdot\gamma_{k+1}(l)$$

where $\gamma_j(l)$ is the weight of a section in $\widetilde{V}_j$, $j=1,\ldots,k+1$. Formula (2) shows that the $\gamma_j(l)$ are, in fact, polynomials in l.

Remark 2.2.3. Let $N\in C$ be a node of C, i.e., a point where two components C_1 and C_2 of C meet. Then, the stalk of a torsion-free sheaf of $\mathcal{Q}$ at N is of the form $\mathcal{O}_{C,N}^{\oplus a_1}\oplus\mathcal{O}_{C_1,N}^{\oplus a_2}\oplus\mathcal{O}_{C_2,N}^{\oplus a_3}$ ([**17**], Huitiême Partie, Proposition 3). Thus, if $\mathcal{E}$ is a torsion-free sheaf and $\mathcal{F}$ is a saturated subsheaf, then the image $\mathcal{F}_1$of $\mathcal{F}$ in $\mathcal{E}_{|C_1}$ is an almost saturated subsheaf, because $\mathrm{Tors}(\mathcal{E}_{|C_1}/\mathcal{F}_1)\cong\mathbb{C}^{\oplus a_3}$ as $\mathcal{O}_{C_1,N}$-module. Therefore, Proposition 1.4.1 allows one to determine the vanishing orders of sections coming from a weight space $W_{\underline{\rho}}$. This observation will be crucial for the following subtle analysis of weights and vanishing orders.

Let us look at some specific $\iota\in\{1,\ldots,c\}$. Then, the space of minimal weight which produces sections which do not vanish on C_ι is $W_{\min}^\iota:=$

$W_{\operatorname{rk}\mathcal{F}_1^\iota,\operatorname{rk}\mathcal{F}_2^\iota-\operatorname{rk}\mathcal{F}_1^\iota,\dots,\operatorname{rk}\mathcal{E}_\iota-\operatorname{rk}\mathcal{F}_k^\iota}$. The associated weight is

$$\begin{aligned} w^{\underline{\alpha}}_{\iota,\min}(l) &:= \sum_{j=1}^{k} \alpha_j \big(\dim V_j \cdot \operatorname{rk}\mathcal{E}_\iota - \chi_l \cdot \operatorname{rk}\mathcal{F}_k^\iota\big) \\ &= \sum_{j=1}^{k} \alpha_j \big(P(\mathcal{F}_j')(l) \cdot \operatorname{rk}\mathcal{E}_\iota - P(\pi_*(\mathcal{E}))(l) \cdot \operatorname{rk}\mathcal{F}_j^\iota\big). \end{aligned}$$

Let $N_1^\iota, \dots, N_{\nu_\iota}^\iota$ be the nodes of C located on C_ι, i.e., the points where C_ι meets other components of C. Note that each $\mathcal{F}_j^\iota$ is a subbundle of $\mathcal{E}_\iota$ outside the above points. This is because $\mathcal{F}_j$ is a saturated subsheaf of $\mathcal{E}$, by Remark 2.2.2, $j = 1, \dots, k$. Let us look at a specific node $N \in \{N_1^\iota, \dots, N_{\nu_\iota}^\iota\}$. Let $\mathbb{E}$ be the fibre of $\mathcal{E}$ at N, and $\mathbb{F}_j$ the image of $\mathcal{F}_j^\iota$ in $\mathbb{E}$, $j = 1, \dots, k$. Set $a_j := \dim \mathbb{F}_j$, $r_j := \operatorname{rk}\mathcal{F}_j^\iota$, and $b_j := \min\{a_j - a_{j-1}, r_j - r_{j-1}\}$, $j = 1, \dots, k+1$. A general section of $W_{\min}^\iota$ will vanish of order $o_N := r - b_N$ at N. This is an immediate consequence of Proposition 1.4.1.

Lemma 2.2.4.

i) *The sections in the subspace $W_{\min}^\iota$ generate the line bundle* $\det\big(\mathcal{E}_{|C_\iota}(l)(-\sum_{s=1}^{\nu_\iota} o_{N_s} N_s)\big)$.

ii) *The image of $W_{\min}^\iota$ in $H^0\big(\det(\mathcal{E}_{|C_\iota}(l)(-\sum_{s=1}^{\nu_\iota} o_{N_s} N_s))\big)$ is a very ample linear system, unless C_ι is rational and attached at only two points and* $\det\big(\mathcal{E}_{|C_\iota}(l)(-\sum_{s=1}^{\nu_\iota} o_{N_s} N_s)\big)$ *is trivial.*

Proof. The assertion i) about global generation results immediately from Assumption 2.2.1 (1). Likewise, 2.2.1 (2) and (3) settle the very ampleness in ii) when C_ι is not a rational component which is attached at only two points. In the remaining case, $\mathcal{E}_\iota$ is of the form $\mathcal{O}_{\mathbb{P}_1}^{\oplus s} \oplus \mathcal{O}_{\mathbb{P}_1}(1)^{\oplus t}$. The $\mathcal{F}_j^\iota$, as globally generated subsheaves of $\mathcal{E}_\iota$, are also of the form $\mathcal{O}_{\mathbb{P}_1}^{\oplus s_j} \oplus \mathcal{O}_{\mathbb{P}_1}(1)^{\oplus t_j}$, $j = 1, \dots, k$. Let $r_1, \dots, r_{b+1}$ be the ranks occurring among the $\mathcal{F}_j^\iota$, and for $\beta = 1, \dots, b+1$, let $\mathcal{G}_\beta$ be the first among the $\mathcal{F}_j^\iota$ to attain that rank. There are now two possibilities: either $H^0(\mathcal{G}_{\beta+1})/H^0(\mathcal{G}_\beta)$ contains a subspace $H^0(\mathcal{O}_{\mathbb{P}_1}(1))$ for some $\beta \in \{0, \dots, b\}$ or not. In the first case, by 2.2.1 (1), it is easy to explicitly construct sections separating points and tangent vectors. The second case, however, can only occur, if all the $\mathcal{G}_\beta$ are trivial, again by 2.2.1. But then, $\det\big(\mathcal{E}_{|C_\iota}(l)(-\sum_{s=1}^{\nu_\iota} o_{N_s} N_s)\big)$ is obviously trivial. q.e.d.

Corollary 2.2.5. *The space $S^m W^\iota_{\min}$ generates*

$$H^0\left(\left(\det(\mathcal{E}(l))_{|C_\iota}\left(-\sum_{s=1}^{\nu_\iota} o_{N_s} N_s\right)\right)^{\otimes m}\right)$$

for all $m \gg 0$.

Let d'_ι be the degree of the line bundle $\det(\mathcal{E}_{|C_\iota}(-\sum_{s=1}^{\nu_\iota} o_{N_s} N_s))$ and $e_\iota := \deg(\omega_{C|C_\iota})$. Because of Corollary 2.2.5, the elements in $S^m W^\iota_{\min}$ will contribute the weight

$$K^{\underline{\alpha}}_\iota(l,m) := m \cdot \big(m \cdot (d'_\iota + l \cdot r \cdot e_\iota) + 1 - g(C_j)\big) \cdot w^{\underline{\alpha}}_{\iota,\min}(l)$$

to a basis of $H^0(C_\iota, \mathcal{L}^\iota_m)$. Thus, we only have to worry about sections vanishing of lower order than $m \cdot o_N$ at N. Note that $W_{\rho_1,\dots,\rho_{k+1}}$ will produce sections which do not vanish on C_ι if and only if the condition

$$\sum_{i=1}^{j} \rho_i \le r_j, \quad j = 1, \dots, k, \tag{3}$$

is satisfied. A tuple $\underline{\rho} = (\rho_1, \dots, \rho_{k+1})$ satisfying (3) will be called *admissible*. Note that there are only finitely many admissible tuples.

Next, we let $\kappa_1 < \cdots < \kappa_s$ be the elements in $\{1, \dots, k+1\}$ with $a_\kappa - a_{\kappa-1} > r_\kappa - r_{\kappa-1}$. Set

$$K := \{\kappa_1, \dots, \kappa_s\} \quad \text{and} \quad K^* := \{1, \dots, k+1\} \setminus K.$$

Lemma 2.2.6. *Fix a vanishing order $o < o_N$, and let $w^{\underline{\alpha}}_{\iota,\underline{\rho}}(l)$ be the minimal weight of a section with vanishing order o. Then, $\underline{\rho}$ may be chosen to satisfy*

$$\begin{array}{rcccll} r_\kappa - r_{\kappa-1} & \le & \rho_\kappa & \le & a_\kappa - a_{\kappa-1} & \text{for } \kappa \in K \\ a_\kappa - a_{\kappa-1} & \le & \rho_\kappa & \le & r_\kappa - r_{\kappa-1} & \text{for } \kappa \in K^*. \end{array}$$

Proof. We begin with the right-hand side inequalities. Suppose ρ_κ violates the right-hand inequality. In particular, $\rho_\kappa > r_\kappa - r_{\kappa-1}$, whence $\sum_{j=1}^{\kappa-1} \rho_j < r_{\kappa-1}$. Define $\underline{\rho}' = (\rho'_1, \dots, \rho'_{k+1})$ by $\rho'_{\kappa-1} := \rho_{\kappa-1} + 1$, $\rho'_\kappa := \rho_\kappa - 1$, and $\rho'_j := \rho_j$ for $j \ne \kappa - 1, \kappa$. Then, $\underline{\rho}'$ is obviously admissible, $w^{\underline{\alpha}}_{\iota,\underline{\rho}'}(l) \le w^{\underline{\alpha}}_{\iota,\underline{\rho}}(l)$, and $W_{\underline{\rho}'}$ will still produce sections of vanishing order o. The latter property results from the fact that ρ_κ was, by assumption, strictly bigger than the maximal number of sections in $\widetilde{V}_\kappa$ with linearly independent images in $\mathbb{E}$.

The other inequality asserts $\rho_j \ge b_j$ for $j = 1, \dots, k+1$. For $k+1$, we have $\sum_{j=1}^{k} \rho_j \le r_k$, so $\rho_{k+1} = r - \sum_{j=1}^{k} \rho \ge r_{k+1} - r_k$. Suppose

$\rho_{j_0} < b_{j_0}$. Then, there is an index $j' > j_0$ with $\rho_{j'} \neq 0$. Otherwise, $\sum_{j=1}^{j_0} \rho_j = r$ and then $\rho_{j_0} \geq r_{j_0} - r_{j_0-1}$ as before, a contradiction. Let j' be minimal with the above properties. Define $\underline{\rho}' = (\rho'_1, \ldots, \rho'_{k+1})$ with $\rho'_{j_0} := \rho_{j_0} + 1$, $\rho'_{j'} := \rho_{j'} - 1$, and $\rho'_j = \rho_j$ for $j \neq j_0, j'$. As $\sum_{j=1}^{j_0-1} \rho_j \leq r_{j_0-1}$ and $\rho_{j_0} < r_{j_0} - r_{j_0-1}$, $\underline{\rho}'$ is again admissible. Moreover, $w^{\underline{\alpha}}_{\iota,\underline{\rho}'}(l) \leq w^{\underline{\alpha}}_{\iota,\underline{\rho}}(l)$ and $W_{\underline{\rho}'}$ will still lead to sections of vanishing order o. This time, the last assertion is the consequence of the assumption that ρ_κ was strictly smaller than the maximal number of sections in $\widetilde{V}_\kappa$ with linearly independent images in $\mathbb{E}$. q.e.d.

Let $I \subset K^* \times K$ be the set of all (i,j) with $i < j$. Fix an order "$\preceq$" on I, such that $(i',j') \preceq (i,j)$ implies $\gamma_{j'}(l) - \gamma_{i'}(l) \leq \gamma_j(l) - \gamma_i(l)$. The idea for the following investigations is the following: suppose we are given an admissible tuple $\underline{\rho}$, satisfying the inequalities of Lemma 2.2.6, and $(i,j) \in I$. Then, we define a new tuple $\underline{\rho}'$ with $\rho'_i := \rho_i - 1$ and $\rho'_j = \rho_j + 1$ and let all other entries of $\underline{\rho}'$ agree with those of $\underline{\rho}$. As $i < j$, $\underline{\rho}'$ is still admissible. However, we will perform this operation only if $\underline{\rho}'$ still satisfies the inequalities of Lemma 2.2.6. In that case, the generic vanishing order of sections in $W_{\underline{\rho}'}$ will be one less than the generic vanishing order in $W_{\underline{\rho}}$. Thus, if we are given a specific vanishing order o, we carry out $s := o_N - o$ operations of the above type as follows: we start with (i,j) which is minimal w.r.t. "$\preceq$" (because the corresponding process will increase the weight the least), perform the operation on (i,j) as many times as possible, say $s_{(i,j)}$ times, then pass to the next pair $(i',j') \in I$ w.r.t. the order "$\preceq$" and so on, until we have performed s such processes in total. Then, we arrive at a tuple $\underline{\rho}'$, such that the generic vanishing order of sections from $W_{\underline{\rho}'}$ is precisely o. The difficult part is to show that the corresponding weight will be, in fact, minimal.

Fix a vanishing order $o < o_N$, let $w^{\underline{\alpha}}_{\iota,\underline{\rho}}(l)$ be the minimal weight of a section which vanishes of order o, and assume that $\underline{\rho}$ satisfies the inequalities of Lemma 2.2.6. Then, we define natural numbers $s_{(i,j)}$ for $(i,j) \in I$ inductively w.r.t. "$\preceq$" as follows: for $(i,j) \in I$, set

$$c_i = \sum_{(i,j') \prec (i,j)} s_{(i,j')}$$

$$c_j = \sum_{(i',j) \prec (i,j)} s_{(i',j)},$$

where empty sums are by definition zero. Then,

$$s_{(i,j)} := \min\big\{r_i - r_{i-1} - \rho_i - c_i, \rho_j - r_j + r_{j-1} - c_j\big\}.$$

Observation 2.2.7.

i) For every index $\kappa \in K^*$ and every admissible tuple $\underline{\rho}$, satisfying the conditions of Lemma 2.2.6, we have

$$\sum_{i\in K: i<\kappa} \big(\rho_i - r_i + r_{i-1}\big) \leq \sum_{j\in K^*: j\leq\kappa} \big(r_j - r_{j-1} - \rho_j\big).$$

From this, one easily infers that

$$\rho_j = r_j - r_{j-1} + \sum_{(i,j)\in I} s_{(i,j)} \qquad \text{for } j \in K$$

$$\rho_i = r_i - r_{i-1} - \sum_{(i,j)\in I} s_{(i,j)} \qquad \text{for } i \in K^*,$$

so that the $s_{(i,j)}$ determine $\underline{\rho}$.

ii) Suppose we are given a tuple $\underline{s} = (s'_{(i,j)}, (i,j) \in I)$ with $0 \leq s'_{(i,j)} \leq s_{(i,j)}$ for all $(i,j) \in I$. Define $\underline{\rho^{\underline{s}}}$ by

$$\rho^{\underline{s}}_j = r_j - r_{j-1} + \sum_{(i,j)\in I} s'_{(i,j)} \qquad \text{for } j \in K$$

$$\rho^{\underline{s}}_i = r_i - r_{i-1} - \sum_{(i,j)\in I} s'_{(i,j)} \qquad \text{for } i \in K^*.$$

The tuple $\underline{\rho^{\underline{s}}}$ is clearly admissible.

Lemma 2.2.8. *Fix a vanishing order $o < o_N$, and let $w^{\alpha}_{\iota,\underline{\rho}}(l)$ be the minimal weight of a section with vanishing order o where $\underline{\rho}$ fulfills the conditions of Lemma* 2.2.6*. Then, $\underline{\rho}$ may be chosen in such a way that the $s_{(i,j)}$ satisfy*

$$s_{(i,j)} = \min\Bigg\{s - \sum_{(i',j')\prec(i,j)} s_{(i',j')}, \qquad r_i - r_{i-1} - a_i + a_{i-1} - c_i, \\ a_j - a_{j-1} - r_j + r_{j-1} - c_j\Bigg\}.$$

Here, $s := \sum s_{(i,j)}$, and c_i and c_j are as before.

Proof. Assume that the assertion were wrong for $s_{(i,j)}$, i.e., $s_{(i,j)}$ is strictly smaller than the right-hand side. Then, there are three cases. In the first case, $s_{(i,j)} = r_i - r_{i-1} - \rho_i - c_i = \rho_j - r_j + r_{j-1} - c_j$. In that case

$$\rho_i \ > \ a_i - a_{i-1} \quad \text{and} \quad \rho_j < a_j - a_{j-1}. \tag{4}$$

Then, there is an $(i', j') \succ (i, j)$ with $s_{(i',j')} > 0$. Set $\underline{\rho}' = (\rho'_1, \dots, \rho'_{k+1})$ with $\rho'_{i'} := \rho_{i'} + 1$, $\rho'_{j'} := \rho_{j'} - 1$, and $\rho'_j = \rho_j$ for $j \neq i', j'$. The tuple $\underline{\rho}'$ is still admissible, by Observation 2.2.7. It is, in fact, defined w.r.t. $\underline{s}$ with $s'_{(i,j)} = s_{(i,j)}$, $(i,j) \neq (i', j')$, and $s'_{(i',j')} = s_{(i',j')} - 1$. Introduce $\underline{\rho}'' = (\rho''_1, \dots, \rho''_{k+1})$ by $\rho''_i := \rho'_i - 1$, $\rho''_j := \rho'_j + 1$, and $\rho''_{j'} := \rho'_{j'}$ for $j' \neq i, j$. This is again admissible, $w^{\underline{\alpha}}_{\iota,\underline{\rho}''}(l) \leq w^{\underline{\alpha}}_{\iota,\underline{\rho}}(l)$, and $W_{\underline{\rho}''}$ still contains sections of vanishing order o, by (4). In other words, we set $s'_{(i,j)} := s_{(i,j)} + 1$.

In the second case, $s_{(i,j)} = r_i - r_{i-1} - \rho_i - c_i < \rho_j - r_j + r_{j-1} - c_j$. Then, as $\rho_j - r_j + r_{j-1} - c_j > 0$, there is an index $(i', j) \succ (i, j)$ with $s_{(i',j)} > 0$. One may now proceed as before. The last case, $s_{(i,j)} = \rho_j - r_j + r_{j-1} - c_j < r_i - r_{i-1} - \rho_i - c_i$, is handled the same way. q.e.d.

Given s, the condition in Lemma 2.2.8 uniquely determines a tuple $\underline{\rho}$ with $\sum s_{(i,j)} = s$ for which $w^{\underline{\alpha}}_{\iota,\underline{\rho}}(l)$ becomes minimal. Note that $W_{\underline{\rho}}$ yields sections with vanishing order $\geq o_N - s$ where "=" is achieved. An immediate consequence is:

Corollary 2.2.9.

i) *Fix a vanishing order $o < o_N$, and let $w^{\underline{\alpha}}_{\iota,\underline{\rho}}(l)$ be the minimal weight of a section with vanishing order o where $\underline{\rho}$ fulfills the conditions of Lemmas* 2.2.6 *and* 2.2.8. *Then, $\sum s_{(i,j)} = o_N - o$.*

ii) *Denote by $w^{N,o}_{\iota,\underline{\alpha}}(l)$ the minimal weight of a section with vanishing order o at N. Then,*

$$w^{N,o-1}_{\iota,\underline{\alpha}}(l) - w^{N,o}_{\iota,\underline{\alpha}}(l) \geq w^{N,o}_{\iota,\underline{\alpha}}(l) - w^{N,o+1}_{\iota,\underline{\alpha}}(l).$$

Define $\mathbb{O}_o := W_{\underline{\rho}}$, where the tuple $\underline{\rho}$ is determined by the conditions of Lemmas 2.2.6 and 2.2.8 and $\sum s_{(i,j)} = o_N - o$, and $\mathbb{O} := \bigoplus_{o=0}^{o_N} \mathbb{O}_o$. We have to find the minimal weights of sections in $H^0(C_\iota, \mathcal{L}^\iota_m)$ vanishing of order $0 \leq o' \leq m \cdot o_N - 1$. We clearly have to look only at sections in

$$S^m\mathbb{O} = \bigoplus_{m_0,\dots,m_{o_N} : \sum m_\nu = m} S^{m_0}\mathbb{O}_0 \otimes \cdots \otimes S^{m_{o_N}}\mathbb{O}_{o_N}.$$

Now, the sections in $S^{m_0}\mathbb{O}_0 \otimes \cdots \otimes S^{m_{o_N}}\mathbb{O}_{o_N}$ vanish of order at least $m_1 + 2 \cdot m_2 + \cdots + o_N \cdot m_{o_N}$, and we can find some with exactly that vanishing order. On the other hand, the weight of sections in that space is

$$\begin{aligned}
&m_0 \cdot w_{\iota,\underline{\alpha}}^{N,0}(l) + \cdots + m_{o_N} \cdot w_{\iota,\underline{\alpha}}^{N,o_N}(l) \\
&= m \cdot w_{\iota,\underline{\alpha}}^{N,0}(l) - m_0\big(w_{\iota,\underline{\alpha}}^{N,0}(l) - w_{\iota,\underline{\alpha}}^{N,1}(l)\big) - \cdots - \\
&\quad - m_{o_N}\big(w_{\iota,\underline{\alpha}}^{N,0}(l) - w_{\iota,\underline{\alpha}}^{N,o_N}(l)\big) \\
&= m \cdot w_{\iota,\underline{\alpha}}^{N,0}(l) - m_0\big(w_{\iota,\underline{\alpha}}^{N,0}(l) - w_{\iota,\underline{\alpha}}^{N,1}(l)\big) - \cdots - \\
&\quad - m_{o_N}\big((w_{\iota,\underline{\alpha}}^{N,0}(l) - w_{\iota,\underline{\alpha}}^{N,1}(l)) + \cdots + (w_{\iota,\underline{\alpha}}^{N,o_N-1}(l) - w_{\iota,\underline{\alpha}}^{N,o_N}(l))\big).
\end{aligned}$$

It follows easily from Corollary 2.2.9 ii) that the elements in $S^m\mathbb{O}$ producing sections of minimal weight vanishing of order o with $(t-1)\cdot m \leq o \leq t \cdot m - 1$ lie in

$$\bigoplus_{i=1}^{m} S^i\mathbb{O}_{t-1} \otimes S^{m-i}\mathbb{O}_t, \qquad t = 1, \ldots, o_N.$$

These contribute the weight

$$m \cdot w_{\iota,\underline{\alpha}}^{N,t}(l) + \frac{m(m-1)}{2}\big(w_{\iota,\underline{\alpha}}^{N,t-1}(l) - w_{\iota,\underline{\alpha}}^{N,t}(l)\big)$$

to a basis for $\mathbb{H} := H^0(C_\iota, \mathcal{L}_m^\iota)/H^0\Big(\big(\det(\mathcal{E}(l))_{|C_\iota}(-\sum_{s=1}^{\nu_\iota} o_{N_s} N_s)\big)^{\otimes m}\Big)$. The total contribution to a basis for $\mathbb{H}$, coming from the node N, thus amounts to

$$\begin{aligned}
C_\iota^{N,\underline{\alpha}}(l,m) := m\big(w_{\iota,\underline{\alpha}}^{N,0}(l) + \cdots + w_{\iota,\underline{\alpha}}^{N,o_N}(l)\big) \\
+ \frac{m(m-1)}{2}\big(w_{\iota,\underline{\alpha}}^{N,0}(l) - w_{\iota,\underline{\alpha}}^{N,o_N}(l)\big).
\end{aligned}$$

All in all, a basis for $H^0(C_\iota, \mathcal{L}_m^\iota)$ will have minimal weight

$$C_\iota^{\underline{\alpha}}(l,m) := K_\iota^{\underline{\alpha}}(l,m) + \sum_{n=1}^{\nu_\iota} C_\iota^{N_n,\underline{\alpha}}(l,m).$$

Let N be an intersection of two components C_ι and $C_{\iota'}$ of C. Then,

$$w_{\iota,\underline{\alpha}}^{N,0}(l) = w_{\iota',\underline{\alpha}}^{N,0}(l) =: w_{\underline{\alpha}}^{N}(l).$$

Let $\mathcal{N}$ be the set of nodes of C. For large m, there is the exact sequence

$$0 \longrightarrow H^0(C, \det \mathcal{E}^{\otimes m}) \longrightarrow \bigoplus_{\iota=1}^{c} H^0(C_\iota, \mathcal{L}_m^\iota) \longrightarrow \bigoplus_{N \in \mathcal{N}} \mathbb{C} \cdot e_N \longrightarrow 0.$$

This shows that

$$H^0(C, \det \mathcal{E}^{\otimes m}) = \bigoplus_{\iota=1}^{c} H^0\left(C_\iota, \mathcal{L}_m^\iota\left(-\sum_{n=1}^{\nu_\iota} N_n^\iota\right)\right) \oplus \bigoplus_{N\in\mathcal{N}} \mathbb{C}\cdot e_N.$$

Thus, we see that the minimal weight of a basis for $H^0(C, \det \mathcal{E}^{\otimes m})$ is

$$P^{\underline{\alpha}}_{\mathcal{F}_\bullet}(l,m) := \sum_{\iota=1}^{c} C^{\underline{\alpha}}_\iota(l,m) - m\cdot \sum_{N\in\mathcal{N}} w^N_{\underline{\alpha}}(l).$$

Note that this polynomial is intrinsically defined in terms of the curve C, the filtration $\mathcal{F}_\bullet$, and $\underline{\alpha}$.

Case B). *C* has nodal irreducible components. In this case, we pass to the semistable curve $\pi\colon C' \longrightarrow C$, where we introduce a projective line for every node at which C is irreducible and the filtration by the $\mathcal{F}_j$ is not a filtration by subbundles, and pullback $\mathcal{E}$ to $\mathcal{E}'$ on C'. Now, let C'_ι be any irreducible component of C'. Then, $\nu_\iota := \pi_{|C'_\iota}\colon C'_\iota \longrightarrow C_\iota$ is a partial normalization. For any node $N \in C_\iota$ which is resolved by ν_ι, $W^\iota_{\min}$ will produce sections of $\det(\mathcal{E}'(l))$ which vanish at both points $p_{N,1}$ and $p_{N,2}$ in $\nu_\iota^{-1}(N)$. As the space of sections of $\det(\mathcal{E}(l))$ vanishing at N identifies with the space of sections of $\det(\mathcal{E}'(l))$ vanishing at both $p_{N,1}$ and $p_{N,2}$, it is easy to see that the analogs of Lemma 2.2.4 and Corollary 2.2.5 continue to hold. The rest of the considerations clearly go through as before.

***H*-semistability.** We are now ready to define our semistability concept.

Definition 2.2.10. A pair $(C, \mathcal{E})$, consisting of a semistable curve C and a vector bundle $\mathcal{E}$ of rank r on C with $\chi(\mathcal{E}) = \chi$, will be called *H-(semi)stable*, if it satisfies the following conditions:

1) The push forward $\pi_*(\mathcal{E})$ to the stable model via $\pi\colon C \longrightarrow C'$ is a semistable torsion-free sheaf (see the remark in the introduction).
2) For every maximal filtration of $\mathcal{F}_\bullet$ of $\pi_*(\mathcal{E})$ by destabilizing subsheaves and every vector $\underline{\alpha}$ of nonnegative rational numbers, there exists an index l^*, such that for all $l \geq l^*$

$$P^{\underline{\alpha}}_{\mathcal{F}_\bullet}(l,m)\ (\preceq)\ 0 \quad \text{as polynomial in } m.$$

Note that this semistability concept has all the properties that were asserted in the introduction. Part ii) of the main theorem is a direct consequence of:

Theorem 2.2.11. *There exist an index l_0 and for every $l \geq l_0$ an index $m(l)$, such that for every $l \geq l_0$, $m' \geq m(l)$, and every pair $(C, \mathcal{E})$, consisting of a semistable curve C and a vector bundle $\mathcal{E}$ of rank r on C with $\chi(\mathcal{E}) = \chi$, which satisfies* (1) *and* (2) *of Definition* 2.2.10

$$P^{\underline{\alpha}}_{\mathcal{F}_\bullet}(l, m') \ (\leq)\ 0 \quad \Longleftrightarrow \quad P^{\underline{\alpha}}_{\mathcal{F}_\bullet}(l, m) \ (\preceq)\ 0 \quad \textit{as polynomial in } m$$

for every filtration $\mathcal{F}_\bullet$ and every tuple $\underline{\alpha}$.

Proof. First note that, given $\underline{\alpha}$, $P^{\underline{\alpha}}_{\mathcal{F}^\bullet}$ depends only on the following data:

- The tuples $(\operatorname{rk} \mathcal{F}^\iota_1, \ldots, \operatorname{rk} \mathcal{F}^\iota_k)$, $\iota = 1, \ldots, c$. These determine all the Hilbert polynomials of the $\mathcal{F}'_j$, $j = 1, \ldots, k$, because these are destabilizing sheaves.
- The tuples $(a^N_1, \ldots, a^N_k)$, N a node of C, and a^N_j the dimension of the image of $\mathcal{F}_j$ in the fibre of $\mathcal{E}$ at N, $j = 1, \ldots, k$.

By boundedness, the sets of data of the above type is in fact finite. Therefore, we will be done, once we have shown that, for a given set of such data, we have to take only finitely many vectors $\underline{\alpha}$ into account.

Given tuples $(r^\iota_1, \ldots, r^\iota_k)$, $\iota = 1, \ldots, c$, and $(a^N_1, \ldots, a^N_k)$, N a node of C, we define sets $K_{N,\iota}$ and $K^*_{N,\iota}$ as before. Note that in our construction before, we had to look at the quantities $\gamma_j(l) - \gamma_i(l)$, $(i, j) \in K^*_{N,\iota} \times K_{N,\iota}$. By Formula (2),

$$\gamma_j(l) - \gamma_i(l) = \sum_{t=i}^{j-1} \alpha_t \cdot \chi_l.$$

For every ordering "$\preceq_{N,\iota}$" of $K^*_{N,\iota} \times K_{N,\iota}$, we get the set of inequalities

$$(*)_{\preceq_{N,\iota}} \qquad \sum_{t=i'}^{j'-1} \alpha_t \leq \sum_{t=i}^{j-1} \alpha_t, \qquad (i', j') \preceq_{N,\iota} (i, j).$$

Let $Q \subset \mathbb{R}^k$ be the quadrant of vectors all the entries of which are nonnegative. This is a rational polyhedral cone. For a given ordering "$\preceq_{N,\iota}$", the inequalities $(*)_{\preceq_{N,\iota}}$ define a proper rational polyhedral subcone of Q. Given two distinct orderings, the resulting cones will meet only along faces, i.e., if we let "$\preceq_{N,\iota}$" vary over all possible orderings, we get a fan decomposition $Q = \bigcup_{\beta=1}^{B_{N,\iota}} Q^{N,\iota}_\beta$ of Q.

We have seen that once the ordering "$\preceq_{N,\iota}$" is fixed, every given vanishing order o uniquely determines a vector $\underline{\rho}$ with $w^{\underline{\alpha}}_{\iota,\underline{\rho}}(l) = w^{N,o}_{\iota,\underline{\alpha}}(l)$ for

all $\underline{\alpha}$ in the cone $Q_\beta^{N,\iota}$ cut out by the inequalities $(*)_{\preceq_{N,\iota}}$. In particular, for $\underline{\alpha}, \underline{\alpha}' \in Q_\beta^{N,\iota}$

$$w^{N,o}_{\iota,\underline{\alpha}+\underline{\alpha}'}(l) = w^{N,o}_{\iota,\underline{\alpha}}(l) + w^{N,o}_{\iota,\underline{\alpha}'}(l) \tag{5}$$

for all possible vanishing orders.

As the intersection of two rational polyhedral cones is again a rational polyhedral cone, we can form the rational polyhedral cones of the form $Q_{\beta_1}^{N_1,\iota_1} \cap \cdots \cap Q_{\beta_\nu}^{N_\nu,\iota_\nu}$. Here, $N_1, \ldots, N_\nu$ are the nodes of C (or, in Case B, the nodes of the corresponding partial normalization), ι_i is an index such that C_{ι_i} contains N_i, and $\beta_i \in \{1, \ldots, B_{N,\iota_i}\}$, $i = 1, \ldots, \nu$. This defines a fan decomposition $Q = \bigcup_{\beta=1}^{B} Q_B$.

Let $\mathcal{F}_\bullet$ be a maximal filtration, realizing the data $(r_1^\iota, \ldots, r_k^\iota)$, $\iota = 1, \ldots, c$, and $(a_1^N, \ldots, a_k^N)$, N a node of Q. Then, for every $\beta \in \{1, \ldots, N\}$, and any $\underline{\alpha}, \underline{\alpha}' \in Q_\beta$, we have, by (5),

$$P^{\underline{\alpha}+\underline{\alpha}'}_{\mathcal{F}_\bullet}(l, m) = P^{\underline{\alpha}}_{\mathcal{F}_\bullet}(l, m) + P^{\underline{\alpha}'}_{\mathcal{F}_\bullet}(l, m). \tag{6}$$

For every edge e of the cone Q_β, denote the minimal integral generator by $\underline{\alpha}_{e,\beta}$. Then, by (6), we have to verify the inequalities in Definition 2.2.10 only for $\underline{\alpha}$ in the finite set

$$\big\{ \underline{\alpha}_{e,\beta} \,|\, \beta = 1, \ldots, B,\ e \text{ an edge of } Q_\beta \big\}.$$

The theorem is now settled. q.e.d.

2.3. The Hilbert compactification as a moduli space. Introduce the functors

$$\underline{\mathrm{HC}}^{(\mathrm{s})\mathrm{s}}(g; \chi, r) \colon \underline{\mathrm{Schemes}}_{\mathbb{C}} \longrightarrow \underline{\mathrm{Sets}}$$

which assign to every scheme S the equivalence classes of pairs $(\mathcal{C}_S, \mathcal{E}_S)$ where $\pi \colon \mathcal{C}_S \longrightarrow S$ is a flat family of semistable curves, and $\mathcal{E}_S$ is an S-flat sheaf, such that, for every closed point $s \in S$, the restriction $\mathcal{E}_{S|\pi^{-1}(s)}$ is an H-(semi)stable vector bundle of uniform rank r and Euler characteristic χ. Two families $(\mathcal{C}_S, \mathcal{E}_S)$ and $(\mathcal{C}'_S, \mathcal{E}'_S)$ are *equivalent*, if there are an isomorphism $\varphi_S \colon \mathcal{C}'_S \longrightarrow \mathcal{C}_S$ and a line bundle $\mathcal{L}_S$ on S, such that

$$\varphi_S^*\big(\mathcal{E}_S \otimes \pi^*(\mathcal{L}_S)\big) \cong \mathcal{E}'_S.$$

Then, our considerations imply:

Theorem 2.3.1.

i) *There is a natural transformation*

$$\vartheta\colon \underline{\mathrm{HC}}^{\mathrm{ss}}(g;\chi,r) \longrightarrow h_{\mathfrak{H}\mathfrak{C}(g;\chi,r)},$$

such that for every other scheme $\mathfrak{S}$ *and every other natural transformation* $\vartheta'\colon \underline{\mathrm{HC}}^{\mathrm{ss}}(g;\chi,r) \longrightarrow h_{\mathfrak{S}}$*, one has a unique morphism*

$$t\colon \mathfrak{H}\mathfrak{C}(g;\chi,r) \longrightarrow \mathfrak{S}$$

with $\vartheta' = h(t)\circ\vartheta$*.*

ii) *The space* $\mathfrak{H}\,\mathfrak{C}\,(g;\,\chi,r)$ *contains an open subscheme* $\mathfrak{H}\mathfrak{C}(g;\chi,r)^{\mathrm{s}}$ *which is a coarse moduli scheme for* $\underline{\mathrm{HC}}^{\mathrm{s}}(g;\chi,r)$*.*

3. Properties of the Hilbert compactification

3.1. Dimension and smooth points. We will call a pair $(C,\mathcal{E})$ with C a semistable curve and $\mathcal{E}$ a vector bundle on C *strictly H-stable*, if it is H-stable and there is no automorphism $\varphi\colon C \longrightarrow C$ with $\varphi^*\mathcal{E} \cong \mathcal{E}$.

Remark 3.1.1. If $(C,\mathcal{E})$ is strictly H-stable, then $\mathcal{E}$ must be a simple bundle, i.e., $\mathrm{End}(\mathcal{E}) \cong \mathbb{C}\cdot \mathrm{id}_{\mathcal{E}}$. In fact, the universal bundle on $\widehat{\mathfrak{H}} := \widehat{\mathfrak{H}}(g;\chi_l,r)$ possesses a $\mathrm{GL}(V^{\chi_l})$-linearization whence the $\mathrm{GL}(V^{\chi_l})$-stabilizers of a point in $\widehat{\mathfrak{H}}$ corresponding to a strictly H-stable pair $(C,\mathcal{E})$ identify with the automorphisms of $\mathcal{E}$ on C which form a dense set in the space of endomorphisms. Therefore, $\mathrm{End}(\mathcal{E})$ can have dimension at most one, because the $\mathrm{GL}(V^{\chi_l})$-stabilizer may have dimension at most one.

Let $\mathfrak{H}\mathfrak{C}(g;\chi,r)^{\star} \subset \mathfrak{H}\mathfrak{C}(g;\chi,r)$ be the open subset parameterizing the strictly H-stable curves.

Theorem 3.1.2.

i) *The Hilbert compactification is a normal variety of dimension* $3g-3+r^2(g-1)+1$*.*

ii) *The subset* $\mathfrak{H}\mathfrak{C}(g;\chi,r)^{\star}$ *is smooth.*

Proof. Let $\kappa_g\colon \mathfrak{H}\mathfrak{C}(g;\chi,r) \longrightarrow \overline{\mathfrak{M}}_g$ be the natural morphism. The irreducibility and the dimension statement in i) are clear, because they are known for the preimage $\mathcal{U}$ of the moduli space $\mathfrak{M}_g$ of smooth curves under κ_g (see [**14**]), and we have seen in Theorem 2.1.2 that $\mathcal{U}$ is dense in the Hilbert compactification.

Let $\mathfrak{H}^0(g;\chi,r)$ be as in the introduction, and $\mathfrak{H}^{\dagger}(g;\chi,r) \subset \mathfrak{H}^0(g;\chi,r)$ the open subset of those $(C, q_C\colon V^{\chi}\otimes\mathcal{O}_C \longrightarrow \mathcal{E})$ for which $H^1(\mathcal{E}_C)$

vanishes. As our considerations in Chapter 2 show, the Hilbert compactification is a quotient of an open subset of $\mathfrak{H}^\dagger(g;\chi,r)$. Please accept for the moment the following statement:

Proposition 3.1.3. *The scheme $\mathfrak{H}^\dagger(g;\chi,r)$ is smooth.*

This proposition settles i). Let $\mathfrak{H}^\star(g;\chi,r)$ be the open part of the Hilbert scheme which parameterizes the strictly H-stable objects. Now, statement ii) follows, because the quotient morphism $\mathfrak{H}^\star(g;\chi,r) \longrightarrow \mathfrak{HC}(g;\chi,r)^\star$ is a principal $\mathrm{PGL}(V^{\chi_l})$-bundle.

We now turn to the proof of Proposition 3.1.3. Let $\mathfrak{M}_g^\star$ be the moduli space of automorphism free smooth curves, and set

$$\mathcal{U}^\star := \kappa_g^{-1}\big(\mathfrak{M}_g^\star\big) \cap \mathfrak{HC}^{\mathrm{s}}.$$

Then, $\mathcal{U}^\star$ is a smooth quasi-projective variety of dimension $3g-3+r^2(g-1)+1$. Since the quotient morphism is over $\mathcal{U}^\star$ a principal $\mathrm{PGL}(V^{\chi_l})$-bundle, the preimage of $\mathcal{U}^\star$ under the quotient morphism is a smooth quasi-projective variety of dimension $3g-3+r^2(g-1)+\chi_l^2$. Moreover, by Theorem 2.1.2, it is dense in $\mathfrak{H}^\dagger(g;\chi,r)$, whence the latter is an irreducible scheme of the same dimension. To prove smoothness, we have to determine the dimension of the tangent spaces. If $x \in \mathfrak{H}^\dagger(g;\chi,r)$ corresponds to the curve $C_x \hookrightarrow \mathfrak{G}$, the tangent space to $\mathfrak{H}^\dagger(g;\chi,r)$ at x is given by $\mathrm{Hom}\big(\mathcal{I}_{C_x}/\mathcal{I}_{C_x}^2, \mathcal{O}_{C_x}\big)$. Since C is a local complete intersection (which is an intrinsic property by [**10**], Prop. 3.2.1 and Cor. 3.2.2), the conormal sheaf $\mathcal{I}_{C_x}/\mathcal{I}_{C_x}^2$ is locally free, and we have the exact sequence

$$0 \longrightarrow \mathcal{I}_{C_x}/\mathcal{I}_{C_x}^2 \longrightarrow \Omega^1_{\mathfrak{G}|C_x} \longrightarrow \Omega^1_{C_x} \longrightarrow 0.$$

Here, the left exactness follows, because (a) the sequence is in any case exact away from the nodes of C_x and (b) since $\mathcal{I}_{C_x}/\mathcal{I}_{C_x}^2$ is torsion-free, it does not contain any subsheaf the support of which has dimension strictly less than one. We derive the exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}(\Omega^1_{C_x}, \mathcal{O}_{C_x}) &\longrightarrow H^0(T_{\mathfrak{G}|C_x}) \longrightarrow \\ \longrightarrow \mathrm{Hom}\big(\mathcal{I}_{C_x}/\mathcal{I}_{C_x}^2, \mathcal{O}_{C_x}\big) &\longrightarrow \mathrm{Ext}^1(\Omega^1_{C_x}, \mathcal{O}_{C_x}) \longrightarrow H^1(T_{\mathfrak{G}|C_x}). \end{aligned}$$

We claim that $H^1(T_{\mathfrak{G}|C_x})$ vanishes. For this, we use the exact sequence

$$0 \longrightarrow \mathcal{E}nd(\mathcal{E}_{\mathfrak{G}}) \longrightarrow \mathcal{E}_{\mathfrak{G}}^{\oplus\chi_l} \longrightarrow T_{\mathfrak{G}} \longrightarrow 0.$$

Let $\mathcal{E}$ be the restriction of $\mathcal{E}_{\mathfrak{G}}$ to the curve C_x, so that we obtain the exact sequence

$$0 \longrightarrow \mathcal{E}nd(\mathcal{E}) \longrightarrow \mathcal{E}^{\oplus\chi_l} \longrightarrow T_{\mathfrak{G}|C_x} \longrightarrow 0.$$

Now, our assumption is that $H^1(\mathcal{E})^{\oplus\chi_l} = H^1(\mathcal{E}^{\oplus\chi_l})$ vanishes, and, for dimension reasons, $H^2(\mathcal{E}nd(\mathcal{E})) = 0$, whence also $H^1(T_{\mathfrak{G}|C_x}) = 0$, as asserted. We also see that

$$h^0(T_{\mathfrak{G}|C_x}) = \chi_l \cdot h^0(\mathcal{E}) - \chi(\mathcal{E}nd(\mathcal{E})) = \chi_l^2 + r^2(g-1).$$

Next, by Serre duality

$$\dim\big(\mathrm{Hom}(\Omega^1_{C_x}, \mathcal{O}_{C_x})\big) - \dim\big(\mathrm{Ext}^1(\Omega^1_{C_x}, \mathcal{O}_{C_x})\big) = \chi(\Omega^1_{C_x} \otimes \omega_{C_x}) = 3g-3.$$

The exact sequence above thus shows

$$\begin{aligned}\dim\big(\mathrm{Hom}(\mathcal{I}_{C_x}/\mathcal{I}^2_{C_x}, \mathcal{O}_{C_x})\big) &= h^0(T_{\mathfrak{G}|C_x}) + \chi(\Omega^1_{C_x} \otimes \omega_{C_x}) \\ &= \chi_l^2 + r^2(g-1) + 3g - 3.\end{aligned}$$

This proves that x is a smooth point of $\mathfrak{H}^\dagger(g;\chi,r)$. q.e.d.

Remark 3.1.4.

i) Deligne and Mumford [**3**] applied Schlessinger's deformation theory [**15**] in order to show that any semistable curve C admits a *miniversal deformation* over the base scheme $\mathcal{M} := \mathrm{Spec}\mathbb{C}[\![t_1, \ldots, t_N]\!]$ with $N := \dim\big(\mathrm{Ext}^1(\Omega_C, \mathcal{O}_C)\big)$. This means that there is a family $\mathcal{C}_\mathcal{M} \longrightarrow \mathcal{M}$ of curves parameterized by $\mathcal{M}$ with C as the fibre over the origin, such that for any flat family of curves $\mathcal{C}_B \longrightarrow B$ with B the spectrum of a local Artin algebra and C as the fibre over the closed point there is a morphism $\varphi\colon B \longrightarrow \mathcal{M}$ with $\mathcal{C}_B \cong \mathcal{C}_\mathcal{M} \times_\mathcal{M} B$. Moreover, φ is unique in case $B = \mathrm{Spec}\big(\mathbb{C}[\varepsilon]/\langle\varepsilon^2\rangle\big)$. The tangent space to $0 \in \mathcal{M}$ thus identifies with $\mathrm{Ext}^1(\Omega_C, \mathcal{O}_C)$. Finally, suppose C has M nodes, then the space of *local deformations* is the deformation space of this set of nodes and, thus, identifies with $\mathcal{M}_{\mathrm{loc}} = \mathrm{Spec}\mathbb{C}[\![u_1, \ldots, u_M]\!]$. Here, one can arrange the generators $t_1, \ldots, t_N$ and $u_1, \ldots, u_M$ in such a way that the natural morphism $\mathcal{M} \longrightarrow \mathcal{M}_{\mathrm{loc}}$ comes from the homomorphism $\mathbb{C}[\![u_1, \ldots, u_M]\!] \longrightarrow \mathbb{C}[\![t_1, \ldots, t_N]\!]$, $u_i \longmapsto t_i$, $i = 1, \ldots, M$.

Now, let $x \in \mathfrak{H}^\dagger(g;\chi,r)$, and let $\mathfrak{U}$ be its formal neighborhood. By the smoothness of $\mathcal{M}$ and its versality, the universal curve over the Hilbert scheme $\mathfrak{H}^\dagger(g;\chi,r)$ provides us with a morphism $\varphi\colon \mathfrak{U} \longrightarrow \mathcal{M}$ the differential of which is the map

$$\mathrm{Hom}\big(\mathcal{I}_{C_x}/\mathcal{I}^2_{C_x}, \mathcal{O}_{C_x}\big) \longrightarrow \mathrm{Ext}^1(\Omega^1_{C_x}, \mathcal{O}_{C_x}).$$

As we have seen before, this map is surjective, so that φ is a submersion whence a smooth morphism.

ii) If we work in the setting of Artin and Deligne-Mumford stacks, we can sharpen the second statement of Theorem 3.1.2. We will do this in Section 3.4 below.

3.2. Existence of universal families. The aim of this section is to prove:

Theorem 3.2.1. *Suppose χ and r are coprime. Then, every point $x \in \mathfrak{HC}(g;\chi,r)^\star$ possesses an étale neighborhood U, such that there exists a universal family over U.*

3.2.1. The universal curve over $\mathfrak{HC}(g;\chi,r)^{\mathrm{s}}$. Let $\mathcal{C}_{\widehat{\mathfrak{H}}} \hookrightarrow \widehat{\mathfrak{H}} \times \mathfrak{G}$ be the universal closed subscheme. It is clearly invariant under the $\mathrm{SL}(V^{\chi_l})$-action on $\widehat{\mathfrak{H}} \times \mathfrak{G}$. On $\widehat{\mathfrak{H}}$, we choose a line bundle $\mathcal{O}_{\widehat{\mathfrak{H}}}(1) := \mathfrak{L}_{a,m}$ where m and $a \gg 0$ are chosen in such a way that the conclusion of the main theorem holds, and on $\mathfrak{G}$ the usual ample line bundle $\mathcal{O}_{\mathfrak{G}}(1)$. Then, for positive integers s and t,

$$\mathcal{L}_{s,t} := \Big(\pi^*_{\widehat{\mathfrak{H}}}\mathcal{O}_{\widehat{\mathfrak{H}}}(s) \otimes \pi^*_{\mathfrak{G}}\mathcal{O}_{\mathfrak{G}}(t)\Big)_{|\mathcal{C}_{\widehat{\mathfrak{H}}}}$$

is an ample $\mathrm{SL}(V^{\chi_l})$-linearized line bundle on $\mathcal{C}_{\widehat{\mathfrak{H}}}$. If we choose s/t large enough, Proposition 1.1.1 grants that points in the preimage $\mathcal{C}^{\mathrm{s}}_{\widehat{\mathfrak{H}}}$ of the points in $\widehat{\mathfrak{H}}$ which are stable w.r.t. the linearization in $\mathfrak{L}_{a,m}$ are stable w.r.t. the linearization in $\mathcal{L}_{s,t}$. Therefore, the geometric quotient

$$\mathcal{C}_{\mathfrak{HC}^{\mathrm{s}}} := \mathcal{C}^{\mathrm{s}}_{\widehat{\mathfrak{H}}} /\!\!/ \mathrm{SL}(V^{\chi_l})$$

exists. Here, we have set $\mathfrak{HC}^{\mathrm{s}} := \mathfrak{HC}(g;\chi,r)^{\mathrm{s}}$. Moreover, there is a natural morphism

$$\sigma\colon\ \mathcal{C}_{\mathfrak{HC}^{\mathrm{s}}} \longrightarrow \mathfrak{HC}^{s}.$$

We call $\mathcal{C}_{\mathfrak{HC}^{s}}$ — abusively — the *universal curve*. For an H-stable pair $(C,\mathcal{E})$, define

$$\mathrm{Aut}(C,\mathcal{E}) := \big\{\alpha\colon C \xrightarrow{\cong} C \,|\, \alpha^*\mathcal{E} \cong \mathcal{E}\,\big\}.$$

From the GIT set up, it follows that the $\mathrm{PGL}(V^{\chi_l})$-stabilizer of a stable point $(C, q\colon V^{\chi_l} \otimes \mathcal{O}_C \longrightarrow \mathcal{E})$ in $\widehat{\mathfrak{H}}$ identifies with the group $\mathrm{Aut}(C,\mathcal{E})$. Thus, we see:

Corollary 3.2.2. *For any point $[C,\mathcal{E}] \in \mathfrak{HC}^{s}$, the fibre $\sigma^{-1}[C,\mathcal{E}]$ is isomorphic to the curve $C/\mathrm{Aut}(C,\mathcal{E})$.*

In particular, we may hope for a universal family only over the open subset $\mathfrak{HC}(g;\chi,r)^\star$.

3.2.2. Proof of Theorem 3.2.1. Now, let $x \in \mathfrak{HC}^\star := \mathfrak{HC}(g;\chi,r)^\star$. Then, x has an étale neighborhood U, such that the family $\sigma_U\colon \mathcal{C}_U := \mathcal{C}_{\mathfrak{HC}^\star} \times_{\mathfrak{HC}^\star} U \longrightarrow U$ possesses a section which meets every fibre in a smooth point. Then, $\mathcal{L}_U := \mathcal{O}_{\mathcal{C}_U}\big(\sigma_U(U)\big)$ is a relative ample invertible sheaf. Let $\mathfrak{U}^\star \subset \widehat{\mathfrak{H}}$ be the $\mathrm{SL}(V^{\chi_l})$-invariant open subset which parameterizes the strictly H-stable points, and let $\psi\colon \mathfrak{U}^\star \longrightarrow \mathfrak{HC}^\star$ be the quotient morphism. Then, since Mumford's GIT supplies universal geometric quotients, the map $\psi_U\colon \mathcal{U} := \mathfrak{U}^\star \times_{\mathfrak{HC}^\star} U \longrightarrow U$ is a geometric quotient, too. For the same reason, the vertical maps in the following Cartesian diagram are both geometric quotients:

$$\begin{array}{ccc} \mathcal{C}_{\mathcal{U}} := \mathcal{C}_{\mathfrak{HC}^\star} \times_{\mathfrak{HC}^\star} \mathcal{U} & \xrightarrow{\sigma_{\mathcal{U}}} & \mathcal{U} \\ \downarrow{\scriptstyle \psi_{\mathcal{C}_U}} & & \downarrow{\scriptstyle \psi_U} \\ \mathcal{C}_U & \longrightarrow & U. \end{array}$$

Define $\mathcal{L}_{\mathcal{U}}$ as the pullback of $\mathcal{L}_U$ under the quotient morphism $\psi_{\mathcal{C}_U}$. This is obviously an $\mathrm{SL}(V^{\chi_l})$-linearized relative ample invertible sheaf.

We have seen that the $\mathrm{SL}(V^{\chi_l})$-stabilizer of a point $u \in \mathcal{U}$ corresponding to a strictly H-stable object consists exactly of the scalar matrices (Remark 3.1.1). The same holds for the points $x \in \mathcal{C}_{\mathcal{U}}$. Let $\mu_{\chi_l} \subset \mathbb{C}^*$ be the subgroup generated by a primitive χ_l-th root of unity ζ. For any point $u \in \mathcal{U}$ or $x \in \mathcal{C}_{\mathcal{U}}$, the $\mathrm{SL}(V^{\chi_l})$-stabilizer now identifies with μ_{χ_l}. If $\mathcal{F}$ is an $\mathrm{SL}(V^{\chi_l})$-linearized sheaf on $\mathcal{U}$ or $\mathcal{C}_{\mathcal{U}}$, we will say that it is of *weight* k, if μ_{χ_l} acts by $\zeta^k \cdot \mathrm{id}_{\mathcal{F}\langle x\rangle}$ for all $x \in \mathcal{U}$ or $x \in \mathcal{C}_{\mathcal{U}}$, respectively. For example, $\mathcal{L}_{\mathcal{U}}$ is of weight zero.

By construction, we have the quotient $q_{\mathcal{U}}\colon V^{\chi_l} \otimes \mathcal{O}_{\mathcal{U}} \longrightarrow \mathcal{E}_{\mathcal{U}}$, and the question we have to answer is whether $\mathcal{E}_{\mathcal{U}}$ descends — possibly after tensorizing it with the pullback of a line bundle on $\mathcal{U}$ — to the quotient $\mathcal{C}_U$. By Kempf's descent lemma (see [**4**]), an $\mathrm{SL}(V^{\chi_l})$-linearized vector bundle $\mathcal{E}$ on $\mathcal{C}_{\mathcal{U}}$ descends to the quotient, if and only if it is of weight zero. Now, the sheaf $\mathcal{E}_{\mathcal{U}}$ is of weight one. Thus, our task will be to find an $\mathrm{SL}(V^{\chi_l})$-linearized invertible sheaf $\mathcal{A}_{\mathcal{U}}$ of weight one. Then, $\mathcal{E}_{\mathcal{U}} \otimes \sigma_{\mathcal{U}}^*\big(\mathcal{A}_{\mathcal{U}}^\vee\big)$ will descend to $\mathcal{C}_U$, and we will be done.

For any m, the sheaf $\mathcal{E}_{\mathcal{U}}[m] := \mathcal{E}_{\mathcal{U}} \otimes \mathcal{L}_{\mathcal{U}}^{\otimes m}$ is an $\mathrm{SL}(V^{\chi_l})$-linearized vector bundle of weight one, and, if m is sufficiently large,

$$\mathcal{F}_m := \sigma_{\mathcal{U}*}\big(\mathcal{E}_{\mathcal{U}}[m]\big)$$

will be an $\mathrm{SL}(V^{\chi_l})$-linearized vector bundle of rank $\chi_l + r \cdot m$ and weight one. Then, for $m \gg 0$,

$$\mathcal{N}_{\mathcal{U}} := \det(\mathcal{F}_{m+1}) \otimes \det(\mathcal{F}_m)^{\vee}$$

is a line bundle of weight r. Since χ and r and thus also $c := \chi_l + r \cdot m$ and r are coprime, we may find integers α and β with $\alpha \cdot c + \beta \cdot r = 1$, so that

$$\mathcal{A}_{\mathcal{U}} := \det(\mathcal{F}_m)^{\otimes \alpha} \otimes \mathcal{N}_{\mathcal{U}}^{\otimes \beta}$$

will indeed have weight one. q.e.d.

3.3. The fibres of the morphism κ_g. As before, let $\kappa_g \colon \mathfrak{H}\mathfrak{C}(g; \chi, r) \longrightarrow \overline{\mathfrak{M}}_g$ be the natural morphism. By $\overline{\mathfrak{M}}_g^{\star}$, we denote the quasi-projective moduli space of automorphism free stable curves. In this section, we want to establish:

Theorem 3.3.1. *For any stable curve C_0 without automorphisms, the variety*

$$\kappa_g^{-1}\big([C_0]\big) \cap \mathfrak{H}\mathfrak{C}^{\mathrm{s}}(g; \chi, r)$$

has only analytical normal crossings as singularities.

We will follow the strategy of Gieseker's paper [**6**] in order to prove the result.

3.3.1. A family of semistable curves with fixed stable model. Let C_0 be a fixed stable curve. Then, by the results of Deligne and Mumford [**3**], C_0 has a universal deformation over $\mathcal{M} := \mathrm{Spec}\mathbb{C}[\![t_1, \ldots, t_N]\!]$ with $N := \dim\big(\mathrm{Ext}^1(\Omega_{C_0}, \mathcal{O}_{C_0})\big) = 3g - 3$. Moreover, let $\mathcal{M}_{\mathrm{loc}} = \mathrm{Spec}\mathbb{C}[\![u_1, \ldots, u_M]\!]$ be the deformation space of the nodes of C_0. Finally, there is the morphism $\mathcal{M} \longrightarrow \mathcal{M}_{\mathrm{loc}}$ normalized in such a way that $\mathbb{C}[\![u_1, \ldots, u_M]\!] \longrightarrow \mathbb{C}[\![t_1, \ldots, t_N]\!]$ is given by $u_i \longmapsto t_i$, $i = 1, \ldots, M$, and $t_i = 0$ is the equation of the i-th node of C_0, $i = 1, \ldots, M$.

Next, let C be a semistable curve the stable model of which is C_0, and let $\pi \colon C \longrightarrow C_0$ be the contraction map. Let $R_1, \ldots, R_S$ be the maximal connected chains of rational curves which are contracted by π. We label the nodes $c_1, \ldots, c_M$ of C_0 in such a way that $\{c_i\} = \pi(R_i)$, $i = 1, \ldots, S$. We then define

$$\mathcal{N} := \big\{ t_i = 0,\ i = S+1, \ldots, N \big\} \subset \mathcal{M}.$$

By restriction of the universal family over $\mathcal{M}$ to $\mathcal{N}$, we obtain the family $\sigma_{\mathcal{N}} \colon \mathcal{C}_{\mathcal{N}} \longrightarrow \mathcal{N}$ which is smooth outside the nodes "which don't move",

i.e., outside the nodes $c_{S+1},\dots,c_M$. For $i=1,\dots,S$, let $d_{i,j}$, $j=1,\dots,\iota_i$, be the nodes of C mapping under π to c_i. Define

$$\mathcal{Q} := \operatorname{Spec}\bigl(\mathbb{C}[\![\, x_{i,j};\ i=1,\dots,S, j=1,\dots,\iota_i \,]\!]\bigr).$$

The homomorphism

$$\begin{aligned}\varphi^*\colon\ \mathbb{C}[\![\, t_1,\dots,t_S \,]\!] &\longrightarrow \mathbb{C}[\![\, x_{i,j};\ i=1,\dots,S, j=1,\dots,\iota_i \,]\!]\\ t_i &\longmapsto x_{i,1}\cdot\dots\cdot x_{i,\iota_i},\quad i=1,\dots,S,\end{aligned}$$

defines a morphism $\varphi\colon \mathcal{Q}\longrightarrow\mathcal{N}$. The pull back of the family $\mathcal{C}_{\mathcal{N}}$ provides us with the family $\sigma_{\mathcal{Q}}\colon \mathcal{C}_{\mathcal{Q}}\longrightarrow\mathcal{Q}$ of stable curves. Near the i-th node, the family $\mathcal{C}_{\mathcal{Q}}$ is defined by the equation

$$y_i\cdot z_i - x_{i,1}\cdot\dots\cdot x_{i,\iota_i}=0,$$

for appropriate parameters y_i and z_i, $i=1,\dots,S$.

Now, let

$$\vartheta\colon\quad \widehat{\mathcal{C}}_{\mathcal{Q}}\longrightarrow\mathcal{Q}$$

be the blow up of the curve $\mathcal{C}_{\mathcal{Q}}$ along the ideal generated by y_1 and $x_{1,1}$. Near the node c_1, we may embed $\mathcal{C}_{\mathcal{Q}}$ into

$$\mathbb{A} = \operatorname{Spec}\bigl(\mathbb{C}[\![\, y_1, z_1, x_{i,j};\ i=1,\dots,S, j=1,\dots,\iota_i \,]\!]\bigr).$$

The blow up $\widehat{\mathbb{A}}$ of $\mathbb{A}$ along y_1 and $x_{1,1}$ is the scheme

$$\widehat{\mathbb{A}} := \Bigl\{ \bigl((y_1, z_1, x_{i,j};\ i=1,\dots,S), [w_0:w_1]\bigr) \,\Big|\, y_1\cdot w_0 = x_{1,1}\cdot w_1 \Bigr\} \subset \mathbb{A}\times\mathbb{P}_1.$$

One checks that the strict transform of $\mathcal{C}_{\mathcal{Q}}$ is given in the chart $w_0=1$ by the equation

$$w_1\cdot z_1 - x_{i,2}\cdot\dots\cdot x_{i,\iota_1}=0$$

and in the chart $w_1=1$ by

$$w_0\cdot y_1 - x_{i,1}=0.$$

We may now iterate the blow up, i.e., blow up $\widehat{\mathcal{C}}_{\mathcal{Q}}$ at the ideal generated by w_1 and $x_{i,2}$ and so on and perform the same procedure at the other nodes, too, in order to construct a flat family

$$\widetilde{\sigma}_{\mathcal{Q}}\colon\ \widetilde{\mathcal{C}}_{\mathcal{Q}}\longrightarrow\mathcal{Q}$$

with C as the fibre over the origin. By construction, $\widetilde{\mathcal{C}}_{\mathcal{Q}}$ is given near the node $d_{i,j}$ by the equation

$$y_{i,j}\cdot z_{i,j} - x_{i,j}=0$$

for suitable local parameters $y_{i,j}$ and $z_{i,j}$, $i=1,\dots,S$, $j=1,\dots,\iota_i$. In particular, it is near $d_{i,j}$ isomorphic to the miniversal deformation

of that node, and $x_{i,j} = 0$ is the locus where the node $d_{i,j}$ "is kept", $i = 1, \dots, S$, $j = 1, \dots, \iota_i$.

By $\mathcal{X} \hookrightarrow \mathcal{Q}$, we denote the subscheme defined by the equations

$$x_{i,1} \cdot \dots \cdot x_{i,\iota_i} = 0, \quad i = 1, \dots, S.$$

The scheme $\mathcal{X}$ obviously has only analytical normal crossing singularities.

3.3.2. The versality property of $\widetilde{\mathcal{C}}_{\mathcal{Q}}$. The family $\widetilde{\sigma}_{\mathcal{Q}} \colon \widetilde{\mathcal{C}}_{\mathcal{Q}} \longrightarrow \mathcal{Q}$ together with the $\mathcal{Q}$-morphism $\pi_{\mathcal{Q}} \colon \widetilde{\mathcal{C}}_{\mathcal{Q}} \longrightarrow \mathcal{C}_{\mathcal{Q}}$ has the following property:

Proposition 3.3.2. *Let $\tau \colon \mathcal{S} := \operatorname{Spec}(A) \longrightarrow \mathcal{N}$ be an $\mathcal{N}$-scheme where A is a local Artin algebra. Suppose that there is a flat family $\sigma_{\mathcal{S}} \colon \mathcal{C}_{\mathcal{S}} \longrightarrow \mathcal{S}$ of semistable curves over $\mathcal{S}$ together with an $\mathcal{S}$-morphism $\pi_{\mathcal{S}} \colon \mathcal{C}_{\mathcal{S}} \longrightarrow \tau^*\mathcal{C}_{\mathcal{N}}$. Suppose that the closed point s of $\mathcal{S}$ maps to the origin of $\mathcal{N}$ and that $\pi_{\mathcal{S}|\sigma_{\mathcal{S}}^{-1}(s)}$ equals the map π.*

Then, there is an $\mathcal{N}$-morphism $\psi \colon \mathcal{S} \longrightarrow \mathcal{Q}$, such that $\mathcal{C}_{\mathcal{S}}$ is over $\tau^*\mathcal{C}_{\mathcal{N}} = \psi^*\mathcal{C}_{\mathcal{Q}}$ isomorphic to $\psi^*\widetilde{\mathcal{C}}_{\mathcal{Q}}$.

Proof. First note that the homomorphism

$$H^1\big(\mathcal{H}om(\Omega^1_C, \mathcal{O}_C)\big) \longrightarrow H^1\big(\mathcal{H}om(\pi^*\Omega^1_{C_0}, \mathcal{O}_C)\big)$$

is injective. In fact, as the computations in [**6**] used for proving the analogous statement (Corollary 4.4) are completely local, they apply to our situation, too. The rest of the proof may now be copied from [**6**], proof of Proposition 4.5. q.e.d.

3.3.3. Proof of Theorem 3.3.1. Let $\mathfrak{H}^\dagger(g; \chi, r)$ be as in Section 3.1. There is a morphism

$$\kappa^\dagger \colon \ \mathfrak{H}^\dagger(g; \chi, r) \longrightarrow \overline{\mathfrak{M}}_g,$$

and we define

$$\mathfrak{H}_{C_0} := \kappa^{\dagger -1}\big([C_0]\big).$$

Let $\sigma_{\mathfrak{H}_{C_0}} \colon \mathcal{C}_{\mathfrak{H}_{C_0}} \longrightarrow \mathfrak{H}_{C_0}$ be the restriction of the universal family. A suitably high power of the relative dualizing sheaf $\omega_{\mathcal{C}_{\mathfrak{H}_{C_0}}/\mathfrak{H}_{C_0}}$ will yield a morphism

$$\begin{array}{ccc} \mathcal{C}_{\mathfrak{H}_{C_0}} & \xrightarrow{\ \iota\ } & \mathcal{P} \\ {\scriptstyle \sigma_{\mathfrak{H}_{C_0}}}\big\downarrow & & \big\downarrow \\ \mathfrak{H}_{C_0} & = & \mathfrak{H}_{C_0}, \end{array}$$

where $\mathcal{P}$ is some projective bundle. The image of ι is a flat family of stable curves, all of which are isomorphic to C_0. As C_0 does not have any automorphisms, this family is trivial. Let

$$\varphi_{\mathfrak{H}_{C_0}}\colon\ \mathcal{C}_{\mathfrak{H}_{C_0}} \longrightarrow C_0 \times \mathfrak{H}_{C_0}$$

be the induced morphism. Let $x \in \mathfrak{H}_{C_0}$ be a point and $\mathfrak{U}$ its formal neighborhood. Denote the fibre of the family $\mathcal{C}_{\mathfrak{H}_{C_0}}$ over x by C, and let $\tau\colon \mathfrak{U} \longrightarrow \mathcal{N}$ be the constant map to the origin. Finally, define

$$\sigma_{\mathfrak{U}}\colon\ \mathcal{C}_{\mathfrak{U}} \longrightarrow \mathfrak{U}$$

as the restriction of the family $\mathcal{C}_{\mathfrak{H}_{C_0}}$. By Proposition 3.3.2, there is a morphism

$$\psi\colon\ \mathfrak{U} \longrightarrow \mathcal{X} \hookrightarrow \mathcal{Q}.$$

Our observation in Remark 3.1.4 i) implies that the morphism ψ is smooth. Therefore, $\mathfrak{H}_{C_0}$ has only analytic normal crossing singularities at x. Finally, as C_0 does not have any automorphisms, the quotient morphism

$$\mathfrak{H}_{C_0} \cap \mathfrak{H}^0(g;\chi,r)^{\mathrm{s}} \longrightarrow \kappa_g^{-1}\big([C_0]\big) \cap \mathfrak{H}\mathfrak{C}^{\mathrm{s}}(g;\chi,r)$$

is a principal $\mathrm{PGL}(V^{\chi_\iota})$-bundle. This proves the theorem. q.e.d.

Remark 3.3.3. If the automorphism group of C_0 is nontrivial, the same arguments show that the fibre $\kappa_g^{-1}\big([C_0]\big) \cap \mathfrak{H}\mathfrak{C}^{\mathrm{s}}(g;\chi,r)$ is the quotient of a variety with analytical normal crossings by the automorphism group of C_0. However, even if C_0 is a smooth curve, the action of the group $\mathrm{Aut}(C_0)$ on the moduli space of semistable bundles has not been thoroughly studied, so far. We refer the reader to the paper [**1**] for information concerning the action of a single automorphism.

3.4. The moduli stacks. In the following, let $\underline{\mathrm{Schemes}}_{\mathbb{C}}$ be the category of schemes of finite type over $\mathbb{C}$, viewed as a 2-category, and $\underline{\mathrm{Groupoids}}$ the 2-category of groupoids, that is the 2-category whose objects are groupoids, i.e., categories in which all morphisms are isomorphisms, the 1-morphisms are functors and the 2-morphisms are natural transformations between functors.

As usual, defining presheaves of groupoids and establishing isomorphisms between them involves many schemes characterized by some universal property, such as fibre products. However, these schemes will be defined only up to canonical isomorphy and there is no equivalence relation which compensates for this. Thus, we have to fix *a priori* a representative for every such isomorphy class. In the following, we assume to have done this.

Next, we introduce the 2-functors

$$\mathcal{HC}^{(\mathrm{s})\mathrm{s}}_{g/\chi/r}\colon\ \underline{\mathrm{Schemes}}_{\mathbb{C}} \longrightarrow \underline{\mathrm{Groupoids}}.$$

For any scheme S of finite type, the objects of $\mathcal{HC}^{(\mathrm{s})\mathrm{s}}_{g/\chi/r}(S)$ are families $(\mathcal{C}_S, \mathcal{E}_S)$ of H-(semi)stable vector bundles as before, the morphisms between $(\mathcal{C}'_S, \mathcal{E}'_S)$ and $(\mathcal{C}_S, \mathcal{E}_S)$ are pairs (φ_S, ψ_S), consisting of an S-isomorphism $\varphi_S\colon \mathcal{C}'_S \longrightarrow \mathcal{C}_S$ and an isomorphism $\psi_S\colon \mathcal{E}'_S \longrightarrow \varphi_S^*\mathcal{E}_S$. For any morphism $f\colon T \longrightarrow S$ pullback of families defines a natural transformation

$$\mathcal{HC}^{(\mathrm{s})\mathrm{s}}_{g/\chi/r}(f)\colon\ \mathcal{HC}^{(\mathrm{s})\mathrm{s}}_{g/\chi/r}(S) \longrightarrow \mathcal{HC}^{(\mathrm{s})\mathrm{s}}_{g/\chi/r}(T).$$

On the other hand, there are the quotient stacks $\left[\mathfrak{H}^{\mathrm{ss}}/\operatorname{GL}(V^{\chi_l})\right]$ and $\left[\mathfrak{H}^{\mathrm{s}}/\operatorname{GL}(V^{\chi_l})\right]$. Here, $\mathfrak{H}^{(\mathrm{s})\mathrm{s}}$ is the open part of $\mathfrak{H}^0(g;\chi_l,r)$ which parameterizes the (semi)stable objects. For any scheme S, the objects of $\left[\mathfrak{H}^{(\mathrm{s})\mathrm{s}}/\operatorname{GL}(V^{\chi_l})\right]$ are pairs $\left(\vartheta_S\colon \mathcal{P} \longrightarrow S, \eta_S\colon \mathcal{P} \longrightarrow \mathfrak{H}^{(\mathrm{s})\mathrm{s}}\right)$ where $\vartheta_S\colon \mathcal{P} \longrightarrow S$ is a principal $\operatorname{GL}(V^{\chi_l})$-bundle and η_S is an equivariant morphism. One has a natural notion of isomorphism and, as before, pullback defines the functor associated with a morphism $f\colon T \longrightarrow S$.

Theorem 3.4.1. *The presheaves $\mathcal{HC}^{(\mathrm{s})\mathrm{s}}_{g/\chi/r}$ and $\left[\mathfrak{H}^{(\mathrm{s})\mathrm{s}}/\operatorname{GL}(V^{\chi_l})\right]$ are isomorphic.*

Proof. The assertions amount to prove that, for every scheme S, the groupoids $\mathcal{HC}^{(\mathrm{s})\mathrm{s}}_{g/\chi/r}(S)$ and $\left[\mathfrak{H}^{(\mathrm{s})\mathrm{s}}/\operatorname{GL}(V^{\chi_l})\right](S)$ are equivalent.

First, suppose $\left(\vartheta_S\colon \mathcal{P} \longrightarrow S, \eta_S\colon \mathcal{P} \longrightarrow \mathfrak{H}^{(\mathrm{s})\mathrm{s}}\right)$ is an object of the groupoid $\left[\mathfrak{H}^{(\mathrm{s})\mathrm{s}}/\operatorname{GL}(V^{\chi_l})\right](S)$. Then, by means of pullback, the morphism η_S yields a $\operatorname{GL}(V^{\chi_l})$-invariant, $\mathcal{P}$-flat family of semistable curves of genus g

$$\mathcal{C}_{\mathcal{P}} \hookrightarrow \mathcal{P} \times \mathfrak{G}$$

and a $\operatorname{GL}(V^{\chi_l})$-linearized vector bundle $\mathcal{E}_{\mathcal{P}}$ on $\mathcal{C}_{\mathcal{P}}$. Now, as S is the geometric quotient of $\mathcal{P}$ by the $\operatorname{GL}(V^{\chi_l})$-action, the same arguments which were used in Section 3.2 show that we have the curve $\pi_S\colon \mathcal{C}_S \longrightarrow S$. This time, as there are no stabilizers present, every fibre of π_S is indeed a semistable curve of genus g. We have to check that the family $\mathcal{C}_S$ is indeed S-flat. For this, choose a Zariski-open set $U \subset S$ over which the principal bundle $\mathcal{P}$ is trivial (this is possible, since we are dealing with $\operatorname{GL}(V^{\chi_l})$.) Set $V := U \times \operatorname{GL}(V^{\chi_l})$. We will show that $\mathcal{C}_{S|V}$ is in fact $\operatorname{GL}(V^{\chi_l})$-equivariantly a product $\mathcal{C}_U \times \operatorname{GL}(V^{\chi_l})$ for some U-flat

family $\mathcal{C}_U$. This clearly settles the affair. By the $\mathrm{GL}(V^{\chi_l})$-equivariance of ϑ_S, we have the commutative diagram

$$\begin{array}{ccccc}
(x,g,h) \in & U \times \mathrm{GL}(V^{\chi_l}) \times \mathrm{GL}(V^{\chi_l}) & \xrightarrow{\eta_{S|V} \times \mathrm{id}} & \mathfrak{H}^{(\mathrm{s})\mathrm{s}} \times \mathrm{GL}(V^{\chi_l}) \\
\downarrow & \downarrow & & \downarrow \\
(x, g \cdot h) \in & U \times \mathrm{GL}(V^{\chi_l}) & \xrightarrow{\eta_{S|V}} & \mathfrak{H}^{(\mathrm{s})\mathrm{s}}.
\end{array}$$

Now, consider the map

$$\begin{array}{rcl}
V = U \times \mathrm{GL}(V^{\chi_l}) & \longrightarrow & \big(U \times \mathrm{GL}(V^{\chi_l})\big) \times \mathrm{GL}(V^{\chi_l}) \\
(x,g) & \longmapsto & (x, \mathrm{id}_{V^{\chi_l}}, g).
\end{array}$$

Define $\eta^0 \colon U \longrightarrow \mathfrak{H}^{(\mathrm{s})\mathrm{s}}$ by $\eta^0(x) := \eta_{S|V}(x, \mathrm{id}_{V^{\chi_l}})$. The content of the diagram before may then be summarized by the suggestive formula

$$\eta_{S|V}(x,g) = \eta^0(x) \cdot g.$$

Finally, the morphism $\widetilde{\eta} \colon V \longrightarrow \mathfrak{H}^{(\mathrm{s})\mathrm{s}}$, $(x,g) \longmapsto \eta^0(x) \cdot g$, is by definition of the group action obtained in the following manner: let $(\mathcal{C}_U, \mathcal{E}_U)$ be the family induced by the morphism η^0. Note that we get even a quotient

$$q_U \colon V^{\chi_l} \otimes \mathcal{O}_{\mathcal{C}_U} \longrightarrow \mathcal{E}_U.$$

Let

$$\Gamma \colon V \otimes \mathcal{O}_{\mathrm{GL}(V^{\chi_l})} \longrightarrow V \otimes \mathcal{O}_{\mathrm{GL}(V^{\chi_l})}$$

be the tautological automorphism. Then, define the following quotient on $\mathcal{C}_U \times \mathrm{GL}(V^{\chi_l})$:

$$q_V \colon \; V \otimes \mathcal{O}_{\mathcal{C}_U \times \mathrm{GL}(V^{\chi_l})} \xrightarrow{\pi^*_{\mathrm{GL}(V^{\chi_l})}(\Gamma)} V \otimes \mathcal{O}_{\mathcal{C}_U \times \mathrm{GL}(V^{\chi_l})} \xrightarrow{\pi^*_{\mathcal{C}_U}(q_U)} \pi^*_{\mathcal{C}_U} \mathcal{E}_U.$$

This quotient defines an embedding $\mathcal{C}_U \times \mathrm{GL}(V^{\chi_l}) \longrightarrow V \times \mathfrak{G}$, and the resulting morphism $V \longrightarrow \mathfrak{H}^{(\mathrm{s})\mathrm{s}}$ is just $\widetilde{\eta} = \eta_{S|V}$. In particular, $\mathcal{C}_{S|V}$ is $\mathrm{GL}(V^{\chi_l})$-equivariantly isomorphic to $\mathcal{C}_U \times \mathrm{GL}(V^{\chi_l})$, as asserted. By Kempf's descent lemma, the bundle $\mathcal{E}_{\mathcal{P}}$ descends to $\mathcal{C}_S$, so that $(\mathcal{C}_S, \mathcal{E}_S)$ is an object of $\mathcal{HC}^{(\mathrm{s})\mathrm{s}}_{g/\chi/r}(S)$. An isomorphism in $\big[\mathfrak{H}^{(\mathrm{s})\mathrm{s}} / \mathrm{GL}(V^{\chi_l})\big](S)$ will clearly lead to a unique isomorphism in $\mathcal{HC}^{(\mathrm{s})\mathrm{s}}_{g/\chi/r}(S)$.

Now, suppose we are given a scheme S and a family $(\pi_S\colon \mathcal{C}_S \longrightarrow S, \mathcal{E}_S)$ of H-(semi)stable vector bundles. Then, we know that

$$\vartheta_S\colon \mathcal{I}som(V^{\chi_l}\otimes\mathcal{O}_S, \pi_{S*}\mathcal{E}_S) \longrightarrow S$$

is a principal $\mathrm{GL}(V^{\chi_l})$-bundle. On $\mathcal{P} := \mathcal{I}som(V^{\chi_l}\otimes\mathcal{O}_S, \pi_{S*}\mathcal{E}_S)$, there is the tautological isomorphism

$$\tau_{\mathcal{P}}\colon\ V^{\chi_l}\otimes\mathcal{O}_{\mathcal{P}} \longrightarrow \vartheta_S^*\pi_{S*}\mathcal{E}_S.$$

Now, form the cartesian diagram

$$\begin{array}{ccc} \mathcal{C}_{\mathcal{P}} & \xrightarrow{\psi} & \mathcal{C}_S \\ {\scriptstyle\pi_{\mathcal{P}}}\downarrow & & \downarrow{\scriptstyle\pi_S} \\ \mathcal{P} & \xrightarrow{\vartheta_S} & S. \end{array}$$

By flat base change

$$\vartheta_S^*\pi_{S*}\mathcal{E}_S \cong \pi_{\mathcal{P}*}\psi^*\mathcal{E}_S.$$

If we set $\mathcal{E}_{\mathcal{P}} := \psi^*\mathcal{E}_S$, then

$$V^{\chi_l}\otimes\mathcal{O}_{\mathcal{C}_{\mathcal{P}}} \xrightarrow{\pi_{\mathcal{P}}^*\tau_{\mathcal{P}}} \pi_{\mathcal{P}}^*\pi_{\mathcal{P}*}\mathcal{E}_{\mathcal{P}} \xrightarrow{\mathrm{ev}} \mathcal{E}_{\mathcal{P}}$$

defines a morphism $\mathcal{P} \longrightarrow \mathfrak{H}^{(\mathrm{s})\mathrm{s}}$ which is by construction $\mathrm{GL}(V^{\chi_l})$-equivariant. Again, isomorphisms in the category $\mathcal{HC}^{(\mathrm{s})\mathrm{s}}_{g/\chi/r}(S)$ will lead canonically to isomorphisms in the groupoid $\big[\mathfrak{H}^{(\mathrm{s})\mathrm{s}}/\,\mathrm{GL}(V^{\chi_l})\big](S)$.

The two operations just introduced clearly establish the desired equivalence of categories. q.e.d.

Now, by the results of Section 3.1, we know that the schemes $\mathfrak{H}^{\mathrm{ss}}$ and $\mathfrak{H}^{s}$ are smooth. Moreover, the quotient map $\mathfrak{H}^{(\mathrm{s})\mathrm{s}} \longrightarrow \big[\mathfrak{H}^{(\mathrm{s})\mathrm{s}}/\,\mathrm{GL}(V^{\chi_l})\big]$ is smooth, whence $\big[\mathfrak{H}^{\mathrm{ss}}/\,\mathrm{GL}(V^{\chi_l})\big]$ and $\big[\mathfrak{H}^{\mathrm{s}}/\,\mathrm{GL}(V^{\chi_l})\big]$ are a smooth Artin stack and a smooth Deligne-Mumford stack, respectively.

Corollary 3.4.2. *The Hilbert compactification $\mathcal{HC}^{\mathrm{ss}}_{g/\chi/r}$ is a smooth Artin stack, and its open substack $\mathcal{HC}^{\mathrm{s}}_{g/\chi/r}$ is a smooth Deligne-Mumford stack.*

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Universität Duisburg-Essen
Fachbereich 6: Mathematik und Informatik
D-45117 Essen, Deutschland
E-mail address: alexander.schmitt@uni-essen.de