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# A PROOF OF A CONJECTURE OF MARIÑO-VAFA ON HODGE INTEGRALS

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## Abstract

We prove a remarkable formula for Hodge integrals conjectured by Mariño and Vafa, 2002, based on large $N$ duality, using functorial virtual localization on certain moduli spaces of relative stable morphisms.

## 1. Introduction

Let $\overline{\mathcal{M}}_{g,n}$ denote the Deligne-Mumford moduli stack of stable curves of genus $g$ with $n$ marked points. Let $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ be the universal curve, and let $\omega_\pi$ be the relative dualizing sheaf. The Hodge bundle

$$\mathbb{E} = \pi_* \omega_\pi$$

is a rank $g$ vector bundle over $\overline{\mathcal{M}}_{g,n}$ whose fiber over $[(C, x_1, \ldots, x_n)] \in \overline{\mathcal{M}}_{g,n}$ is $H^0(C, \omega_C)$. Let $s_i : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n+1}$ denote the section of $\pi$ which corresponds to the $i$-th marked point, and let

$$\mathbb{L}_i = s_i^* \omega_\pi$$

be the line bundle over $\overline{\mathcal{M}}_{g,n}$ whose fiber over $[(C, x_1, \ldots, x_n)] \in \overline{\mathcal{M}}_{g,n}$ is the cotangent line $T^*_{x_i} C$ at the $i$-th marked point $x_i$. A Hodge integral is an integral of the form

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{j_1} \cdots \psi_n^{j_n} \lambda_1^{k_1} \cdots \lambda_g^{k_g}$$

where $\psi_i = c_1(\mathbb{L}_i)$ is the first Chern class of $\mathbb{L}_i$, and $\lambda_j = c_j(\mathbb{E})$ is the $j$-th Chern class of the Hodge bundle.

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Hodge integrals arise in the calculations of Gromov-Witten invariants by localization techniques [14, 8]. The explicit evaluation of Hodge integrals is a difficult problem. The Hodge integrals involving only $\psi$ classes can be computed recursively by Witten's conjecture [26] proved by Kontsevich [13]. Algorithms of computing Hodge integrals are described in [3].

In [23], M. Mariño and C. Vafa obtained a closed formula for a generating function of certain open Gromov-Witten invariants, some of which has been reduced to Hodge integrals by localization techniques which are not fully clarified mathematically. This leads to a conjectural formula of Hodge integrals by comparing with the calculations in [12]. To state this formula, we introduce some notation. Let

$$\Lambda_g^\vee(u) = u^g - \lambda_1 u^{g-1} + \cdots + (-1)^g \lambda_g$$

be the Chern polynomial of $\mathbb{E}^\vee$, the dual of the Hodge bundle. For a partition $\mu$ given by

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{l(\mu)} > 0,$$

let $|\mu| = \sum_{i=1}^{l(\mu)} \mu_i$, and define

$$\begin{aligned}
\mathcal{C}_{g,\mu}(\tau) = &-\frac{\sqrt{-1}^{|\mu|+l(\mu)}}{|\operatorname{Aut}(\mu)|}(\tau(\tau+1))^{l(\mu)-1}\prod_{i=1}^{l(\mu)}\frac{\prod_{a=1}^{\mu_i-1}(\mu_i\tau+a)}{(\mu_i-1)!} \\
&\cdot\int_{\overline{\mathcal{M}}_{g,l(\mu)}}\frac{\Lambda_g^\vee(1)\Lambda_g^\vee(-\tau-1)\Lambda_g^\vee(\tau)}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)}, \\
\mathcal{C}_\mu(\lambda;\tau) = &\sum_{g\geq 0}\lambda^{2g-2+l(\mu)}\mathcal{C}_{g,\mu}(\tau)
\end{aligned}$$

Note that

$$\int_{\overline{\mathcal{M}}_{0,l(\mu)}}\frac{\Lambda_0^\vee(1)\Lambda_0^\vee(-\tau-1)\Lambda_0^\vee(\tau)}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)} = \int_{\overline{\mathcal{M}}_{0,l(\mu)}}\frac{1}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)} = |\mu|^{l(\mu)-3}$$

for $l(\mu) \geq 3$, and we use this expression to extend the definition to the case $l(\mu) < 3$.

Introduce formal variables $p = (p_1, p_2, \ldots, p_n, \ldots)$, and define

$$p_\mu = p_{\mu_1}\cdots p_{\mu_{l(\mu)}}$$

for a partition $\mu$. Define generating functions

$$\mathcal{C}(\lambda;\tau;p) = \sum_{|\mu|\geq 1} \mathcal{C}_\mu(\lambda;\tau)p_\mu,$$
$$\mathcal{C}(\lambda;\tau;p)^\bullet = e^{\mathcal{C}(\lambda;\tau;p)}.$$

As pointed out in [23], by comparing computations in [23] with computations in [12], one obtains a conjectural formula for $\mathcal{C}_\mu(\tau)$. This formula is explicitly written down in [28]:

$$\mathcal{C}(\lambda;\tau;p) = \sum_{n\geq 1}\frac{(-1)^{n-1}}{n}\sum_{\mu}\left(\sum_{\cup_{i=1}^n \mu^i=\mu}\prod_{i=1}^n\sum_{|\nu^i|=|\mu^i|}\frac{\chi_{\nu^i}(C(\mu^i))}{z_{\mu^i}} \cdot e^{\sqrt{-1}(\tau+\frac{1}{2})\kappa_{\nu^i}\lambda/2}V_{\nu^i}(\lambda)\right)p_\mu, \tag{1}$$

$$\mathcal{C}(\lambda;\tau;p)^\bullet = \sum_{|\mu|\geq 0}\left(\sum_{|\nu|=|\mu|}\frac{\chi_\nu(C(\mu))}{z_\mu}e^{\sqrt{-1}(\tau+\frac{1}{2})\kappa_\nu\lambda/2}V_\nu(\lambda)\right)p_\mu, \tag{2}$$

where

$$V_\nu(\lambda) = \prod_{1\leq a<b\leq l(\nu)}\frac{\sin[(\nu_a-\nu_b+b-a)\lambda/2]}{\sin[(b-a)\lambda/2]} \cdot \frac{1}{\prod_{i=1}^{l(\nu)}\prod_{v=1}^{\nu_i}2\sin[(v-i+l(\nu))\lambda/2]}. \tag{3}$$

The right-hand side of (1) is actually some truncated version of the more general formula [23, (5.6)] given by Mariño and Vafa.

We now explain the notation on the right-hand sides of (1) and (2). For a partition $\mu$, $\chi_\mu$ denotes the character of the irreducible representation of $S_d$ indexed by $\mu$, where $d = |\mu| = \sum_{i=1}^{l(\mu)}\mu_i$. The number $\kappa_\mu$ is defined by

$$\kappa_\mu = |\mu| + \sum_i(\mu_i^2 - 2i\mu_i).$$

For each positive integer $i$,

$$m_i(\mu) = |\{j : \mu_j = i\}|.$$

Denote by $C(\nu)$ the conjugacy class of $S_d$ corresponding to the partition $\nu$, and by $\chi_\mu(C(\nu))$ the value of the character $\chi_\mu$ on the conjugacy class $C(\nu)$. Finally,

$$z_\mu = \prod_j m_j(\mu)! j^{m_j(\mu)}.$$

In this paper, we will call (1) the Mariño-Vafa formula. The third author proved in [27] some special cases of the Mariño-Vafa formula and found some applications [29, 30].

We now describe our approach to the Mariño-Vafa formula (1). Denote the right-hand sides of (1) and (2) by $R(\lambda;\tau;p)$ and $R(\lambda;\tau;p)^\bullet$ respectively. In [28], the third author proved the following two equivalent cut-and-join equations similar to the one satisfied by Hurwitz numbers (see [7], [15], [11, Section 15.2]):

**Theorem 1.**

(4)
$$\frac{\partial R}{\partial \tau} = \frac{\sqrt{-1}\lambda}{2} \sum_{i,j\geq 1} \left( ijp_{i+j} \frac{\partial^2 R}{\partial p_i \partial p_j} + ijp_{i+j} \frac{\partial R}{\partial p_i} \frac{\partial R}{\partial p_j} + (i+j)p_i p_j \frac{\partial R}{\partial p_{i+j}} \right),$$

(5)
$$\frac{\partial R^\bullet}{\partial \tau} = \frac{\sqrt{-1}\lambda}{2} \sum_{i,j\geq 1} \left( ijp_{i+j} \frac{\partial^2 R^\bullet}{\partial p_i \partial p_j} + (i+j)p_i p_j \frac{\partial R^\bullet}{\partial p_{i+j}} \right).$$

Here is a crucial observation: One can rewrite (5) as a sequence systems of ordinary equations, one for each positive integer $d$, hence if $\mathcal{C}(\lambda;\tau;p)^\bullet$ satisfies (5), then it is determined by the initial value $\mathcal{C}(\lambda;0;p)^\bullet$. To prove (1) or (2), it suffices to prove the following two statements:

(a) Equation (4) is satisfied by $\mathcal{C}(\lambda;\tau;p)$.

(b) $\mathcal{C}(\lambda;0;p) = R(\lambda;0;p)$.

Or equivalently,

(a′) Equation (5) is satisfied by $\mathcal{C}(\lambda;\tau;p)^\bullet$.

(b′) $\mathcal{C}(\lambda;0;p)^\bullet = R(\lambda;0;p)^\bullet$.

The generating function $\mathcal{C}(\lambda;0;p)$ of Hodge integrals has a closed formula [4, Theorem 2]. This closed formula is shown to be equal to $R(\lambda;0;p)$ in [28]. Therefore, the Mariño-Vafa formula (1) follows from the following theorem.

**Theorem 2.**

$$\frac{\partial \mathcal{C}}{\partial \tau} = \frac{\sqrt{-1}\lambda}{2} \sum_{i,j\geq 1} \left( ijp_{i+j} \frac{\partial^2 \mathcal{C}}{\partial p_i \partial p_j} + ijp_{i+j} \frac{\partial \mathcal{C}}{\partial p_i} \frac{\partial \mathcal{C}}{\partial p_j} + (i+j)p_i p_j \frac{\partial \mathcal{C}}{\partial p_{i+j}} \right). \tag{6}$$

Our earlier paper [21] contains an essentially complete proof of the Mariño-Vafa formula based on the approach described above. The purpose of this paper is to supply various computational details, and to present some related results. See [25] for another approach to the Mariño-Vafa formula.

The rest of this paper is organized as follows. In Section 2, we give a proof of the initial condition (b). In Section 3, we give a proof of Theorem 1. The materials in these two sections are already contained in [28] and some in [21], and included here for the convenience of the readers. In Section 4, we recall the moduli spaces of relative stable morphisms and obstruction bundles which will be used in the proof of Theorem 2. In Section 5, we use the graph notation to describe the torus fixed points in the moduli spaces introduced in Section 4. In Section 6, we prove Theorem 2 by functorial virtual localization. Details of virtual localization are given in Appendix A. In Section 7, we reproduce the cut-and-join equation of Hurwitz numbers (proved in [7], [15], [11, Section 15.2]) by virtual localization.

## 2. Initial condition

We recall some notations for partitions. For a partition $\nu$, let $\nu'$ denote its transpose. Define

$$n(\eta) = \sum_i (i-1)\eta_i = \sum_i \binom{\eta_i'}{2}. \tag{7}$$

The hook length of $\nu$ at the square $x$ located at the $i$-th row and $j$-th column is defined to be:

$$h(x) = \nu_i + \nu_j' - i - j + 1.$$

Then one has the following two identities [22, pp. 10 -11]:

$$\prod_{x\in\nu}(1-t^{h(x)}) = \frac{\prod_{i=1}^{l(\nu)}\prod_{j=1}^{\nu_i-i+l(\nu)}(1-t^j)}{\prod_{i<j}(1-t^{\nu_i-\nu_j-i+j)}}, \tag{8}$$

$$\sum_{x\in\nu} h(x) = n(\nu)+n(\nu')+|\nu|. \tag{9}$$

We first obtain a simple expression for $V_\nu(\lambda)$.

**Proposition 2.1.**

$$V_\nu(\lambda) = \frac{1}{2^l\prod_{x\in\nu}\sin[h(x)\lambda/2]}.$$

*Proof.* We rewrite the right-hand side of (8) as follows.

$$\begin{aligned}
RHS &= \frac{\prod_{i=1}^{l(\nu)}\prod_{j=1}^{-i+l(\nu)}(1-t^j)}{\prod_{i<j}(1-t^{\nu_i-\nu_j-i+j)}}\cdot\prod_{i=1}^{l(\nu)}\prod_{j=1-i+l(\nu)}^{\nu_i-i+l(\nu)}(1-t^j)\\
&= \frac{\prod_{i<j}(1-t^{j-i})}{\prod_{i<j}(1-t^{\nu_i-\nu_j-i+j)}}\cdot\prod_{i=1}^{l(\nu)}\prod_{j=1}^{\nu_i}(1-t^{j-i+l(\nu)})\\
&= t^{\left(\sum_{i<j}(j-i)-\sum_{i<j}(\nu_i-\nu_j-i+j)+\sum_{i=1}^{l(\nu)}\sum_{j=1}^{\nu_i}(j-i+l(\nu))\right)/2}\\
&\quad\cdot\frac{\prod_{i<j}(t^{-(j-i)/2}-t^{(j-i)/2})}{\prod_{i<j}(t^{-(\nu_i-\nu_j-i+j)/2}-t^{(\nu_i-\nu_j-i+j)/2})}\\
&\quad\cdot\prod_{i=1}^{l(\nu)}\prod_{j=1}^{\nu_i}(t^{-(j-i+l(\nu))/2}-t^{(j-i+l(\nu))/2}).
\end{aligned}$$

Now

$$\begin{aligned}
&\sum_{i<j}(j-i)-\sum_{i<j}(\nu_i-\nu_j-i+j)+\sum_{i=1}^{l(\nu)}\sum_{j=1}^{\nu_i}(j-i+l(\nu))\\
&= -\sum_{i<j}\nu_i+\sum_{i<j}\nu_j+\sum_{i=1}^{l(\nu)}\sum_{j=1}^{\nu_i}j-\sum_{i=1}^{l(\nu)}\sum_{j=1}^{\nu_i}i+\sum_{i=1}^{l(\nu)}\sum_{j=1}^{\nu_i}l(\nu)\\
&= -\sum_{i=1}^{l(\nu)}(l(\nu)-i)\nu_i+\sum_{j=1}^{l(\nu)}(j-1)\nu_j+\sum_{i=1}^{l(\nu)}\frac{\nu_i(\nu_i+1)}{2}-\sum_{i=1}^{l(\nu)}i\nu_i+|\nu|l(\nu)
\end{aligned}$$

$$
\begin{aligned}
&= -|\nu| l(\nu) + \sum_{i=1}^{l(\nu)} i\nu_i \\
&\quad + \sum_{j=1}^{l(\nu)} (j-1)\nu_j + \sum_{i=1}^{l(\nu)} \frac{\nu_i(\nu_i - 1)}{2} + |\nu| - \sum_{i=1}^{l(\nu)} i\nu_i + |\nu| l(\nu) \\
&= \sum_{j=1}^{l(\nu)} (j-1)\nu_j + \sum_{i=1}^{l(\nu)} \frac{\nu_i(\nu_i - 1)}{2} + |\nu| \\
&= n(\nu) + n(\nu') + |\nu| \\
&= \sum_{x \in \nu} h(x).
\end{aligned}
$$

Comparing with the left-hand side, one then gets:

$$
\begin{aligned}
&\prod_{x \in \nu} (t^{-h(x)/2} - t^{h(x)/2}) \\
&= \frac{\prod_{i<j} (t^{-(j-i)/2} - t^{(j-i)/2})}{\prod_{i<j} (t^{-(\nu_i - \nu_j - i + j)/2} - t^{(\nu_i - \nu_j - i + j)/2})} \\
&\quad \cdot \prod_{i=1}^{l(\nu)} \prod_{j=1}^{\nu_i} (t^{-(j-i+l(\nu))/2} - t^{(j-i+l(\nu))/2}).
\end{aligned}
$$

The proof is completed by taking $t = e^{-\sqrt{-1}\lambda}$. q.e.d.

We next compute the initial values $\mathcal{C}(\lambda; 0; p)$.

**Proposition 2.2.**

$$
\mathcal{C}(\lambda; 0; p) = -\sum_{d>0} \frac{\sqrt{-1}^{d+1} p_d}{2d \sin(d\lambda/2)}.
$$

*Proof.* When $l(\mu) > 1$, we clearly have

$$
\mathcal{C}_\mu(\lambda; 0) = 0.
$$

When $\mu = (d)$ we have

$$\begin{aligned}\mathcal{C}_{(d)}(\lambda;0) &= -\sum_{g\geq 0}\lambda^{2g-1}\sqrt{-1}^{d+1}\frac{\prod_{a=1}^{d-1}(d\cdot 0+a)}{(d-1)!}\\ &\quad\cdot\int_{\overline{\mathcal{M}}_{g,1}}\frac{\Lambda_g^\vee(1)\Lambda_g^\vee(0)\Lambda_g^\vee(-1)}{1-d\psi_1}\\ &= -\frac{\sqrt{-1}^{d+1}}{d}\left((d\lambda)^{-1}+\sum_{g\geq 1}(d\lambda)^{2g-1}\int_{\overline{\mathcal{M}}_{g,1}}\lambda_g\psi_1^{2g-2}\right)\\ &= -\frac{\sqrt{-1}^{d+1}}{d^2\lambda}\cdot\frac{d\lambda/2}{\sin(d\lambda/2)}\\ &= -\frac{\sqrt{-1}^{d+1}}{2d\sin(d\lambda/2)}.\end{aligned}$$

In the second equality we have used the Mumford's relations [24, 5.4]:

$$\Lambda_g^\vee(1)\Lambda_g^\vee(-1) = (-1)^g.$$

In the third equality we have used [4, Theorem 2]. This completes the proof. q.e.d.

**Proposition 2.3.** *We have the following identity:*

$$\log\left(\sum_{n\geq 0}\sum_{|\rho|=n}\sum_{|\eta|=|\rho|}\frac{e^{\frac{1}{4}\kappa_\rho\sqrt{-1}\lambda}}{\prod_{e\in\rho}2\sin(h(e)\lambda/2)}\frac{\chi_\rho(\eta)}{z_\eta}p_\eta\right) = -\sum_{d\geq 1}\frac{\sqrt{-1}^{d+1}p_d}{2d\sin(d\lambda/2)}.$$

To prove Proposition 2.3, we need the following two lemmata.

**Lemma 2.4.** *Introducing formal variables $x_1,\dots,x_n,\dots$ such that*

$$p_i(x_1,\dots,x_n,\dots) = x_1^i+\cdots+x_n^i+\cdots.$$

*Then we have*

$$\sum_{n\geq 0}t^n\sum_{|\rho|=n}\sum_{|\eta|=|\rho|}\frac{q^{n(\rho)}}{\prod_{e\in\rho}(1-q^{h(e)})}\frac{\chi_\rho(\eta)}{z_\eta}p_\eta = \frac{1}{\prod_{i,j}(1-tx_iq^{j-1})}. \tag{10}$$

*Proof.* Recall the following facts about Schur polynomials:

$$s_\rho(x) = \sum_{|\eta|=|\rho|} \frac{\chi_\rho(\eta)}{z_\eta} p_\eta(x), \tag{11}$$

$$s_\rho(1, q, q^2, \dots) = \frac{q^{n(\rho)}}{\prod_{e\in\rho}(1-q^{h(e)})}, \tag{12}$$

$$\sum_{n\geq 0} t^n \sum_{|\rho|=n} s_\rho(x) s_\rho(y) = \frac{1}{\prod_{i,j}(1-tx_iy_j)}. \tag{13}$$

Combining the last two identities, one gets:

$$\sum_{n\geq 0} t^n \sum_{|\rho|=n} \frac{q^{n(\rho)}}{\prod_{e\in\rho}(1-q^{h(e)})} s_\rho(x) = \frac{1}{\prod_{i,j}(1-tx_iq^{j-1})}.$$

The proof is completed by (11). q.e.d.

**Lemma 2.5.** *For any partition $\rho$ we have*

$$\frac{1}{2}\sum_{e\in\rho} h(e) - n(\rho) = \frac{1}{4}\kappa_\rho + \frac{1}{2}|\rho|. \tag{14}$$

*Proof.*

$$\begin{aligned}
&\frac{1}{2}\sum_{e\in\rho} h(e) - n(\rho) \\
&= \frac{1}{2}(n(\rho') - n(\rho) + |\rho|) \\
&= \frac{1}{2}\left(\sum_i \binom{\rho_i}{2} - \sum_i (i-1)\rho_i + |\rho|\right) \\
&= \frac{1}{4}\left(\sum_i \rho_i(\rho_i - 1) - 2\sum_i i\rho_i + 4|\rho|\right) \\
&= \frac{1}{4}\kappa_\rho + \frac{1}{2}|\rho|.
\end{aligned}$$

q.e.d.

*Proof of Proposition* 2.3. Let $q = e^{-\sqrt{-1}\lambda}$, and $t = \sqrt{-1}q^{1/2}$. Then

$$\begin{aligned}
&\sum_{n\geq 0} t^n \sum_{|\rho|=n} \sum_{|\eta|=|\rho|} \frac{q^{n(\rho)}}{\prod_{e\in\rho}(1-q^{h(e)})} \frac{\chi_\rho(\eta)}{z_\eta} p_\eta \\
&= \sum_{n\geq 0} \sqrt{-1}^n q^{n/2} \sum_{|\rho|=n} \sum_{|\eta|=|\rho|} \frac{q^{n(\rho)-\frac{1}{2}\sum_{e\in\rho} h(e)}}{\prod_{e\in\rho}(q^{-h(e)/2}-q^{h(e)/2})} \frac{\chi_\rho(\eta)}{z_\eta} p_\eta \\
&= \sum_{n\geq 0} \sqrt{-1}^n q^{n/2} \sum_{|\rho|=n} \sum_{|\eta|=|\rho|} \frac{q^{-\frac{1}{4}\kappa_\rho - \frac{1}{2}n}}{\prod_{e\in\rho}(q^{-h(e)/2}-q^{h(e)/2})} \frac{\chi_\rho(\eta)}{z_\eta} p_\eta \\
&= \sum_{n\geq 0} \sum_{|\rho|=n} \sum_{|\eta|=|\rho|} \frac{e^{\frac{1}{4}\kappa_\rho\sqrt{-1}\lambda}}{\prod_{e\in\rho} 2\sin(h(e)\lambda/2)} \frac{\chi_\rho(\eta)}{z_\eta} p_\eta.
\end{aligned}$$

Hence by (10),

$$\begin{aligned}
&\log\left(\sum_{n\geq 0} \sum_{|\rho|=n} \sum_{|\eta|=|\rho|} \frac{e^{\frac{1}{4}\kappa_\rho\sqrt{-1}\lambda}}{\prod_{e\in\rho} 2\sin(h(e)\lambda/2)} \frac{\chi_\rho(\eta)}{z_\eta} p_\eta\right) \\
&= \log\left(\sum_{n\geq 0} t^n \sum_{|\rho|=n} \sum_{|\eta|=|\rho|} \frac{q^{n(\rho)}}{\prod_{e\in\rho}(1-q^{h(e)})} \frac{\chi_\rho(\eta)}{z_\eta} p_\eta\right) \\
&= \log \frac{1}{\prod_{i,j}(1-tx_i q^{j-1})} = \sum_{i,j\geq 1} \sum_{d\geq 1} \frac{1}{d} t^d q^{d(j-1)} x_i^d \\
&= \sum_{j\geq 1} \sum_{d\geq 1} \frac{1}{d} t^d q^{d(j-1)} p_d = \sum_{d\geq 1} \frac{p_d}{d} \frac{t^d}{1-q^d} \\
&= -\sum_{d\geq 0} \frac{\sqrt{-1}^{d+1} p_d}{2d\sin(d\lambda/2)}.
\end{aligned}$$

q.e.d.

By Proposition 2.1 and Proposition 2.3, we have

$$\begin{aligned}
R(\lambda;0;p) &= \log\left(\sum_{n\geq 0} \sum_{|\rho|=n} \sum_{|\eta|=|\rho|} \frac{\chi_\rho(\eta)}{z_\eta} e^{\frac{1}{4}\kappa_\rho\sqrt{-1}\lambda} V_\rho(\lambda) p_\eta\right) \\
&= \log\left(\sum_{n\geq 0} \sum_{|\rho|=n} \sum_{|\eta|=|\rho|} \frac{e^{\frac{1}{4}\kappa_\rho\sqrt{-1}\lambda}}{\prod_{e\in\rho} 2\sin(h(e)\lambda/2)} \frac{\chi_\rho(\eta)}{z_\eta} p_\eta\right) \\
&= -\sum_{d\geq 1} \frac{\sqrt{-1}^{d+1} p_d}{2d\sin(d\lambda/2)}.
\end{aligned}$$

By Proposition 2.2, we have

$$\mathcal{C}(\lambda;0;p) = -\sum_{d\geq 1} \frac{\sqrt{-1}^{d+1} p_d}{2d\sin(d\lambda/2)}.$$

So the initial condition (b) holds.

## 3. Proof of Theorem 1

Let $\mu, \eta$ be two partitions, both represented by Young diagrams. We write $\eta \in J_{i,j}(\mu)$ and $\mu \in C_{i,j}(\eta)$ if $\eta$ is obtained from $\mu$ by removing a row of length $i$ and a row of length $j$, then adding a row of length $i+j$. It is easy to see that

$$\frac{m_{i+j}(\eta)}{\prod_k m_k(\eta)!} = \begin{cases} \frac{m_i(\mu)m_j(\mu)}{\prod_k m_k(\mu)!}, & i \neq j, \\ \frac{m_i(\mu)(m_i(\mu)-1)}{\prod_k m_k(\mu)!}, & i = j. \end{cases} \tag{15}$$

Recall

$$c_\mu = \sum_{g\in C_\mu} g$$

lies in the center of the group algebra $\mathbb{C}S_d$, hence it acts as a scalar $f_\nu(\mu)$ on any irreducible representation $R_\nu$. In other words, let $\rho : S_d \to \operatorname{End} R_\nu$ be the representation indexed by $\nu$, then

$$\sum_{g\in C(\mu)} \rho_\nu(g) = f_\nu(\mu)\,\mathrm{id}\,.$$

We need the following interpretation of $\kappa_\nu$ in terms of character.

**Lemma 3.1.** *We have*

$$\kappa_\nu = 2f_\nu(C(2)),$$

*where we use $C(2)$ to denote the class of transpositions.*

*Proof.* By [22, p. 118, Example 7],

$$\begin{aligned} f_\nu(C(2)) &= |C(2)|\frac{\chi_\nu(C(2))}{\dim R_\nu} = n(\nu') - n(\nu) \\ &= \sum_{i=1}^{l(\nu)} \binom{\nu_i}{2} - \sum_{i=1}^{l(\nu)} (i-1)\nu_i \\ &= \frac{1}{2}\sum_{i=1}^{l(\nu)} (\nu_i^2 - 2i\nu_i + \nu_i) = \frac{1}{2}\kappa_\nu. \end{aligned}$$

In the above we have used (7). q.e.d.

We need the following result:

**Lemma 3.2.** *Suppose $h \in S_d$ has cycle type $\mu$. The product $c_{(2)} \cdot h$ is a sum of elements of $S_d$ whose type is either a cut or a join of $\mu$. More precisely, there are $ijm_i(\mu)m_j(\mu)$ (when $i < j$) or $i^2 m_i(\mu)(m_i(\mu)-1)/2$ (when $i = j$) elements obtained from $h$ by joining an $i$-cycle in $h$ to a $j$-cycle in $h$, and there are $(i+j)m_{i+j}(\mu)$ (when $i < j$) or $im_{2i}(\mu)$ (when $i = j$) elements obtained from $h$ by cutting an $(i+j)$-cycle into an $i$-cycle and a $j$-cycle.*

*Proof.* Denote by $[s_1, \ldots, s_k]$ a $k$-cycle. Then

$$[s,t] \cdot [s, s_2, \ldots, s_i, t, t_2, \ldots, t_j] = [s, s_2, \ldots, s_i][t, t_2, \ldots, t_j],$$

i.e., an $i+j$-cycle is cut into an $i$-cycle and a $j$-cycle. Conversely,

$$[s,t] \cdot [s, s_2, \ldots, s_i][t, t_2, \ldots, t_j] = [s, s_2, \ldots, s_i, t, t_2, \ldots, t_j],$$

i.e., an $i$-cycle and a $j$-cycle is joined to an $i+j$-cycle. Hence, for a permutation $h$ of type $\mu$, $c_{(2)} \cdot h$ is a sum of all elements obtained from $h$ by either a cut or a join. Fix a pair of $i$-cycle and $j$-cycle of $h$, there are $i \cdot j$ different ways to join them to an $(i+j)$-cycle. Taking into the account of $m_i(\mu)$ choices of $i$-cycles, and $m_j(\mu)$ choices of $j$-cycles, we get the number of different ways to obtain an element from $h$ by joining an $i$-cycle in $h$ to a $j$-cycle in $h$ is

$$\begin{cases} ijm_i(\mu)m_j(\mu), & i < j \\ i^2 m_i(\mu)(m_i(\mu)-1)/2, & i = j. \end{cases}$$

Similarly, fix an $(i+j)$-cycle of $h$, there are $i+j$ different ways to cut it into an $i$-cycle and a disjoint $j$-cycle when $i < j$. When $i = j$, there are

only $i$ different ways to cut it into two $i$-cycles. And taking into account the number of $(i+j)$-cycles in $h$, we get the number of different ways to obtain an element from $h$ by cutting an $(i+j)$-cycle into an $i$-cycle and a $j$-cycle is

$$\begin{cases} (i+j)m_{i+j}(\mu), & i<j, \\ im_{2i}(\mu), & i=j. \end{cases}$$

q.e.d.

For any $h \in S_d$ of cycle type $\mu$ we have

$$\begin{aligned}
&\sum_\mu f_\nu(2)\frac{\chi_\nu(\mu)}{z_\mu}p_\mu \\
&= \sum_\mu \operatorname{tr}[f_\nu(2)\,\mathrm{id}\cdot\rho_\nu(h)]\cdot\prod_i \frac{p_i^{m_i(\mu)}}{i^{m_i(\mu)}m_i(\mu)!} \\
&= \sum_\mu \operatorname{tr}\left[\sum_{g\in C(2)}\rho_\nu(g)\cdot\rho_\nu(h)\right]\cdot\prod_i \frac{p_i^{m_i(\mu)}}{i^{m_i(\mu)}m_i(\mu)!} \\
&= \sum_\mu \operatorname{tr}\rho_\nu\left(\sum_{g\in C(2)} g\cdot h\right)\cdot\prod_i \frac{p_i^{m_i(\mu)}}{i^{m_i(\mu)}m_i(\mu)!} \\
&= \sum_\mu \left(\sum_{i<j}\left(\sum_{\eta\in J_{i,j}(\mu)} ijm_i(\mu)m_j(\mu)\chi_\nu(\eta)\right.\right. \\
&\qquad \left. + \sum_{\eta\in C_{i,j}(\mu)}(i+j)m_{i+j}(\mu)\chi_\nu(\eta)\right) \\
&\quad + \sum_i\left(\sum_{\eta\in J_{i,i}(\mu)}\frac{1}{2}i^2m_i(\mu)(m_i(\mu)-1)\chi_\nu(\eta)\right. \\
&\qquad \left.\left. + \sum_{\eta\in C_{i,i}(\mu)} im_{2i}(\mu)\chi_\nu(\eta)\right)\right)\prod_i \frac{p_i^{m_i(\mu)}}{i^{m_i(\mu)}m_i(\mu)!} \\
&= \frac{1}{2}\sum_{i,j}\left((i+j)p_ip_j\frac{\partial}{\partial p_{i+j}} + ijp_{i+j}\frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j}\right)\sum_\eta \frac{\chi_\nu(\eta)}{z_\eta}p_\eta.
\end{aligned}$$

In the last equality we have used (15). It follows that

$$\begin{aligned}
&\frac{\partial R(\lambda;\tau;p)^\bullet}{\partial\tau}\\
&=\frac{\sqrt{-1}\lambda}{2}\sum_{\mu,\nu}\left(f_\nu(2)\frac{\chi_\nu(C(\mu))}{z_\mu}p_\mu\right)e^{\sqrt{-1}(\tau+\frac{1}{2})\kappa_\nu\lambda/2}V_\nu(\lambda)\\
&=\frac{\sqrt{-1}\lambda}{2}\left(ijp_{i+j}\frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j}+(i+j)p_ip_j\frac{\partial}{\partial p_{i+j}}\right)\\
&\quad\cdot\sum_\eta\frac{\chi_\nu(\eta)}{z_\eta}p_\eta e^{\sqrt{-1}(\tau+\frac{1}{2})\kappa_\nu\lambda/2}V_\nu(\lambda).
\end{aligned}$$

This finishes the proof of Theorem 1.

## 4. Moduli spaces of relative stable morphisms

In this section, we introduce the geometric objects involved in the proof of Theorem 2.

### 4.1 Moduli space of relative morphisms

We first describe the moduli space of relative stable morphisms to $\mathbb{P}^1$ used in [19]. The moduli spaces of algebraic relative stable morphisms are constructed by J. Li [16].

We introduce some notations. For any nonnegative integer $m$, let

$$\mathbb{P}^1[m]=\mathbb{P}^1_{(0)}\cup\mathbb{P}^1_{(1)}\cup\cdots\cup\mathbb{P}^1_{(m)}$$

be a chain of $m+1$ copies $\mathbb{P}^1$, where $\mathbb{P}^1_{(l)}$ is glued to $\mathbb{P}^1_{(l+1)}$ at $p_1^{(l)}$ for $0\le l\le m-1$. The irreducible component $\mathbb{P}^1_{(0)}$ will be referred to as the root component, and the other irreducible components will be called the bubble components. A point $p_1^{(m)}\neq p_1^{(m-1)}$ is fixed on $\mathbb{P}^1_{(m)}$. Denote by $\pi[m]:\mathbb{P}^1[m]\to\mathbb{P}^1$ the map which is identity on the root component and contracts all the bubble components to $p_1^{(0)}$. For $m>0$, let

$$\mathbb{P}^1(m)=\mathbb{P}^1_{(1)}\cup\cdots\cup\mathbb{P}^1_{(m)}$$

denote the union of bubble components of $\mathbb{P}^1[m]$.

Let $\mu$ be a partition of $d>0$. Let $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1,\mu)$ be the moduli space of morphisms

$$f:(C,x_1,\ldots,x_{l(\mu)})\to(\mathbb{P}^1[m],p_1^{(m)}),$$

such that:

1. $(C, x_1, \ldots, x_{l(\mu)})$ is a prestable curve of genus $g$ with $l(\mu)$ marked points. For convenience, we assume the marked points are *unordered*.

2. $f^{-1}(p_1^{(m)}) = \sum_{i=1}^{l(\mu)} \mu_i x_i$ as Cartier divisors, and $\deg(\pi[m] \circ f) = d$.

3. The preimage of each node in $\mathbb{P}^1[m]$ consists of nodes of $C$. If $f(y) = p_1^{(l)}$ and $C_1$ and $C_2$ are two irreducible components of $C$ which intersect at $y$, then $f|_{C_1}$ and $f|_{C_2}$ have the same contact order to $p_1^{(l)}$ at $y$.

4. The automorphism group of $f$ is finite.

Two such morphisms are isomorphic if they differ by an isomorphism of the domain and an automorphism of the pointed curve $(\mathbb{P}^1(m), p_1^{(0)}, p_1^{(m)})$. In particular, this defines the automorphism group in the stability condition (4) above.

In [16, 17], J. Li showed that $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu)$ is a separated, proper Deligne-Mumford stack with a perfect obstruction theory of virtual dimension

$$r = 2g - 2 + d + l(\mu),$$

so it has a virtual fundamental class of degree $r$.

## 4.2 Torus action

Consider the $\mathbb{C}^*$-action

$$t \cdot [z^0 : z^1] = [tz^0 : z^1]$$

on $\mathbb{P}^1$. It has two fixed points $p_0 = [0 : 1]$ and $p_1 = [1 : 0]$. This induces an action on $\mathbb{P}^1[m]$ by the action on the root component induced by the isomorphism to $\mathbb{P}^1$, and the trivial actions on the bubble components. This in turn induces an action on $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu)$.

## 4.3 The branch morphism

There is a branch morphism

$$\mathrm{Br} : \overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu) \to \mathrm{Sym}^r \mathbb{P}^1 \cong \mathbb{P}^r.$$

Note that $\mathbb{P}^r$ can be identified with $\mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}(r))$, and the isomorphism

$$\mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}(r)) \cong \mathrm{Sym}^r \mathbb{P}^1$$

is given by $[s] \mapsto \mathrm{div}(s)$. The $\mathbb{C}^*$-action on $\mathbb{P}^1$ induces a $\mathbb{C}^*$-action on $H^0(\mathbb{P}^1, \mathcal{O}(r))$ by

$$t \cdot (z^0)^k (z^1)^{r-k} = t^{-k} (z^0)^k (z^1)^{r-k}.$$

So $\mathbb{C}^*$ acts on $\mathbb{P}^r$ by

$$t \cdot [a_0 : a_1 : \cdots : a_r] = [a_0 : t^{-1} a_1 : \cdots : t^{-r} a_r],$$

where $[a_0 : a_1 : \cdots : a_r]$ corresponds to $\sum_{k=0}^r a_k (z^0)^k (z^1)^{r-k} \in H^0(\mathbb{P}^1, \mathcal{O}(r))$. With this action, the branch morphism is $\mathbb{C}^*$-equivariant. See [6, 9] for more details.

The $\mathbb{C}^*$-action on $\mathbb{P}^r$ has $r+1$ fixed points $p_0, \ldots, p_r$, where $p_k \in \mathbb{P}^r$ corresponds to the complex line $\mathbb{C}(z^0)^k (z^1)^{r-k} \subset H^0(\mathbb{P}^1, \mathcal{O}(r))$.

### 4.4 The obstruction bundle

In [19], J. Li and Y. Song constructed an obstruction bundle over the stratum where the target is $\mathbb{P}^1[0] = \mathbb{P}^1$, and proposed an extension over the entire $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu)$. Here we use a different extension which is defined in [1, Section 3].

Let

$$\pi : \mathcal{U}_{g,\mu} \to \overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu)$$

be the universal domain curve, and let

$$P : \mathcal{T}_{g,\mu} \to \overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu)$$

be the universal target. There is an evaluation map

$$F : \mathcal{U}_{g,\mu} \to \mathcal{T}_{g,\mu}$$

and a contraction map

$$\widetilde{\pi} : \mathcal{T}_{g,\mu} \to \mathbb{P}^1.$$

Let $\mathcal{D}_{g,\mu} \subset \mathcal{U}_{g,\mu}$ be the divisor corresponding to the $l(\mu)$ marked points. Define

$$V_D = R^1 \pi_* (\mathcal{O}_{\mathcal{U}_{g,\mu}}(-\mathcal{D}_{g,\mu}))$$
$$V_{D_d} = R^1 \pi_* \widetilde{F}^* \mathcal{O}_{\mathbb{P}^1}(-1),$$

where $\widetilde{F} = \widetilde{\pi} \circ F : \mathcal{U}_{g,\mu} \to \mathbb{P}^1$. The fibers of $V_D$ and $V_{D_d}$ at

$$\big[ f : (C, x_1, \ldots, x_{l(\mu)}) \to \mathbb{P}^1[m] \ \big] \in \overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu)$$

are $H^1(C, \mathcal{O}_C(-D))$ and $H^1(C, \widetilde{f}^* \mathcal{O}_{\mathbb{P}^1}(-1))$, respectively, where $D = x_1 + \cdots + x_{l(\mu)}$, and $\widetilde{f} = \pi[m] \circ f$. Note that

$$H^0(C, \mathcal{O}_C(-D)) = H^0(C, \widetilde{f}^* \mathcal{O}_{\mathbb{P}^1}(-1)) = 0,$$

so $V_D$ and $V_{D_d}$ are vector bundles of ranks $l(\mu) + g - 1$ and $d + g - 1$, respectively. The obstruction bundle

$$V = V_D \oplus V_{D_d}$$

is a vector bundle of rank $r = 2g - 2 + d + l(\mu)$.

We lift the $\mathbb{C}^*$-action on $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu)$ to $V_D$ and $V_{D_d}$ as follows. The action on $V_{D_d}$ comes from an action on $\mathcal{O}_{\mathbb{P}^1}(-1) \to \mathbb{P}^1$ with weights $-\tau - 1$ and $-\tau$ at the two fixed points $p_0$ and $p_1$, respectively, where $\tau \in \mathbb{Z}$. The fiber of $V_D$ does not depend on the map $f$, so the fibers over two points in the same orbit of the $\mathbb{C}^*$-action can be canonically identified. The action of $\lambda \in \mathbb{C}^*$ on $V_D$ is multiplication by $\lambda^\tau$.

## 5. Fixed points of torus action

### 5.1 Graph notation

Similar to the case of $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, d)$, the connected components of the $\mathbb{C}^*$ fixed points set $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu)^{\mathbb{C}^*}$ are parameterized by labeled graphs.

Given a morphism

$$f : (C, x_1, \ldots, x_{l(\mu)}) \to \mathbb{P}^1[m]$$

which represents a fixed point of the $\mathbb{C}^*$-action on $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu)$, let

$$\widetilde{f} = \pi[m] \circ f : C \to \mathbb{P}^1.$$

The restriction of $\widetilde{f}$ to an irreducible component of $C$ is either a constant map to one of the $\mathbb{C}^*$ fixed points $p_0, p_1$ or a cover of $\mathbb{P}^1$ which is fully ramified over $p_0$ and $p_1$. We associate a labeled graph $\Gamma$ to the $\mathbb{C}^*$ fixed point

$$\big[ f : (C, x_1, \ldots, x_{l(\mu)}) \to \mathbb{P}^1[m] \ \big]$$

as follows:

1. We assign a vertex $v$ to each connected component $C_v$ of $\widetilde{f}^{-1}(\{p_0, p_1\})$, a label $i(v) = i$ if $\widetilde{f}(C_v) = p_i$, where $i = 0, 1$, and a label $g(v)$ which is the arithmetic genus of $C_v$ (We define $g(v) = 0$ if $C_v$ is a point). Denote by $V(\Gamma)^{(i)}$ the set of vertices with $i(v) = i$, where $i = 0, 1$. Then the set $V(\Gamma)$ of vertices of the graph $\Gamma$ is a disjoint union of $V(\Gamma)^{(0)}$ and $V(\Gamma)^{(1)}$.

2. We assign an edge $e$ to each rational irreducible component $C_e$ of $C$ such that $\widetilde{f}|_{C_e}$ is not a constant map. Let $d(e)$ be the degree of $\widetilde{f}|_{C_e}$. Then $\widetilde{f}|_{C_e}$ is fully ramified over $p_0$ and $p_1$. Let $E(\Gamma)$ denote the set of edges of $\Gamma$.

3. The set of flags of $\Gamma$ is given by

$$F(\Gamma) = \{(v, e) : v \in V(\Gamma), e \in E(\Gamma), C_v \cap C_e \neq \emptyset\}.$$

4. For each $v \in V(\Gamma)$, define

$$d(v) = \sum_{(v,e)\in F(\Gamma)} d(e),$$

and let $\nu(v)$ be the partition of $d(v)$ determined by $\{d(e) : (v, e) \in F(\Gamma)\}$. When the target is $\mathbb{P}^1[m]$, where $m > 0$, we assign an additional label for each $v \in V(\Gamma)^{(1)}$: let $\mu(v)$ be the partition of $d(v)$ determined by the ramification of $f|_{C_v} : C_v \to \mathbb{P}^1(m)$ over $p_1^{(m)}$.

Note that for $v \in V(\Gamma)^{(1)}$, $\nu(v)$ coincides with the partition of $d(v)$ determined by the ramification of $f|_{C_v} : C_v \to \mathbb{P}^1(m)$ over $p_1^{(0)}$.

## 5.2 Fixed points

Let $G_{g,0}(\mathbb{P}^1, \mu)$ be the set of all the graphs associated to the $\mathbb{C}^*$ fixed points in $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu)$, as described in Section 5.1. In this section, we describe the set of fixed points associated to a given graph $\Gamma \in G_{g,0}(\mathbb{P}^1, \mu)$.

### 5.2.1 The target is $\mathbb{P}^1$

Any $\mathbb{C}^*$ fixed point in $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu)$ which is represented by a morphism to $\mathbb{P}^1$ is associated to the graph $\Gamma^0$, where

$$V(\Gamma^0)^{(0)} = \{v_0\}, \quad V(\Gamma^0)^{(1)} = \{v_1, \ldots, v_{l(\mu)}\}, \quad E(\Gamma^0) = \{e_1, \ldots, e_{l(\mu)}\},$$

and

$$g(v) = g, \quad g(v_i) = 0, \quad d(e_i) = \mu_i$$

for $i = 1, \ldots, l(\mu)$. The two end points of the edge $e_i$ are $v_0$ and $v_i$. Let $\mathrm{Aut}(\mu)$ denote the automorphism group of the partition $\mu$ of $d$. Any morphism associated to the graph $\Gamma^0$ has automorphism group $A_{\Gamma^0}$, where

$$1 \to \prod_{i=1}^{l(\mu)} \mathbb{Z}_{\mu_i} \to A_{\Gamma^0} \to \mathrm{Aut}(\mu) \to 1.$$

Let

$$\overline{\mathcal{M}}_{\Gamma^0} = \begin{cases} \{\text{point}\}, & (g, l(\mu)) = (0,1), (0,2), \\ \overline{\mathcal{M}}_{g,l(\mu)}, & (g, l(\mu)) \neq (0,1), (0,2). \end{cases}$$

There is a morphism

$$i_{\Gamma^0} : \overline{\mathcal{M}}_{\Gamma^0} \to \overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu)$$

whose image is the fixed locus $F_{\Gamma^0}$ associated to $\Gamma_0$. The morphism $i_{\Gamma_0}$ induces an isomorphism

$$\overline{\mathcal{M}}_{\Gamma^0}/A_{\Gamma^0} \cong F_{\Gamma^0}.$$

The dimension of $F_{\Gamma^0}$ is

$$d_{\Gamma^0} = \begin{cases} 0, & (g, l(\mu)) = (0,1), (0,2), \\ 3g - 3 + l(\mu), & (g, l(\mu)) \neq (0,1), (0,2). \end{cases}$$

### 5.2.2 The target is $\mathbb{P}^1[m]$, $m > 0$

Let $\Gamma \in G_{g,0}(\mathbb{P}^1, \mu)$ be a graph associated to a $\mathbb{C}^*$ fixed point represented by some morphism to $\mathbb{P}^1[m]$, $m > 0$. Let $\overline{\mathcal{M}}_\Gamma^{(1)}$ denote the moduli space of morphisms

$$\hat{f} : \hat{C} \to \mathbb{P}^1(m)$$

such that:

(a) $\hat{C}$ is the disjoint union of $\{C_v : v \in V(\Gamma)^{(1)}\}$.

(b) $(C_v, x_{v,1}, \ldots, x_{v,l(\mu(v))}, y_{v,1}, \ldots, y_{v,l(\nu(v))})$ is a prestable curve of genus $g(v)$ with $l(\mu(v)) + l(\nu(v))$ marked points. Here the marked points are *ordered*.

(c) As Cartier divisors,

$$(\hat{f}|_{C_v})^{-1}(p_1^{(0)}) = \sum_{i=1}^{l(\mu(v))} \mu(v)_i \, x_{v,i},$$
$$(\hat{f}|_{C_v})^{-1}(p_1^{(m)}) = \sum_{j=1}^{l(\nu(v))} \nu(v)_i \, y_{v,j}.$$

The morphism $(\hat{f}|_{C_v})^{-1}(B) \to B$ is of degree $d(v)$ for each irreducible component $B$ of $\mathbb{P}^1(m)$.

(d) The automorphism group of $\hat{f}$ is finite.

Two such morphisms are isomorphic if they differ by an isomorphism of the domain and an automorphism of the pointed curve $(\mathbb{P}^1(m), p_1^{(0)}, p_1^{(m)})$, which is an element of $(\mathbb{C}^*)^m$. In particular, this defines the automorphism group in the stability condition (d) above.

The moduli space $\overline{\mathcal{M}}_\Gamma^{(1)}$ is a variant of J. Li's moduli spaces of stable relative morphisms [16, 17]. It is a separated, proper Deligne-Mumford stack with a perfect obstruction theory.

Given

$$f : (C, x_1, \ldots, x_{l(\mu)}) \to \mathbb{P}^1[m]$$

associated to the graph $\Gamma$, let $\hat{C}$ be the disjoint union of $\{C_v : v \in V(\Gamma)^{(1)}\}$. Let $\hat{f}$ be the restriction of $f$ to $\hat{C}$. Then

$$\hat{f} : \hat{C} \to \mathbb{P}^1(m)$$

represents a point in $\overline{\mathcal{M}}_\Gamma^{(1)}$.

Define

$$r_0(v) = 2g(v) - 2 + \mathrm{val}(v), \qquad v \in V(\Gamma)^{(0)},$$
$$r_1(v) = 2g(v) - 2 + l(\mu(v)) + l(\nu(v)), \quad v \in V(\Gamma)^{(1)},$$

$$V^I(\Gamma)^{(0)} = \{v \in V(\Gamma)^{(0)} : r_0(v) = -1\},$$
$$V^{II}(\Gamma)^{(0)} = \{v \in V(\Gamma)^{(0)} : r_0(v) = 0\},$$
$$V^S(\Gamma)^{(0)} = \{v \in V(\Gamma)^{(0)} : r_0(v) > 0\},$$
$$V^{II}(\Gamma)^{(1)} = \{v \in V(\Gamma)^{(1)} : r_1(v) = 0\},$$
$$V^S(\Gamma)^{(1)} = \{v \in V(\Gamma)^{(1)} : r_1(v) > 0\}.$$

Note that $V^S(\Gamma)^{(1)} \neq \emptyset$ by the stability condition (d).

Let $\mathrm{Aut}(\Gamma)$ denote the automorphism of the labeled graph $\Gamma$. The automorphism group of any morphism associated to the graph $\Gamma$ is $A_\Gamma$, where

$$1 \to \prod_{e \in E(\Gamma)} \mathbb{Z}_{d(e)} \to A_\Gamma \to \mathrm{Aut}(\Gamma) \to 1.$$

Let $\overline{\mathcal{M}}_\Gamma = \overline{\mathcal{M}}_\Gamma^{(0)} \times \overline{\mathcal{M}}_\Gamma^{(1)}$, where

$$\overline{\mathcal{M}}_\Gamma^{(0)} = \prod_{v \in V^S(\Gamma)^{(0)}} \overline{\mathcal{M}}_{g(v),\mathrm{val}(v)}.$$

There is a morphism

$$i_\Gamma : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu)$$

whose image is the fixed locus $F_\Gamma$ associated to the graph $\Gamma$. The morphism $i_\Gamma$ induces an isomorphism $\overline{\mathcal{M}}_\Gamma / A_\Gamma \cong F_\Gamma$.

The dimension of $\overline{\mathcal{M}}_\Gamma^{(0)}$ is given by

$$d_\Gamma^{(0)} = \sum_{v \in V^S(\Gamma)^{(0)}} (3g(v) - 3 + \mathrm{val}(v)),$$

and the virtual dimension of $\overline{\mathcal{M}}_\Gamma^{(1)}$ is given by

$$d_\Gamma^{(1)} = \left( \sum_{v \in V(\Gamma)^{(1)}} r_1(v) \right) - 1.$$

So the virtual dimension of $F_\Gamma$ is given by

$$\begin{aligned} d_\Gamma &= d_\Gamma^{(0)} + d_\Gamma^{(1)} \\ &= \sum_{v \in V^S(\Gamma)^{(0)}} (3g(v) - 3 + \mathrm{val}(v)) + \sum_{v \in V(\Gamma)^{(1)}} r_1(v) \ - 1 \\ &= \sum_{v \in V(\Gamma)^{(0)}} (3g(v) - 3 + \mathrm{val}(v)) + |V^{II}(\Gamma)^{(0)}| + 2|V^I(\Gamma)^{(0)}| \\ &\quad + \sum_{v \in V(\Gamma)^{(1)}} (2g(v) - 2 + l(\mu(v)) + l(\nu(v))) - 1 \end{aligned}$$

$$
\begin{aligned}
&= 3 \sum_{v \in V(\Gamma)^{(0)}} g(v) - 3|V(\Gamma)^{(0)}| + |E(\Gamma)| + |V^{II}(\Gamma)^{(0)}| + 2|V^{I}(\Gamma)^{(0)}| \\
&\quad + 2 \sum_{v \in V(\Gamma)^{(1)}} g(v) - 2|V(\Gamma)^{(1)}| + l(\mu) + |E(\Gamma)| - 1 \\
&= 2 \left( \sum_{v \in V(\Gamma)} g(v) - |V(\Gamma)| + |E(\Gamma)| + 1 \right) - 3 + l(\mu) \\
&\quad + \sum_{v \in V(\Gamma)^{(0)}} (g(v) - 1) + |V^{II}(\Gamma)^{0}| + 2|V^{I}(\Gamma)^{0}| \\
&= 2g - 3 + l(\mu) + \sum_{v \in V^{S}(\Gamma)^{(0)}} (g(v) - 1) + |V^{I}(\Gamma)^{0}|.
\end{aligned}
$$

The last equality comes from the following identity:

$$g = \sum_{v \in V(\Gamma)} g(v) + b_1(\Gamma) = \sum_{v \in V(\Gamma)} g(v) - |V(\Gamma)| + |E(\Gamma)| + 1,$$

where $b_1(\Gamma)$ is the first betti number of the graph $\Gamma$.

## 6. Proof of Theorem 2

### 6.1 Functorial localization

Let $T = \mathbb{C}^*$. We have seen in Section 4.3 that the branch morphism

$$\mathrm{Br} : \overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu) \to \mathbb{P}^r$$

is $T$-equivariant. We will compute

$$\mathrm{Br}_* e_T(V) = \sum_{l=0}^{r} a_l(\tau) H^l u^{r-l}.$$

by virtual functorial localization [20], where $H \in H^2(\mathbb{P}^r; \mathbb{Z})$ is the hyperplane class, and $a_l(\tau)$ is a polynomial in $\tau$. Recall that $\tau \in \mathbb{Z}$ parametrizes torus actions on the obstruction bundle, as described in Section 4.4.

Let $p_0, \ldots, p_r \in \mathbb{P}^r$ be the torus fixed points defined as in Section 4.3, and let $f_k : p_k \to \mathbb{P}^r$ be the inclusion. From the torus action on $\mathbb{P}^r$

described in Section 4.3, one gets

$$\frac{f_k^*\mathrm{Br}_* e_T(V)}{e_T(T_{p_k}\mathbb{P}^r)} = \frac{F(\tau,k)}{(-1)^{r-k}k!(r-k)!},$$

where

$$F(\tau,x) = \sum_{l=0}^{r} a_l(\tau)x^l.$$

By functorial localization, we have

$$\int_{p_k} \frac{f_k^*\mathrm{Br}_* e_T(V)}{e_T(T_{p_k}\mathbb{P}^r)} = \sum_{F_\Gamma \subset \mathrm{Br}^{-1}(p_k)} \frac{1}{|A_\Gamma|} \int_{[\overline{\mathcal{M}}_\Gamma]^{\mathrm{vir}}} \frac{i_\Gamma^* e_T(V)}{e_T(N_\Gamma^{\mathrm{vir}})}$$

for $k = 0, \ldots, r$, where $N_\Gamma^{\mathrm{vir}} \to \overline{\mathcal{M}}_\Gamma$ is the pull-back of the virtual normal bundle of $F_\Gamma$ in $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu)$. Note that $\mathrm{Br}(F_{\Gamma^0}) = p_r$, and

$$\mathrm{Br}(F_\Gamma) = p_{r-d_\Gamma^{(1)}-1}$$

for $\Gamma \in G_{g,0}(\mathbb{P}^1, \mu)$, $\Gamma \neq \Gamma^0$. Recall that $d_\Gamma^{(1)}$ is the virtual dimension of $\overline{\mathcal{M}}_\Gamma^{(1)}$, and $0 \leq d_\Gamma^{(1)} \leq r-1$. So

$$\begin{aligned} F(\tau,x) &= \sum_{k=0}^{r} \frac{F(\tau,k)}{(-1)^{r-k}k!(r-k)!} x(x-1)\cdots \\ &\qquad (x-k+1)(x-k-1)\cdots(x-r) \\ &= \sum_{k=0}^{r} I_{g,\mu}^{r-k}(\tau)x(x-1)\cdots(x-k+1)(x-k-1)\cdots(x-r), \end{aligned}$$

where

$$I_{g,\mu}^0(\tau) = \frac{1}{|A_{\Gamma^0}|} \int_{\overline{\mathcal{M}}_{\Gamma^0}} \frac{i_{\Gamma^0}^* e_T(V)}{e_T(N_{\Gamma^0}^{\mathrm{vir}})},$$

and

$$I_{g,\mu}^k(\tau) = \sum_{\Gamma \in G_{g,0}(\mathbb{P}^1,\mu), \Gamma \neq \Gamma^0, d_\Gamma + 1 = k} \frac{1}{|A_\Gamma|} \int_{[\overline{\mathcal{M}}_\Gamma]^{\mathrm{vir}}} \frac{i_\Gamma^* e_T(V)}{e_T(N_\Gamma^{\mathrm{vir}})}$$

for $k = 1, \ldots, r$.

## 6.2 Contribution from each graph

### 6.2.1 The target is $\mathbb{P}^1$

Consider the graph $\Gamma^0 \in G_{g,0}(\mathbb{P}^1, \mu)$. We first consider the stable case, i.e., $(g, l(\mu)) \neq (0,1), (0,2)$. Let $d = |\mu|$ as before. Using the Feynman rules derived in Appendix A, we obtain

$$\begin{aligned}
I^0_{g,\mu}(\tau) &= \frac{1}{|A_{\Gamma^0}|} \int_{\overline{\mathcal{M}}_{\Gamma^0}} \frac{i^*_{\Gamma^0} e_T(V)}{e_T(N^{\mathrm{vir}}_{\Gamma^0})} \\
&= \frac{(-1)^{d-1}}{|\mathrm{Aut}(\mu)|} (\tau(\tau+1))^{l(\mu)-1} \left( \prod_{i=1}^{l(\mu)} \frac{\prod_{a=1}^{\mu_i-1}(\mu_i \tau + a)}{(\mu_i - 1)!} \right) \\
&\quad \cdot \int_{\overline{\mathcal{M}}_{g,l(\mu)}} \frac{\Lambda^\vee_g(u)\Lambda^\vee_g(\tau u)\Lambda^\vee_g((-\tau-1)u) u^{2l(\mu)-3}}{\prod_{i=1}^{l(\mu)}(u - \mu_i \psi_i)} \\
&= \frac{(-1)^{d-1}}{|\mathrm{Aut}(\mu)|} (\tau(\tau+1))^{l(\mu)-1} \left( \prod_{i=1}^{l(\mu)} \frac{\prod_{a=1}^{\mu_i-1}(\mu_i \tau + a)}{(\mu_i - 1)!} \right) \\
&\quad \cdot \int_{\overline{\mathcal{M}}_{g,l(\mu)}} \frac{\Lambda^\vee_g(1)\Lambda^\vee_g(\tau)\Lambda^\vee_g(-\tau-1)}{\prod_{i=1}^{l(\mu)}(1 - \mu_i \psi_i)} \\
&= \sqrt{-1}^{d-l(\mu)} \mathcal{C}_{g,\mu}(\tau).
\end{aligned}$$

In particular,

$$\begin{aligned}
I^0_{0,\mu}(\tau) &= \frac{(-1)^{d-1}}{|\mathrm{Aut}(\mu)|} (\tau(\tau+1))^{l(\mu)-1} \left( \prod_{i=1}^{l(\mu)} \frac{\prod_{a=1}^{\mu_i-1}(\mu_i \tau + a)}{(\mu_i - 1)!} \right) \\
&\quad \cdot \int_{\overline{\mathcal{M}}_{0,l(\mu)}} \frac{1}{\prod_{i=1}^{l(\mu)}(1 - \mu_i \psi_i)} \\
&= \frac{(-1)^{d-1}}{|\mathrm{Aut}(\mu)|} (\tau(\tau+1))^{l(\mu)-1} \left( \prod_{i=1}^{l(\mu)} \frac{\prod_{a=1}^{\mu_i-1}(\mu_i \tau + a)}{(\mu_i - 1)!} \right) d^{l(\mu)-3}.
\end{aligned}$$

The contribution from $\Gamma^0$ for the two unstable cases is also given by the above formula:

$$\begin{aligned}
I^0_{0,(d)}(\tau) &= (-1)^{d-1} \left( \frac{\prod_{a=1}^{d-1}(d\tau + a)}{(d-1)!} \right) \frac{1}{d^2}, \\
I^0_{0,(\mu_1,\mu_2)}(\tau) &= \frac{(-1)^{d-1}}{|\mathrm{Aut}((\mu_1,\mu_2))|} \tau(\tau+1) \left( \prod_{i=1}^{2} \frac{\prod_{a=1}^{\mu_i-1}(\mu_i \tau + a)}{(\mu_i - 1)!} \right) \frac{1}{d}
\end{aligned}$$

Note that $I^0_{g,\mu}(\tau)$ is a degree $r = 2g-2+d+l(\mu)$ polynomial in $\tau$ with rational coefficients, and

$$I^0_{g,\mu}(-\tau-1) = (-1)^{d-l(\mu)} I^0_{g,\mu}(\tau).$$

### 6.2.2 The target is $\mathbb{P}^1[m]$, $m > 0$

Consider $\Gamma \in G_{g,0}(\mathbb{P}^1,\mu)$, $\Gamma \neq \Gamma_0$. Using the Feynman rules derived in Appendix A, we obtain

$$I^\Gamma = \frac{1}{|A_\Gamma|}\int_{[\overline{\mathcal{M}}_\Gamma]^{\mathrm{vir}}} \frac{i_\Gamma^* e_T(V)}{e_T(N_\Gamma^{\mathrm{vir}})} = \frac{1}{|A_\Gamma|}\int_{[\overline{\mathcal{M}}_\Gamma]^{\mathrm{vir}}} \frac{\prod_{v\in V(\Gamma)} B_v}{-u-\psi^t}$$

where

$$B_v = \begin{cases} A_v A_v^V \prod_{(v,e)\in F(\Gamma)} (A_e A_e^V), & v \in V(\Gamma)^{(0)}, \\ A_v A_v^V, & v \in V(\Gamma)^{(1)}. \end{cases}$$

More explicitly, in the notation of Appendix A,

$$\begin{aligned} B_v = {} & \left(\prod_{(v,e)\in F(\Gamma)} d(e)\right)(-1)^{d(v)-1}(\tau(\tau+1))^{\mathrm{val}(v)-1} \\ & \cdot \left(\prod_{(v,e)\in F(\Gamma)} \frac{\prod_{a=1}^{d(e)-1}(d(e)\tau+a)}{(d(e)-1)!}\right) \\ & \cdot \frac{\Lambda^\vee_{g(v)}(u)\Lambda^\vee_{g(v)}(\tau u)\Lambda^\vee_{g(v)}(-(\tau+1)u)u^{2\,\mathrm{val}(v)-3}}{\prod_{(v,e)\in F(\Gamma)}\big(u-d(e)\psi_{(v,e)}\big)}, \quad v \in V^S(\Gamma)^{(0)}, \\ & d(v)(-1)^{d(v)-1}\left(\frac{\prod_{a=1}^{d(v)-1}(d(v)\tau+a)}{(d(e)-1)!}\right)\frac{1}{d(v)^2}, \quad v \in V^I(\Gamma)^{(0)}, \\ & d(e_1)d(e_2)(-1)^{d(v)-1}(\tau(\tau+1))\left(\prod_{i=1}^{2}\frac{\prod_{a=1}^{d(e_i)-1}(d(e_i)\tau+a)}{(d(e_i)-1)!}\right)\frac{1}{d(v)}, \\ & \quad v \in V^{II}(\Gamma)^{(0)}, (v,e_1), (v,e_2) \in F(\Gamma), \\ & (-1)^{g(v)+\mathrm{val}(v)-1}(\tau u)^{r_1(v)}\prod_{(v,e)\in F(\Gamma)} d(e), \quad v \in V(\Gamma)^{(1)}. \end{aligned}$$

Recall that $r_1(v) = 2g(v)-2+l(\mu(v))+\mathrm{val}(v)$ for $v \in V(\Gamma)^{(1)}$, and

$$d_\Gamma^{(1)} = \left(\sum_{v\in V(\Gamma)^{(1)}} r_1(v)\right) - 1$$

is the virtual dimension of $\overline{\mathcal{M}}_\Gamma^{(1)}$.

We have

$$I^\Gamma(\tau)=\frac{1}{|A_\Gamma|}\prod_{v\in V(\Gamma)^{(0)}}\Big(|\mathrm{Aut}(\nu(v))|I^0_{g(v),\nu(v)}(p)\Big)\left(\prod_{e\in E(\Gamma)}d(e)\right)^2 \cdot\prod_{v\in V(\Gamma)^{(1)}}(-1)^{g(v)+\mathrm{val}(v)-1}\int_{[\overline{\mathcal{M}}_\Gamma^{(1)}]^{\mathrm{vir}}}\frac{(\tau u)^{d_\Gamma^{(1)}+1}}{-u-\psi^t}.$$

Here $\nu(v)$ is the partition of $d(v)$ determined by $\{d(e):(v,e)\in F(\Gamma)\}$, as in Section 5.1.

Recall that

$$|A_\Gamma|=|\mathrm{Aut}(\Gamma)|\prod_{e\in E}d(e),$$

so

$$\begin{aligned}I^\Gamma(\tau)&=\frac{1}{|\mathrm{Aut}(\Gamma)|}\prod_{v\in V(\Gamma)^{(0)}}(|\mathrm{Aut}(\nu(v))|I^0_{g(v),\nu(v)}(\tau))\\&\quad\cdot\prod_{v\in V(\Gamma)^{(1)}}\left((-1)^{g(v)+\mathrm{val}(v)-1}\prod_{(v,e)\in F(\Gamma)}d(e)\right)\\&\quad\cdot\int_{[\overline{\mathcal{M}}_\Gamma^{(1)}]^{\mathrm{vir}}}\frac{(\tau u)^{d_\Gamma^{(1)}+1}}{-u-\psi^t}\\&=(-\tau)^{d_\Gamma^{(1)}+1}\frac{1}{|\mathrm{Aut}(\Gamma)|}\prod_{v\in V(\Gamma)^{(0)}}(|\mathrm{Aut}(\nu(v))|I^0_{g(v),\nu(v)}(p))\\&\quad\cdot\prod_{v\in V(\Gamma)^{(1)}}\left((-1)^{g(v)+\mathrm{val}(v)-1}\prod_{(v,e)\in F}d(e)\right)\int_{[\overline{\mathcal{M}}_\Gamma^{(1)}]^{\mathrm{vir}}}(\psi^t)^{d_\Gamma^{(1)}}\\&=\tau^{d_\Gamma^{(1)}+1}J^\Gamma(\tau).\end{aligned}$$

Note that $J^\Gamma(\tau)$ is a degree $r-d_\Gamma^{(1)}-1$ polynomial in $\tau$, and

$$J^\Gamma(-\tau-1)=(-1)^{d-l(\mu)+d_\Gamma^{(1)}+1}J^\Gamma(\tau).$$

### 6.3 Sum over graphs

We have

$$I^k_{g,\mu}(\tau)=\tau^k J^k_{g,\mu}(\tau),$$

where $J^0_{g,\mu}(\tau) = I^0_{g,\mu}(\tau)$, and

$$J^k_{g,\mu}(\tau) = \sum_{\Gamma\in G_{g,0}(\mathbb{P}^1,\mu),\Gamma\neq\Gamma_0 d^{(1)}_\Gamma+1=k} J^\Gamma(\tau)$$

for $k = 1,\dots,r$. Note that $J^k_{g,\mu}(\tau)$ is a degree $r-k$ polynomial in $\tau$, and

$$J^k_{g,\mu}(-\tau-1) = (-1)^{|\mu|-l(\mu)+k} J^k_{g,\mu}(\tau).$$

For $k = 1$, we have

$$d^{(1)}_\Gamma = \sum_{v\in V(\Gamma)^{(1)}} r_1(v) - 1 = 0,$$

so $V^S(\Gamma)^{(1)}$ consists of one single vertex $\bar{v}$ with $r_1(\bar{v}) = 1$. In particular, $g(\bar{v}) = 0$, and we have the following two cases:

Case 1: $\mu(\bar{v}) = (\mu_i)$, $\nu(\bar{v}) = (j,k)$, where $j+k = \mu_i$. In this case, we say $\nu \in C(\mu)$ (cut) and define $a_{\mu,\nu} = jk$. Note that

$$\begin{aligned}
&\prod_{v\in V(\Gamma)^{(1)}} \left( (-1)^{g(v)+\mathrm{val}(v)-1} \prod_{(v,e)\in F} d(e) \right) \int_{[\overline{\mathcal{M}}^{(1)}_\Gamma]^{\mathrm{vir}}} 1 \\
&= \left( -\mu_1\cdots\mu_{l(\mu)} \frac{jk}{\mu_i} \right) \frac{\mu_i}{\mu_1\cdots\mu_{l(\mu)}} \\
&= -a_{\mu,\nu}.
\end{aligned}$$

Case 2: $\mu(\bar{v}) = (\mu_i,\mu_j)$, $\nu(\bar{v}) = (\mu_i+\mu_j)$. In this case, we say $\nu \in J(\mu)$ (join) and define $b_{\mu,\nu} = \mu_i+\mu_j$. Note that

$$\begin{aligned}
&\prod_{v\in V(\Gamma)^{(1)}} \left( (-1)^{g(v)+\mathrm{val}(v)-1} \prod_{(v,e)\in F} d(e) \right) \int_{[\overline{\mathcal{M}}^{(1)}_\Gamma]^{\mathrm{vir}}} 1 \\
&= \left( \mu_1\cdots\mu_{l(\mu)} \frac{\mu_i+\mu_j}{\mu_i\mu_j} \right) \frac{\mu_i\mu_j}{\mu_1\cdots\mu_{l(\mu)}} \\
&= b_{\mu,\nu}.
\end{aligned}$$

So

$$\begin{aligned}J^1_{g,\mu}(\tau) &= -\sum_{\nu\in J(\mu)}\frac{|\operatorname{Aut}(\nu)|}{|\operatorname{Aut}(\Gamma)|}b_{\mu,\nu}I^0_{g,\nu}(\tau)+\sum_{\nu\in C(\mu)}\frac{|\operatorname{Aut}(\nu)|}{|\operatorname{Aut}(\Gamma)|}a_{\mu,\nu}I^0_{g-1,\nu}(\tau)\\ &\quad+\sum_{g_1+g_2=g,\nu^1\cup\nu^2\in C(\mu)}\frac{|\operatorname{Aut}(\nu^1)||\operatorname{Aut}(\nu^2)|}{|\operatorname{Aut}(\Gamma)|}a_{\mu,\nu^1\cup\nu^2}I^0_{g_1,\nu^1}I^0_{g_2,\nu_2}(\tau)\\ &= -\sum_{\nu\in J(\mu)}I_1(\nu)I^0_{g,\nu}(\tau)+\sum_{\nu\in C(\mu)}I_2(\nu)I^0_{g-1,\nu}(\tau)\\ &\quad+\sum_{g_1+g_2=g,\nu^1\cup\nu^2\in C(\mu)}I_3(\nu^1,\nu^2)I^0_{g_1,\nu^1}I^0_{g_2,\nu_2}(\tau),\end{aligned}$$

where $I_1$, $I_2$, and $I_3$ are defined as in [15].

We have

$$\begin{aligned}J^0_{g,\mu}(\tau) &= \sqrt{-1}^{|\mu|-l(\mu)}\mathcal{C}_{g,\mu}(\tau),\\ J^1_{g,\mu}(\tau) &= \sqrt{-1}^{|\mu|-l(\mu)-1}\left(\sum_{\nu\in J(\mu)}I_1(\nu)\mathcal{C}_{g,\nu}(\tau)+\sum_{\nu\in C(\mu)}I_2(\nu)\mathcal{C}_{g-1,\nu}(\tau)\right.\\ &\qquad\left.+\sum_{g_1+g_2=g,\nu^1\cup\nu^2\in C(\mu)}I_3(\nu^1,\nu^2)\mathcal{C}_{g_1,\nu^1}(\tau)\mathcal{C}_{g_2,\nu^2}(\tau)\right).\end{aligned}$$

It follows from the definition that (6) in Theorem 2 is equivalent to

$$\frac{d}{d\tau}J^0_{g,\mu}(\tau) = -J^1_{g,\mu}(\tau).$$

## 6.4 Final Calculations

We have

$$\begin{aligned}F(\tau,x) &= \sum_{k=0}^{r}I^{r-k}_{g,\mu}(\tau)x(x-1)\cdots(x-k+1)(x-k-1)\cdots(x-r)\\ &= \sum_{k=0}^{r}\tau^{r-k}J^{r-k}_{g,\mu}(\tau)x(x-1)\cdots(x-k+1)(x-k-1)\cdots(x-r)\\ &= \sum_{k=0}^{r}\tau^{k}J^{k}_{g,\mu}(\tau)x(x-1)\cdots\\ &\qquad(x-(r-k-1))(x-(r-k+1))\cdots(x-r).\end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{Br}_* e_T(V) = \sum_{k=0}^{r} \tau^k J^k_{g,\mu}(\tau) H(H-u)\cdots(H-(r-k-1)u)\\ \cdot (H-(r-k+1)u)\cdots(H-ru).\end{aligned}$$

For $i = 0, \ldots, r-1$, we have

$$\begin{aligned}&H^i H(H-u)\cdots(H-(r-k-1)u)(H-(r-k+1)u)\cdots(H-ru)\\ &= ((H-(r-k)u)+(r-k)u)^i H(H-u)\cdots(H-(r-k-1)u)\\ &\quad\cdot(H-(r-k+1)u)\cdots(H-ru)\\ &= ((r-k)u)^i H(H-u)\cdots(H-(r-k-1)u)\\ &\quad\cdot(H-(r-k+1)u)\cdots(H-ru)\end{aligned}$$

since

$$H(H-u)\cdots(H-ru) = 0.$$

Therefore,

$$\int_{\mathbb{P}^r} \mathrm{Br}_* e_T(V) H^i = u^i \sum_{k=0}^{r} (r-k)^i \tau^k J^k_{g,\mu}(\tau).$$

Let $J^k_{g,\mu}(\tau) = \sum_{j=0}^{r-k} a^k_j \tau^j$. We have

$$u^{-i} \int_{\mathbb{P}^r} \mathrm{Br}_* e_T(V) H^i = \sum_{l=0}^{r} \left( \sum_{j+k=l} (r-k)^i a^k_j \right) \tau^l.$$

Here is a crucial observation: as a polynomial in $\tau$, $u^{-i}\int_{\mathbb{P}^r} \mathrm{Br}_* e_T(V) H^i$ is of degree no more than $i$. Therefore,

$$\sum_{j+k=l} (r-k)^i a^k_j = 0$$

for $0 \le i < l \le r$. Now fix $l$ such that $1 \le l \le r$. We have

$$\sum_{k=0}^{l} (r-k)^i a^k_{l-k} = 0, \qquad 0 \le i < l, \tag{16}$$

which is a system of $l$ linear equations of the $l+1$ variables $\{a^k_{l-k} : k = 0, \ldots, l\}$.

Both

$$\{(r-t)^i : i = 0, \dots, l-1\}$$

and

$$\{1, t, t(t-1), \dots, t(t-1)\dots(t-l+2)\}$$

are bases of the vector space

$$\{f(t) \in \mathbb{Q}[t] : \deg(f) \le l-1\},$$

so there exists an invertible $l \times l$ matrix $(A_{ij})_{0 \le i,j \le l-1}$ such that

$$t(t-1)\cdots(t-i+1) = \sum_{j=0}^{l-1} A_{ij}(r-t)^j.$$

In particular,

$$k(k-1)\cdots(k-i+1) = \sum_{j=0}^{l-1} A_{ij}(r-k)^j.$$

for $k = 0, 1, \dots, l$, so (16) is equivalent to

$$\sum_{k=0}^{l} k(k-1)\cdots(k-i+1)a_{l-k}^k = 0, \quad 0 \le i < l,$$

i.e.,

$$\sum_{k=i}^{l} \frac{k!}{(k-i)!} a_{l-k}^k = 0, \quad 0 \le i < l.$$

The above equations can be rewritten as

$$\begin{pmatrix} 1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 \\ 0 & 1! & 2 & \cdots & \cdots & \cdots & l \\ 0 & 0 & 2! & 3\cdot 2 & \cdots & \cdots & l(l-1) \\ 0 & 0 & 0 & 3! & \cdots & \cdots & l(l-1)(l-2) \\ 0 & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & (l-1)! & l(l-1)\cdots 2 \end{pmatrix} \begin{pmatrix} a_l^0 \\ a_{l-1}^1 \\ \vdots \\ \vdots \\ a_0^l \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{pmatrix}.$$

The kernel is clearly one dimensional. One can check that the kernel is given by

$$a_{l-k}^k = (-1)^k \frac{l!}{k!(l-k)!} a_l^0. \tag{17}$$

Note that (17) for $l = 1, \ldots, r$ is equivalent to

$$J_{g,\mu}^k(\tau) = \frac{(-1)^k}{k!} \frac{d^k}{d\tau^k} J_{g,\mu}^0(\tau) \tag{18}$$

for $k = 0, \ldots, r$. In particular,

$$J_{g,\mu}^1(\tau) = -\frac{d}{d\tau} J_{g,\mu}^0(\tau)$$

which is equivalent to the cut-and-join equation (6) in Theorem 2. Equation (18) and the cut-and-join equation imply that $J_{g,\mu}^k(\tau)$ can be obtained from $J_{g,\mu}^0(\tau)$ by repeating the cut-and-join operation $k$-times.

## 7. Hurwitz numbers

In this section, inspired by T. Graber and R. Vakil [9], we use virtual localization on moduli spaces of relative stable morphisms to recover some results on Hurwitz numbers. We give a unified proof of the ELSV formula and the cut-and-join equation for Hurwitz numbers.

The Hirwitz numbers can be defined as

$$H_{g,\mu} = \int_{[\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1,\mu)]^{\mathrm{vir}}} \mathrm{Br}^* H^r. \tag{19}$$

We lift $H^r \in H^{2r}(\mathbb{P}^r;\mathbb{Z})$ to $\prod_{k=1}^r (H - w_k u) \in H_T^{2r}(\mathbb{P}^r;\mathbb{Z})$, where $w_k \in \mathbb{Z}$, and compute

$$H_{g,\mu} = \int_{[\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1,\mu)]^{\mathrm{vir}}} \mathrm{Br}^* \left( \prod_{k=1}^r (H - w_k u) \right)$$

by virtual localization.

Let $p_k \in \mathbb{P}^r$ be the $\mathbb{C}^*$ fixed point defined as in Section 4.3, and $f_k : p_k \to \mathbb{P}^r$ be the inclusion. Then

$$f_k^* \Big( \prod_{l=1}^r (H - w_k u) \Big) = \left( \prod_{l=1}^r (k - w_l) \right) u^r \in H_T^{2r}(p_k).$$

**Remark 7.1.** In the definition of $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu)$ given in Section 4.1, if we order the $l(\mu)$ marked points on the domain as in [16, 17], there will be an extra factor $1/|\operatorname{Aut}(\mu)|$ on the right-hand side of (19).

## 7.1 Contribution from each graph

### 7.1.1 The target is $\mathbb{P}^1$

Consider the graph $\Gamma^0 \in \overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu)$. We have $\mathrm{Br}(F_{\Gamma^0}) = \{p_r\}$. We first consider the stable case. By the Feynman rules derived in Appendix A, the contribution from $\Gamma^0$ is given by

$$\prod_{l=1}^{r}(r - w_l)\widetilde{I}^0_{g,\mu},$$

where

$$\begin{aligned}\widetilde{I}^0_{g,\mu} &= \frac{1}{|A_\Gamma|}\int_{\overline{\mathcal{M}}_{g,l(\mu)}} \frac{u^r}{e(N_\Gamma^{\mathrm{vir}})} \\ &= \frac{1}{|\mathrm{Aut}(\mu)|}\prod_{i=1}^{l(\mu)}\frac{\mu_i^{\mu_i}}{\mu_i!}\int_{\overline{\mathcal{M}}_{g,l(\mu)}} \frac{\Lambda_g^\vee(u)u^{2g+2l(\mu)-3}}{\prod_{i=1}^{l(\mu)}(u-\mu_i\psi_i)} \\ &= \frac{1}{|\mathrm{Aut}(\mu)|}\prod_{i=1}^{l(\mu)}\frac{\mu_i^{\mu_i}}{\mu_i!}\int_{\overline{\mathcal{M}}_{g,l(\mu)}} \frac{\Lambda_g^\vee(1)}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)}\end{aligned}$$

In particular,

$$\begin{aligned}\widetilde{I}^0_{0,\mu} &= \frac{1}{|\mathrm{Aut}(\mu)|}\prod_{i=1}^{l(\mu)}\frac{\mu_i^{\mu_i}}{\mu_i!}\int_{\overline{\mathcal{M}}_{0,l(\mu)}} \frac{1}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)} \\ &= \frac{1}{|\mathrm{Aut}(\mu)|}\left(\prod_{i=1}^{l(\mu)}\frac{\mu_i^{\mu_i}}{\mu_i!}\right)d^{l(\mu)-3}.\end{aligned}$$

The contribution from $\Gamma^{(0)}$ for the two unstable cases is also given by the above formula:

$$\begin{aligned}\widetilde{I}^0_{0,(d)} &= \frac{d^d}{d!}\frac{1}{d^2} = \frac{d^{d-2}}{d!}. \\ \widetilde{I}^0_{0,(\mu_1,\mu_2)} &= \frac{1}{|\mathrm{Aut}((\mu_1,\mu_2))|}\frac{\mu_1^{\mu_1}\mu_2^{\mu_2}}{\mu_1!\mu_2!}\frac{1}{d}.\end{aligned}$$

### 7.1.2 The target is $\mathbb{P}^1[m]$, $m > 0$

Consider $\Gamma \in G_{g,0}(\mathbb{P}^1, \mu)$, $\Gamma \neq \Gamma^0$. We have $\mathrm{Br}(F_\Gamma) = \{p_{r-d_\Gamma^{(1)}-1}\}$. The contribution from $\Gamma$ is given by

$$\prod_{l=1}^{r}(r - d_\Gamma^{(1)} - 1 - w_l)\widetilde{I}^\Gamma,$$

where

$$\widetilde{I}^{\Gamma} = \frac{1}{|A_\Gamma|}\int_{[\overline{\mathcal{M}}_\Gamma]^{\mathrm{vir}}} \frac{u^r}{e_T(N_\Gamma^{\mathrm{vir}})} = \frac{1}{|A_\Gamma|}\int_{[\overline{\mathcal{M}}_\Gamma]^{\mathrm{vir}}} \frac{\prod_{v\in V(\Gamma)}\widetilde{B}_v}{-u-\psi^t},$$

$$\widetilde{B}_v = \begin{cases} u^{2g(v)-2+\mathrm{val}(v)}, A_v\prod_{(v,e)\in F(\Gamma)}(A_e u^{d(e)}) & v\in V(\Gamma)^{(0)},\\ u^{r_1(v)}A_v, & v\in V(\Gamma)^{(1)}. \end{cases}$$

More explicitly, in the notation of Appendix A,

$$\begin{aligned}\widetilde{B}_v = &\left(\prod_{(v,e)\in F(\Gamma)} d(e)\right)\left(\prod_{(v,e)\in F(\Gamma)}\frac{d(e)^{d(e)}}{d(e)!}\right)\\ &\cdot\frac{\Lambda^\vee_{g(v)}(u)u^{2g(v)+\mathrm{val}(v)-3}}{\prod_{(v,e)\in F(\Gamma)}\big(u-d(e)\psi_{(v,e)}\big)}, \quad v\in V^S(\Gamma)^{(0)},\\ &d(e)\frac{d(e)^{d(e)-2}}{d(e)!}, \quad v\in V^I(\Gamma)^{(0)}, (v,e)\in F(\Gamma),\\ &d(e_1)d(e_2)\frac{d(e_1)^{d(e_1)}d(e_2)^{d(e)_2}}{d(e_1)!d(e_2)!}\frac{1}{d(v)},\\ &\qquad\qquad v\in V^{II}(\Gamma)^{(0)}, (v,e_1),(v,e_2)\in F(\Gamma),\\ &u^{r_1(v)}\prod_{(v,e)\in F(\Gamma)} d(e), \quad v\in V(\Gamma)^{(1)}.\end{aligned}$$

So

$$\widetilde{I}^{\Gamma} = \frac{1}{|A_\Gamma|}\prod_{v\in V(\Gamma)^{(0)}}\Big(|\mathrm{Aut}(\nu(v))|\widetilde{I}^0_{g(v),\nu(v)}\Big) \cdot\left(\prod_{e\in E(\Gamma)} d(e)\right)^2\int_{[\overline{\mathcal{M}}^{(1)}_\Gamma]^{\mathrm{vir}}}\frac{u^{d^{(1)}_\Gamma+1}}{-u-\psi^t}.$$

Recall that

$$|A_\Gamma| = |\mathrm{Aut}(\Gamma)|\prod_{e\in E} d(e)\prod_{v\in V^{II}(\Gamma)^{(1)}} d(v),$$

so

$$\widetilde{I}^{\Gamma} = \frac{(-1)^{d_\Gamma^{(1)}+1}}{|\mathrm{Aut}(\Gamma)|} \prod_{v\in V(\Gamma)^{(0)}} (|\mathrm{Aut}(\nu(v))| I^0_{g(v),\nu(v)}) \cdot \left( \prod_{v\in V^S(\Gamma)^{(1)}} \prod_{(v,e)\in F} d(e) \right) \int_{[\overline{\mathcal{M}}_\Gamma^{(1)}]^{\mathrm{vir}}} (\psi^t)^{d_\Gamma^{(1)}}.$$

## 7.2 Sum over Graphs

For $k = 1, \ldots, r$ define

$$\widetilde{I}^k_{g,\mu} = \sum_{\Gamma\in G_{g,0}(\mathbb{P}^1,\mu), \Gamma\neq\Gamma^0, d_\Gamma^{(1)}+1=k} \widetilde{I}^{\Gamma}.$$

Then

$$H_{g,\mu} = \sum_{k=0}^{r} \prod_{l=1}^{r} (r-k-w_l) \widetilde{I}^k_{g,\mu}.$$

for any $w_1, \ldots, w_r \in \mathbb{Z}$.

Setting $(w_1, \ldots, w_r) = (0, 1, \ldots, r-k-1, r-k+1, \ldots, r)$ yields

$$H_{g,\mu} = (-1)^k k!(r-k)! \widetilde{I}^k_{g,\mu}.$$

The $k = 0$ case implies the following formula due to Ekedahl, Lando, Shapiro, and Vainshtein [2], also proved by T. Graber and R. Vakil [9]:

$$H_{g,\mu} = r! \widetilde{I}^0_{g,\mu} = \frac{r!}{|\mathrm{Aut}(\mu)|} \prod_{i=1}^{l(\mu)} \frac{\mu_i^{\mu^i}}{\mu_i!} \int_{\overline{\mathcal{M}}_{g,l(\mu)}} \frac{\Lambda_g^\vee(1)}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)}. \tag{20}$$

For $k = 1$, we have

$$H_{g,\mu} = -(r-1)! \widetilde{I}^1_{g,\mu},$$

which is, by a derivation similar to that in Section 6.3, equivalent to

$$\frac{H_{g,\mu}}{(r-1)!} = \sum_{\nu\in J(\mu)} I_1(\nu) \widetilde{I}^0_{g,\nu} + \sum_{\nu\in C(\mu)} I_2(\nu) \widetilde{I}^0_{g-1,\nu} + \sum_{g_1+g_2=g} \sum_{\nu^1\cup\nu^2\in C(\mu)} I_3(\nu^1,\nu^2) \widetilde{I}^0_{g_1,\nu^1} \widetilde{I}^0_{g_2,\nu_2}. \tag{21}$$

Combining (20) and (21), one obtains

$$\begin{aligned}\frac{H_{g,\mu}}{(r-1)!} = &\sum_{\nu\in J(\mu)} I_1(\nu)\frac{H_{g,\nu}}{(r-1)!} + \sum_{\nu\in C(\mu)} I_2(\nu)\frac{H_{g-1,\nu}}{(r-1)!}\\ &+ \sum_{g_1+g_2=g}\ \sum_{\nu^1\cup\nu^2\in C(\mu)} I_3(\nu^1,\nu^2)\\ &\cdot \frac{H_{g_1,\nu^1}}{(2g_1-2+|\nu^1|+l(\nu^1))!}\frac{H_{g_2,\nu^2}}{(2g_2-2+|\nu^2|+l(\nu^2))!}\end{aligned}$$

which is equivalent to the cut-and-join equation

$$\begin{aligned}H_{g,\mu} = &\sum_{\nu\in J(\mu)} I_1(\nu)H_{g,\nu} + \sum_{\nu\in C(\mu)} I_2(\nu)H_{g-1,\nu}\\ &+ \sum_{g_1+g_2=g}\ \sum_{\nu^1\cup\nu^2\in C(\mu)} \begin{pmatrix} r-1 \\ 2g_1-2+|\nu^1|+l(\nu^1)\end{pmatrix}\\ &\cdot I_3(\nu^1,\nu^2)H_{g_1,\nu^1}H_{g_2,\nu^2}.\end{aligned} \tag{22}$$

The above cut-and-join equation was first proved using combinatorics by Goulden, Jackson and Vainstein in [7] and later proved using symplectic sum formula by Li-Zhao-Zheng [15] and Ionel-Parker [11, Section 15.2].

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## Appendix A. Localization

In this appendix, we provide details of localization in our particular case. Related results are discussed in [10]. We first introduce some notation.

Let $(w)$ be the 1-dimensional representation of $\mathbb{C}^*$ given by $\lambda\cdot z = \lambda^w z$ for $\lambda\in\mathbb{C}^*$, $z\in\mathbb{C}$. We do calculations on $\overline{\mathcal{M}}_\Gamma$ which is a finite cover of $F_\Gamma\subset\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1,\mu)$, so the weight $w$ of $\mathbb{C}^*$-action can be fractional.

Given $\Gamma \in G_{g,0}(\mathbb{P}^1,\mu)$, let

$$\big[f:(C,x_1,\ldots,x_{l(\mu)})\to\mathbb{P}^1[m]\ \big]$$

be a fixed point of the $\mathbb{C}^*$-action on $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1,\mu)$ associated to $\Gamma$. Given $(v,e)\in F(\Gamma)$, where $v\in V^S(\Gamma)^{(0)}\cup V(\Gamma)^{(1)}$, let $q_{(v,e)}\in C$ denote the node at which $C_v$ and $C_e$ intersect. Let $\psi_{(v,e)}$ denote the first Chern class of the line bundle over $\overline{\mathcal{M}}_\Gamma$ whose fiber at

$$\big[f:(C,x_1,\ldots,x_{l(\mu)})\to\mathbb{P}^1[m]\ \big]$$

is $T^*_{q_{(v,e)}}C_v$. Given $v\in V^{II}(\Gamma)^{(0)}$, let $q_v$ denote the node at which $C_{e_1}$ and $C_{e_2}$ intersect, where $\{e_1,e_2\}=\{e\in E(\Gamma):(v,e)\in F(\Gamma)\}$.

## A.1 Virtual Normal Bundle

The tangent space $T^1$ and the obstruction space $T^2$ of $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1,\mu)$ at

$$\big[f:(C,x_1,\ldots,x_{l(\mu)})\to\mathbb{P}^1[m]\ \big]\in\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1,\mu)$$

are given by the following two exact sequences [17, Section 5.1]:

$$\begin{aligned}
&0\to \mathrm{Ext}^0(\Omega_C(D),\mathcal{O}_C)\to H^0(\mathbf{D}^\bullet)\to T^1\\
&\quad\to \mathrm{Ext}^1(\Omega_C(D),\mathcal{O}_C)\to H^1(\mathbf{D}^\bullet)\to T^2\to 0\\
&0\to H^0(C,f^*(\omega_{\mathbb{P}^1[m]}(\log p_1^{(m)}))^\vee)\to H^0(\mathbf{D}^\bullet)\to \oplus_{l=0}^{m-1}H^0_{\mathrm{et}}(\mathbf{R}_l^\bullet)\\
&\quad\to H^1(C,f^*(\omega_{\mathbb{P}^1[m]}(\log p_1^{(m)}))^\vee)\to H^1(\mathbf{D}^\bullet)\to \oplus_{l=0}^{m-1}H^1_{\mathrm{et}}(\mathbf{R}_l^\bullet)\to 0
\end{aligned}$$

where $D=x_1+\cdots+x_{l(\mu)}$, $\omega_{\mathbb{P}^1[m]}$ is the dualizing sheaf of $\mathbb{P}^1[m]$,

$$\begin{aligned}
H^0_{\mathrm{et}}(\mathbf{R}_l^\bullet)&\cong\bigoplus_{q\in f^{-1}(p_1^{(l)})}T_q\left(f^{-1}(\mathbb{P}^1_{(l)})\right)\otimes T_q^*\left(f^{-1}(\mathbb{P}^1_{(l)})\right)\cong\mathbb{C}^{\oplus n_l},\\
H^1_{\mathrm{et}}(\mathbf{R}_l^\bullet)&\cong(T_{p_1^{(l)}}\mathbb{P}^1_{(l)}\otimes T_{p_1^{(l)}}\mathbb{P}^1_{(l+1)})^{\oplus(n_l-1)},
\end{aligned}$$

and $n_l$ is the number of nodes over $p_1^{(l)}$.

Let $\widetilde{f}=\pi[m]\circ f:C\to\mathbb{P}^1$. Then

$$f^*(\omega_{\mathbb{P}^1[m]}(\log p_1^{(m)}))^\vee\cong\widetilde{f}^*\mathcal{O}_{\mathbb{P}^1}(1).$$

Let

$$\begin{aligned}
B_1&=\mathrm{Ext}^0(\Omega_C(D),\mathcal{O}_C), & B_2&=H^0(C,\widetilde{f}^*\mathcal{O}_{\mathbb{P}^1}(1)),\\
B_3&=\oplus_{l=0}^{m-1}H^0_{\mathrm{et}}(\mathbf{R}_l^\bullet), & B_4&=\mathrm{Ext}^1(\Omega_C(D),\mathcal{O}_C),\\
B_5&=H^1(C,\widetilde{f}^*\mathcal{O}_{\mathbb{P}^1}(1)), & B_6&=\oplus_{l=0}^{m-1}H^1_{\mathrm{et}}(\mathbf{R}_l^\bullet).
\end{aligned}$$

We now assume that

$$[f : (C, x_1, \dots, x_{l(\mu)}) \to \mathbb{P}^1[m]] \in F_\Gamma \subset \overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu),$$

where $\Gamma \in G_{g,0}(\mathbb{P}^1, \mu)$, and $F_\Gamma$ is the set of fixed points associated to $\Gamma$. The $\mathbb{C}^*$-action on $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu)$ induces $\mathbb{C}^*$-actions on $B_1, \dots, B_6$. In particular, the $\mathbb{C}^*$-actions on $B_2$ and $B_5$ come from the $\mathbb{C}^*$-action on $\mathcal{O}_{\mathbb{P}^1}(1)$ which acts on $\mathcal{O}_{\mathbb{P}^1}(1)_{p_0}$ and $\mathcal{O}_{\mathbb{P}^1}(1)_{p_1}$ by (1) and (0), respectively. Let $\widetilde{B}_i$ denote the moving part of $B_i$ under the $\mathbb{C}^*$-action. Then each $\widetilde{B}_i$ form a vector bundle over $\overline{\mathcal{M}}_\Gamma$. We will use the same notation $\widetilde{B}_i$ to denote the vector bundle.

Note that $\widetilde{B}_3 = 0$, and

$$\widetilde{B}_6 = \begin{cases} 0, & m = 0, \\ H^1_{\text{et}}(\mathrm{R}^\bullet_0) = (T_{p_1^{(0)}}\mathbb{P}^1_{(0)} \otimes T_{p_1^{(0)}}\mathbb{P}^1_{(1)})^{\oplus(n_0-1)}, & m > 0. \end{cases}$$

We have

$$\frac{1}{e_T(N_\Gamma^{\text{vir}})} = \frac{e_T(\widetilde{T}^2)}{e_T(\widetilde{T}^1)} = \frac{e_T(\widetilde{B}_1)e_T(\widetilde{B}_5)e_T(\widetilde{B}_6)}{e_T(\widetilde{B}_2)e_T(\widetilde{B}_4)}$$

where $\widetilde{T}^1$, $\widetilde{T}^2$ are the moving parts of $T^1$, $T^2$, respectively.

### A.1.1 The target is $\mathbb{P}^1$

We have seen that there is only one graph $\Gamma^0$. Recall that

$$\overline{\mathcal{M}}_{\Gamma^0} = \begin{cases} \{\text{point}\} & (g, l(\mu)) = (0,1), (0,2), \\ \overline{\mathcal{M}}_{g,l(\mu)} & (g, l(\mu)) \neq (0,1), (0,2). \end{cases}$$

We first compute $e_T(\widetilde{B}_1)/e_T(\widetilde{B}_4)$. When $(g, l(\mu)) \neq (0,1), (0,2)$. The domain is

$$C = C_{v_0} \cup C_{e_1} \cup \cdots C_{e_{l(\mu)}}.$$

We have

$$\widetilde{B}_1 = 0, \quad \widetilde{B}_4 = \bigoplus_{i=1}^{l(\mu)} T_{q_i} C_{v_0} \otimes T_{q_i} C_{e_i},$$

where $q_i = q_{(v_0,e_i)}$. So

$$\frac{e_T(\widetilde{B}_1)}{e_T(\widetilde{B}_4)} = \frac{1}{\prod_{i=1}^{l(\mu)} \left(\frac{u}{\mu_i} - \psi_i\right)},$$

where $\psi_i = \psi_{(v_0,e_i)}$.

When $(g, l(\mu)) = (0, 1)$, we have

$$\widetilde{B}_1 = \left(\frac{1}{d}\right), \quad \widetilde{B}_4 = 0,$$

so

$$\frac{e_T(\widetilde{B}_1)}{e_T(\widetilde{B}_4)} = \frac{u}{d}.$$

When $(g, l(\mu)) = (0, 2)$, we have

$$\widetilde{B}_1 = 0, \quad \widetilde{B}_4 = \left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right).$$

so

$$\frac{e_T(\widetilde{B}_1)}{e_T(\widetilde{B}_4)} = \frac{1}{\frac{u}{\mu_1} + \frac{u}{\mu_2}}.$$

We next compute $e_T(\widetilde{B}_5)/e_T(\widetilde{B}_2)$. When $(g, l(\mu)) \neq (0, 1), (0, 2)$, consider the normalization sequence

$$\begin{aligned} 0 \to \widetilde{f}^*\mathcal{O}_{\mathbb{P}^1}(1) \to (\widetilde{f}|_{C_{v_0}})^*\mathcal{O}_{\mathbb{P}^1}(1) \oplus \bigoplus_{i=1}^{l(\mu)} (\widetilde{f}|_{C_{e_i}})^*\mathcal{O}_{\mathbb{P}^1}(1) \\ \to \bigoplus_{i=1}^{l(\mu)} \mathcal{O}_{\mathbb{P}^1}(1)_{p_0} \to 0. \end{aligned}$$

The corresponding long exact sequence reads

$$\begin{aligned} 0 &\to H^0(C, \widetilde{f}^*\mathcal{O}_{\mathbb{P}^1}(1)) \\ &\to H^0(C_{v_0}, (\widetilde{f}|_{C_{v_0}})^*\mathcal{O}_{\mathbb{P}^1}(1)) \oplus \bigoplus_{i=1}^{l(\mu)} H^0(C_{e_i}, (\widetilde{f}|_{C_{e_i}})^*\mathcal{O}_{\mathbb{P}^1}(1)) \\ &\to \bigoplus_{i=1}^{l(\mu)} \mathcal{O}_{\mathbb{P}^1}(1)_{p_0} \to H^1(C, \widetilde{f}^*\mathcal{O}_{\mathbb{P}^1}(1)) \\ &\to H^1(C_{v_0}, (\widetilde{f}|_{C_{v_0}})^*\mathcal{O}_{\mathbb{P}^1}(1)) \oplus \bigoplus_{i=1}^{l(\mu)} H^1(C_{e_i}, (\widetilde{f}|_{C_{e_i}})^*\mathcal{O}_{\mathbb{P}^1}(1)) \to 0. \end{aligned}$$

The representations of $\mathbb{C}^*$ are

$$\begin{aligned} 0 \to H^0(C, \widetilde{f}^*\mathcal{O}_{\mathbb{P}^1}(1)) \to H^0(C_{v_0}, \mathcal{O}_{C_{v_0}}) \otimes (1) \oplus \bigoplus_{i=1}^{l(\mu)} \left(\bigoplus_{a=1}^{\mu_i} \left(\frac{a}{\mu_i}\right)\right) \\ \to \bigoplus_{i=1}^{l(\mu)} (1) \to H^1(C, \widetilde{f}^*\mathcal{O}_{\mathbb{P}^1}(1)) \to H^1(C_{v_0}, \mathcal{O}_{C_{v_0}}) \otimes (1) \to 0 \end{aligned}$$

So we have

$$\frac{e_T(\widetilde{B}_5)}{e_T(\widetilde{B}_2)} = \Lambda_g^\vee(u)u^{l(\mu)-1}\prod_{i=1}^{l(\mu)}\left(\frac{\mu_i^{\mu_i}}{\mu_i!}u^{-\mu_i}\right). \tag{23}$$

One can check that (23) is also valid for $(g, l(\mu)) = (0,1), (0,2)$.

Finally, $\widetilde{B}_6 = 0$, so $e_T(\widetilde{B}_6) = 1$. We obtain the following Feynman rules:

$$\frac{1}{e_T(N_{\Gamma^0}^{\mathrm{vir}})} = A_{v_0}\prod_{e\in E(\Gamma^0)} A_e,$$

where

$$A_{v_0} = \begin{cases} \frac{u}{d}, & (g,l(\mu)) = (0,1) \\ \frac{u}{\frac{u}{\mu_1}+\frac{u}{\mu_2}}, & (g,l(\mu)) = (0,2) \\ \frac{\Lambda_g^\vee(u)}{u}\prod_{i=1}^{l(\mu)}\frac{u}{\frac{u}{\mu_i}-\psi_i}, & (g,l(\mu)) \neq (0,1),(0,2) \end{cases}$$

$$A_e = \frac{d(e)^{d(e)}}{d(e)!}u^{-d(e)}.$$

### A.1.2 The target is $\mathbb{P}^1[m]$, $m > 0$

Let $\Gamma \in G_{g,0}(\mathbb{P}^1, \mu)$, $\Gamma \neq \Gamma^0$. We first compute $e_T(\widetilde{B}_1)/e_T(\widetilde{B}_4)$. We have

$$\begin{aligned}\widetilde{B}_1 &= \bigoplus_{v\in V^I(\Gamma)^{(0)}}\left(\frac{1}{d(v)}\right) \\ \widetilde{B}_4 &= \bigoplus_{v\in V^{II}(\Gamma)^{(0)}}\left(\sum_{(v,e)\in F(\Gamma)}\frac{1}{d(e)}\right) \\ &\quad\oplus \bigoplus_{v\in V^S(\Gamma)^{(0)}}\left(\bigoplus_{(v,e)\in F(\Gamma)} T_{q_{(v,e)}}C_v \otimes T_{q_{(v,e)}}C_e\right) \\ &\quad\oplus \bigoplus_{v\in V(\Gamma)^{(1)}}\left(\bigoplus_{(v,e)\in F(\Gamma)} T_{q_{(v,e)}}C_v \otimes T_{q_{(v,e)}}C_e\right)\end{aligned}$$

So

$$\frac{e_T(\widetilde{B}_1)}{e_T(\widetilde{B}_4)} = \prod_{v\in V^I(\Gamma)^{(0)}} \frac{u}{d(v)} \prod_{v\in V^{II}(\Gamma)^{(0)}} \left(\sum_{(v,e)\in F(\Gamma)} \frac{u}{d(e)}\right)^{-1} \cdot \prod_{v\in V^S(\Gamma)^{(0)}} \left(\sum_{(v,e)\in F(\Gamma)} \frac{1}{\frac{u}{d(e)} - \psi_{(v,e)}}\right) \cdot \prod_{v\in V(\Gamma)^{(1)}} \left(\prod_{(v,e)\in F(\Gamma)} \frac{1}{\frac{-u}{d(e)} - \psi_{(v,e)}}\right)$$

We next compute $e_T(\widetilde{B}_5)/e_T(\widetilde{B}_2)$. Consider the normalization sequence

$$0 \to \widetilde{f}^*\mathcal{O}_{\mathbb{P}^1}(1) \to \bigoplus_{v\in V^S(\Gamma)^{(0)}\cup V(\Gamma)^{(1)}} (\widetilde{f}|_{C_v})^*\mathcal{O}_{\mathbb{P}^1}(1) \oplus \bigoplus_{e\in E(\Gamma)} (\widetilde{f}|_{C_e})^*\mathcal{O}_{\mathbb{P}^1}(1)$$
$$\to \bigoplus_{v\in V^{II}(\Gamma)^{(0)}} \mathcal{O}_{\mathbb{P}^1}(1)_{p_0} \oplus \bigoplus_{v\in V^S(\Gamma)^{(0)}} \left(\bigoplus_{(v,e)\in F} \mathcal{O}_{\mathbb{P}^1}(1)_{p_0}\right)$$
$$\oplus \bigoplus_{v\in V(\Gamma)^{(1)}} \left(\bigoplus_{(v,e)\in F} \mathcal{O}_{\mathbb{P}^1}(1)_{p_1}\right) \to 0.$$

The corresponding long exact sequence reads

$$0 \to H^0(C, \widetilde{f}^*\mathcal{O}_{\mathbb{P}^1}(1))$$
$$\to \bigoplus_{v\in V^S(\Gamma)^{(0)}\cup V(\Gamma)^{(1)}} H^0(C_v, (\widetilde{f}|_{C_v})^*\mathcal{O}_{\mathbb{P}^1}(1))$$
$$\oplus \bigoplus_{e\in E(\Gamma)} H^0(C_e, (\widetilde{f}|_{C_e})^*\mathcal{O}_{\mathbb{P}^1}(1))$$
$$\to \bigoplus_{v\in V^{II}(\Gamma)^{(0)}} \mathcal{O}_{\mathbb{P}^1}(1)_{p_0} \oplus \bigoplus_{v\in V^S(\Gamma)^{(0)}} \left(\bigoplus_{(v,e)\in F} \mathcal{O}_{\mathbb{P}^1}(1)_{p_0}\right)$$
$$\oplus \bigoplus_{v\in V(\Gamma)^{(1)}} \left(\bigoplus_{(v,e)\in F} \mathcal{O}_{\mathbb{P}^1}(1)_{p_1}\right)$$
$$\to H^1(C, \widetilde{f}^*\mathcal{O}_{\mathbb{P}^1}(1))$$

$$\to \bigoplus_{v\in V^S(\Gamma)^{(0)}\cup V(\Gamma)^{(1)}} H^0(C_v,(\widetilde{f}|_{C_v})^*\mathcal{O}_{\mathbb{P}^1}(1))$$
$$\oplus \bigoplus_{e\in E(\Gamma)} H^0(C_e,(\widetilde{f}|_{C_e})^*\mathcal{O}_{\mathbb{P}^1}(1))\to 0.$$

The representations of $\mathbb{C}^*$ are

$$0\to H^0(C,\widetilde{f}^*\mathcal{O}_{\mathbb{P}^1}(1))$$
$$\to \bigoplus_{v\in V^S(\Gamma)^{(0)}} H^0(C_v,\mathcal{O}_{C_v})\otimes(1)\oplus \bigoplus_{v\in V(\Gamma)^{(1)}} H^0(C_v,\mathcal{O}_{C_v})\otimes(0)$$
$$\oplus \bigoplus_{e\in E(\Gamma)}\left(\bigoplus_{a=1}^{d(e)}\left(\frac{a}{d(e)}\right)\right)$$
$$\to \bigoplus_{v\in V^{II}(\Gamma)^{(0)}}(1)\oplus \bigoplus_{v\in V^S(\Gamma)^{(0)}}\left(\bigoplus_{(v,e)\in F}(1)\right)\oplus \bigoplus_{v\in V(\Gamma)^{(1)}}\left(\bigoplus_{(v,e)\in F}(0)\right)$$
$$\to H^1(C,\widetilde{f}^*\mathcal{O}_{\mathbb{P}^1}(1))$$
$$\to \bigoplus_{v\in V^S(\Gamma)^{(0)}} H^1(C_v,\mathcal{O}_{C_v})\otimes(1)\oplus \bigoplus_{v\in V(\Gamma)^{(1)}} H^1(C_v,\mathcal{O}_{C_v})\otimes(0)\to 0$$

So

$$\frac{e_T(\widetilde{B}_5)}{e_T(\widetilde{B}_2)}=\prod_{v\in V(\Gamma)^{(0)}}\left(\Lambda^\vee_{g(v)}(u)u^{\mathrm{val}(v)-1}\right)\prod_{e\in E(\Gamma)}\left(\frac{d(e)^{d(e)}}{d(e)!}u^{-d(e)}\right).$$

Finally,

$$\widetilde{B}_6=\left(T_{p_1^{(0)}}\mathbb{P}^1_{(0)}\otimes T_{p_1^{(0)}}\mathbb{P}^1_{(1)}\right)^{|E(\Gamma)|-1},$$

so

$$e_T(\widetilde{B}_6)=(-u-\psi^t)^{|E(\Gamma)|-1},$$

where $\psi^t$ is the first Chern class of the line bundle over $\overline{\mathcal{M}}_\Gamma^{(1)}$ whose fiber at

$$[\hat{f}:\hat{C}\to\mathbb{P}^1(m)]$$

is $T^*_{p_1^{(0)}}\mathbb{P}^1(m)$. Note that $\psi^t=d(e)\psi_{(v,e)}$ for $v\in V(\Gamma)^{(1)}$, $(v,e)\in F$.

We have the following Feynman rules:

$$\frac{1}{e_T(N_\Gamma^{\mathrm{vir}})}=\frac{1}{-u-\psi^t}\prod_{v\in V(\Gamma)}A_v\prod_{e\in E(\Gamma)}A_e,$$

where

$$A_v = \begin{cases} \prod_{(v,e)\in F(\Gamma)} \frac{-u-\psi^t}{\frac{-u}{d(e)}-\psi_{(v,e)}} & \\ = \prod_{(v,e)\in F(\Gamma)} d(e), & v \in V(\Gamma)^{(1)}, \\ \frac{\Lambda^\vee_{g(v)}(u)}{u} \prod_{(v,e)\in F(\Gamma)} \frac{u}{\frac{u}{d(e)}-\psi_{(v,e)}}, & v \in V^S(\Gamma)^{(0)}, \\ \frac{u}{d(v)}, & v \in V^I(\Gamma)^{(0)}, \\ \frac{1}{\frac{1}{d(e_1)}+\frac{1}{d(e_2)}}, & v \in V^{II}(\Gamma)^{(0)}, \\ & (v,e_1),(v,e_2) \in F(\Gamma), \end{cases}$$

$$A_e = \frac{d(e)^{d(e)}}{d(e)!} u^{-d(e)}.$$

The degree of $e_T(N_\Gamma^{\mathrm{vir}})$ is

$$\begin{aligned} &1 + \sum_{v\in V^S(\Gamma)^{(0)}} (1-g(v)) - |V^I(\Gamma)^{(0)}| + \sum_{e\in E(\Gamma)} d(e) \\ &= 1 + \sum_{v\in V^S(\Gamma)^{(0)}} (1-g(v)) - |V^I(\Gamma)^{(0)}| + d. \end{aligned}$$

We have seen that the virtual dimension of $F_\Gamma$ is

$$2g-3+l(\mu) + \sum_{v\in V^S(\Gamma)^{(0)}} (g(v)-1) + |V^I(\Gamma)^0|.$$

We have

$$\begin{aligned} &2g-3+l(\mu) + \sum_{v\in V^S(\Gamma)^{(0)}} (g(v)-1) + |V^I(\Gamma)^0| \\ &\quad + 1 + \sum_{v\in V^S(\Gamma)^{(0)}} (1-g(v)) - |V^I(\Gamma)^{(0)}| + d \\ &= 2g-2+d+l(\mu) \end{aligned}$$

as expected.

## A.2 The bundle $V_D$

The short exact sequence

$$0 \to \mathcal{O}_C(-D) \to \mathcal{O}_C \to \mathcal{O}_D \to 0$$

gives rise to a long exact sequence

$$0 \to H^0(C, \mathcal{O}_C(-D)) \to H^0(C, \mathcal{O}_C) \to \bigoplus_{i=1}^{l(\mu)} \mathcal{O}_{x_i}$$
$$\to H^1(C, \mathcal{O}_C(-D)) \to H^1(C, \mathcal{O}_C) \to 0$$

The representations of $\mathbb{C}^*$ are

$$0 \to (\tau) \to \bigoplus_{i=1}^{l(\mu)} (\tau) \to H^1(C, \mathcal{O}_C(-D)) \otimes (\tau) \to H^1(C, \mathcal{O}_C) \otimes (\tau) \to 0.$$

So

$$e_T(V_D) = e_T(V_0)(\tau u)^{l(\mu)-1},$$

where

$$V_0 = R^1\pi_*\mathcal{O}_{\mathcal{U}_{g,\mu}}.$$

Recall that $\pi : \mathcal{U}_{g,\mu} \to \overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, \mu)$ is the universal curve.

### A.2.1 The target is $\mathbb{P}^1$

There is only one graph $\Gamma^0$ in this case. When $(g, l(\mu)) \neq (0,1), (0,2)$, consider the normalization sequence

$$0 \to \mathcal{O}_C \to \mathcal{O}_{C_{v_0}} \oplus \bigoplus_{i=1}^{l(\mu)} \mathcal{O}_{C_{e_i}} \to \bigoplus_{i=1}^{l(\mu)} \mathcal{O}_{q_i} \to 0.$$

The corresponding long exact sequence reads

$$0 \to H^0(C, \mathcal{O}_C) \to H^0(C_{v_0}, \mathcal{O}_{C_{v_0}}) \oplus \bigoplus_{i=1}^{l(\mu)} H^0(C_{e_i}, \mathcal{O}_{C_{e_i}}) \to \bigoplus_{i=1}^{l(\mu)} \mathcal{O}_{q_i}$$
$$\to H^1(C, \mathcal{O}_C) \to H^1(C_{v_0}, \mathcal{O}_{C_{v_0}}) \oplus \bigoplus_{i=1}^{l(\mu)} H^1(C_{e_i}, \mathcal{O}_{C_{e_i}}) \to 0.$$

The representations of $\mathbb{C}^*$ are

$$0 \to (\tau) \to (\tau) \oplus \bigoplus_{i=1}^{l(\mu)} (\tau) \to \bigoplus_{i=1}^{l(\mu)} (\tau)$$
$$\to H^1(C, \mathcal{O}_C) \otimes (\tau) \to H^1(C_{v_0}, \mathcal{O}_{C_{v_0}}) \otimes (\tau) \to 0.$$

So

$$i_\Gamma^* e_T(V_0) = \Lambda_g^\vee(\tau u)$$

which is clearly also valid for $(g, l(\mu)) = (0,1), (0,2)$.

We have

$$i_{\Gamma^0}^* e_T(V_D) = A_{v_0}^D,$$

where

$$A_{v_0}^D = \Lambda_g^\vee(\tau u) \cdot (\tau u)^{l(\mu)-1}.$$

Note that the degree of $i_{\Gamma^0}^* e_T(V_D)$ is $l(\mu) + g - 1$, as expected.

### A.2.2 The target is $\mathbb{P}^1[m]$, $m > 0$

Given a graph $\Gamma \in G_{g,0}(\mathbb{P}^1, \mu)$, $\Gamma \neq \Gamma^0$, consider the normalization sequence

$$0 \to \mathcal{O}_C \to \bigoplus_{v \in V^S(\Gamma)^{(0)} \cup V(\Gamma)^{(1)}} \mathcal{O}_{C_v} \oplus \bigoplus_{e \in E(\Gamma)} \mathcal{O}_{C_e}$$

$$\to \bigoplus_{v \in V^{II}(\Gamma)^{(0)}} \mathcal{O}_{q_v} \oplus \bigoplus_{v \in V^S(\Gamma)^{(0)} \cup V(\Gamma)^{(1)}} \left( \bigoplus_{(v,e) \in F} \mathcal{O}_{q_{(v,e)}} \right) \to 0.$$

The corresponding long exact sequence reads

$$0 \to H^0(C, \mathcal{O}_C) \to \bigoplus_{v \in V^S(\Gamma)^{(0)} \cup V(\Gamma)^{(1)}} H^0(C_v, \mathcal{O}_{C_v}) \oplus \bigoplus_{e \in E(\Gamma)} H^0(C_e, \mathcal{O}_{C_e})$$

$$\to \bigoplus_{v \in V^{II}(\Gamma)^{(0)}} \mathcal{O}_{q_v} \oplus \bigoplus_{v \in V^S(\Gamma)^{(0)} \cup V(\Gamma)^{(1)}} \left( \bigoplus_{(v,e) \in F} \mathcal{O}_{q_{(v,e)}} \right)$$

$$\to H^1(C, \mathcal{O}_C)$$

$$\to \bigoplus_{v \in V^S(\Gamma)^{(0)} \cup V(\Gamma)^{(1)}} H^1(C_v, \mathcal{O}_{C_v}) \oplus \bigoplus_{e \in E(\Gamma)} H^1(C_e, \mathcal{O}_{C_e}) \to 0.$$

The representations of $\mathbb{C}^*$ are

$$0 \to (\tau) \to \bigoplus_{v \in V^S(\Gamma)^{(0)} \cup V(\Gamma)^{(1)}} (\tau) \oplus \bigoplus_{e \in E(\Gamma)} (\tau)$$

$$\to \bigoplus_{v \in V^{II}(\Gamma)^{(0)}} (\tau) \oplus \bigoplus_{v \in V^S(\Gamma)^{(0)} \cup V(\Gamma)^{(1)}} \left( \bigoplus_{(v,e) \in F} (\tau) \right)$$

$$\to H^1(C, \mathcal{O}_C) \otimes (\tau) \to \bigoplus_{v \in V^S(\Gamma)^{(0)} \cup V(\Gamma)^{(1)}} H^1(C_v, \mathcal{O}_{C_v}) \otimes (\tau) \to 0.$$

So

$$i_\Gamma^* e_T(V_0) = (\tau u)^{|E(\Gamma)|-|V(\Gamma)|+1} \prod_{v\in V(\Gamma)} \Lambda^\vee_{g(v)}(\tau u)$$
$$i_\Gamma^* e_T(V_D) = (\tau u)^{|E(\Gamma)|-|V(\Gamma)|+l(\mu)} \prod_{v\in V(\Gamma)} \Lambda^\vee_{g(v)}(\tau u).$$

We have the following Feynman rules:

$$i_\Gamma^* e_T(V_D) = \prod_{v\in V(\Gamma)} A_v^D,$$

where

$$A_v^D = \begin{cases} \Lambda^\vee_{g(v)}(\tau u)\cdot(\tau u)^{\mathrm{val}(v)-1}, & v\in V(\Gamma)^{(0)}, \\ \Lambda^\vee_{g(v)}(\tau u)\cdot(\tau u)^{l(\mu(v))-1} & v\in V(\Gamma)^{(1)}, \end{cases}$$

Note that the degree of $i_\Gamma^* e_T(V_D)$ is

$$\begin{aligned} &\sum_{v\in V(\Gamma)^{(0)}} (g(v)+\mathrm{val}(v)-1) + \sum_{v\in V(\Gamma)^{(1)}} (g(v)+l(\mu(v))-1) \\ &= \sum_{v\in V(\Gamma)} g(v) + |E(\Gamma)| - |V(\Gamma)| + l(\mu(v)) \\ &= l(\mu) + \left(\sum_{v\in V(\Gamma)} g(v) + |E(\Gamma)| - |V(\Gamma)| + 1\right) - 1 \\ &= l(\mu) + g - 1 \end{aligned}$$

as expected.

## A.3 The bundle $V_{D_d}$

### A.3.1 The target is $\mathbb{P}^1$

There is only one graph $\Gamma^0$ in this case. When $(g, l(\mu)) \neq (0,1), (0,2)$, consider the normalization sequence

$$\begin{aligned} 0 \to \widetilde{f}^*\mathcal{O}_{\mathbb{P}^1}(-1) \to (\widetilde{f}|_{C_{v_0}})^*\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \bigoplus_{i=1}^{l(\mu)} (\widetilde{f}|_{C_{e_i}})^*\mathcal{O}_{\mathbb{P}^1}(-1) \\ \to \bigoplus_{i=1}^{l(\mu)} \mathcal{O}_{\mathbb{P}^1}(-1)_{p_0} \to 0. \end{aligned}$$

The corresponding long exact sequence reads

$$\begin{aligned}
0 \to\ & H^0(C, \widetilde{f}^*\mathcal{O}_{\mathbb{P}^1}(-1)) \\
& \to H^0(C_{v_0}, (\widetilde{f}|_{C_{v_0}})^*\mathcal{O}_{\mathbb{P}^1}(-1)) \oplus \bigoplus_{i=1}^{l(\mu)} H^0(C_{e_i}, (\widetilde{f}|_{C_{e_i}})^*\mathcal{O}_{\mathbb{P}^1}(-1)) \\
& \to \bigoplus_{i=1}^{l(\mu)} \mathcal{O}_{\mathbb{P}^1}(-1)_{p_0} \to H^1(C, \widetilde{f}^*\mathcal{O}_{\mathbb{P}^1}(-1)) \\
& \to H^1(C_{v_0}, (\widetilde{f}|_{C_{v_0}})^*\mathcal{O}_{\mathbb{P}^1}(-1)) \oplus \bigoplus_{i=1}^{l(\mu)} H^1(C_{e_i}, (\widetilde{f}|_{C_{e_i}})^*\mathcal{O}_{\mathbb{P}^1}(-1)) \to 0.
\end{aligned}$$

The representations of $\mathbb{C}^*$ are

$$\begin{aligned}
0 \to 0 \to\ & H^0(C_{v_0}, \mathcal{O}_{C_{v_0}}) \otimes (-\tau - 1) \to \bigoplus_{i=1}^{l(\mu)} (-\tau - 1) \\
& \to H^1(C, \widetilde{f}^*\mathcal{O}_{\mathbb{P}^1}(-1)) \\
& \to H^1(C_{v_0}, \mathcal{O}_{C_{v_0}}) \otimes (-\tau - 1) \oplus \bigoplus_{i=1}^{l(\mu)} \left( \bigoplus_{a=1}^{d(e)-1} \left( -\tau - \frac{a}{d(e)} \right) \right) \to 0.
\end{aligned}$$

We have the following Feynman rules:

$$i_{\Gamma^0}^* e_T(V_{D_d}) = A_{v_0}^{D_d} \prod_{e \in E(\Gamma)} A_e^{D_d},$$

where

$$\begin{aligned}
A_{v_0}^{D_d} &= \Lambda_g^\vee((-\tau - 1)u) \cdot ((-\tau - 1)u)^{l(\mu)-1} \\
A_e^{D_d} &= \frac{\prod_{a=1}^{d(e)-1}(d(e)\tau + a)}{d(e)^{d(e)-1}}(-u)^{d(e)-1}.
\end{aligned}$$

It is easily checked that the above Feynmann rules are also valid for $(g, l(\mu)) = (0, 1), (0, 2)$.

Note that the degree of $i_\Gamma^* e_T(V_{D_d})$ is $d + g - 1$, as expected.

### A.3.2 The target is $\mathbb{P}^1[m]$, $m > 0$

Given a graph $\Gamma \in G_{g,0}(\mathbb{P}^1, \mu)$, $\Gamma \neq \Gamma^0$, consider the normalization sequence

$$
\begin{aligned}
0 \to\ & \widetilde{f}^*\mathcal{O}_{\mathbb{P}^1}(-1) \\
\to\ & \bigoplus_{v \in V^S(\Gamma)^{(0)} \cup V(\Gamma)^{(1)}} (\widetilde{f}|_{C_v})^*\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \bigoplus_{e \in E(\Gamma)} (\widetilde{f}|_{C_e})^*\mathcal{O}_{\mathbb{P}^1}(-1) \\
\to\ & \bigoplus_{v \in V^{II}(\Gamma)^{(0)}} \mathcal{O}_{\mathbb{P}^1}(-1)_{p_0} \oplus \bigoplus_{v \in V^S(\Gamma)^{(0)}} \left( \bigoplus_{(v,e) \in F} \mathcal{O}_{\mathbb{P}^1}(-1)_{p_0} \right) \\
& \oplus \bigoplus_{v \in V(\Gamma)^{(1)}} \left( \bigoplus_{(v,e) \in F} \mathcal{O}_{\mathbb{P}^1}(-1)_{p_1} \right) \to 0.
\end{aligned}
$$

The corresponding long exact sequence reads

$$
\begin{aligned}
0 \to\ & H^0(C, \widetilde{f}^*\mathcal{O}_{\mathbb{P}^1}(-1)) \\
\to\ & \bigoplus_{v \in V^S(\Gamma)^{(0)} \cup V(\Gamma)^{(1)}} H^0(C_v, (\widetilde{f}|_{C_v})^*\mathcal{O}_{\mathbb{P}^1}(1)) \\
& \oplus \bigoplus_{e \in E(\Gamma)} H^0(C_e, (\widetilde{f}|_{C_e})^*\mathcal{O}_{\mathbb{P}^1}(-1)) \\
\to\ & \bigoplus_{v \in V^{II}(\Gamma)^{(0)}} \mathcal{O}_{\mathbb{P}^1}(-1)_{p_0} \oplus \bigoplus_{v \in V^S(\Gamma)^{(0)}} \left( \bigoplus_{(v,e) \in F} \mathcal{O}_{\mathbb{P}^1}(-1)_{p_0} \right) \\
& \oplus \bigoplus_{v \in V(\Gamma)^{(1)}} \left( \bigoplus_{(v,e) \in F} \mathcal{O}_{\mathbb{P}^1}(-1)_{p_1} \right) \\
\to\ & H^1(C, \widetilde{f}^*\mathcal{O}_{\mathbb{P}^1}(-1)) \\
\to\ & \bigoplus_{v \in V^S(\Gamma)^{(0)} \cup V(\Gamma)^{(1)}} H^1(C_v, (\widetilde{f}|_{C_v})^*\mathcal{O}_{\mathbb{P}^1}(1)) \\
& \oplus \bigoplus_{e \in E(\Gamma)} H^1(C_e, (\widetilde{f}|_{C_e})^*\mathcal{O}_{\mathbb{P}^1}(-1)) \to 0.
\end{aligned}
$$

The representations of $\mathbb{C}^*$ are

$$\begin{aligned}
&0 \to H^0(C, \widetilde{f}^*\mathcal{O}_{\mathbb{P}^1}(-1))\\
&\quad\to \bigoplus_{v\in V^S(\Gamma)^{(0)}} H^0(C_v,\mathcal{O}_{C_v})\otimes(-\tau-1)\oplus \bigoplus_{v\in V(\Gamma)^{(1)}} H^0(C_v,\mathcal{O}_{C_v})\otimes(-\tau)\\
&\quad\to \bigoplus_{v\in V^{II}(\Gamma)^{(0)}} (-\tau-1)\oplus \bigoplus_{v\in V^S(\Gamma)^{(0)}} \left(\bigoplus_{(v,e)\in F}(-\tau-1)\right)\\
&\qquad\oplus \bigoplus_{v\in V(\Gamma)^{(1)}} \left(\bigoplus_{(v,e)\in F}(-\tau)\right)\\
&\quad\to H^1(C, \widetilde{f}^*\mathcal{O}_{\mathbb{P}^1}(-1))\\
&\quad\to \bigoplus_{v\in V^S(\Gamma)^{(0)}} H^1(C_v,\mathcal{O}_{C_v})\otimes(-\tau-1)\oplus \bigoplus_{v\in V(\Gamma)^{(1)}} H^1(C_v,\mathcal{O}_{C_v})\otimes(-\tau)\\
&\qquad\oplus \bigoplus_{e\in E(\Gamma)} \left(\prod_{a=1}^{d(e)-1}\left(-\tau-\frac{a}{d(e)}\right)\right)\to 0.
\end{aligned}$$

We have the following Feynman rules:

$$i_\Gamma^* e_T(V_{D_d}) = \prod_v A_v^{D_d}\cdot \prod_{e\in E(\Gamma)} A_e^{D_d},$$

where

$$\begin{aligned}
A_v^{D_d} &= \begin{cases} \Lambda^\vee_{g(v)}((-\tau-1)u)\cdot((-\tau-1)u)^{\mathrm{val}(v)-1}, & v\in V(\Gamma)^{(0)},\\ \Lambda^\vee_{g(v)}(-\tau u)\cdot(-\tau u)^{\mathrm{val}(v)-1}, & v\in V(\Gamma)^{(1)},\end{cases}\\
A_e^{D_d} &= \frac{\prod_{a=1}^{d(e)-1}(d(e)\tau+a)}{d(e)^{d(e)-1}}(-u)^{d(e)-1}.
\end{aligned}$$

Note that the degree of $i_\Gamma^* e_T(V_{D_d})$ is

$$\begin{aligned}
&\sum_{v\in V(\Gamma)}(g(v)+\mathrm{val}(v)-1)+\sum_{e\in E(\Gamma)}(d(e)-1)\\
&=\sum_{v\in V(\Gamma)} g(v)+2|E(\Gamma)|-|V(\Gamma)|+d-|E|\\
&=d+\left(\sum_{v\in V(\Gamma)} g(v)+|E(\Gamma)|-|V(\Gamma)|+1\right)-1\\
&=d+g-1
\end{aligned}$$

as expected.

## A.4 The obstruction bundle

Combining results in Section A.3 and Section A.2, we obtain Feynman rules for the obstruction bundle

$$V = V_D \oplus V_{D_d}.$$

### A.4.1 The target is $\mathbb{P}^1$

We have

$$i^*_{\Gamma^0}(e_T(V)) = A^V_{v_0} \prod_{e\in E(\Gamma^0)} A^V_e,$$

where

$$A^V_{v_0} = A^D_{v_0} A^{D_d}_{v_0} = \Lambda^\vee_g(\tau u)\Lambda^\vee_g(-(\tau+1)u)\cdot(\tau u(-\tau-1)u)^{l(\mu)-1}$$
$$A^V_e = A^D_e A^{D_d}_e = \frac{\prod_{a=1}^{d(e)-1}(d(e)p+a)}{d(e)^{d(e)-1}}(-u)^{d(e)-1}.$$

### A.4.2 The target is $\mathbb{P}^1[m]$, $m > 0$

For any $\Gamma \in G_{g,0}(\mathbb{P}^1,\mu)$, $\Gamma \neq \Gamma^0$, we have

$$i^*_\Gamma(e_T(V)) = \prod_{v\in V(\Gamma)} A^V_v \cdot \prod_{e\in E(\Gamma)} A^V_e,$$

where

$$A^V_v = A^D_v A^{D_d}_v = \begin{cases} \Lambda^\vee_{g(v)}(\tau u)\Lambda^\vee_{g(v)}(-(\tau+1)u) \\ \cdot(\tau u(-\tau-1)u)^{\mathrm{val}(v)-1}, & v\in V(\Gamma)^{(0)}, \\ (-1)^{g(v)+\mathrm{val}(v)-1}(\tau u)^{r_1(v)}, & v\in V(\Gamma)^{(1)} \end{cases}$$
$$A^V_e = A^D_e A^{D_d}_e = \frac{\prod_{a=1}^{d(e)-1}(d(e)\tau+a)}{d(e)^{d(e)-1}}(-u)^{d(e)-1}.$$

Recall that $r_1(v) = 2g(v) - 2 + l(\mu(v)) + \mathrm{val}(v)$ for $v \in V(\Gamma)^{(1)}$.

## References

[1] J. Bryan & R. Pandharipande, *Curves in Calabi-Yau 3-folds and topological quantum field theory*, preprint, math.AG/0306316.

[2] T. Ekedahl, S. Lando, M. Shapiro & A. Vainshtein, *Hurwitz numbers and intersections on moduli spaces of curves*, Invent. Math. **146**(**2**) (2001) 297–327, MR 1864018.

[3] C. Faber, *Algorithms for computing intersection numbers on moduli spaces of curves, with an application to the class of the locus of Jacobians*, in 'New trends in algebraic geometry' (Warwick, 1996), 93–109, London Math. Soc., Lecture Note Ser., **264**, Cambridge Univ. Press, Cambridge, 1999, MR 1714822, Zbl 0952.14042.

[4] C. Faber & R. Pandharipande, *Hodge integrals and Gromov-Witten theory*, Invent. Math. **139**(**1**) (2000) 173–199, MR 1728879, Zbl 0960.14031.

[5] C. Faber & R. Pandharipande, *Relative maps and tautological classes*, preprint, 2000, no. 1, 173-199.

[6] B. Fantecchi & R. Pandharipande, *Stable maps and branch divisors*, Compositio Math. **130**(**3**) (2002) 345–364, MR 1887119.

[7] I.P. Goulden, D.M. Jackson & A. Vainshtein, *The number of ramified coverings of the sphere by the torus and surfaces of higher genera*, Ann. of Comb. **4** (2000) 27–46, MR 1763948, Zbl 0957.58011.

[8] T. Graber & R. Pandharipande, *Localization of virtual classes*, Invent. Math. **135**(**2**) (1999) 487–518, MR 1666787, MR 0953.14035.

[9] T. Graber & R. Vakil, *Hodge integrals and Hurwitz numbers via virtual localization.* Compositio Math. **135** (2003) 25–36, MR 1955162.

[10] T. Graber & R. Vakil, *Relative virtual localization and vanishing of tautological classes on moduli spaces of curves*, in preparation.

[11] E.-N. Ionel & T. Parker *The symplectic sum formula for Gromov-Witten invariants*, preprint, math.SG/0010207.

[12] S. Katz & C.-C. Liu, *Enumerative geometry of stable maps with Lagrangian boundary condtions and multiple covers of the disc*, Adv. Theor. Math. Phys. **5** (2001) 1–49, MR 1894336.

[13] M. Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function*, Comm. Math. Phys. **147**(**1**) (1992) 1–23, MR 1171758, Zbl 0756.35081.

[14] M. Kontsevich, *Enumeration of rational curves via torus actions*, in 'The moduli space of curves' (Texel Island, 1994), 335–368, Progr. Math., **129**, Birkhäuser Boston, Boston, MA, 1995, MR 1363062, Zbl 0885.14028.

[15] A.M. Li, G. Zhao & Q. Zheng, *The number of ramified coverings of a Riemann surface by Riemann surface*, Comm. Math. Phys. **213**(**3**) (2000) 685–696, MR 1785434.

[16] J. Li, *Stable morphisms to singular schemes and relative stable morphisms*, J. Differential Geom. **57** (2001) 509–578, MR 1882667.

[17] J. Li, *A degeneration formula of GW-invariants*, J. Differential Geom. **60** (2002) 199–293, MR 1938113.

[18] J. Li, *Lecture notes on relative GW-invariants*, preprint.

[19] J. Li & Y.S. Song, *Open string instantons and relative stable morphisms*, Adv. Theor. Math. Phys. **5**(**1**) (2001) 67–91, MR 1894338, Zbl 1015.14026.

[20] B. Lian, K. Liu & S.-T. Yau, *Mirror principle* III, Asian J. Math. **3**(**4**) (1999), 771–800, MR 1797578, Zbl 1009.14006.

[21] C.-C. Liu, K. Liu & J. Zhou, *On a proof of a conjecture of Marino-Vafa on Hodge integrals*, preprint, math.AG/0306257.

[22] I.G. MacDonald, *Symmetric functions and Hall polynomials*, 2nd edition, Claredon Press, 1995, MR 1354144, Zbl 0899.05068.

[23] M. Mariño & C. Vafa, *Framed knots at large $N$*, in 'Orbifolds in mathematics and physics' (Madison, WI, 2001), 185–204, Contemp. Math., **310**, Amer. Math. Soc., Providence, RI, 2002, MR 1950947.

[24] D. Mumford, *Towards an enumerative geometry of the moduli space of curves*, in 'Arithmetic and Geometry' (M. Artin, J. Tate, eds.), Part II, Birkhäuser, 1983, 271–328, MR 0717614, Zbl 0554.14008.

[25] A. Okounkov & R. Pandharipande, *Hodge integrals and invariants of the unknots*, preprint, math.AG/0307209.

[26] E. Witten, *Two-dimensional gravity and intersection theory on moduli space*, in 'Surveys in differential geometry' (Cambridge, MA, 1990), 243–310, Lehigh Univ., Bethlehem, PA, 1991, MR 1144529, Zbl 0757.53049.

[27] J. Zhou, *Some closed formulas and conjectures for Hodge integrals*, Math. Res. Lett. **10** (2003) 275–286, MR 1981904.

[28] J. Zhou, *Hodge integrals, Hurwitz numbers, and symmetric groups*, preprint.

[29] J. Zhou *Marino-Vafa formula and BPS numbers in local $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ geometry*, preprint, submitted.

[30] J. Zhou, *Proof and interpretation of a string duality*, preprint, submitted.

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