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CIRCLE PACKINGS ON SURFACES WITH PROJECTIVE STRUCTURES

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Abstract

The Koebe-Andreev-Thurston theorem states that for any triangulation of a closed orientable surface Σ_g of genus g which is covered by a simple graph in the universal cover, there exists a unique metric of curvature $1, 0$ or -1 on the surface depending on whether $g = 0, 1$ or ≥ 2 such that the surface with this metric admits a circle packing with combinatorics given by the triangulation. Furthermore, the circle packing is essentially rigid, that is, unique up to conformal automorphisms of the surface isotopic to the identity.

In this paper, we consider projective structures on the surface where circle packings are also defined. We show that the space of projective structures on a surface of genus $g \geq 2$ which admits a circle packing contains a neigborhood of the Koebe-Andreev-Thurston structure homeomorphic to $\mathbb{R}^{6g-6}$. We furthemore show that if a circle packing consists of one circle, then the space is globally homeomorphic to $\mathbb{R}^{6g-6}$ and that the circle packing is rigid.

1. Introduction

This paper is motivated by the interplay between Mostow rigidity and hyperbolic Dehn surgery theory and is concerned with rigidity properties as well as deformation theory on the space of projective structures on surfaces. Mostow rigidity concerns the rigidity property of say, complete hyperbolic structures on manifolds of dimension ≥ 3 based solely on the topological data of the manifold. On the other hand, if we relax

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the completeness condition in dimension three in particular, there is a nontrivial deformation theory. Hyperbolic Dehn surgery theory basically tells us how nice deformations are parameterized.

Mostow rigidity fails in dimension two and the deformation space of hyperbolic structures on a closed orientable surface Σ_g of genus $g \geq 2$ is homeomorphic to $\mathbb{R}^{6g-6}$. However the addition of certain "tight" topological data corresponding to circle packings can impose some rigidity properties as follows:

Roughly, a circle packing on Σ_g is a collection of closed disks on the surface such that the interiors of any two distinct disks are disjoint and the complement of the disks in Σ_g consists of disjoint, triangular interstices. One obtains a cellular decomposition of Σ_g by assigning a vertex for each disk, an edge joining two vertices for each point of tangency of two (not necessarily distinct) disks and a triangle for each triangular interstice. We call the graph on Σ_g consisting of the vertices and edges obtained in this way the nerve of the circle packing. It triangulates Σ_g in a generalized sense.

The preimage of the nerve in the universal cover contains no loops by one or two edges, because otherwise, one would find a self-tangent circle or two circles with two points of tangency on the plane, which both would be absurd. In other words, the nerve must be covered by a simple graph in the graph theoretic sense.

Now, the Koebe-Andreev-Thurston theorem states that the surface Σ_g together with a graph τ with the property above determines a unique metric of constant curvature on Σ_g such that the surface with this metric admits a circle packing whose nerve is isotopic to τ. Furthermore, this circle packing is rigid, that is, it is unique up to conformal automorphisms of the surface isotopic to the identity. This result was proved by Koebe in [13] and rediscovered by Andreev in [2] for the sphere and proved by Thurston in [18] (see also [3] and [8] for other proofs) for higher genus. Such a rigidity property implies that the set of points in the Teichmüller space which can be circle-packed is a countable set (Brooks proved in [7] that this set is dense). We are interested in finding some relaxation of the conditions so that nontrivial deformations can occur.

A priori, the notion of a circle on a surface appears to be a metric notion, that is, a circle is the set of points equidistant from the center, and metrics of constant curvature form a very natural context for the study of circle packings. However there is a more general class of geometric structures on surfaces called the complex projective structures

in which the notion of a circle still makes sense. A complex projective structure (henceforth abbreviated to projective structure in this paper) is a (G, X) structure (see [18]) with the group $G = \mathrm{PSL}_2(\mathbb{C})$ and the model space $X = \hat{\mathbb{C}}$, the Riemann sphere. Two structures (Σ_g, μ_1) and (Σ_g, μ_2) are equivalent if there is a projective isomorphism isotopic to the identity between the two. The space of projective structures up to equivalence is denoted by $\mathcal{P}_g$ and is known to be homeomorphic to $\mathbb{R}^{12g-12}$ $(g \geq 2)$. The uniformization defines a bundle projection

$$u : \mathcal{P}_g \to \mathcal{T}_g$$

to the Teichmüller space $\mathcal{T}_g$, the space of all conformal structures on Σ_g. The uniformization admits a canonical section

$$s : \mathcal{T}_g \to \mathcal{P}_g$$

which assigns to each conformal structure the equivalent hyperbolic structure, see for example [12]. Since any projective transformation $\gamma \in \mathrm{PSL}_2(\mathbb{C})$ maps a circle on the Riemann sphere to a circle, the circle (and disk) makes sense as a geometric object on a surface with a projective structure. Moreover, it is obvious that the nerve of a packing in a surface with projective structure again must be covered by a simple graph in the universal cover. The two main questions we are interested in are:

Question 1. For a fixed triangulation τ on Σ_g which is covered by a simple graph in the universal cover, what is the deformation space of projective structures which admit a circle packing with nerve isotopic to τ?

Question 2. For a surface with a fixed projective structure which admits a circle packing with nerve τ, is the circle packing rigid?

Note that the Koebe-Andreev-Thurston theorem implies that the intersection of this deformation space with $s(\mathcal{T}_g)$ is a point. The main results in this paper are the complete answer to the two questions above in the case where τ has only one vertex, that is, the circle packing of the surface consists of only one circle, and the partial answer for general τ.

To attack our problem, we make use of a cross ratio. The cross ratio type invariant is suitable in the context with circle packings and linear fractional transformations, and has been used by He and Schramm in [17] and [10]. The cross ratio type invariant is a positive real number,

for a certain configuration of 4 circles as in [10]. It determines a configuration up to multiplication by elements of $\mathrm{PSL}_2(\mathbb{C})$. Suppose that we have a circle packing P on a surface with projective structure (Σ_g, μ) with nerve τ. Then we define a map

$$\mathbf{c}_P : E(\tau) \to \mathbb{R}_+$$

by assigning to each edge the cross ratio type invariant of its neighboring 4 circles in the universal cover, where $E(\tau)$ is the set of edges of τ. We call $\mathbf{c}_P$ a cross ratio parameter of the packing P on a surface with projective structure (Σ_g, μ). It will be clear by defining the above more precisely in §2 that the cross ratio parameter is a complete projective invariant of the pair of a projective structure μ and a packing P.

The problem then is to find suitable conditions which must be satisfied by the cross ratio parameters. To do this, for each map $\mathbf{c} : E(\tau) \to \mathbb{R}_+$ which does not necessarily come from a circle packing, we assign to each edge e an associated matrix $A = \begin{pmatrix} 0 & 1 \\ -1 & \mathbf{c}(e) \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$. Furthermore, for each vertex $v \in V(\tau)$ with valence n, one reads off the edges incident to v in the clockwise direction and form a word W_v of length n from the corresponding associated matrices (W_v is defined up to cyclic permutation). The main general result is the following:

Lemma 1 (Lemma 2.17)**.** *Suppose $\mathbf{c}_P$ is the cross ratio parameter of the packing P on (Σ_g, μ) with nerve τ and let W_v be the associated word of the associated matrices for each $v \in V(\tau)$. Then, we have:*

(1) *$W_v = -I$, where I is the identity matrix, and*

(2) *if the length of W_v is n, then every subword of W_v of length $\leq n-1$ is admissible, and every subword of length $\leq n-2$ is strictly admissible (admissibility and strict admissibility are defined by some inequalities in the cross ratio type invariants which will be precisely defined in §2).*

Conversely, if a map $\mathbf{c} : E(\tau) \to \mathbb{R}_+$ satisfies (1) *and* (2)*, then there is a unique projective structure μ on Σ_g together with a circle packing P with nerve isotopic to τ such that the cross ratio parameter $\mathbf{c}_P$ is equal to $\mathbf{c}$.*

Note that for circle packings on the plane, He and Schramm have introduced associated matrices and have obtained the similar condition for a cross ratio parameter to realize a circle packing in [10].

We call the set of all maps in $\mathbb{R}^{E(\tau)}$ satisfying Conditions (1) and (2) for each vertex $v \in V(\tau)$ in the Main Lemma the cross ratio parameter space and denote it by

$$\mathcal{C}_\tau = \{\mathbf{c} : E(\tau) \to \mathbb{R}_+ \,|\, \mathbf{c} \text{ satisfies (1) and (2) for each vertex}\}.$$

By this lemma, $\mathcal{C}_\tau$ can be identified with the set of all pairs (μ, P) and hence contains a special element obtained by the Koebe-Andreev-Thurston theorem.

Clearly, $\mathcal{C}_\tau$ is a real semi-algebraic set, however this does not say much about its true structure. It is a priori possible that $\mathcal{C}_\tau$ has singularities or may even reduce to a single point represented by the Koebe-Andreev-Thurston solution. Also, each point in $\mathcal{C}_\tau$ determines not just a projective structure but a circle packing on the surface with nerve τ. It is quite possible that there is a nontrivial family of different circle packings with nerve τ on the surface with a fixed projective structure (that is, the circle packings are not rigid).

Such problems can be well formulated using the map

$$f : \mathcal{C}_\tau \to \mathcal{P}_g$$

defined by forgetting the packing P. For example, the rigidity question is equivalent to the injectivity of f.

In §3, we will count the formal dimension of $\mathcal{C}_\tau$ which is just the number of variables minus the number of equations which appear in the definition, and show that it is $6g - 6$ when $g \geq 1$. Then we show using the deformation theory of Kleinian groups that this turns out to give the correct dimension at least near the Koebe-Andreev-Thurston solution when $g \geq 2$. When $g = 1$, we will see by hyperbolic Dehn surgery theory that the formal dimension count overlooks some dependence in the equations. The results in §3 can be summarized by:

Main Theorem 1 (Lemma 3.2, Lemma 3.3)**.** *The Koebe-Andreev-Thurston solution has a neighborhood in $\mathcal{C}_\tau$ which is homeomorphic to the euclidean space of dimension $6g - 6$ when $g \geq 2$ and 2 when $g = 1$, and to which the restriction of the forgetting map f is an embedding.*

The other main result describes a global picture of $\mathcal{C}_\tau$ for packings by one circle. In this case, τ has exactly one vertex. Note that a graph with one vertex which triangulates Σ_g is necessarily covered by a simple graph in the universal cover.

Main Theorem 2 (Lemma 4.7, Lemma 5.1)**.** *Suppose that the triangulation τ of Σ_g $(g \geq 2)$ has exactly one vertex. Then:*

(i) *$\mathcal{C}_\tau$ is homeomorphic to the euclidean space of dimension $6g-6$, and*

(ii) *the forgetting map $f : \mathcal{C}_\tau \to \mathcal{P}_g$ is an embedding.*

Remarks.

1. When $g = 1$, the same conclusion (replacing the dimension by 2) can be deduced from a previous work by Mizushima [14]. His methods, different from ours, provides much more information about the deformation space, as we see in the third remark, but do not generalize easily to the higher genus case. Our method works also for this case, and it is discussed in the Appendix.

2. We conjecture that Main Theorem 2 holds for any triangulation τ of Σ_g which is covered by a simple graph in the universal cover.

3. In [14], Mizushima showed that the composition $u \circ f$ of the forgetting map and the uniformization is a homeomorphism when $g = 1$ (stated in a slightly different way). We conjecture that this holds for genus $g \geq 2$ as well. If true, this would give a uniformization theorem that every complex structure can be uniformized by a projective structure circle-packed by one circle. However, again, the methods of [14] do not generalize. In the torus case, it is relatively easy to uniformize complex affine structures on the torus to Euclidean structures via the log function. The general uniformization map is much more complicated and so different methods would have to be employed.

We here outline the proof of Main Theorem 2 and show how the rather strong condition that τ has exactly one vertex plays a crucial role. In this case, τ has $6g-3$ edges so that there are $6g-3$ variables. Condition (1) of the Main Lemma then corresponds to 3 relations on these variables. To show nonsingularity of $\mathcal{C}_\tau$, we will find a specific set of three variables such that the remaining $6g-6$ variables are free but the chosen three are uniquely determined by these free variables. Then we will see that the range of free variables consists of a convex set. This is enough to see that $\mathcal{C}_\tau$ is homeomorphic to the euclidean space of dimension $6g-6$.

To see the injectivity of the forgetting map, suppose that $\mathbf{c}$ and $\mathbf{c}'$ are two cross ratio parameters which correspond to the same projective structure on Σ_g (but possibly different circle packings). Then we would like to show that $\mathbf{c} = \mathbf{c}'$ which implies rigidity of the circle packing. A priori, we only know that the holonomy representations corresponding to $\mathbf{c}$ and $\mathbf{c}'$ are conjugate since they give rise to the same projective structure. We show that the holonomy representation of the specific set of three side-pairings which appear in showing nonsingularity of $\mathcal{C}_\tau$ must be actually the same for $\mathbf{c}$ and $\mathbf{c}'$, and not just conjugate. We then show that for such a triple, the holonomy image is always non-elementary so that the holonomy representation of the full fundamental group $\pi_1(\Sigma_g)$ is exactly the same for both $\mathbf{c}$ and $\mathbf{c}'$ and hence $\mathbf{c} = \mathbf{c}'$.

Remark. The argument above for the injectivity of the forgetting map f implies a rather remarkable conclusion that the composition of f with the map $\chi : \mathcal{P}_g \to \mathrm{Hom}\,(\pi_1(\Sigma_g), \mathrm{PSL}_2(\mathbb{C}))/\mathrm{PSL}_2(\mathbb{C})$ which assigns to each projective structure the conjugacy class of its holonomy representation is injective. This contrasts with the fact that χ, which is a local homeomorphism, is in fact never injective, see for instance [11].

The rest of this paper is organized as follows: In §2, we give some preliminaries, define the cross ratio and associated matrix, and prove the general result that the cross ratio parameter space associated to τ is a real semi-algebraic set. We also show how to obtain the holonomy representation from the cross ratios. In §3, we prove Main Theorem 1 using well developed machinery from 3-dimensional deformation theory. In §4, we study the cross ratio parameter space $\mathcal{C}_\tau$ when τ has one vertex, which corresponds to a circle packing by one circle. In particular, we show that $\mathcal{C}_\tau$ can be parameterized by coordinates lying in a full convex subset of $\mathbb{R}^{6g-6}$. In §5, we prove that the circle packing is rigid for the one circle packing, hence completing the proof of our Main Theorem 2. Finally, we work out in detail the case of the one circle packing on the torus ($g = 1$) in the Appendix, where although the results are similar, the methods are slightly different.

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2. Cross ratio parameter

2.1 Preliminaries

A projective structure on a closed orientable surface Σ_g of genus g is a maximal collection of local charts modeled on the Riemann sphere $\hat{\mathbb{C}}$ such that locally, the transition functions are restrictions of $\gamma \in \mathrm{PSL}_2(\mathbb{C})$. Another point of view is that a projective structure is a structure modeled on the boundary at ∞ of the three-dimensional hyperbolic space $\mathbb{H}^3$ under the action of the isometries of $\mathbb{H}^3$. Associated to such a projective structure is a developing map $\mathrm{dev} : \widetilde{\Sigma}_g \longrightarrow \hat{\mathbb{C}}$, defined up to composition with elements of $\mathrm{PSL}_2(\mathbb{C})$ and a holonomy representation $\rho : \pi_1(\Sigma_g) \longrightarrow \mathrm{PSL}_2(\mathbb{C})$ defined up to conjugation by elements of $\mathrm{PSL}_2(\mathbb{C})$ such that the holonomy representation is equivariant with respect to the developing map.

If μ is a projective structure on Σ_g, a disk in (Σ_g, μ) is the closure of a simply connected region $\mathcal{D}^i$ in Σ_g such that the lifts of the closure $\mathcal{D}$ in $\widetilde{\Sigma}_g$ are mapped homeomorphically to closed disks in $\hat{\mathbb{C}}$ by the developing map. This makes sense since circles are preserved by the linear fractional transformations $\gamma \in \mathrm{PSL}_2(\mathbb{C})$.

Definition 2.1. A circle packing on (Σ_g, μ) is a collection of closed disks $\{\mathcal{D}_i\}$ in Σ_g such that the interiors of any two distinct disks are disjoint and $\Sigma_g \backslash \cup_i \mathcal{D}_i$ consists of a finite number of triangular interstices each bounded by three circular arcs.

Definition 2.2. For a circle packing on Σ_g, we assign a vertex to each circle and an edge joining two vertices for each tangency point between two circles. The graph τ on Σ_g obtained thus triangulates Σ_g and this is called the nerve of the circle packing.

2.2 Cross ratio and associated matrix

We define a cross ratio type invariant on the edges of τ, which has the property that if an edge separating two interstices is assigned a given positive real number, then knowing the developing image of one of the interstices determines the developing image of the other interstice. We start with a collection of 4 circles in $\hat{\mathbb{C}}$ whose nerve is a graph consisting of 4 vertices and 5 edges such that the edges bound two triangles with a common edge. The cross ratio type invariant (henceforth the cross ratio for short) is defined on the common edge (see Figure 1a).

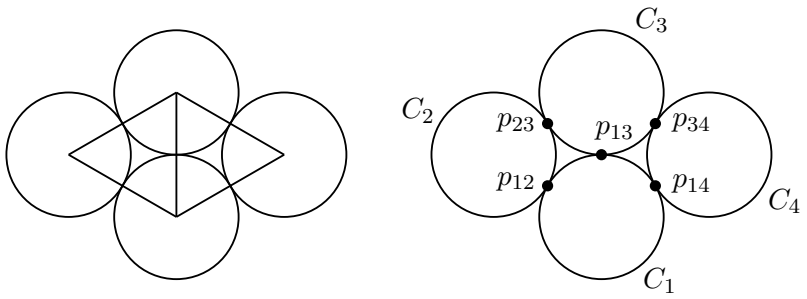


Figure 1.

Proposition 2.3. *Let C_1, C_2, C_3, C_4 be 4 circles in $\hat{\mathbb{C}}$ such that C_1 is tangent to C_2, C_3 and C_4, C_2 is tangent to C_3, and C_3 is tangent to C_4 (see Figure* 1b*). Denote the tangency points between C_i and C_j by p_{ij}. We also further assume that p_{12}, p_{13} and p_{14} lie in a clockwise direction on C_1. Then the cross ratio $(p_{14}, p_{23}, p_{12}, p_{13}) = x\sqrt{-1}$, where $x \in \mathbb{R}_+$.*

Proof. We can find a linear fractional transformation $\gamma \in \mathrm{PSL}_2(\mathbb{C})$ mapping p_{23} to 1, p_{12} to 0 and p_{13} to ∞ so that $\gamma(p_{14}) = (p_{14}, p_{23}, p_{12}, p_{13})$ using the definition of the cross ratio of 4 points given in Ahlfors [1]. Clearly, $\gamma(C_1) = \{z \in \mathbb{C} | Re(z) = 0\} \cup \{\infty\}$, $\gamma(C_2) = \{z \in \mathbb{C} |\ |z - 1/2| = 1/2\}$ and $\gamma(C_3) = \{z \in \mathbb{C} | Re(z) = 1\} \cup \{\infty\}$. It follows that $\gamma(p_{14})$ is a purely imaginary number since it lies on $\gamma(C_1)$. The condition p_{12}, p_{13} and p_{14} lie in a clockwise direction on C_1 now implies that $\gamma(p_{14}) = x\sqrt{-1}$ where $x > 0$, see Figure 2. q.e.d.

Definition 2.4 (Cross Ratio of Configuration)**.** Given a configuration of 4 circles as defined in Proposition 2.3 above, the cross ratio of the configuration, or alternatively, of the interior edge of the nerve of the configuration is the positive real number x such that $(p_{14}, p_{23}, p_{12}, p_{13}) = x\sqrt{-1}$.

Remarks.

1. It is easy to see that the cross ratio of (C_1, C_2, C_3, C_4) is the same as that for (C_3, C_4, C_1, C_2) so that the cross ratio is defined on the undirected interior edge.

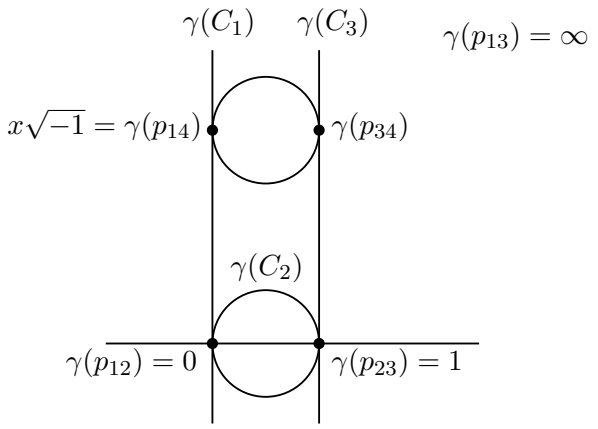


Figure 2.

2. The cross ratio is clearly an invariant of the configuration under the action of $\mathrm{PSL}_2(\mathbb{C})$. It measures the distance between C_2 and C_4 in the following sense: Consider the circles C_2 and C_4 as the boundaries at infinity of two hyperbolic planes in $\mathbb{H}^3$. If the two circles are disjoint or tangent, the planes are at some nonnegative distance d from each other. Then the cross ratio $x = \cosh(d/2)$. If the two circles intersect at angle θ, then $x = \cos(\theta/2)$. It follows that C_2 and C_4 are disjoint if and only if $x > 1$.

3. The cross ratio, together with C_1, C_2 and C_3 determines C_4. Alternatively, we can say that knowing the position of one interstice and the cross ratio determines the neighboring one.

Definition 2.5 (Associated Matrix)**.** To each configuration of 4 circles having the cross ratio x is associated the matrix $A = \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix}$.

We note that $A \in \mathrm{SL}_2(\mathbb{R})$ rather than $\mathrm{PSL}_2(\mathbb{R})$. This is important as it will allow us to keep track of overlapping in a configuration of one circle surrounded by several circles.

We next explain the role of the associated matrix A. Basically, if we put the configuration of 4 circles into a standard position by some linear fractional transformation, then A maps C_1 to itself and one interstice to the other. As much of the subsequent arguments depend on putting a configuration into some standard position, we start with the following

definition:

Definition 2.6 (Standard Interstice)**.** The standard interstice is defined to be the interstice with vertices at 0, $\sqrt{-1}$ and ∞. It is bounded by the 3 circles (or straight lines) $C_1 = \{z \in \hat{\mathbb{C}} : \mathrm{Im}(z) = 0\}$, $C_2 = \{z \in \hat{\mathbb{C}} : \mathrm{Im}(z) = 1\}$ and $C_3 = \{z \in \hat{\mathbb{C}} : \ |z - \sqrt{-1}/2| = 1/2\}$. We denote the standard interstice by $\mathcal{I}_s$.

Proposition 2.7. *Let C_1, C_2, C_3 and C_4 be a configuration of 4 circles as in Proposition 2.3 with cross ratio x and associated matrix $A = \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix}$ as defined in Definition 2.5. Furthermore, suppose that C_1, C_2 and C_3 are as defined in Definition 2.6 so they bound the standard interstice (see Figure 3). Then $A(C_1) = C_1$, $A(C_2) = C_3$, $A(C_3) = C_4$. In other words, A maps the interstice $\mathcal{I}_s$ to the other interstice of the configuration, with the boundary adjacent to C_1 being mapped to the boundary adjacent to C_1.*

Proof. First note that by a simple computation, since the cross ratio is x, $p_{14} = 1/x$. Clearly, $A(C_1) = C_1$ since entries of A are real. Next $A(C_2) = C_3$ since $A(\infty) = 0$ and $A(x + \sqrt{-1}) = \sqrt{-1}$ and A maps tangency points to tangency points. Finally, $A(C_3) = C_4$ since $A(0) = 1/x$. q.e.d.

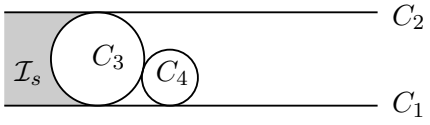


Figure 3.

For convenience, we state in the next proposition various configurations of four circles C_1, C_2, C_3, C_4 all of which have cross ratio x.

Proposition 2.8. *The configurations of four circles C_1, C_2, C_3, C_4 defined in each of the following cases all have cross ratio x (see Figure 4):*

(i) $C_1 = \{z \in \hat{\mathbb{C}} : \mathrm{Im}(z) = 0\}$, $C_2 = \{z \in \hat{\mathbb{C}} : \ |z - (x + \sqrt{-1}/2)| = 1/2\}$, $C_3 = \{z \in \hat{\mathbb{C}} : \mathrm{Im}(z) = 1\}$, $C_4 = \{z \in \hat{\mathbb{C}} : \ |z - \sqrt{-1}/2| = 1/2\}$.

(ii) $C_1 = \{z \in \hat{\mathbb{C}} : \mathrm{Im}(z) = 0\}$, $C_2 = \{z \in \hat{\mathbb{C}} : \mathrm{Im}(z) = 1\}$, $C_3 = \{z \in \hat{\mathbb{C}} : \ |z - \sqrt{-1}/2| = 1/2\}$, $p_{14} = 1/x$.

(iii) $C_1 = \{z \in \hat{\mathbb{C}} : \mathrm{Im}(z) = 0\}$, $p_{12} < p_{13} < p_{14}$, $\sqrt{\frac{r_3}{r_2}} + \sqrt{\frac{r_3}{r_4}} = x$, *where* r_i *is the radius of* C_i, $i = 2, 3, 4$.

Proof. The proof follows from some elementary calculations and will be omitted. q.e.d.

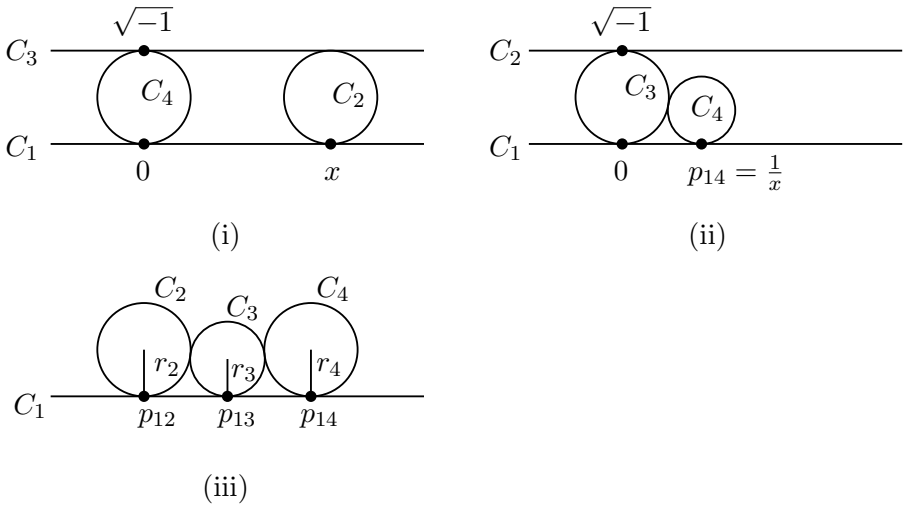


Figure 4.

2.3 Admissibility

In the next set of results, we consider a configuration of circles with a central circle $\overline{C}$ and $n+1$ circles $C_0, \ldots, C_n$ surrounding it such that:

(i) $\overline{C}$ is tangent to C_i for $i = 0, \ldots, n$;

(ii) C_i is tangent to C_{i-1} and C_{i+1}, for $i = 1, \ldots, n-1$;

(iii) The points of tangency p_i between $\overline{C}$ and C_i are arranged in clockwise direction around $\overline{C}$ (see Figure 5a).

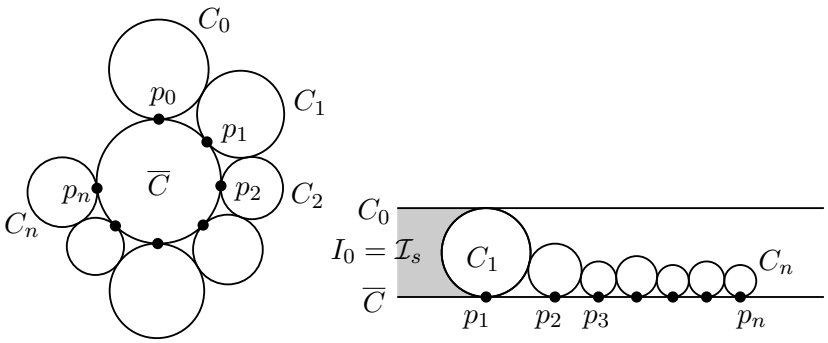
Let τ be the nerve of this configuration, with vertices $\overline{v}, v_0, \ldots, v_n$ corresponding to the circles $\overline{C}, C_0, \ldots, C_n$ respectively. Denote by e_i the edge joining $\overline{v}$ to v_i and $e_{\{i,i+1\}}$ the edge joining v_i to v_{i+1}.

Definition 2.9 (Cross Ratio Vector and Associated Word)**.** For such a configuration, let x_i be the cross ratio of the edge e_i and A_i be the associated matrix, for $i = 1, \ldots, n-1$. $\mathbf{x} = (x_1, x_2, \ldots, x_{n-1})$ will be called the *cross ratio vector* of the configuration and $W_{\mathbf{x}} = A_1 A_2 \ldots A_n$ the *word of associated matrices* of the configuration.

We have the following result:

Proposition 2.10. *Suppose that $\{\overline{C}, C_0, \ldots, C_n\}$ is a configuration of circles as defined above. Let I_i be the interstice bounded by the circles $\overline{C}$, C_i and C_{i+1}. We say that the configuration is in standard position if I_0 is the standard interstice $\mathcal{I}_s$ and $\overline{C}$ is the real line (see Figure* 5b*). Let $W_k = A_1 A_2 \ldots A_k$ for $k = 1, \ldots, n-1$. If the configuration is in standard position, then $W_k(I_0) = I_k$ with the side of I_0 bounded by $\overline{C}$ being mapped to the side of I_k bounded by $\overline{C}$, so that $W_k(p_0) = p_k$, $W_k(p_1) = p_{k+1}$.*

Proof. We prove by induction on k. For $k = 1$, the result is just Proposition 2.7. Suppose the result is true for k. Then $W_k A_{k+1} W_k^{-1}(I_k) = I_{k+1} \Longrightarrow W_k A_{k+1} W_k^{-1} W_k(I_0) = I_{k+1} \Longrightarrow W_{k+1}(I_0) = I_{k+1}$. Also, since $W_k \in \mathrm{SL}_2(\mathbb{R})$, W_k maps $\overline{C}$ to itself so the second condition holds. q.e.d.



(a) (b)

Figure 5.

In the configuration of one central circle and $n+1$ surrounding circles defined in Proposition 2.10, it is possible that some of the surrounding circles $C_0, \ldots, C_n$ not adjacent to each other may overlap. This is permissible from the definition, in general, from the point of view of circle

packings on surfaces with projective structures, this would imply that the developing map from $\widetilde{\Sigma}_g$ to $\hat{\mathbb{C}}$ is not a homeomorphism onto its image. More importantly, it may be possible that boundaries of the interstices lying on the central circle $\overline{C}$ overlap as they wrap around $\overline{C}$. We are interested in the case where they do not overlap, that is, they do not go more than once around $\overline{C}$. In standard position, this just means that the points of tangency p_i between $\overline{C}$ and C_i satisfy the inequalities

$$p_0 = -\infty < p_1 = 0 < p_2 < \cdots < p_n.$$

Definition 2.11 (Admissibility)**.** The circles $\{\overline{C}, C_0, \ldots, C_n\}$ as given in Proposition 2.10 is *admissible* if in standard position, the points of tangency p_i between $\overline{C}$ and C_i satisfy

$$p_0 = -\infty < p_1 = 0 < p_2 < \cdots < p_n.$$

It is *strictly admissible* if $p_n \neq \infty$, that is, if in addition, $p_n \neq p_0$. The $(n-1)$-tuple $\mathbf{x} = (x_1, \ldots, x_{n-1}) \in \mathbb{R}_+^{n-1}$ is said to be (strictly) admissible if it is the cross ratio vector of a (strictly) admissible configuration. Similarly, a word $W_{\mathbf{x}} = A_1 A_2 \ldots A_{n-1}$ is (strictly) admissible if it is the word of associated matrices of a (strictly) admissible configuration.

Geometrically, a strictly admissible configuration is one where the boundaries of the interstices lying on the central circle do not overlap, an admissible configuration is one where there is either no overlap or overlap at only one point. It is clear from the definition that a proper subword of an admissible word is always strictly admissible.

The condition of admissibility of $(x_1, \ldots, x_{n-1}) \in \mathbb{R}_+^{n-1}$ can be easily translated into a condition on the subwords of the word $W_{\mathbf{x}} = A_1 A_2 \ldots A_{n-1}$ in the associated matrices as seen below:

Proposition 2.12. *Let* $(x_1, \ldots, x_{n-1}) \in \mathbb{R}_+^{n-1}$ *and* $A_i = \left(\begin{smallmatrix} 0 & 1 \\ -1 & x_i \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$*,* $i = 1, \ldots, n-1$*. Then* $(x_1, \ldots, x_{n-1}) \in \mathbb{R}_+^{n-1}$ *is admissible if and only if for all subwords* $W_k^i = \begin{pmatrix} a_{ik} & b_{ik} \\ c_{ik} & d_{ik} \end{pmatrix} = A_i A_{i+1} \ldots A_k$ *of* $W_{\mathbf{x}}$*,* $a_{ik} \leq 0$ *(with equality only when* $i = k$*),* $b_{ik} > 0$*,* $c_{ik} < 0$*, and* $d_{ik} \geq 0$*. Furthermore, the last inequality is always strict for all proper subwords, and if it is also strict for the word* $W_{\mathbf{x}}$*, then the word is strictly admissible.*

Proof. For convenience, we consider only the case of strict admissibility. The remaining case is easily deduced.

($\Longleftarrow$) Consider the configuration $\{\overline{C}, C_0, \ldots, C_n\}$ in standard position with cross ratio vector $(x_1, \ldots, x_{n-1}) \in \mathbb{R}_+^{n-1}$. Recall that p_i is the point of tangency of $\overline{C}$ with C_i and $p_0 = -\infty$, $p_1 = 0$ since the configuration is in standard position. To simplify notation, let $W_k^1 = \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix}$. By Proposition 2.10, $W_k^1(0) = b_k/d_k = p_{k+1}$, $W_k^1(-\infty) = a_k/c_k = p_k$. Since $b_k, d_k > 0$, $p_{k+1} > 0$ for $k = 1, \ldots, n-1$. Furthermore, since $a_k d_k - b_k c_k = 1$, $p_k = p_{k+1} + 1/(c_k d_k) < p_{k+1}$ since $c_k < 0, d_k > 0$. Hence,

$$p_0 = -\infty < p_1 = 0 < p_2 < p_3 < \cdots < p_n < \infty$$

so that the configuration is strictly admissible.

($\Longrightarrow$) We first note that if $\{\overline{C}, C_0, C_1, \ldots, C_n\}$ is strictly admissible, then so is the sub-configuration $\{\overline{C}, C_i, C_{i+1}, \ldots, C_n\}$, so $(x_1, \ldots, x_{n-1})$ is admissible implies $(x_i, x_{i+1}, \ldots, x_{n-1})$ is admissible for $0 < i \leq n-1$. Hence it suffices to prove that $W_k^1 = \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix}$ satisfies the conditions stated in the proposition for $k = 1, \ldots, n-1$. We do this by induction on k. For $k = 1$, $W_1^1 = \begin{pmatrix} 0 & 1 \\ -1 & x_1 \end{pmatrix}$ which clearly satisfies the conditions. Now suppose that W_k^1 satisfies the conditions. Then

$$W_{k+1}^1 = \begin{pmatrix} a_{k+1} & b_{k+1} \\ c_{k+1} & d_{k+1} \end{pmatrix} = \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & x_{k+1} \end{pmatrix}$$

$$= \begin{pmatrix} -b_k & a_k + b_k x_{k+1} \\ -d_k & c_k + d_k x_{k+1} \end{pmatrix}.$$

Hence $a_{k+1} = -b_k < 0$, $c_{k+1} = -d_k < 0$. Furthermore, since the configuration is admissible, $\dfrac{b_{k+1}}{d_{k+1}} = p_{k+1} > p_k = \dfrac{a_{k+1}}{c_{k+1}} > 0$. Using the same argument as in the previous case, we have

$$a_{k+1} d_{k+1} - b_{k+1} c_{k+1} = 1$$

$$\Longrightarrow \frac{a_{k+1}}{c_{k+1}} - \frac{b_{k+1}}{d_{k+1}} = \frac{1}{c_{k+1} d_{k+1}} < 0.$$

Since $c_{k+1} < 0$, this implies $d_{k+1} > 0$ and hence $b_{k+1} > 0$. q.e.d.

Example 2.13. It is easily checked that the vector $(\sqrt{2}, \sqrt{2}, \sqrt{2}, \sqrt{2})$ and corresponding word $\left(\begin{smallmatrix} 0 & 1 \\ -1 & \sqrt{2} \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ -1 & \sqrt{2} \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ -1 & \sqrt{2} \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ -1 & \sqrt{2} \end{smallmatrix}\right)$ are admissible (but not strictly admissible) whereas the vector $(\sqrt{2}, \sqrt{2}, \sqrt{2}, \sqrt{2}, \sqrt{2})$ and corresponding word are not admissible.

The next proposition tells us when a strictly admissible word W remains admissible when we multiply it by an associated matrix either on the left or right, or both.

Proposition 2.14. *Suppose* $W = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A_1 A_2 \dots A_n$ *is a strictly admissible word and* $X = \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix}$. *Then:*

(i) XW *is admissible if and only if* $x \geq b/d$;

(ii) WX *is admissible if and only if* $x \geq -c/d$;

(iii) XWX *is admissible if and only if*

$$x \geq \frac{b - c + \sqrt{(b+c)^2 + 4}}{2d}$$

with strict admissibility if the inequalities are strict.

Proof.

(i) Let $\{\overline{C}, C_0, \dots, C_{n+1}\}$ be the configuration in standard position corresponding to W. The configuration corresponding to AW is obtained by adding another circle C_{-1} tangent to $\overline{C}$ and C_0. This configuration is admissible if and only if $p_{-1} \geq p_{n+1}$, see Figure 6, with strict admissibility if the inequality is strict. By Proposition 2.8, $p_{-1} = x$, by Proposition 2.10, $p_{n+1} = b/d$ so (i) follows.

(ii) The proof is similar to that of Proposition 2.12 and will be omitted.

(iii) Let $\{\overline{C}, C_0, \dots, C_{n+1}\}$ be the configuration in standard position corresponding to W and let $\{\overline{C}, C_{-1}, C_0, \dots, C_{n+1}, C_{n+2}\}$ be the augmented configuration corresponding to XWX. Clearly, the configuration is admissible if and only if $p_{-1} \geq p_{n+2}$ and Conditions (i) and (ii) are satisfied. Since $p_{-1} = x$ and $p_{n+2} = \dfrac{a + bx}{c + dx}$, this is easily shown to be equivalent to $x \geq \alpha$ where α is the larger of the two roots of the equation $dx^2 + (c - b)x - a = 0$ which is equivalent to (iii) using $ad - bc = 1$. q.e.d.

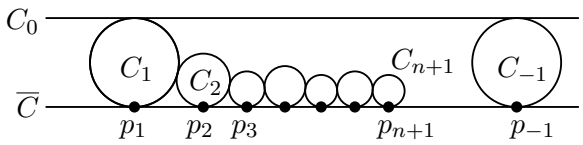


Figure 6.

2.4 Convexity

The next result shows the convexity of the admissibility condition, that is, fixing some positive n, the set of all strictly admissible cross ratio vectors of length n is convex.

Let $C^n_{SA} \subset \mathbb{R}^n$ be the set of all strictly admissible cross ratio vectors of length n and let $\mathbf{x} = (x_1, \ldots, x_n) \in C^n_{SA}$. Suppose $W_{\mathbf{x}} = A_1 A_2 \ldots A_n = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the word of associated matrices and let $p = b/d$. We now regard the entries a, b, c, d of $W_{\mathbf{x}}$ and the ratio p as real valued functions of $\mathbf{x}$.

Proposition 2.15. *We have the following:*

(i) *C^n_{SA} is convex and p is a positive convex function of C^n_{SA}, that is, $\alpha p(\mathbf{x}) + (1-\alpha)p(\mathbf{y}) \geq p(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) > 0$, for any $\mathbf{x}, \mathbf{y} \in C^n_{SA}$ and $\alpha \in [0,1]$.*

(ii) *If $(x_1, \ldots, x_n) \in C^n_{SA}$ and $x'_i \geq x_i$ for all $i = 1, \ldots, n$, then $(x'_1, \ldots, x'_n) \in C^n_{SA}$.*

Proof.

(i) We prove by induction on n. For $n = 1$, $C^1_{SA} = (0, \infty)$ is clearly convex. The convexity and positivity of p amounts to showing that for $x, y > 0$, $\alpha \in [0,1]$,

$$\alpha \frac{1}{x} + (1-\alpha)\frac{1}{y} \geq \frac{1}{\alpha x + (1-\alpha)y} > 0.$$

This follows from the positivity and convexity of the function $g(x) = 1/x$ for $x > 0$. Now suppose that (i) holds for C^{n-1}_{SA} and $W'_{\mathbf{x}} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = A_2 A_3 \ldots A_n$. Let $p' = b'/d'$. We again regard the entries a', b', c', d'

and the ratio p' as real valued functions of $\mathbf{x}$. Then since

$$A_1 = \begin{pmatrix} 0 & 1 \\ -1 & x_1 \end{pmatrix},$$

we have

$$W_{\mathbf{x}} = \begin{pmatrix} c'(\mathbf{x}) & d'(\mathbf{x}) \\ -a'(\mathbf{x}) + c'(\mathbf{x})x_1 & -b'(\mathbf{x}) + d'(\mathbf{x})x_1 \end{pmatrix}. \tag{2.1}$$

Since $W_{\mathbf{x}}$ is strictly admissible, by Proposition 2.12,

$$-b'(\mathbf{x}) + d'(\mathbf{x})x_1 > 0 \Longrightarrow x_1 > \frac{b'(\mathbf{x})}{d'(\mathbf{x})} = p'(\mathbf{x}).$$

Similarly, for $\mathbf{y} = (y_1, \ldots, y_n)$ we have

$$y_1 > \frac{b'(\mathbf{y})}{d'(\mathbf{y})} = p'(\mathbf{y}).$$

Hence

$$\begin{aligned} \alpha x_1 + (1-\alpha)y_1 &> \alpha p'(\mathbf{x}) + (1-\alpha)p'(\mathbf{y}) \\ &\geq p'(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) \qquad \text{(by the hypothesis)} \\ &= \frac{b'(\alpha\mathbf{x} + (1-\alpha)\mathbf{y})}{d'(\alpha\mathbf{x} + (1-\alpha)\mathbf{y})}. \end{aligned}$$

By Proposition 2.14 (i) this implies that $W_{\alpha\mathbf{x}+(1-\alpha)\mathbf{y}}$ is strictly admissible, hence C^n_{SA} is convex. Also, Proposition 2.12 implies that $p(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) > 0$.

Now by (2.1),

$$p(\mathbf{x}) = \frac{d'(\mathbf{x})}{-b'(\mathbf{x}) + d'(\mathbf{x})x_1} = \frac{1}{-p'(\mathbf{x}) + x_1},$$

similarly,

$$p(\mathbf{y}) = \frac{1}{-p'(\mathbf{y}) + y_1}$$

and

$$p(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) = \frac{1}{-p'(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) + \alpha x_1 + (1-\alpha)y_1}.$$

We have

$$
\begin{aligned}
& p(\alpha\mathbf{x}+(1-\alpha)\mathbf{y}) \\
&= \frac{1}{-p'(\alpha\mathbf{x}+(1-\alpha)\mathbf{y})+\alpha x_1+(1-\alpha)y_1} \\
&< \frac{1}{-\alpha p'(\mathbf{x})+(\alpha-1)p'(\mathbf{y})+\alpha x_1+(1-\alpha)y_1}, \\
& \qquad\qquad\qquad\qquad \text{by induction hypothesis} \\
&= \frac{1}{\alpha\frac{1}{p(\mathbf{x})}+(1-\alpha)\frac{1}{p(\mathbf{y})}} \\
&\le \alpha p(\mathbf{x})+(1-\alpha)p(\mathbf{y}), \qquad \text{by convexity of } g(x)=\frac{1}{x}.
\end{aligned}
$$

Hence p is a convex function of C^n_{SA} and the proof by induction is complete.

(ii) Clearly, it suffices to show that if we increase any entry of a strictly admissible cross ratio vector, the resulting vector remains strictly admissible. This is easily seen by using a suitably normalized configuration of circles for the vector. If the kth entry of the vector $(x_1,\dots,x_n)$ is increased, we use the configuration $\{\overline{C}, C_0, C_1, \dots, C_{n+1}\}$ with $\overline{C}$ the real line, C_k the line $\mathrm{Im}(z)=1$ and C_{k+1} the circle $|z-\sqrt{-1}/2| = 1/2$. If p_i is the point of tangency of C_i with $\overline{C}$, then we have: $p_{k+1}=0$, $p_k=\infty$, $p_{k-1}=x_k$, (by our normalization), $p_{k+1} < p_{k+2} < \cdots < p_{n+1}$, $\quad p_{n+1} < p_0 < p_1 < \cdots < p_{k-1}$, since $(x_1,\dots,x_n)$ is strictly admissible. The effect of increasing the entry x_k by a positive constant k on the configuration is then to shift the circles $C_0, C_1, \dots, C_{k-1}$ to the right by k and to leave all the other circles fixed, if $p_{n+1} < p_0$ to start with, then the inequality remains after this shift and so the configuration remains strictly admissible. q.e.d.

2.5 Main Lemma

In this subsection, we prove our main lemma by gathering the results stated above.

Suppose we are given a surface with projective structure (Σ_g, μ) on which a circle packing P lies. Let τ be the nerve of the packing P. The circle packing P lifts to a packing on the universal cover $\widetilde{\Sigma}_g$, similarly, τ lifts to a triangulation $\widetilde{\tau}$ of $\widetilde{\Sigma}_g$. For each edge of $\widetilde{\tau}$, the related circles

form a configuration on $\widetilde{\Sigma}_g$. Developing them to the Riemann sphere, we obtain a configuration of 4 circles on $\hat{\mathbb{C}}$ to which we assign the cross ratio. This assignment is equivariant with respect to the action of the covering transformations.

Definition 2.16 (Cross Ratio Parameter and Associated Words)**.** The map $\mathbf{c}_P : E(\tau) \to \mathbb{R}_+$ in the introduction, which we call a cross ratio parameter of P, is the assignment defined above. Moreover for each vertex $v \in V(\tau)$ with valence n, the associated word W_v of the associated matrices of length n is obtained by reading off the associated matrices of the edges incident to v in the clockwise direction.

Here we have the Main Lemma:

Lemma 2.17 (Main Lemma)**.** *Suppose $\mathbf{c}_P$ is a cross ratio parameter of some packing P on a surface with projective structure and let W_v be the associated word for each $v \in V(\tau)$. Then, we have:*

(1) *$W_v = -I$, where I is the identity matrix, and*

(2) *if the length of W_v is n, then every subword of W_v of length $\leq n-1$ is admissible and every subword of length $\leq n-2$ is strictly admissible.*

Conversely, if a map $\mathbf{c} : E(\tau) \to \mathbb{R}_+$ satisfies (1) *and* (2)*, then there is a unique projective structure on Σ_g together with a circle packing P with nerve isotopic to τ such that the cross ratio parameter $\mathbf{c}_P$ is equal to $\mathbf{c}$.*

Proof. The proof follows almost directly from Propositions 2.10 and 2.12. We first prove the first half. Let v be any vertex of τ with valence n and let C be the circle corresponding to v. The developed image of C is surrounded by n circles $C_1, \ldots, C_n$ (see Figure 7). Let $(x_1, \ldots, x_n)$ be the corresponding cross ratio vector, and $W_v = W_1 \ldots W_n$ the word of associated matrix. By Proposition 2.10, $W_v(I_0) = I_0$ where I_0 is the interstice between C_n, C and C_1 when the configuration is in standard position. Hence $W_v = \pm I$. Furthermore, as the interstices wrap around the central circle C exactly once (see Figure 7), any proper subword of W_v is admissible and any subword of length $\leq n-2$ is strictly admissible. It follows that $W_{n-1} = A_1 A_2 \ldots A_{n-1} = -A_n^{-1}$ since $W_{n-1} A_n = \pm I$ and the $(1,2)$ entry of W_{n-1} is negative. Hence $W_v = -I$.

For the last half, given a map $\mathbf{c} : E(\tau) \to \mathbb{R}_+$, the conditions of the lemma are exactly what we need to uniquely define a developing map

from $\widetilde{\Sigma}_g$ to $\hat{\mathbb{C}}$ which is a local homeomorphism and which gives a circle packing on $\widetilde{\Sigma}_g$ up to the composition of elements of $\mathrm{PSL}_2(\mathbb{C})$. This gives the required projective structure and circle packing. q.e.d.

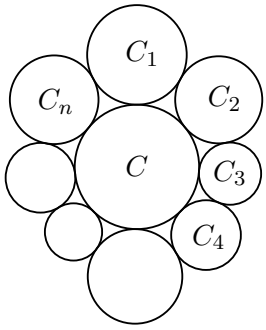


Figure 7.

We recall the following definition of the cross ratio parameter space mentioned in the introduction:

Definition 2.18 (Cross Ratio Parameter Space)**.** The set of all maps in $\mathbb{R}^{E(\tau)}$ satisfying Conditions (1) and (2) for each vertex of τ in the Main Lemma will be called the cross ratio parameter space and denoted by

$$\mathcal{C}_\tau = \{\mathbf{c} : E(\tau) \to \mathbb{R}_+ \,|\, \mathbf{c} \text{ satisfies (1) and (2) for each vertex}\}.$$

Remark. It is worth noting that if τ is covered by a non-simple graph in the universal cover, then there are no maps of $E(\tau)$ to $\mathbb{R}_+$ satisfying Conditions (1) and (2) and hence $\mathcal{C}_\tau$ is an empty set.

2.6 Holonomy representation

Given a surface with projective structure together with a circle packing with nerve τ, we have a cross ratio parameter defined on the edges of τ. An extension of Proposition 2.10 allows us to find the holonomy representation using the cross ratios and the combinatorics of τ (compare with the Fenchel-Nielsen coordinates where it is generally quite difficult to obtain the holonomy representation from the coordinates). To begin, we fix the developing map and the holonomy by developing a fixed interstice to the standard interstice. The holonomy representation of each

covering transformation $\gamma \in \pi_1(\Sigma_g)$ is given by a word in the alphabet $\{A_i^{\pm 1}, R^{\pm 1}\}$ where A_i are the associated matrices of the cross ratios and $R = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 1 \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{C})$ fixes the standard interstice $\mathcal{I}_s$ and permutes its vertices in an anti-clockwise direction ($R(0) = \sqrt{-1}$, $R(\sqrt{-1}) = \infty$, $R(\infty) = 0$). We start with some definitions:

Definition 2.19. A *marked triangle* (Δ, v) of $\widetilde{\Sigma_g} - \widetilde{\tau}$ is a triangle with a marked vertex. This corresponds to an interstice of the circle packing on $\widetilde{\Sigma_g}$ with a marked edge. There are two types of basic moves on the set of marked triangles defined as follows:

(i) A Type 1 move on the set of marked triangles is a rotation either clockwise or anti-clockwise about the marked vertex of a marked triangle to an adjacent triangle. This moves a marked triangle to an adjacent marked triangle with marked vertex in the same position. If x is the cross ratio of the edge common to the two triangles, we associate the matrix $A = \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix}$ to a clockwise rotation, and A^{-1} to an anti-clockwise rotation.

(ii) A Type 2 move fixes the triangle but changes its marking by rotating the vertices either anti-clockwise or clockwise. We associate the matrix $R = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 1 \end{pmatrix}$ for an anti-clockwise rotation of the marking and R^{-1} for a clock-wise rotation.

Suppose that $\gamma(\Delta_1, v_1) = (\Delta_2, v_2)$ for some $\gamma \in \pi_1(\Sigma_g)$. Then the holonomy representation $\rho(\gamma)$ of γ acts on the developing image of (Δ_1, v_1) by $\rho(\gamma)(\mathrm{dev}(\Delta_1, v_1)) = (\mathrm{dev}(\Delta_2, v_2))$. Fix a marked triangle (Δ_0, v_0) and develop it to the standard interstice $\mathcal{I}_s$ such that v_0 corresponds to the real line. We have the following result:

Proposition 2.20. *Let $m_1, m_2, \ldots, m_{k_i}$ be a sequence of moves that maps the marked triangle (Δ_0, v_0) to (Δ_i, v_i) in $\widetilde{\Sigma_g} - \widetilde{\tau}$ and let $B_1, \ldots, B_{k_i}$ be the associated matrices of these moves. Suppose further that* dev $(\Delta_0, v_0) = \mathcal{I}_s$ *with v_0 corresponding to the real line. Then the transformation in* $\mathrm{PSL}_2(\mathbb{C})$ *taking* $\mathrm{dev}(\Delta_0, v_0) = \mathcal{I}_s$ *to* $\mathrm{dev}(\Delta_i, v_i)$ *is $W_i = B_1 B_2 \ldots B_{k_i}$. Hence if $\gamma \in \pi_1(\Sigma_g)$ and $\gamma(\Delta_1, v_1) = (\Delta_2, v_2)$, then the holonomy representation ρ of γ is given by $\rho(\gamma) = W_2 W_1^{-1}$.*

The proof of the above is essentially the same as the proof of Proposition 2.10 and will be omitted.

Example 2.21. Consider the triangulation τ of the torus Σ_1 with one vertex and three edges, e_1, e_2 and e_3 and suppose we have a projective structure on the torus which admits a circle packing with nerve τ. A fundamental domain for the torus consists of two triangles, see Figure 8 for a portion of the triangulation of $\widetilde{\Sigma_1}$ where all horizontal edges are lifts of e_1, all edges of angle $2\pi/3$ with the positive x-axis are lifts of e_2 and all edges of angle $\pi/3$ with the positive x-axis are lifts of e_3. Let x_1, x_2 and x_3 be the cross ratios and A_1, A_2 and A_3 be the associated matrices of e_1, e_2 and e_3 respectively. The fundamental group of Σ_1 is generated by three elements γ_1, γ_2 and γ_3 where $\gamma_1(\Delta_0, v_0) = (\Delta_2, v_2)$, and $\gamma_2(\Delta_1, v_1) = (\Delta_3, v_3)$, $\gamma_3(\Delta_2, v_2) = (\Delta_4, v_4)$, and $\gamma_3\gamma_2^{-1}\gamma_1 = id$ (see Figure 8). It follows from Proposition 2.20 that

$$\begin{aligned} \rho(\gamma_1) &= A_3A_1R \\ \rho(\gamma_2) &= A_3A_1A_2RA_3^{-1} \\ \rho(\gamma_3) &= A_3A_1A_2A_3RA_1^{-1}A_3^{-1}. \end{aligned}$$

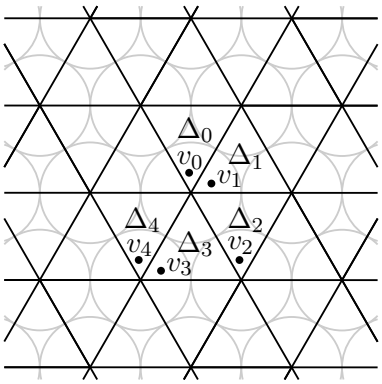


Figure 8.

3. Around the Koebe-Andreev-Thurston solution

3.1 Formal dimension count

Given a triangulation τ on a surface Σ_g, we easily conclude from the Main Lemma in the previous section that:

Proposition 3.1. *The cross ratio parameter space $\mathcal{C}_\tau$ is a real semi-algebraic set of formal dimension $6g-6$ when $g \geq 1$.*

Proof. Since the two conditions in the Main Lemma are expressed in terms of the identities and inequalities in cross ratios, $\mathcal{C}_\tau$ is a real semi-algebraic set. It remains to do a formal dimension count. An easy application of the Gauss-Bonnet theorem shows that $|E(\tau)| = 6g-6+3|V(\tau)|$. For each vertex v, the condition $W_v = -I$ from Lemma 2.17 gives 3 equations in the cross ratios. Since there are $|V(\tau)|$ vertices, this gives $3|V(\tau)|$ equations. The formal dimension is, by definition, the number of variables minus the number of equations and we thus have $|E(\tau)| - 3|V(\tau)| = 6g-6$. q.e.d.

Remark. The formal dimension count works even for τ which is covered by a non-simple graph in the universal cover, because the computation uses only euler characteristic argument. However, the starting point of the argument in the subsequent subsections is the Koebe-Andreev-Thurston solution, so that τ will have to be covered by a simple graph in the universal cover in the sequel.

3.2 When $g \geq 2$

In this subsection, we will see, in the case $g \geq 2$, that the formal dimension of $\mathcal{C}_\tau$ is the correct one at least near the Koebe-Andreev-Thurston solution and that the solution has a nice manifold neighborhood. The arguments are based heavily on the well developed deformation theory of Kleinian groups by Ahlfors, Bers and other authors, see for instance [5].

When $g \geq 2$, the image of the canonical section $s : \mathcal{T}_g \to \mathcal{P}_g$ has a tubular neighborhood which consists of projective structures whose developing map is injective and whose developed image is surrounded by a topological circle embedded in $\hat{\mathbb{C}}$. This neighborhood is called the quasi Fuchsian space and is denoted by $\mathcal{QF}_g$. The projective structure in $\mathcal{QF}_g$ is sometimes called quasi Fuchsian. The bounding circle of the developed image of a quasi Fuchsian structure bounds a disk on the other side in $\hat{\mathbb{C}}$ by the Jordan curve theorem and the action of the holonomy image there is still properly discontinuous. Hence by taking the quotient, we obtain another Riemann surface homeomorphic to Σ_g. By assigning to each quasi Fuchsian structure, its conformal structure with opposite orientation and the conformal structure induced on the

other side, we obtain a map

$$\beta : \mathcal{QF}_g \to \overline{\mathcal{T}}_g \times \mathcal{T}_g.$$

Here we adopt the convention so that the original projective structure with opposite orientation projects down to the first component. Bers' simultaneous uniformization theorem states that β is in fact a homeomorphism.

Lemma 3.2. *Suppose $g \geq 2$, and let β_2 be the composition of β with the projection to the second factor. Then*

$$\beta_2 \circ f : f^{-1}(\mathcal{QF}_g) \to \mathcal{T}_g$$

is a homeomorphism, where f is the "forgetting map" from $\mathcal{C}_\tau$ to $\mathcal{P}_g$ defined in the introduction. In particular, $f^{-1}(\mathcal{QF}_g) \subset \mathcal{C}_\tau$, which includes the Koebe-Andreev-Thurston solution, is homeomorphic to the euclidean space of dimension $6g - 6$.

Proof. We first construct a hyperbolic 3-manifold using the Koebe-Andreev-Thurston solution. Given τ, the Koebe-Andreev-Thurston solution gives a hyperbolic structure on Σ_g and a packing P_0. Developing the hyperbolic surface, we obtain a packing on the upper half plane $\mathbb{H}^2$. Let R be its mirror image on the lower half plane $\overline{\mathbb{H}}^2$. The dual packing R^* which consists of the circumscribed circles of interstices in R fills $\overline{\mathbb{H}}^2$. Each member of both packings bounds a hemisphere in $\mathbb{H}^3$ and such a hemisphere bounds a hemiball facing $\overline{\mathbb{H}}^2$. Chopping off those hemiballs from $\mathbb{H}^3$, we obtain a hyperbolic 3-manifold L with ideal polygonal boundary. The point of tangency in the packing R becomes an ideal vertex with rectangular section, and its modulus is the cross ratio of a corresponding point of tangency in P_0 times $\sqrt{-1}$.

The holonomy image of the Koebe-Andreev-Thurston solution acts properly discontinuously on $\mathbb{H}^3$ by the Poincaré extension and L is invariant under its action. Hence taking quotient, we obtain a hyperbolic 3-manifold M with ideal polygonal boundary and with one end homeomorphic to Σ_g. The polygonal faces on the boundary are divided into two classes, one extends to the hemisphere bounded by a circle in R and the other in R^*. Take the double of L first along faces in the first class and then again take the double of the result of the first doubling along faces in the second class. This double doubling construction gives us a geometrically finite hyperbolic 3-manifold N with 4 ends, each homeomorphic to Σ_g, and $|E(\tau)|$ cusps corresponding to the points of tangency

in P_0. It is obvious by the construction that the modulus of each cusp in N is the cross ratio of the corresponding point of tangency in P_0 times $\sqrt{-1}$. Moreover, N naturally admits a $\mathbb{Z}/2 \times \mathbb{Z}/2$ $(= G)$ symmetry generated by two reflections coming from the double doubling.

We now let $\mathrm{Def}^G(N)$ be the space of geometrically finite deformations of N preserving the G action. Again by the theorem of Bers, it can be identified with the space of G-invariant quasi conformal deformations of the end of N and thus we have a homeomorphism

$$q : \mathrm{Def}^G(N) \to \mathcal{T}_g.$$

In any geometrically finite deformation of N, the cusps remain cusps and there are $|E(\tau)|$ cusps. Moreover, since the deformation is chosen to admit a G action generated by two reflections, each cusp must admit the same symmetry so that the fundamental domain of the cusp remains rectangular. Thus for each deformation of N, we have a map assigning to each edge of τ a positive real number obtained by the modulus of the corresponding cusp over $\sqrt{-1}$. This gives a map $m : \mathrm{Def}^G(N) \to \mathbb{R}^{E(\tau)}$.

Let N' be a G-invariant geometrically finite deformation of N. Then the quotient M' of N' by the G action has an ideal polygonal boundary having the same combinatorial pattern with that of M. The universal cover L' of M' also has an ideal polygonal boundary having the same combinatorial pattern with that of L. The faces corresponding to R extend to hemispheres which provide a circle packing on the domain bounded by a Jordan curve in $\hat{\mathbb{C}}$. The packing is obviously invariant by the action of the holonomy image of N' and thus by taking quotient we obtain a projective structure on Σ_g together with a circle packing with nerve isotopic to τ. The image of a deformation N' by m is nothing but the cross ratio parameter of the packing so obtained. Thus we have squeezed the image of m by

$$m : \mathrm{Def}^G(N) \to \mathcal{C}_\tau.$$

Now it is not hard to see that the composition of $m \circ q^{-1}$ and $\beta_2 \circ f$, is the identity and so is the composition with the reversed order. q.e.d.

3.3 When $g = 1$

In this subsection, we show that the Koebe-Andreev-Thurston solution has a nice manifold neighborhood of dimension 2 when $g = 1$, in contrast with the formal dimension count of $\mathcal{C}_\tau$ in §3.1. The arguments are based

heavily on the hyperbolic Dehn surgery theory developed by Thurston [18].

The fundamental group of Σ_1 is isomorphic to $\mathbb{Z} \times \mathbb{Z}$ and is mapped by the holonomy representation of a projective structure to an elementary subgroup in $\mathrm{PSL}_2(\mathbb{C})$ which fixes a single point or two points on $\hat{\mathbb{C}}$ globally. Thus since $\mathcal{C}_\tau$ is the space of deformations of projective structures on Σ_1 with a specific type of packings, we in particular obtain a family of such representations of $\mathbb{Z} \times \mathbb{Z}$ in $\mathrm{PSL}_2(\mathbb{C})$ parameterized by $\mathcal{C}_\tau$. In this setting, the hyperbolic Dehn surgery theory defines a continuous map

$$d : \mathcal{C}_\tau \to \mathbb{R}^2 \cup \{\infty\}/\sim$$

which generalizes the classical Dehn surgery coefficients in the knot theory, such that the Koebe-Andreev-Thurston solution is mapped to ∞ by d. The map d is in fact described in terms of the complex affine structure in [18] and the image lies in $\mathbb{R}^2 \cup \{\infty\}$, but $\mathcal{C}_\tau$ here is described in terms of the projective structure which is generically in one to two correspondence with the complex affine structure, so that we divide the image by the antipodal involution $\sim$ of $\mathbb{R}^2$.

Lemma 3.3. *Suppose that $g = 1$. Then there is a neighborhood U of the Koebe-Andreev-Thurston solution in $\mathcal{C}_\tau$ which is homeomorphic to the euclidean space of dimension 2, and the restriction of d to which is an embedding.*

Proof. Given τ, the Koebe-Andreev-Thurston solution gives an euclidean torus and a packing P_0. Developing the torus, we obtain a packing R on the complex plane $\mathbb{C}$. Using the same dualizing, chopping off, taking quotients and double doubling construction in the previous subsection, we obtain a dual packing R^* on $\mathbb{C}$ and the hyperbolic 3-manifolds L, M, N respectively, where N has 4 old cusps and $|E(\tau)|$ new cusps corresponding to the points of tangency in P_0. It is again obvious by the construction that the modulus of each cusp is the cross ratio of the corresponding point of tangency in P_0 times $\sqrt{-1}$. Moreover, N admits a $\mathbb{Z}/2 \times \mathbb{Z}/2$ $(= G)$ symmetry generated by the two reflections coming from the double doubling.

Let $\mathrm{Def}^G(N)$ be in this case a small neighborhood of N in the space of hyperbolic Dehn surgery deformations of N preserving the G action. Then since the deformations preserve the G action, the cusps created by the double doubling must admit the G-symmetry generated by the two reflections, so they remain cusps. Thus by Thurston's theorem, the

deformations of N which are small enough are parameterized by the G-invariant generalized Dehn surgery coefficients of the other old cusps and we have an embedding

$$q : \mathrm{Def}^G(N) \to \mathbb{R}^2 \cup \{\infty\}/\sim$$

where the image is a neighborhood of $\infty = q(N)$.

In any Dehn surgery deformation of N in $\mathrm{Def}^G(N)$, there are $|E(\tau)|$ cusps corresponding to the points of tangency in P_0. Moreover since such cusps admit reflectional symmetry, their fundamental domains remain rectangular. Thus for each deformation of N, we have a map assigning to each edge of τ a positive real number obtained by the modulus of the corresponding cusp over $\sqrt{-1}$. Then the assignment of this map to N defines a map $m : \mathrm{Def}^G(N) \to \mathbb{R}^{E(\tau)}$.

Let N' be a G-invariant incomplete Dehn surgery deformation of N whose completion lies in $\mathrm{Def}^G(N)$. Thus N' is still homeomorphic to N. Then the quotient M' of N' by the G action has an ideal polygonal boundary having the same combinatorial pattern with that of M. The universal cover L' of M' also has an ideal polygonal boundary having the same combinatorial pattern with that of L. Since L' carries a hyperbolic metric though incomplete, it can be developed to $\mathbb{H}^3$. The developed image of a face corresponding to R extends to a hemisphere whose boundary defines a circle on $\hat{\mathbb{C}}$. The coherent collection of such circles provide a developed image of a certain circle packing on Σ_g. Pulling back the projective structure on $\hat{\mathbb{C}}$ to the universal cover of Σ_g, and taking quotient by the covering transformations, we obtain a projective structure on Σ_1 together with a circle packing with nerve isotopic to τ. The image of a deformation N' by m is nothing but the cross ratio parameter of the packing so obtained. We have thus again squeezed the image of m by

$$m : \mathrm{Def}^G(N) \to \mathcal{C}_\tau.$$

Now it is not hard to see that $m \circ q^{-1}$ is the inverse of d in a small neighborhood of ∞. q.e.d.

This lemma together with Lemma 3.2 gives Main Theorem 1 in the Introduction.

4. Topology of $\mathcal{C}_\tau$ for one circle packings

4.1 Combinatorics

For this section and the next, we assume that $g \geq 2$, and τ is a graph on Σ_g with one vertex which triangulates Σ_g. By the Gauss-Bonnet, τ has $6g-3$ edges and triangulates Σ_g into $4g-2$ triangles. Let τ' be the dual graph of τ. τ' is a trivalent graph (all vertices have valence 3) with $6g-3$ edges $e'_1, \ldots, e'_{6g-3}$. The cross ratios defined on the edges of τ clearly define a map of $E(\tau')$ to $\mathbb{R}_+$ which we also call a cross ratio parameter, hence the cross ratio parameter space $\mathcal{C}_\tau$ can also be interpreted as being defined on the edges of τ', and we will adopt this point of view whenever appropriate.

Cutting Σ_g along the edges of τ' gives a polygon $\mathcal{P}$ with $12g-6$ sides $s_1, s_2, \ldots, s_{12g-6}$ and $6g-3$ side pairings $s_i \leftrightarrow s_j$, each paired side corresponding to some edge e'_s of τ'. When $g=2$ there are essentially 8 different possible side-pairing patterns: see Figure 9, where each 18-gon represents the unique face of the cell division corresponding to τ', and the arcs there indicate the side-pairing pattern. When $g=3$, a computer search produces over nine hundred different side-pairing patterns.

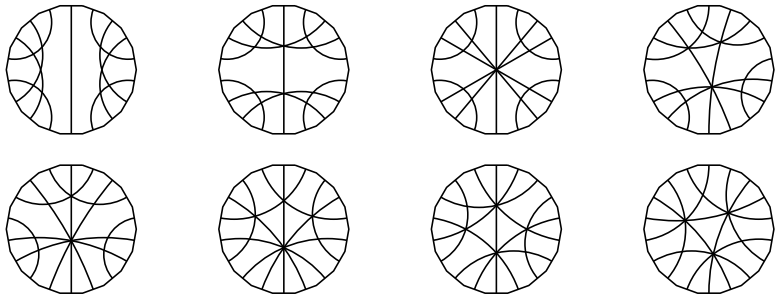
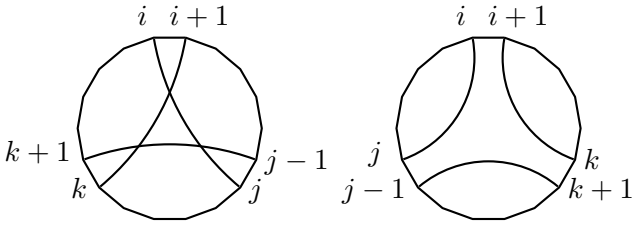


Figure 9.

Since each vertex of τ' has valence 3, it lifts to 3 vertices of $\mathcal{P}$ and the three edges incident to the vertex corresponds to 3 pairs of sides of $\mathcal{P}$. It follows that if $s_i \leftrightarrow s_j$ and $s_{i+1} \leftrightarrow s_k$, then $s_{j-1} \leftrightarrow s_{k+1}$ (indices are taken mod $12g-6$). $\{s_i \leftrightarrow s_j, s_{i+1} \leftrightarrow s_k, s_{j-1} \leftrightarrow s_{k+1}\}$ is called the triple of side-pairings associated to the vertex. We can divide the triples to 2 types according to whether the pairs of sides separate each other as follows:

Definition 4.1. The triple of side pairings $\{s_i \leftrightarrow s_j, s_{i+1} \leftrightarrow s_k, s_{j-1} \leftrightarrow s_{k+1}\}$ is *separating* if $i, i+1, j-1, j, k, k+1$ is in cyclic order mod $12g-6$ and *non-separating* if $i, i+1, k, k+1, j-1, j$ is in cyclic order mod $12g-6$ (see Figures 10a and 10b). Similarly, a vertex of τ' and the triple of edges incident to the vertex are called separating and non-separating if they give rise to separating and non-separating triples of side-pairings respectively.



(a) separating (b) non-separating

Figure 10.

Remark. In the case of a non-separating triple, a simple topological argument shows that s_{i+1} and s_k are not adjacent sides of $\mathcal{P}$, similarly s_{k+1} and s_{j-1} are not adjacent, and s_j and s_i are not adjacent.

In the case of genus 1, there is only one triple and it is separating. It will be crucial in our analysis of the one circle packing case for higher genus that there exists a non-separating triple. We have the following result:

Proposition 4.2. *If $g \geq 2$ and τ' is a trivalent graph on Σ_g cutting Σ_g into one polygon, then both separating and non-separating vertices of τ' must occur.*

Proof. We first show that there exists non-separating triples of side-pairings. If $s_1 \leftrightarrow s_i$ and $s_2 \leftrightarrow s_j$ with $i > j$, then $\{s_1 \leftrightarrow s_i, s_2 \leftrightarrow s_j, s_{j+1} \leftrightarrow s_{i-1}\}$ is a non-separating triple and we are done. So we may assume that $i < j$ and $\{s_1 \leftrightarrow s_i, s_2 \leftrightarrow s_j, s_{j+1} \leftrightarrow s_{i-1}\}$ is a separating triple. Furthermore, by a cyclic permutation if necessary, we may assume that $i \geq 5$ since $6g-3 > 6$. Now consider the two side-pairings $s_2 \leftrightarrow s_j$ and $s_3 \leftrightarrow s_k$. Again, if $k < j$ this gives rise to a non-separating triple $\{s_2 \leftrightarrow s_j, s_3 \leftrightarrow s_k, s_{k+1} \leftrightarrow s_{j-1}\}$, so we may assume again that $k > j$. Continuing inductively, we either get a non-

separating triple or we eventually have $s_{i-2} \leftrightarrow s_l$ where $l > j+2$. This gives us the non-separating triple $\{s_{i-2} \leftrightarrow s_l, s_{i-1} \leftrightarrow s_{j+1}, s_{j+2} \leftrightarrow s_{l-1}\}$ and we are done.

To show that there exists a separating triple, we use a similar method. We may assume that $\{s_1 \leftrightarrow s_i, s_2 \leftrightarrow s_j, s_{j+1} \leftrightarrow s_{i-1}\}$ is a non-separating triple so that $j < i$ and furthermore $j > 3$ by the remark following Definition 4.1. Now if $s_3 \leftrightarrow s_k$, either $\{s_2 \leftrightarrow s_j, s_3 \leftrightarrow s_k, s_{k+1} \leftrightarrow s_{j-1}\}$ is a separating triple or $k < j$. After less than $j-1$ such steps, we either get a separating triple or we reach a stage where $s_m \leftrightarrow s_p$, $s_{m+1} \leftrightarrow s_q$ with $q > p$. This gives the separating triple $\{s_m \leftrightarrow s_p, s_{m+1} \leftrightarrow s_q, s_{p-1} \leftrightarrow s_{q+1}\}$ as required. q.e.d.

4.2 Free cross ratios

In this subsection, we will show that we can choose a set of $6g-6$ independent cross ratios on the set of edges of τ, (equivalently of τ') such that they determine the remaining three cross ratios. The idea is to choose a non-separating vertex v' of τ' (always possible by Proposition 4.2) and use the cross ratios of the 3 edges of τ' incident to v' as the set of dependent variables and the remaining $6g-6$ variables as the free variables.

We start with a local argument. Recall from §2 that apart from the admissibility conditions which are open conditions, locally, the cross ratio parameter space $\mathcal{C}_\tau$ is given by the matrix identity $W_v := \begin{pmatrix} a & b \\ c & d \end{pmatrix} = -I$ or equivalently, by the 3 equations

$$\begin{aligned} a(x_1, \ldots, x_{6g-3}) &= -1 \\ b(x_1, \ldots, x_{6g-3}) &= 0 \\ c(x_1, \ldots, x_{6g-3}) &= 0. \end{aligned}$$

(Note that the entries a, b, c and d are regarded as functions of the variables and that $d = -1$ follows from the above since $\det(W_v) = 1$.) We will show that we can find a set of 3 variables x_i, x_j and x_k such that the Jacobian

$$\begin{vmatrix} \frac{\partial a}{\partial x_i} & \frac{\partial a}{\partial x_j} & \frac{\partial a}{\partial x_k} \\ \frac{\partial b}{\partial x_i} & \frac{\partial b}{\partial x_j} & \frac{\partial b}{\partial x_k} \\ \frac{\partial c}{\partial x_i} & \frac{\partial c}{\partial x_j} & \frac{\partial c}{\partial x_k} \end{vmatrix} \neq 0.$$

By the implicit function theorem, this implies that locally, $\mathcal{C}_\tau$ is determined by the remaining $6g-6$ variables and hence is locally of dimension $6g-6$.

Recall that if v is the vertex of τ and $e_1, \ldots, e_{6g-3}$ the set of edges with cross ratios $x_1, \ldots, x_{6g-3}$, reading off the edges incident to v in a clockwise direction, we get the cross ratio vector $\mathbf{x} = (x_{n_1}, x_{n_2}, \ldots, x_{n_{12g-6}})$ and the associated word $W_{\mathbf{x}} = A_{n_1} A_{n_2} \ldots A_{n_{12g-6}}$. Alternatively, we can think of the cross ratio vector as the the vector obtained by reading off the cross ratios of the sides of the polygon obtained by cutting Σ_g along the dual graph τ'.

Example 4.3. For the polygon with side pairing given in the first pattern of Figure 9, the cross ratio vector is given by

$$\mathbf{x} = (x_1, x_2, x_3, x_4, x_2, x_5, x_3, x_4, x_5, x_1, x_6, x_7, x_8, x_6, x_9, x_7, x_8, x_9).$$

By Proposition 4.2, and by doing a cyclic permutation if necessary, we may assume that $\{s_{12g-6} \leftrightarrow s_i, s_1 \leftrightarrow s_j, s_{j+1} \leftrightarrow s_{i-1}\}$ is a non-separating triple so that $j < i$. Let x, y and z be the cross ratios of the edges of τ' corresponding to the pairs of sides $s_1 \leftrightarrow s_j$, $s_{j+1} \leftrightarrow s_{i-1}$ and $s_{12g-6} \leftrightarrow s_i$ respectively and let $x_1, x_2, \ldots, x_{6g-6}$ be the cross ratios of the remaining edges of τ' corresponding to the other remaining pairs of sides numbered in some fixed order. The cross ratio vector $\mathbf{x}$ is then of the form

$$\mathbf{x} = (x, x_{n_2}, \ldots, x_{n_{j-1}}, x, y, x_{n_{j+2}}, \ldots, x_{n_{i-2}}, y, z, x_{n_{i+1}}, \ldots, x_{n_{12g-7}}, z)$$

with associated word W in the associated matrices given by

$$\begin{aligned} W_{\mathbf{x}} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & y \end{pmatrix}^U \begin{pmatrix} 0 & 1 \\ -1 & y \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & z \end{pmatrix}^V \begin{pmatrix} 0 & 1 \\ -1 & z \end{pmatrix} \end{aligned}$$

where T, U and V are subwords of W of lengths $j-2$, $i-j-3$ and $12g-i-7$ respectively which depend only on the variables $x_1, \ldots, x_{6g-6}$.

Proposition 4.4. *With the notation above, let* $W_k := \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix} = A_{n_1} A_{n_2} \ldots A_{n_k}$ *be the subword of* $W_{\mathbf{x}}$ *given by the product of the first*

k associated matrices. Then

$$\begin{aligned}
\begin{pmatrix} a_x & b_x \\ c_x & d_x \end{pmatrix} &= \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} -b_{j-1}d_{j-1} & b_{j-1}^2 \\ -d_{j-1}^2 & b_{j-1}d_{j-1} \end{pmatrix} \\
\begin{pmatrix} a_y & b_y \\ c_y & d_y \end{pmatrix} &= \begin{pmatrix} -b_jd_j & b_j^2 \\ -d_j^2 & b_jd_j \end{pmatrix} + \begin{pmatrix} -b_{i-2}d_{i-2} & b_{i-2}^2 \\ -d_{i-2}^2 & b_{i-2}d_{i-2} \end{pmatrix} \\
\begin{pmatrix} a_z & b_z \\ c_z & d_z \end{pmatrix} &= \begin{pmatrix} -b_{i-1}d_{i-1} & b_{i-1}^2 \\ -d_{i-1}^2 & b_{i-1}d_{i-1} \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\end{aligned}$$

where a_x, a_y, a_z etc. denotes the partial derivatives of the entries of W with respect to the variables x, y and z.

Proof. We will show the second formula, the first and third are similar. We apply the product rule, and using the fact that $W_{\mathbf{x}} = -I$ since the cross ratio vector corresponds to a point in $\mathcal{C}_\tau$, we get

$$\begin{aligned}
\begin{pmatrix} a_y & b_y \\ c_y & d_y \end{pmatrix} &= W_j \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} (-W_{j+1})^{-1} + W_{i-2} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} (-W_{i-1})^{-1} \\
&= \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -d_{j+1} & b_{j+1} \\ c_{j+1} & -a_{j+1} \end{pmatrix} \\
&\quad + \begin{pmatrix} a_{i-2} & b_{i-2} \\ c_{i-2} & d_{i-2} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -d_{i-1} & b_{i-1} \\ c_{i-1} & -a_{i-1} \end{pmatrix} \\
&= \begin{pmatrix} b_jc_{j+1} & -b_ja_{j+1} \\ d_jc_{j+1} & -d_ja_{j+1} \end{pmatrix} + \begin{pmatrix} b_{i-2}c_{i-1} & -b_{i-2}a_{i-1} \\ d_{i-2}c_{i-1} & -d_{i-2}a_{i-1} \end{pmatrix} \\
&= \begin{pmatrix} -b_jd_j & b_j^2 \\ -d_j^2 & b_jd_j \end{pmatrix} + \begin{pmatrix} -b_{i-2}d_{i-2} & b_{i-2}^2 \\ -d_{i-2}^2 & b_{i-2}d_{i-2} \end{pmatrix}
\end{aligned}$$

since

$$\begin{pmatrix} a_{j+1} & b_{j+1} \\ c_{j+1} & d_{j+1} \end{pmatrix} = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & x_{n_{j+1}} \end{pmatrix}$$

so that $c_{j+1} = -d_j$, $a_{j+1} = -b_j$ and similarly $c_{i-1} = -d_{i-2}$, $a_{i-1} = -b_{i-2}$. q.e.d.

We are now ready to prove the local result for the cross ratio parameter space $\mathcal{C}_\tau$.

Lemma 4.5. *The cross ratio parameter space $\mathcal{C}_\tau$ for a triangulation τ of Σ_g ($g \geq 2$) with one vertex is locally homeomorphic to $\mathbb{R}^{6g-6}$. Furthermore, the local parameters can be taken to be the cross ratios on*

a set of $6g-6$ *edges of* τ' *such that the remaining* 3 *edges are incident to a non-separating vertex of* τ'.

Proof. Adopting the notation of Proposition 4.4, it suffices to prove that the Jacobian

$$\begin{vmatrix} \frac{\partial a}{\partial x} & \frac{\partial a}{\partial y} & \frac{\partial a}{\partial z} \\ \frac{\partial b}{\partial x} & \frac{\partial b}{\partial y} & \frac{\partial b}{\partial z} \\ \frac{\partial c}{\partial x} & \frac{\partial c}{\partial y} & \frac{\partial c}{\partial z} \end{vmatrix} \neq 0$$

where x, y and z are the parameters of the edges incident to a non-separating vertex. By Proposition 4.4, this is equivalent to showing that

$$\begin{vmatrix} -b_{j-1}d_{j-1} & -b_jd_j - b_{i-2}d_{i-2} & -b_{i-1}d_{i-1} \\ b_{j-1}^2 & b_j^2 + b_{i-2}^2 & b_{i-1}^2 + 1 \\ -1 - d_{j-1}^2 & -d_j^2 - d_{i-2}^2 & -d_{i-1}^2 \end{vmatrix} \neq 0.$$

We will show that the 3 columns ϕ_1, ϕ_2 and ϕ_3 of the above Jacobian are linearly independent. By Proposition 2.12,

$$b_{j-1}, d_{j-1}, b_j, d_j, b_{i-2}, d_{i-2}, b_{i-1}, d_{i-1} > 0, \text{ and}$$

$$p_{j-1} = b_{j-1}/d_{j-1} < p_j = b_j/d_j < p_{i-2} = b_{i-2}/d_{i-2} < p_{i-1} = b_{i-1}/d_{1-1}$$

since the subwords W_k of $W_{\mathbf{x}}$ are all strictly admissible for $k \leq 12g-4$. Let

$$\vec{v}_1 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \qquad \vec{v}_2 = \begin{pmatrix} -p_{j-1} \\ p_{j-1}^2 \\ -1 \end{pmatrix}, \qquad \vec{v}_3 = \begin{pmatrix} -p_j \\ p_j^2 \\ -1 \end{pmatrix},$$

$$\vec{v}_4 = \begin{pmatrix} -p_{i-2} \\ p_{i-2}^2 \\ -1 \end{pmatrix}, \qquad \vec{v}_5 = \begin{pmatrix} -p_{i-1} \\ p_{i-1}^2 \\ -1 \end{pmatrix}, \qquad \vec{v}_6 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

We have $\phi_1 = \vec{v}_1 + d_{j-1}^2\vec{v}_2$, $\quad \phi_2 = d_j^2\vec{v}_3 + d_{i-2}^2\vec{v}_4$, $\quad \phi_3 = d_{i-1}^2\vec{v}_5 + \vec{v}_6$.

Note that the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5$, and $\vec{v}_6$ all lie on the circular cone passing through the origin and the parabola $\{y = x^2,\ z = -1\}$ and that furthermore they are in anti-clockwise order. Since ϕ_1 is a positive linear combination of $\vec{v}_1$ and $\vec{v}_2$, its intersection with the horizontal plane $\{z = -1\}$ lies on the chord joining $(0, 0, -1)$ and $(-p_{j-1}, p_{j-1}^2, -1)$. Similarly, the intersection of ϕ_2 with the horizontal plane $\{z = -1\}$ lies on the chord joining $(-p_j, p_j^2, -1)$ and $(-p_{i-2}, p_{i-2}^2, -1)$ and the intersection of ϕ_3 with the horizontal plane $\{z = -1\}$ lies on the half infinite line segment $\{(x, y, z) \mid x = -p_{i-1},\ y \geq p_{i-1}^2,\ z = -1\}$. The

convexity of the curve $y = x^2$ and the arrangement of the points now implies that the three vectors are linearly independent. q.e.d.

Remark. The proofs of Proposition 4.4 and Lemma 4.5 show that more general results hold. For example, for Lemma 4.5, we need not choose a triple of edges corresponding to a non-separating vertex to be the ones with the dependent variables. In fact, we can use any 3 edges such that the three sets of paired sides in the fundamental polygon $\mathcal{P}$ corresponding to these 3 edges are mutually non-separating. Indeed, one sees how the proof breaks down if the non-separating property is not present, which is why the result does not hold for the genus 1 case. Similarly, for more general triangulations of Σ_g, if a vertex with valence n is chosen such that no edge is incident to it more than once, then the 3 equations corresponding to that vertex are always independent and we can choose the cross ratios on any set of $n-3$ edges as the free variables for these equations.

4.3 Topology of $\mathcal{C}_\tau$

Lemma 4.5 shows that $\mathcal{C}_\tau$ is a manifold of dimension $6g-6$ but it does not give information about the topology of $\mathcal{C}_\tau$. Before proving a global version of Lemma 4.5, we first state and prove the following preliminary result:

Proposition 4.6. *Let A, B and C be three strictly admissible words of associated matrices. Then*

$$\begin{aligned} & ABC = -I \\ \Leftrightarrow \quad & (2,2) \text{ entries of } \begin{cases} AB = -C^{-1} \\ BC = -A^{-1} \\ CA = -B^{-1} \end{cases} . \end{aligned}$$

Furthermore, if the above holds, then ABC also satisfies Condition (2) *of the Main Lemma, that is, every proper subword of ABC is admissible.*

Proof. ($\Rightarrow$) is clear.

($\Leftarrow$) Let $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$, $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$ and $C = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}$. By Proposition 2.12, since A, B and C are strictly admissible words, $a_1, b_1, c_1 \leq 0$, $a_2, b_2, c_2 > 0$, $a_3, b_3, c_3 < 0$ and $a_4, b_4, c_4 > 0$. Then the

conditions on the $(2,2)$ entries implies

$$-c_1 = a_3b_2 + a_4b_4 \tag{4.1}$$
$$-a_1 = b_3c_2 + b_4c_4 \tag{4.2}$$
$$-b_1 = c_3a_2 + c_4a_4. \tag{4.3}$$

It suffices to show that

$$\begin{aligned} -c_4 &= a_1b_1 + a_2b_3 \\ c_2 &= a_1b_2 + a_2b_4 \\ c_3 &= a_3b_1 + a_4b_3 \end{aligned}$$

as together with (4.1) this implies $AB = -C^{-1}$, hence $ABC = -I$.

From $\det C = 1$, we have

$$\begin{aligned} c_2c_3 - c_1c_4 + 1 &= 0 \\ \implies \quad a_2b_3c_2c_3 - a_2b_3c_1c_4 + a_2b_3 &= 0. \end{aligned} \tag{4.4}$$

From (4.2) and (4.3),

$$\begin{aligned} -b_3c_2 &= a_1 + b_4c_4 \\ -a_2c_3 &= b_1 + a_4c_4 \\ \implies \quad a_2b_3c_2c_3 &= a_1b_1 + (a_1a_4 + b_1b_4)c_4 + a_4b_4c_4^2. \end{aligned}$$

Substituting into (4.4), we get

$$a_4b_4c_4^2 + (a_1a_4 + b_1b_4 - a_2b_3c_1)c_4 + (a_1b_1 + a_2b_3) = 0. \tag{4.5}$$

We claim that

$$a_1a_4 + b_1b_4 - a_2b_3c_1 = 1 + a_4b_4(a_1b_1 + a_2b_3). \tag{4.6}$$

Substituting $-c_1 = a_3b_2 + a_4b_4$ from (4.1), this is equivalent to

$$\begin{aligned} a_1a_4 + b_1b_4 + a_2b_3a_3b_2 &= 1 + a_4b_4a_1b_1 \\ \iff \quad a_2a_3b_2b_3 &= (1 - a_1a_4)(1 - b_1b_4). \end{aligned}$$

This is obviously true since

$$a_1a_4 - 1 = a_2a_3, \qquad b_1b_4 - 1 = b_2b_3.$$

Now substituting (4.6) into (4.5), we get

$$\begin{aligned}(c_4 + (a_1b_1 + a_2b_3))(a_4b_4c_4 + 1) &= 0\\ \Longrightarrow \qquad -c_4 &= a_1b_1 + a_2b_3\end{aligned}$$

since $a_4b_4c_4 + 1 > 0$.

Now from (4.2), we get

$$\begin{aligned}b_3c_2 &= -a_1 - b_4c_4\\ &= -a_1 + b_4(a_1b_1 + a_2b_3)\\ &= a_1(-1 + b_1b_4) + a_2b_4b_3\\ &= a_1b_2b_3 + a_2b_4b_3\\ \Longrightarrow \quad c_2 &= a_1b_2 + a_2b_4, \quad \text{since } \ b_3 < 0.\end{aligned}$$

Similarly, from (4.3), we get

$$\begin{aligned}a_2c_3 &= -b_1 - a_4c_4\\ &= -b_1 + a_4(a_1b_1 + a_2b_3)\\ &= b_1(-1 + a_1a_4) + a_2a_4b_3\\ &= b_1a_2a_3 + a_2a_4b_3\\ \Longrightarrow \quad c_3 &= a_3b_1 + a_4b_3, \qquad \text{since } \ a_2 \neq 0.\end{aligned}$$

This completes the first part of the proposition. Finally, we note that since A, B and C are strictly admissible, if we consider the configuration of circles associated to A, B and C, the interstices do not go around the central circle completely. Hence for ABC, the interstices go around the central circle less than 3 times. However, since $ABC = -I$, the interstices corresponding to ABC go around the central circle exactly an odd number of times. As it is strictly less than 3, they go around exactly once and ABC is a word that satisfies both Conditions (1) and (2) of the Main Lemma. q.e.d.

We now prove the global version of Lemma 4.5:

Lemma 4.7. *Let τ' be a trivalent graph on Σ_g dual to a triangulation τ of Σ_g with one vertex. Let $e'_1, e'_2, \ldots, e'_{6g-3}$ be a numbering of the edges of τ' such that the last three edges are associated to a non-separating vertex and let $x_1, \ldots, x_{6g-3}$ be the cross ratios of the edges. The projection of the cross ratio parameter space $\mathcal{C}_\tau$ to $\mathbb{R}^{6g-6}$ given by taking the first $6g-6$ variables,*

$$p : \mathcal{C}_\tau \longrightarrow \mathbb{R}^{6g-6},$$

$$p(x_1, \ldots, x_{6g-3}) = (x_1, \ldots, x_{6g-6})$$

is a homeomorphism onto its image. Furthermore, the image of the projection is a strictly convex subset of $\mathbb{R}^{6g-6}$ *with the property that if* $(a_1, \ldots, a_{6g-6}) \in p(\mathcal{C}_\tau)$*, then* $(b_1, \ldots, b_{6g-6}) \in p(\mathcal{C}_\tau)$ *whenever* $b_i \geq a_i$ *for* $i = 1, \ldots, 6g-6$*.*

Proof. We adopt the notation of Proposition 4.4 and Lemma 4.5, and denote the last 3 variables by x, y and z respectively.

If a vector $\mathbf{x} = (x_1, \ldots, x_{6g-6}, x, y, z)$ represents a point in $\mathcal{C}_\tau$, the associated word is

$$W_{\mathbf{x}} = \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix} T \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & y \end{pmatrix} U \begin{pmatrix} 0 & 1 \\ -1 & y \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & z \end{pmatrix} V \begin{pmatrix} 0 & 1 \\ -1 & z \end{pmatrix}$$

where T, U and V are subwords of W of lengths $j-2$, $i-j-3$ and $12g-i-7$ respectively that depend only on the parameters $x_1, \ldots, x_{6g-6}$. By the Main Lemma (Lemma 2.17), T, U and V are strictly admissible.

Now suppose that $(x_1, \ldots, x_{6g-6})$ are chosen so that T, U and V are strictly admissible.

Claim. There exists a unique triple $(x, y, z) \in \mathbb{R}^3_+$ such that $W_{\mathbf{x}}$ satisfies the conditions of Lemma 2.17, that is, $W_{\mathbf{x}} = -I$ and all proper subwords of $W_{\mathbf{x}}$ are admissible.

Proof of claim. Let $T = \begin{pmatrix} t_1 & t_2 \\ t_3 & t_4 \end{pmatrix}$, $U = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix}$, $V = \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 1 \\ -1 & y \end{pmatrix}$, $Z = \begin{pmatrix} 0 & 1 \\ -1 & z \end{pmatrix}$. Note that the entries of each of T, U and V depend only on $x_1, \ldots, x_{6g-6}$ and that $t_1, u_1, v_1 \leq 0$, $t_2, u_2, v_2 > 0$, $t_3, u_3, v_3 < 0$ and $t_4, u_4, v_4 > 0$ since T, U and V are strictly admissible.

By Proposition 2.14 (iii), XTX, YUY and ZVZ are strictly admissible if and only if $x > \alpha$, $y > \beta$, and $z > \gamma$ where

$$\alpha = \frac{t_2 - t_3 + \sqrt{(t_2+t_3)^2+4}}{2t_4}, \qquad \beta = \frac{u_2 - u_3 + \sqrt{(u_2+u_3)^2+4}}{2u_4},$$

$$\gamma = \frac{v_2 - v_3 + \sqrt{(v_2+v_3)^2+4}}{2v_4}.$$

Since we require all subwords of $W_{\mathbf{x}}$ to be admissible, we are only interested in solutions of $W_{\mathbf{x}} = -I$ where $x > \alpha$, $y > \beta$ and $z > \gamma$. We will show that there exists a unique triple (x, y, z) satisfying $x > \alpha$, $y > \beta$

and $z > \gamma$ such that

$$(2,2) \text{ entries of } \begin{cases} XTXYUY = -(ZVZ)^{-1} \\ YUYZVZ = -(XTX)^{-1} \\ ZVZXTX = -(YUY)^{-1}. \end{cases}$$

We start with the first equation. Equating the $(2,2)$ entry of $XTXYUY$ with that of $-(ZVZ)^{-1}$, we get

(4.7)
$$(t_2 - t_4x)(u_3 + u_4y) + (t_4x^2 + (t_3 - t_2)x - t_1)(u_4y^2 + (u_3 - u_2)y - u_1) = v_4.$$

Let S_1 be the surface in $\mathbb{R}^3$ defined by (4.7) and the inequalities $x > \alpha$, $y > \beta$, $z > \gamma$. Since the equation is independent of z, the surface is parallel to the z-axis and it suffices to analyze the intersection of the surface with any horizontal plane $z = z_0 > \gamma$. Denote the left-hand side of (4.7) by $h(x, y)$ and note that it represents the $(2,2)$ entry of $XTXYUY$ and is quadratic in both x and y.

Fix $y = y_0 > \beta$. Then $h(x, y_0)$ is a quadratic polynomial in x with positive leading coefficient so that $\lim_{x\to\infty} h(x, y_0) = \infty$. At $x = \alpha$,

$$\begin{aligned} XTXYUY &= \begin{pmatrix} - & + \\ - & 0 \end{pmatrix} \begin{pmatrix} -u_4 & u_3 + u_4y_0 \\ u_2 - u_4y_0 & -u_1 + (u_3 - u_2)y_0 + u_4y_0^2 \end{pmatrix} \\ &= \begin{pmatrix} * & * \\ * & - \end{pmatrix} \qquad \text{since} \qquad u_3 + u_4y_0 > 0 \end{aligned}$$

so $h(\alpha, y_0) < 0$. It follows that for each $y_0 > \beta$, there exists a unique $x > \alpha$ such that $h(x, y_0) = v_4$ since $v_4 > 0$ (the other solution must be $< \alpha$).

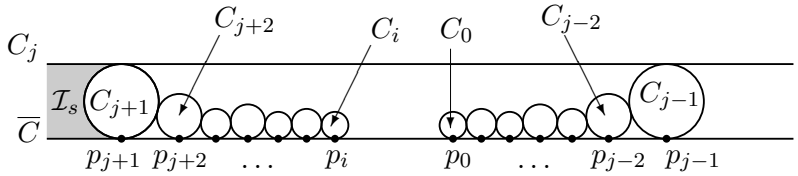
In fact, more is true. For this particular solution, $XTXYUY$ is strictly admissible. We first show that for $y_0 > \beta$ and x sufficiently large, $XTXYUY$ is strictly admissible. It is convenient to expand $XTXYUY$ as a word in the associated matrices $A_1, A_2 \dots$ where $A_1 = X = A_j$, $A_{j+1} = Y = A_{i-1}$, $T = A_{n_2} \dots A_{n_{j-1}}$, $U = A_{n_{j+2}} \dots A_{n_{i-2}}$. Then look at the circle configuration $\{\overline{C}, C_0, \dots, C_i\}$ corresponding to $XTXYUY$ (cf. definition 2.11) with a suitable normalization. We choose the normalization so that the central circle $\overline{C}$ is the real line, C_j is the line $\{\mathrm{Im}(z) = 1\}$ and C_{j+1} is the circle $|z - \sqrt{-1}/2| = 1/2$ so that the interstice enclosed by $\overline{C}$, C_j and C_{j+1} is the standard interstice (see Figure 11). Denote by p_n the tangency between $\overline{C}$ and C_n, note that $p_n \in \mathbb{R} \cup \{\infty\}$. We have:

(i) $p_j = \infty$, $p_{j+1} = 0$, by our normalization;

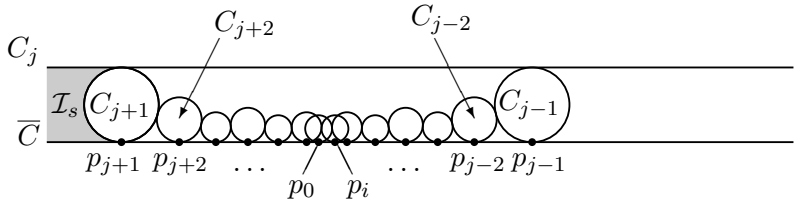
(ii) $p_{j-1} = x$, since $A_j = X$;

(iii) $0 = p_{j+1} < p_{j+2} < \cdots < p_i < \infty$, since $YUY = A_{j+1} \dots A_{i-1}$ is strictly admissible;

(iv) $0 < p_0 < p_1 < \cdots < p_{j-1} = x$, since $XTX = A_1 \dots A_j$ is strictly admissible.



(a) $XTXYUY$ is strictly admissible.



(b) $XTXYUY$ is not admissible.

Figure 11.

The condition that $XTXYUY$ is strictly admissible is the same as saying the interstices do not intersect along $\overline{C}$. This is easily seen to be equivalent to the condition that $p_i < p_0$. Now keeping all other cross ratios fixed, increasing x by some positive number c has the following effect on the points $\{p_k\}$:

(a) $p_j, p_{j+1}, \dots, p_i$ are left unchanged;

(b) $p_1, p_2, \dots, p_{j-1}$ are all shifted to the right by c;

(c) p_0 is shifted to the right by more than c since the radius of the circle C_0 is decreased.

It follows that for x sufficiently large and $y > \beta$, $XTXYUY$ is strictly admissible. Furthermore, the above shows that there is a smallest value x_1 of x when the word first becomes admissible but not strictly admissible, this occurs when $p_0 = p_i$. In this case, $h(x_1, y_0) = 0$. We have $x_1 > \alpha$ since XTX as a proper subword of an admissible word is strictly admissible.

Since $h(x, y_0)$ is continuous and approaches infinity when $x \longrightarrow \infty$, it follows that there exists $x_2 > x_1 > \alpha$ for which $h(x_2, y_0) = v_4$, this must be the value we have already found since it was the unique solution with $x > \alpha$. Since $x_2 > x_1$, $XTXYUY$ is strictly admissible.

We now use the strict admissibility of $XTXYUY$ to show that the curve defined by $h(x, y) = v_4$ with $x > \alpha$ and $y > \beta$ is the graph of a decreasing function with vertical asymptote $x = \alpha$ and horizontal asymptote $y = \beta$. The gradient $\nabla h = \langle h_x, h_y \rangle$ at a point on the curve $h(x, y) = v_4$ can be calculated using the product rule as in the proof of Proposition 4.4. We have h_x given by the $(2, 2)$ entry of the matrix

$$\begin{aligned}
&\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} TXYUY + XT \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} YUY \\
&= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} - & + \\ - & + \end{pmatrix} + \begin{pmatrix} - & + \\ - & + \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} - & + \\ - & + \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ - & + \end{pmatrix} + \begin{pmatrix} - & + \\ - & + \end{pmatrix} = \begin{pmatrix} - & + \\ - & + \end{pmatrix}
\end{aligned}$$

where the $(+)$ and $(-)$ entries follow from the fact that $XTXYUY$ is strictly admissible and hence so are all subwords. Similarly, h_y is given by the $(2, 2)$ entry of the matrix

$$XTX \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} UY + XTXYU \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} - & + \\ - & + \end{pmatrix}.$$

Hence, $\nabla h = \langle h_x, h_y \rangle = \langle +, + \rangle$ so $h(x, y) = v_4$ defines y as a decreasing function of x for $x > \alpha$, $y > \beta$. The fact that $y = \beta$ is a horizontal asymptote and $x = \alpha$ is a vertical asymptote follows from an easy analysis of $h(x, y)$ as $x \longrightarrow \alpha^+$ and $y \longrightarrow \beta^+$.

Clearly, the same analysis can be applied to the other two surfaces S_2 and S_3 given by $x > \alpha$, $y > \beta$, $z > \gamma$ and the equations

$$\text{(2,2) entries of } \begin{cases} YUYZVZ = -(XTX)^{-1} \\ ZVZXTX = -(YUY)^{-1} \end{cases}$$

respectively. S_2 is given by an equation of the form $h_2(y,z) = t_4$ which defines z as a decreasing function of y and S_3 is given by an equation of the form $h_3(z,x) = u_4$ which defines x as a decreasing function of z. The intersection of S_2 and S_3 is a curve parameterized by $z \in (\gamma, \infty)$ whose projection onto the xy-plane is a curve with positive slope which approaches the point (α, β) as $z \longrightarrow \infty$ and whose x and y coordinates approach ∞ as $z \longrightarrow \gamma^+$. Since $h(x,y) = v_4$ has negative slope with asymptotes at $x = \alpha$ and $y = \beta$, there is a unique point of intersection of S_1 with the curve of intersection of S_2 and S_3, hence of the 3 surfaces. To complete the proof of the claim, we apply Proposition 4.6 with $A = XTX$, $B = YUY$ and $C = ZVZ$. q.e.d.

The uniqueness part of the claim now implies that the projection map p is one-to-one. The existence part implies that in fact, $\mathcal{C}_\tau$ is homeomorphic to

$$\Big\{(x_1, \ldots, x_{6g-6}) \in \mathbb{R}_+^{6g-6} \mid T, U, V \text{ are strictly admissible}\Big\}.$$

Now the last part of the lemma follows from the convexity properties proven in 2.15. q.e.d.

5. Rigidity for one circle packings

5.1 Rigidity

In the previous section, we gave a complete description of the cross ratio parameter space $\mathcal{C}_\tau$ for a triangulation τ of Σ_g with one vertex. By definition, each element of $\mathcal{C}_\tau$ determines a projective structure on Σ_g together with a circle packing with nerve τ. The forgetting map $f : \mathcal{C}_\tau \longrightarrow \mathcal{P}_g$ is an embedding if we can show that the circle packings are rigid, that is, given any two circle packings P_1 and P_2 on a surface with projective structure such that their nerves are isotopic, then there exists a projective automorphism of the surface isotopic to the identity taking P_1 to P_2.

There have been many recent papers concerning rigidity results of circle packings, the most general perhaps being [9] which also contains a comprehensive list of references. However, the results do not seem to apply to our problem, indeed, many of the methods do not generalize and many basic results like the ring lemma do not apply in our case.

We prove the following result which implies rigidity of circle packings in the one-circle case. Main Theorem 2 then follows as a corollary of Lemma 4.7 and Lemma 5.1.

Lemma 5.1. *Suppose that τ is a triangulation of Σ_g with one vertex and $\mathbf{c}, \mathbf{c}' \in \mathcal{C}_\tau$. Then if $f(\mathbf{c}) = f(\mathbf{c}')$, in other words, $\mathbf{c}$ and $\mathbf{c}'$ define the same projective structure on Σ_g, then $\mathbf{c} = \mathbf{c}'$. Hence, the circle packings of Σ_g by one circle are rigid.*

Proof. We adopt the notation of Proposition 4.4, Lemma 4.5 and 4.7 so that the 3 values of $\mathbf{c}$ (respectively $\mathbf{c}'$) are denoted by x, y, z (respectively x', y', z') and are the cross ratios of three edges of the dual graph τ' meeting at a non-separating vertex v'. The corresponding words of associated matrices for $\mathbf{c}$ and $\mathbf{c}'$ are given by

$$W = XTXYUYZVZ, \qquad W' = X'T'X'Y'U'Y'Z'V'Z'$$

where $X = \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 1 \\ -1 & y \end{pmatrix}$, $Z = \begin{pmatrix} 0 & 1 \\ -1 & z \end{pmatrix}$, $T = \begin{pmatrix} t_1 & t_2 \\ t_3 & t_4 \end{pmatrix}$, $U = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix}$ and $V = \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix}$ are subwords of W of lengths $j-2$, $i-j-3$ and $12g-i-7$ respectively and depend only on $6g-6$ variables, and similarly for X', Y', Z', T', U' and V'. We note that these matrices are in $\mathrm{SL}_2(\mathbb{C})$ but when we consider the holonomy representation, we consider them also as elements in $\mathrm{PSL}_2(\mathbb{C})$, there should be no confusion as the choice should be clear from the context.

We first fix the developing map and hence the holonomy representation for both $\mathbf{c}$ and $\mathbf{c}'$ by developing the circle to the real line and the interstice containing v' to the standard interstice. We get a configuration of circles $\{\overline{C}, C_1, \ldots, C_{12g-6}\}$ where $\overline{C}$ is the real line, C_1 is the circle $|z - \sqrt{-1}/2| = 1/2$ and C_{12g-6} is the line $\mathrm{Im}(z) = 1$ for $\mathbf{c}$ and a similar configuration for $\mathbf{c}'$. Let $\gamma_1, \gamma_2, \gamma_3 \in \pi_1(\Sigma_g)$ where $\gamma_1 : s_1 \to s_j$, $\gamma_2 : s_{j+1} \to s_{i-1}$, $\gamma_3 : s_i \to s_{12g-6}$. Then $\gamma_3\gamma_2\gamma_1 = id$ and if ρ and ρ' are the holonomy representations corresponding to $\mathbf{c}$ and $\mathbf{c}'$ respectively, then by Proposition 2.20,

$$\begin{aligned} \rho(\gamma_1) &= XTXR^{-1}, & \rho(\gamma_2) &= XTXYUYR^{-1}(XTX)^{-1}, \\ \rho'(\gamma_1) &= X'T'X'R^{-1}, & \rho'(\gamma_2) &= X'T'X'Y'U'Y'R^{-1}(X'T'X')^{-1} \end{aligned}$$

where $R^{-1} = \begin{pmatrix} 1 & -\sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix}$. Now since $\mathbf{c}$ and $\mathbf{c}'$ give the same projective structures, the holonomy representations ρ and ρ' must be

conjugate, that is, there exists some $H \in \mathrm{PSL}_2(\mathbb{C})$ such that for any $\gamma \in \pi_1(\Sigma_\gamma)$, $\rho'(\gamma) = H^{-1}\rho(\gamma)H$. Hence

$$\begin{aligned} tr(\rho(\gamma_1)) &= \pm tr(\rho'(\gamma_1)) \\ \Rightarrow tr(XTXR^{-1}) &= \pm tr(X'T'X'R^{-1}) \\ \Rightarrow -t_4 - (t_2+t_3)\sqrt{-1} &= \pm(-t'_4 - (t'_2+t'_3)\sqrt{-1}). \end{aligned}$$

Equating real and imaginary parts, and using the fact that $t_4, t'_4 > 0$, we get $t_4 = t'_4$ and $t_2+t_3 = t'_2+t'_3$. Note that since $t_4 > 0$, $tr(\rho(\gamma_1)) \neq 0$ and so $\rho(\gamma_1)$ is not an elliptic element of order 2. Let $t'_2 = t_2 - k$, so $t'_3 = t_3 + k$. Then

$$\begin{aligned} X'T'X' &= \begin{pmatrix} 0 & 1 \\ -1 & x' \end{pmatrix}\begin{pmatrix} t'_1 & t'_2 \\ t'_3 & t'_4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & x' \end{pmatrix} \\ &= \begin{pmatrix} -t'_4 & t'_3 + t'_4x' \\ t'_2 - t'_4x' & t'_4x'^2 + (t'_3 - t'_2)x' - t'_1 \end{pmatrix} \\ &= \begin{pmatrix} -t_4 & t_3 + (t_4x' + k) \\ t_2 - (t_4x' + k) & t_4x'^2 + (t_3 - t_2 + 2k)x' - t'_1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & x'' \end{pmatrix}\begin{pmatrix} t_1 & t_2 \\ t_3 & t_4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & x'' \end{pmatrix} \\ &= X''TX'' \qquad \text{where} \quad x'' = x' + \frac{k}{t_4}, \quad X'' = \begin{pmatrix} 0 & 1 \\ -1 & x'' \end{pmatrix}. \end{aligned}$$

$X''TX''$ is strictly admissible since $X'T'X'$ is strictly admissible (admissibility of $X''T$ and TX'' follows from Proposition 2.14 by looking at the $(1,2)$ entry and the $(2,1)$ entries and admissibility of $X''TX''$ then follows from the $(2,2)$ entry). Similarly, we have

$$\begin{aligned} Y'U'Y' &= \begin{pmatrix} 0 & 1 \\ -1 & y'' \end{pmatrix}\begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & y'' \end{pmatrix} = Y''UY'' \\ \text{and} \qquad Z'V'Z' &= \begin{pmatrix} 0 & 1 \\ -1 & z'' \end{pmatrix}\begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & z'' \end{pmatrix} = Z''VZ''. \end{aligned}$$

By the uniqueness part of the claim in the proof of Lemma 4.7, we get that $x'' = x$, $y'' = y$ and $z'' = z$ so that $XTX = X'T'X'$, $YUY = Y'U'Y'$ and $ZVZ = Z'V'Z'$. Hence $\rho(\gamma_1) = \rho'(\gamma_1)$, $\rho(\gamma_2) = \rho'(\gamma_2)$ and $\rho(\gamma_3) = \rho'(\gamma_3)$, that is, the conjugating element H commutes with $\rho(\gamma_1), \rho(\gamma_2)$ and $\rho(\gamma_3)$.

We now claim that $\rho(\gamma_1)$ and $\rho(\gamma_2)$ do not commute. If true, then $H = I$ since it commutes with both $\rho(\gamma_1)$ and $\rho(\gamma_2)$, and they are not elliptics of order 2. Hence $\rho = \rho'$ and the two configurations for $\mathbf{c}$ and $\mathbf{c}'$ are identical and so $\mathbf{c} = \mathbf{c}'$.

It remains to prove that $\rho(\gamma_1)$ and $\rho(\gamma_2)$ do not commute (equivalently, $\rho(\gamma_1)$, $\rho(\gamma_2)$ and $\rho(\gamma_3)$ do not pairwise commute since $\gamma_3\gamma_2\gamma_1 = id$).

Suppose not. Consider the configuration of circles $\{\overline{C}, C_1, \ldots, C_{12g-6}\}$. Recall that the tangency points $\{p_m\}$ of $\overline{C}$ with C_m satisfy

$$0 = p_1 < p_j < p_{j+1} < p_{i-1} < p_i < p_{12g-6} = \infty$$

with $\rho(\gamma_1)(p_1) = p_j$, $\rho(\gamma_2)(p_{j+1}) = p_{i-1}$, $\rho(\gamma_3)(p_i) = p_{12g-6}$.

We have $\rho(\gamma_2)(\overline{C}) = C_{i-1}$ is tangent to $\overline{C}$, hence $\rho(\gamma_1)(\rho(\gamma_2)(\overline{C}))$ is tangent to $\rho(\gamma_1)(\overline{C}) = C_j$. By assumption $\rho(\gamma_1)\rho(\gamma_2) = \rho(\gamma_2)\rho(\gamma_1)$ so

$$\rho(\gamma_1)(\rho(\gamma_2)(\overline{C})) = \rho(\gamma_2)\rho(\gamma_1)(\overline{C}) = \rho(\gamma_3^{-1})(\overline{C}) = C_i$$

is tangent to C_j. Similarly, permuting the roles of γ_1, γ_2 and γ_3, the commuting condition implies that C_{j+1} is tangent to C_{12g-6} and C_{i-1} is tangent to C_1. Now C_{j+1} is tangent to C_{12g-6} and $\overline{C}$ implies that $C_{j+1} = \{z \in \mathbb{C} : |z - (a + \sqrt{-1}/2)| = 1/2\}$, the circle of radius $1/2$ with center at $a + \sqrt{-1}/2$ where $p_{j+1} = a > 0$ (see Figure 12). Let r be the radius of the circle in the upper half plane tangent to the real line at the origin and also tangent to C_{j+1}. Since C_j is tangent to C_{j+1} and its point of tangency with the real line lies between 0 and p_{j+1}, the radius of C_j is less than r. Now since C_{i-1} is tangent to C_1 and its point of tangency p_{i-1} with the real line lies between p_{j+1} and ∞, the radius of C_{i-1} is greater than r. Hence the radius of C_j is strictly less than the radius of C_{i-1} with the points of tangencies satisfying $p_j < p_{i-1}$. Now C_i is tangent to C_{i-1} and its tangency point with the real line lies between p_{i-1} and ∞ which implies that it does not intersect p_j, giving a contradiction. Hence $\rho(\gamma_1)$ and $\rho(\gamma_2)$ do not commute. q.e.d.

5.2 Concluding remark

Much of our arguments above rely on the combinatorics of the one circle packing and it remains unsolved whether Main Theorem 2 holds for a general triangulation τ of Σ_g. There are also other aspects to be explored, for example one can look at circle packings on surfaces with

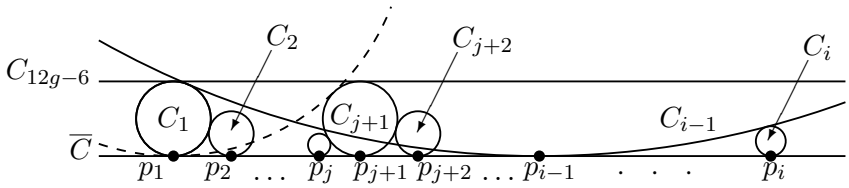


Figure 12.

cone projective structures, where the cone points would be the centers of some of the circles of the packing. Some results have been obtained and details will be given in a future paper. Of particular interest is the case of branched circle packings, this correspond to cone points with cone angles which are multiples of 2π. For example one can look at the deformation space of cone projective structures on the surface admitting a branched circle packing by one circle with branch point of multiplicity one, this would correspond to the condition that $W_v = I$ and that in the developing map, each circle is surrounded by $12g-6$ circles which wrap around the circle exactly twice. It seems that in this case, the deformation space $\mathcal{C}_\tau$ may admit some interesting topology, analogous to the case of the components of the space of representations $\mathrm{Hom}\,(\pi_1(\Sigma_g), \mathrm{PSL}_2(\mathbb{R}))/\mathrm{PSL}_2(\mathbb{R})$ where the Euler class is not $\pm(2-2g)$.

6. Appendix

We consider the case of the torus Σ_1 with a circle packing by one circle. Here the nerve τ consists of one vertex v and 3 edges e_1, e_2 and e_3 with cross ratios $x > 0$, $y > 0$ and $z > 0$ respectively and associated matrices $X = \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 1 \\ -1 & y \end{pmatrix}$ and $Z = \begin{pmatrix} 0 & 1 \\ -1 & z \end{pmatrix}$. The word associated to the vertex is given by $W = XYZXYZ$ and by Condition (1) of Lemma 2.17,

$$\begin{aligned} & XYZXYZ = -I \\ \Leftrightarrow\ & XYZ = -(XYZ)^{-1} \\ \Leftrightarrow\ & \begin{pmatrix} -y & yz-1 \\ 1-xy & xyz-x-z \end{pmatrix} = \begin{pmatrix} x+z-xyz & yz-1 \\ 1-xy & y \end{pmatrix} \\ \Leftrightarrow\ & xyz = x+y+z. \end{aligned} \tag{6.1}$$

Note that three equations derived from the matrix identity reduces to just one equation in this case. Condition (2) of Lemma 2.17 is automatically satisfied if x, y and z are all positive and satisfy (6.1) since $xy-1=(x+y)/z>0$, $yz-1=(y+z)/x>0$, $zx-1=(z+x)/y>0$, and $xyz-x-z=y>0$ hence XYZ is strictly admissible and so $XYZXYZ$ corresponds to a configuration where the surrounding circles wrap around the central circle exactly once. The cross ratio parameter space $\mathcal{C}_\tau$ is hence given by

$$\mathcal{C}_\tau = \{(x,y,z)\in\mathbb{R}^3 |\ xyz = x+y+z,\ x,y,z>0\}$$

and the image of the projection to the xy-plane is the set

$$p(\mathcal{C}_\tau) = \{(x,y)\in\mathbb{R}^2 |\ xy-1>0,\ x,y>0\}$$

which is convex.

To show that the circle packings are rigid, we directly compute the trace of the holonomy representation of a pair of generating elements of $\pi_1(\Sigma_1)$ (note that $\pi_1(\Sigma_1)$ is abelian and so the holonomy image of $\pi_1(\Sigma_1)$ in $\mathrm{PSL}_2(\mathbb{C})$ is necessarily an elementary group unlike the higher genus case). From Example 2.21, $\pi_1(\Sigma_1)$ is generated by γ_1 and γ_2 with

$$\begin{aligned}
\rho(\gamma_1) &= ZXR = \begin{pmatrix} x\sqrt{-1} & x-\sqrt{-1} \\ (xz-1)\sqrt{-1} & (xz-1)-z\sqrt{-1} \end{pmatrix} \\
\rho(\gamma_2) &= ZXYRZ^{-1} = Z\begin{pmatrix} y\sqrt{-1} & y-\sqrt{-1} \\ (yx-1)\sqrt{-1} & (yx-1)-x\sqrt{-1} \end{pmatrix}Z^{-1}
\end{aligned}$$

so $\mathrm{tr}(\rho(\gamma_1)) = \pm(xz-1+(x-z)\sqrt{-1})$, $\mathrm{tr}(\rho(\gamma_2)) = \pm(yx-1+(y-x)\sqrt{-1})$. If (x',y',z') is another point in the cross ratio parameter space which gives the same projective structure, then equating the traces, we get

$$\begin{aligned}
x'z'-1+(x'-z')\sqrt{-1} &= \pm(xz-1+(x-z)\sqrt{-1}), \\
y'x'-1+(y'-x')\sqrt{-1} &= \pm(yx-1+(y-x)\sqrt{-1}).
\end{aligned}$$

Equating real and imaginary parts and taking into account the inequalities satisfied by the variables, we get $x'=x$, $y'=y$ and $z'=z$, hence the circle packing is rigid. It follows that the space of projective structures on the torus admitting a circle packing by one circle is homeomorphic to $\mathbb{R}^2$. Finally, we note that a projective structure on the torus is necessarily an euclidean or similarity structure, in the case of the euclidean

structure, it is the one with fundamental domain a regular hexagon. In the other cases, it can be easily shown that the developing maps of the similarity structures obtained in this way are precisely the well-known Doyle spirals.

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