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SYMPLECTIC CONIFOLD TRANSITIONS

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Abstract

We introduce a symplectic surgery in six dimensions which collapses Lagrangian three-spheres and replaces them by symplectic two-spheres. Under mirror symmetry it corresponds to an operation on complex 3-folds studied by Clemens, Friedman and Tian. We describe several examples which show that there are either many more Calabi-Yau manifolds (e.g., rigid ones) than previously thought or there exist "symplectic Calabi-Yaus" — non-Kähler symplectic 6-folds with $c_1 = 0$. The analogous surgery in four dimensions, with a generalisation to ADE-trees of Lagrangians, implies that the canonical class of a minimal complex surface contains symplectic forms if and only if it has positive square.

1. Introduction

A 3-fold ordinary double point, or *node* as we shall call it, is a complex singularity analytically equivalent to

$$\{xy = zw\} \subset \mathbb{C}^4.$$

By taking the graph of the rational function $x/z = w/y$ from a neighbourhood of the singularity to $\mathbb{P}^1$ we get a *small resolution* of the node; a smooth resolution with exceptional set $\mathbb{P}^1$. This is because away from the origin at least one of x/z or w/y is uniquely defined on $\{xy = zw\}$, so the graph is isomorphic to the domain away from the origin, and replaces the origin with the whole $\mathbb{P}^1$. Similarly using the function $x/w = z/y$ gives another small resolution, the *flop* of the first. Alternatively, smoothing the node, $\{xy - zw = \epsilon\}$, yields a 3-sphere vanishing cycle (described below).

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So given a node on a Kähler 3-fold one may either try to smooth it (producing a Lagrangian S^3 vanishing cycle) or resolve it (producing a holomorphic $\mathbb{P}^1$). Passing from one desingularisation to the other is called a *conifold transition* in the physics literature. There is a *natural complex structure on the resolution*, but *not* a natural Kähler structure. Indeed, locally there is an obvious parameter for any Kähler form, given by the symplectic area of the resolving sphere, and the existence of such choices means that there are also obstructions to patching together the local choice of symplectic form to a global form. Hence the resolution may not admit a symplectic structure compatible with the complex structure.

Conversely the smoothing $\{xy - zw = \epsilon\}$ *is in a natural way symplectic*, but *not* naturally complex. At least in the Calabi-Yau setting, it is the volume of the vanishing cycle, computed via integrating the holomorphic 3-form, or equivalently ϵ, which defines a local parameter for the choice of complex structure, and there is an obstruction to patching these choices to give a global complex structure. However, on any complex smoothing there is a natural compatible symplectic structure, and even if complex smoothings do not exist [8], this "symplectic smoothing" does (Theorem 2.7 below). As explained in Section 2, the elementary but fundamental fact is that there is a symplectomorphism between $\{xy = zw\}\backslash\{0\}$ and $T^*S^3\backslash\{\text{Zero section}\}$, equipped with their standard symplectic structures.

In the Calabi-Yau case, there are necessary and sufficient conditions [8], [35] for the existence of a complex smoothing; in the Kähler case these can be interpreted as saying that the "symplectic smoothing" of Theorem 2.7 admits a compatible complex structure if and only if the conditions of Friedman-Tian are satisfied.

Under mirror symmetry, the mirrors of Calabi-Yau manifolds with nodes are usually also Calabi-Yaus with nodes, and the smoothing and resolution processes get swapped [27]. So there should be a criterion mirror to that of Friedman-Tian giving necessary and sufficient conditions for the resolution of a symplectic manifold with nodes to admit a symplectic structure. We give such a result here (Theorem 2.9), giving a new way to produce symplectic manifolds via the surgery of replacing Lagrangian S^3s with symplectic $\mathbb{P}^1$s (and preserving c_1 in the process).

We give a number of examples of conifold transitions that preserve a symplectic structure. In the complex setting, Lu and Tian [23] produce a complex structure (non-Kähler, with trivial canonical bundle) on $(S^3 \times S^3)^{\#(n\geq 2)}$, and mirror to this we produce "symplectic

Calabi-Yaus" (symplectic manifolds with $c_1 = 0$) with Betti numbers $b_3 = 2, 2 \leq b_2 \leq 25$. One can think of this as mirroring Reid's fantasy [29]. However, it is not clear if one can go further than this, i.e., if there can be symplectic structures with vanishing first Chern class on manifolds with $b_3 = 0$.

To set this in context, note that for simply-connected 4-manifolds X, symplectic geometry is very similar to Kähler geometry when either $c_1(X) > 0$ (i.e., when $c_1(X)$ can be represented by a symplectic form) or $c_1(X) = 0$. In the first case X can be shown to be Fano, and in the second it follows from results of Morgan and Szabo [26] that X is homeomorphic to the K3 surface. Beyond this, symplectic and Kähler geometry diverge, one reason being the existence of symplectic surgeries — fibre connect sums — which are non-Kähler. (The analogue of our surgery in four dimensions, and its cousins, are themselves interesting, as we point out in 3.1.) We expect such a divergence for $c_1 = 0$ in 6-dimensions, where conifold transitions provide a symplectic surgery preserving c_1. However, finding and studying Lagrangian S^3s and their configurations — the geometric input for such a surgery — is much harder than finding holomorphic $\mathbb{P}^1$s, and so it can be hard to find examples for which the surgery gives a non-Kähler result. In particular, controlling the intersections of the Lagrangians can be highly nontrivial — for instance, it may be that there are subtle, Floer-theoretic obstructions to obtaining disjoint families of Lagrangian spheres spanning $b_3/2$-dimensional subspaces of H_3, preventing us from using our surgery to produce symplectic 6-folds with vanishing c_1, b_3 and π_1 (which would be necessarily non-Kähler by (2.13)). For instance, Donaldson has asked [6] if all Lagrangian spheres in algebraic varieties arise as vanishing cycles for complex degenerations, and one can check that a positive answer to this question would exclude the existence of essential Lagrangian spheres in Calabi-Yau 3-folds with $b_3 = 2$.

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2. Smoothings and resolutions

To describe the symplectic versions of degeneration and resolution,

we will begin with some local facts about nodes. Fix once and for all a complex parametrisation $W = \{\sum z_i^2 = 0\}$ of a node. This complex variety has three resolutions of singularities relevant to the discussion. First, one can blow up the singular point to obtain a variety W_b, which can be described as follows. Blowing up the origin of $\mathbb{C}^4$ introduces an exceptional divisor $\mathbb{P}^3$; the closure of $W\backslash\{0\}$ inside the blow-up of $\mathbb{C}^4$ at 0 meets this $\mathbb{P}^3$ in the quadric surface $E = \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ given by the defining equation $\{\sum z_i^2 = 0\}$. This closure W_b is smooth and the normal bundle to $E \subset W_b$ is $\mathcal{O}(-1,-1)$. There are also the two *small* resolutions $W^{\pm}$ mentioned in the Introduction given by blowing down either of the two rulings $E \to \mathbb{P}^1$ of E. Thus each of $W^{\pm}$ has exceptional locus a $\mathbb{P}^1$ over $0 \in W$ (with normal bundle $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$), and blowing up the $\mathbb{P}^1 \subset W^{\pm}$ gives back W_b. The projection maps of the resolutions define canonical isomorphisms $(W_b\backslash E) \cong (W\backslash\{0\}) \cong (W^{\pm}\backslash\mathbb{P}^1)$.

It will be important for us to fix models of $W^{\pm}$ so that the two are distinct and not interchangeable. Changing the coordinates in the introduction by

$$x \mapsto z_1 + iz_2,\ y \mapsto z_1 - iz_2,\ z \mapsto -z_3 - iz_4,\ w \mapsto z_3 - iz_4$$

takes $\{xy = zw\}$ to $\{\sum z_i^2 = 0\}$, so we fix $W^{\pm}$ to be defined via the graphs of the following rational maps

$$W^+ :\ \frac{z_1 + iz_2}{z_3 + iz_4} = -\frac{z_3 - iz_4}{z_1 - iz_2}\ ; \qquad W^- :\ \frac{z_1 + iz_2}{z_3 - iz_4} = -\frac{z_3 + iz_4}{z_1 - iz_2}.$$

In particular changing the choice of coordinates on W by $z_4 \mapsto -z_4$ (which preserves $\sum z_i^2$) swaps $W^{\pm}$.

Now we relate these resolutions to the cotangent bundle of the three-sphere. For the standard oriented $S^3 \subset \mathbb{R}^4$ we can fix coordinates on T^*S^3 as follows:

$$T^*S^3 = \{(u,v) \in \mathbb{R}^4 \times \mathbb{R}^4 \mid |u| = 1, \langle u, v\rangle = 0\}.$$

The key local fact is that there is a symplectomorphism

$$\begin{aligned} (W_b\backslash E \ \cong\ W^{\pm}\backslash\mathbb{P}^1 \ \cong\ ) \\ (W\backslash\{0\}, \omega_{\mathbb{C}^4}) \ \xrightarrow{\ \phi\ }\ (T^*S^3\backslash\{v = 0\}, d(vdu)) \end{aligned} \tag{2.1}$$

which can be given explicitly in coordinates via the map

$$(z_j = x_j + iy_j)_{1\le j\le 4} \ \mapsto\ (x_j/|x|, -|x|y_j)_{1\le j\le 4}.$$

(Here $T^*S^3\backslash\{v=0\}$ is the cotangent bundle of S^3 minus its zero-section, and $|x| = (\sum x_i^2)^{1/2}$ is the norm of the real vector which is the real part of z. One computes directly that $\phi^*(\sum_j dv_j \wedge du_j) = (i/2)\sum_j dz_j \wedge d\bar{z}_j$. For more discussion of this, see [33].)

From this, we have the following important observation. The holomorphic, hence orientation-preserving automorphism of $\mathbb{C}^4$ given by $z_4 \mapsto -z_4$ (that preserves W and interchanges the two small resolutions $W^{\pm}$) acts on the real slice $\mathbb{R}^4 \subset \mathbb{C}^4$ by reflection in a hyperplane, and in particular induces an orientation-reversing diffeomorphism of $\{|u| = 1\} \cong S^3 \subset \mathbb{R}^4$. In other words, *flopping the* $\mathbb{P}^1 \Leftrightarrow$ *changing orientation on the* S^3. A more thorough description of this, involving the relevant homogeneous spaces, moment maps, and the topology of the surgery we are performing, is in the Appendix.

We now globalise the transition from a smoothing to a resolution and back again. To obtain smooth surgeries well-defined up to diffeomorphism it will be important to control the choices involved; this same information will later give us control on the symplectic structures via Moser's theorem.

Thus, we can make the above discussion relevant to more general symplectic manifolds by recalling [38] that if $L \subset X$ is a Lagrangian three-sphere in a symplectic six-manifold, then a neighbourhood of L in X is symplectomorphic to a neighbourhood of the zero-section in T^*L, equipped with its canonical symplectic structure. (An explicit symplectomorphism can be defined by a choice of compatible almost complex structure on X, and these form a contractible space [25].) Such a Lagrangian is not canonically oriented. However, if we pick an orientation, then there is a unique orientation-preserving diffeomorphism to $S^3 \subset \mathbb{R}^4$ up to homotopy, since $\mathrm{Diff}_+(S^3) \simeq \mathrm{SO}(4)$ by a famous result of Hatcher [15]. This induces a symplectomorphism between a neighbourhood of L and T^*S^3.

Similarly, given a symplectic two-sphere $C \subset X$ with normal bundle having first Chern class -2, Weinstein's neighbourhood theorem [38] implies that a neighbourhood of C in X is symplectomorphic to a neighbourhood of the zero-section in the bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$ equipped with a Kähler form giving the zero-section the same area as C. To define such a symplectomorphism, following ([25], p. 94-5), choose an almost complex structure taming ω and making C J-holomorphic, together with a sufficiently small positive number ε so that the exponential map of the metric defined by ω and J is injective on the

ε-disc bundle inside the normal bundle of C. Both C and $\mathbb{P}^1$ are canonically oriented; also fix an orientation-preserving diffeomorphism $C \to \mathbb{P}^1$ and a lift of this to a linear isomorphism of complex normal bundles $\nu_{C/X} \to \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Given all this data, [25] gives an explicit symplectomorphism from a small neighbourhood of C to a small neighbourhood of $\mathbb{P}^1$ inside the $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ bundle. Each of the sets of required choices is connected: the space of tamed almost complex structures is contractible, the choice of diffeomorphism belongs to $\mathrm{Diff}_+(S^2) \simeq \mathrm{SO}(3)$, and any two lifts to bundle maps differ by a change of framing of $\nu_{C/X}$; such framings are parametrised by $\pi_2(\mathrm{SO}(4)) = \{1\}$. Hence, up to homotopy, there is a unique symplectomorphism from a neighbourhood of C in X to a neighbourhood of the exceptional curve $\mathbb{P}^1$ in either small resolution $W^{\pm}$ of a node. Here we have fixed complex co-ordinates with respect to which our surgery is canonically defined. It follows that the operations defined below yield manifolds well-defined up to isotopy (and in particular diffeomorphism):

Definition 2.2. Let X be a symplectic six-manifold and $L \subset X$ a Lagrangian three-sphere. By a *conifold transition* of X in L, we mean the *smooth* manifold $Y = (X \backslash L) \cup_{\phi^{-1}} W^{\pm}$, where we fix a diffeomorphism $L \to S^3$ and hence parametrise a neighbourhood of L by a neighbourhood of the zero-section in T^*S^3; then ϕ^{-1} is the (restriction to this neighbourhood of the) diffeomorphism (2.1) $T^*S^3 \backslash \{v = 0\} \to W^{\pm} \backslash \mathbb{P}^1$.

Let Y be a symplectic manifold and $C \subset X$ a symplectic two-sphere whose normal bundle has Chern class -2. The *reverse conifold transition* of Y in C is the smooth manifold $X = (Y \backslash C) \cup_{\phi} (T^*S^3)$, where ϕ is the (restriction to suitable neighbourhoods of the) diffeomorphism of (2.1) composed with a diffeomorphism of $\nu_{C/Y}$ and $W \backslash \{0\}$ as above.

By the preceeding discussion, then, up to homotopy there are two $\mathbb{Z}/2\mathbb{Z}$ choices in the conifold transition — we orient L, and we choose a small resolution. Swapping both choices gives back the same smooth manifold since, as we have seen, changing orientation on S^3 interchanges the factors of $\mathbb{P}^1 \times \mathbb{P}^1$ and swaps the small resolutions, and vice-versa. Hence, up to diffeomorphism, there are exactly two distinct conifold transitions. The reverse conifold transition is uniquely defined as a smooth manifold, but the obvious embedded three-sphere is *not* canonically oriented, since changing the gluing map via the automorphism $z_4 \mapsto -z_4$ on $W \backslash \{0\}$ changes the orientation on the S^3.

To understand how symplectic structures interact with conifold transitions it will be convenient to use the intermediate space with a node,

and a model symplectic structure at the node. So we make the following definition, in n complex dimensions for now; later n will be 3. We say a continuous map $\phi : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ is *admissible* if it is smooth away from the origin, C^1 at the origin with $d\phi_0 \in \mathrm{Sp}\,(2n+2, \mathbb{R})$, and setwise fixes the quadric $W = \{\sum z_i^2 = 0\}$.

Definition 2.3. A *conifold* is a topological space X covered by an atlas of charts $\{(U_i, \phi_i)\}_{i \in I}$ of the following two types: either $\phi_i : U_i \to D^{2n}$ is a homeomorphism onto an open disc in $\mathbb{R}^{2n}$ or $\phi_j : U_j \to W \cap D^{2n+2}$ is a homeomorphism onto the intersection of an open disc in $\mathbb{C}^{n+1}$ with the quadric $W = \{\sum_{i=1}^{n+1} z_i^2 = 0\} \subset \mathbb{C}^{n+1}$. In the latter case, the point $P = \phi_j^{-1}(0)$ is called a *node* of X.

Moreover, the transition maps $\phi_{ij} = \phi_i \circ \phi_j^{-1}$ must be C^∞ away from nodes, and if $P \in U_i \cap U_j$ is a node then there must be an open subset $V \subset \mathbb{C}^{n+1}$ containing 0 such that $\phi_{ij}|_{V \cap W}$ coincides with the restriction of an admissible homeomorphism.

We will call such charts *smooth admissible coordinates*.

Definition 2.4. A *symplectic structure* on a conifold X is a smooth closed nondegenerate two-form ω_X on $X \backslash \{\text{Nodes}\}$ which, in any set of admissible coordinates around each node, co-incides with the restriction of a closed two-form on $\mathbb{C}^{n+1}$ which is smooth away from $0 \in \mathbb{C}^{n+1}$, and continuous and equal to the standard Kähler 2-form at the origin.

Two such closed forms ω_i define *equivalent* symplectic structures on X if there exists an admissible homeomorphism ϕ of X such that $\phi^* \omega_1 \equiv \omega_2$ on $X \backslash \{\text{Nodes}\}$.

We will call such an (X, ω_X) a symplectic conifold. (Observe that the class of two-forms we consider on $\mathbb{C}^{n+1}$ is preserved by admissible homeomorphisms, and using such homeomorphisms necessitates allowing forms which are only continuous at zero.) We have a "Darboux theorem", asserting that locally the symplectic structure is unique near any node. This would not be possible without some pointwise information at the node, for instance, consider the 2-dimensional case: the 1-fold node is two symplectic two-planes meeting transversely at a point; writing one as a graph $f : \mathbb{R}^2 \to \mathbb{R}^2$ over the symplectic orthogonal complement of the other at the node, the trace and determinant of df are local symplectic invariants.

Proposition 2.5. *Let X be a symplectic conifold and let $P \in X$ be a node. There is some neighbourhood U of P with admissible coordinates* (2.3) *in which ω_X is equivalent to the restriction of $\omega_{\mathbb{C}^{n+1}}$ to a neighbourhood of $0 \in W$.*

Proof. Fixing an admissible chart (2.3), we may assume we are working on a neighbourhood of the origin in the standard node W, with a non-standard symplectic structure ω defined on a ball near the origin in $\mathbb{C}^{n+1}$ and co-inciding with the standard structure $\omega_{\mathbb{C}^{n+1}}$ at the origin. Recall the usual proof of the Darboux theorem. On the ball around $0 \in \mathbb{C}^{n+1}$ we can choose a one-form σ such that $d\sigma = \omega - \omega_{\mathbb{C}^{n+1}}$, and without loss of generality we can suppose σ vanishes (to order two) at the origin. We then define a family of vector fields X_t, all vanishing continuously at the origin, via

$$\sigma + \iota_{X_t}\omega_t = 0 \tag{2.6}$$

where $\omega_t = \omega_{\mathbb{C}^{n+1}} + td\sigma$. The flow of this family of vector fields yields a family of diffeomorphisms $\{f_t\}$ of $\mathbb{C}^{n+1}\backslash\{0\}$, extending as C^1-maps over 0 which fix the origin, such that f_1 pulls back $\omega_{\mathbb{C}^{n+1}}$ to ω.

Note that, at least in a small enough ball around the origin, the linear family of symplectic forms ω_t will all have nondegenerate restriction to W (e.g., all the forms tame the integrable complex structure J in some small ball and W is J-holomorphic). For the same reason, in some ball the forms will all be symplectic on the fibres W_t (where $W_0 = W$) of the map $\pi : z \mapsto \sum z_i^2$. We may therefore define the "horizontal projections" $\{H_t\}$ of the vector fields $\{X_t\}$ as follows. For every $0 \leq t \leq 1$ and $0 \neq z \in \mathbb{C}^{n+1}$ there is a real rank two subbundle of $T_z\mathbb{C}^{n+1}$ which is the ω_t-symplectic orthogonal complement to the tangent bundle of the fibre of π through z. Let H_t denote the projection of X_t to this real rank two subbundle. This is certainly smooth as a vector field on $\mathbb{C}^{n+1}\backslash\{0\}$, and we claim that it has a continuous extension over the origin.

For $t = 0$, this follows by a direct computation. In this case, $\omega_0 = \omega_{\mathbb{C}^{n+1}}$ is the standard symplectic structure, and the symplectic orthogonal complement to $T_z(\pi^{-1}(\pi(z)))$ is generated by the complex conjugate vector, i.e., is $\mathbb{C}\langle\overline{z}\rangle$. It follows that the vector field H_t is given by $H_t(z) = \alpha(z)\overline{z}$ where the function α is defined by the identity

$$\omega_0(X_0(z) - \alpha(z)\overline{z}, \overline{z}) = \langle X_0(z) - \alpha(z)\overline{z}, i\overline{z}\rangle = 0$$

with $\langle\cdot,\cdot\rangle$ the usual metric on $\mathbb{C}^{n+1}$. Although the function α may not be continuous, as not every function vanishing to order two can be written as $|z|^2$ times a continuous function, the vector field $H_0(z) = \alpha(z)\overline{z}$ does vanish continuously at the origin since $\langle X_0(z), \overline{z}\rangle$ vanishes to order two. The vector fields H_t will also vanish continuously at the origin, since these can be determined explicitly from H_0 by functions of the global changes of co-ordinates given by the family of maps $\{f_t\}$.

Integrating up, let $\{F_t\}$ denote the flow of the vector fields $V_t = X_t - H_t$, well-defined in a small ball around the origin. Since the vector fields are tangent to the fibres of π and C^0 at the origin, the maps F_t will all be admissible. Given the definition of equivalence of symplectic forms on conifolds, it is enough to prove that $F_1^*\omega$ and $\omega_{\mathbb{C}^{n+1}}$ have the same restriction to the open quadric $W\backslash\{0\}$. Using the defining equation $\frac{d}{dt}F_t = V_t \circ F_t$ we find that

$$\frac{d}{dt}F_t^*\omega_t = F_t^*\left(\frac{d\omega_t}{dt} + d\iota_{V_t}\omega_t\right);$$

combining this with $d\omega_t/dt = d\sigma$ and 2.6 we have

$$\frac{d}{dt}\left((F_t^*\omega_t)|_{W\backslash\{0\}}\right) = F_t^*\left(\iota_{H_t}\omega_t)_{W\backslash\{0\}}\right) = 0$$

since $\omega_t(H_t, dF_t(u)) = 0$ for any $u \in \ker(d\pi)$. We therefore have that $F_t^*\omega_t$ is constant on restriction to the quadric, and in particular $(F_1^*\omega)|_{W\backslash\{0\}} = \omega_{\mathbb{C}^{n+1}}|_{W\backslash\{0\}}$ with F_1 admissible, as required. q.e.d.

It is very likely that, as for smooth subvarieties Z, one can in fact find a smooth change of co-ordinates taking ω to $\omega_{\mathbb{C}^{n+1}}$ even in the case where there is an isolated singular point. This would require a more substantial analysis. Despite the loss of regularity, the above implies that *any symplectic form near the node which is standard on the node, is symplectomorphic, in a punctured neighbourhood, to the restriction of the form $\omega_{\mathbb{C}^{n+1}}$ to the punctured quadric $W\backslash\{0\}$*. This is all we shall require in the sequel. For later, note also that the proof has an obvious extension to symplectic structures on manifolds with isolated singular points modelled on other singularities, for instance on ADE singularities.

Now, mirror to the fact that a small resolution of a node on a complex variety is again naturally complex, we can show that the smoothing is naturally a symplectic operation. The proof shows that, although we refer to a "symplectic smoothing" by analogy with complex geometry, the surgery is really a symplectic resolution — but with exceptional set a Lagrangian three-sphere. A better name might be a "Lagrangian blow-up".

Theorem 2.7. *Every symplectic conifold (X, ω_X) admits a symplectic smoothing which contains an embedded Lagrangian n-sphere for each node. In particular, the reverse conifold transition $\widetilde{X}$ of any small resolution of a six-dimensional conifold carries a distinguished symplectic structure, well-defined up to symplectomorphism.*

Proof. By Proposition 2.5 a punctured neighbourhood of each node is isomorphic to $(\{\sum_i z_i^2 = 0\}\backslash\{0\}, \omega_{\mathbb{C}^{n+1}})$, which by (2.1) is isomorphic to $T^*S^n\backslash S^n$ with its canonical symplectic structure. Hence we can replace the node $\{z_i = 0 \ \forall i\}$ by the Lagrangian n-sphere and smoothly extend the form keeping it globally symplectic.

For the uniqueness statement when $n = 3$, recall that the set of choices in performing the C^∞ surgery is connected. So the resulting smooth manifold is unique, with two different sets of choices giving two different symplectic forms on it. These forms are connected by a family of symplectic forms, coming from connectedness of the space of gluings, and the cohomology class along the family is constant (since the neighbourhood T^*S^3 has trivial H^2). Moser's theorem [28] then gives the required symplectomorphism. q.e.d.

Remark. In dimension $n = 3$, one could alternatively start with a smooth six-manifold X with a closed two-form η which is nondegenerate except along a two-sphere C, where it coincides with the appropriate local model (pull-back to a small resolution of the standard form on the node). This situation arises naturally as the limit of a path of symplectic forms on X as in the Remarks after Theorem 2.9. Note that finding symplectic two-spheres in a simply-connected symplectic six-manifold X is straightforward; they are governed by an h-principle [13]. However, finding families of symplectic forms which degenerate only along such spheres, yielding conifolds, is more subtle.

Secondly, and more deeply, we want to prove the mirror of the results of [8], [35], which we describe now.

Fix a complex 3-fold X with trivial canonical bundle, nodal singularities only, and such that small resolutions satisfy the $\partial\overline{\partial}$-Lemma (these are called "cohomologically Kähler" by Lu-Tian). This final condition, *which won't concern us in the mirror situation*, is needed to be able to use Hodge theory to relate deformations of complex structure $H^1(TY)$ to 3-cycles $H_3(Y) \cong H^3(Y)$ on a smooth 3-fold Y with trivial canonical bundle. (Since K_Y is trivial there is an isomorphism $TY \cong \Lambda^{2,0}T^*Y$ and so $H^1(TY) \cong H^{2,1}(Y) \subset H^3(Y)$.) Following ideas of Clemens [4], Friedman showed that a necessary condition for the existence of a complex smoothing of X is that there is a relation in homology between the exceptional curves $C_i \cong \mathbb{P}^1$ in a small resolution Y, of the form

$$\sum_i \lambda_i [C_i] = 0 \ \in \ H_2(Y;\mathbb{Z}) \quad \text{with} \ \ \lambda_i \neq 0 \ \text{ for all } \ i. \tag{2.8}$$

(This condition is independent of the choice of small resolution, as flopping a curve C_j simply reverses the sign of λ_j in (2.8).) He also showed that (2.8) is sufficient for a first order infinitesimal smoothing of X; Tian showed that this deformation is always unobstructed, i.e., can be extended to give a genuine smoothing.

Such a *"good"* relation is given by a 3-chain bounding the $\lambda_i C_i$ in Y, which becomes a 3-cycle in X passing through all of its nodes (and a "Poincaré dual" 3-cycle in the C^∞ smoothing of X, which we may think of as a vanishing cycle for the nodes). Intuitively, via the correspondence between 3-cycles and deformations of complex structure on Calabi-Yau manifolds, this gives a global deformation of complex structure that restricts at each node to (λ_i times) the unique standard local smoothing of that node. So for the λ_is nonzero the result is a smooth 3-fold.

The mirror situation for resolutions is perfectly analogous.

Theorem 2.9. *Fix a symplectic* 6*-manifold* X *with a collection of* n *disjoint embedded Lagrangian* 3*-spheres* $L_i \cong S^3$*. There is a* "good" *relation* (*cf.* (2.8))

$$\sum_i \lambda_i [L_i] = 0 \ \in \ H_3(X;\mathbb{Z}) \quad \textit{with} \quad \lambda_i \neq 0 \ \textit{ for all } i \tag{2.10}$$

iff there is a symplectic structure on one of the 2^n *choices of conifold transitions of* X *in the Lagrangians* L_i*, such that the resulting* $\mathbb{P}^1$*s* C_i *are symplectic.*

Proof. Via (2.1) we can replace each Lagrangian sphere by a node and then replace the node by a two-sphere via a small resolution. This gives a manifold (Y,ω) where ω is globally closed, and degenerate along a collection of embedded two-spheres $C_i \subset Y$ (i.e., ω is the form pulled back from the symplectic conifold). We show that the 4-chain giving the homology relation gives rise to a four-cycle σ on a resolution (its S^3 boundaries have been collapsed to S^2s) with $\sigma \,.\, C_i = |\lambda_i| > 0$. Firstly we give the local model.

In our (u,v)-coordinates (2.1) on T^*S^3 a collar neighbourhood of the S^3 zero-section $\{\sum u_i^2 = 1,\ v = 0\}$ is given by the equations defining half of a real line-bundle over S^3 (such a line bundle being necessarily trivial):

$$\Delta = \{(u,v) \mid v_1 = -\lambda u_2,\ v_2 = \lambda u_1,\ v_3 = -\lambda u_4,\ v_4 = \lambda u_3;\ \lambda \geq 0\}.$$

Using quaternionic multiplication (cf. the Appendix) we can write this

as $\{(u,v)\,|\,v=\lambda Iu,\ \lambda\geq 0\}$. One can check that under the diffeomorphism (2.1) Δ is exactly the image of the complex surface

$$S=\{z_1=iz_2, z_3=iz_4\}$$

inside $\mathbb{C}^4$ which lies inside the quadric and contains the node; λ appears as $\sum \mathrm{Re}\,(z_i)^2$. The other "half" of the line bundle, taking $\lambda\leq 0$ in the defining equations above, arises from the second complex surface $\{z_1=-iz_2, z_3=-iz_4\}$. The surface S is smooth, as is its proper transform in either small resolution. Depending on the resolution chosen (see for instance [23]), this proper transform is either isomorphic to S, intersecting the exceptional $\mathbb{P}^1$ transversally in $+1$, or is the blow up $\widehat{S}$ of S at the origin with its exceptional $\mathbb{P}^1$ coinciding with the exceptional set of the small resolution; in this case it is easy to see that $\widehat{S}\,.\,\mathbb{P}^1=-1$.

By definition our global 4-chain is an element of $H_4(X,\cup_i L_i)$ which maps to $\oplus_i\lambda_i[L_i]\in\oplus_i H_3(L_i,\mathbb{Z})$ in the homology exact sequence of the pair. By excision we can replace $\lambda_i[L_i]\in H_3(L_i,\mathbb{Z})$ by $\lambda_i\partial(\Delta_i\cap U_i)$ in $H_3(\Delta_i\cap U_i,\mathbb{Z})\cong H_3(L_i,\mathbb{Z})$, where U_i is a small tubular neighbourhood of L_i in X, and Δ_i is the collar of L_i defined above in local coordinates. Also by excision, our global relation gives a four-chain in $H_4(X,\cup_i(\Delta_i\cap U_i))\cong H_4(X,\cup_i L_i)$ with boundary $\lambda_i\partial(\Delta_i\cap U_i)$. Adding this to the collars $(\Delta_i\cap U_i)$ give a 4-chain (homologous to the original chain) which in local coordinates is exactly λ_i times our collar model around L_i. Thus, choosing the right small resolution (with local intersection of the complex surface with the exceptional $\mathbb{P}^1$ given by $\mathrm{sign}(\lambda_i)$) gives a 4-cycle σ with intersection $|\lambda_i|$ with C_i. Write $\widetilde{\sigma}$ for a two-form in the class Poincaré dual to σ.

Fix some tubular neighbourhood U_i of each curve C_i. Since $H^2(U_i;\mathbb{R})\cong\mathbb{R}$ we know that $\widetilde{\sigma}|_{U_i}$ is cohomologous to $\lambda_i\omega_i$, where ω_i is the standard Kähler form on the total space of $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$ over $C_i\cong\mathbb{P}^1$ for which C_i has area 1.

Write $\widetilde{\sigma}|_{U_i}=\lambda_i\omega_i+d\phi_i$ on U_i, and (via cut-off functions) pick a one-form ϕ on Y such that $\phi|_{U_i}=\phi_i\ \ \forall i$. Then replacing $\widetilde{\sigma}$ by $\widetilde{\sigma}+d\phi$ we may assume that $\widetilde{\sigma}$ restricts to $\lambda_i\omega_i$ in a neighbourhood of each C_i.

By the compactness of $X\setminus\bigcup U_i$, and the openness of the non-degeneracy condition, we may choose N sufficiently large that $\Omega=N\omega+\widetilde{\sigma}$ is a symplectic form on $Y\setminus\bigcup_i U_i$. We claim that $N\omega+\widetilde{\sigma}$ is in fact a global symplectic form. As $\omega|_{C_i}=0$ and $\widetilde{\sigma}|_{U_i}=\lambda_i\omega_i$, Ω is nondegenerate in some smaller neighbourhood $V_i\subset U_i$ of the C_i. The remaining place to check is in $U_i\setminus V_i$. Now in general, convex combinations of symplectic

forms are not symplectic, and certainly the forms we have here are not directly proportional (for instance, ω is induced from a form on T^*S^3 and is exact, whereas each ω_i is nontrivial in cohomology).

However, as Gromov first pointed out, if two symplectic forms ω, ω_i both tame some fixed almost complex structure, then convex combinations are necessarily symplectic. We are in just such a situation. Since non-degeneracy is local, using Proposition 2.5 we may as well work on a neighbourhood of the origin in the standard node W with its standard symplectic structure. This gives us a complex structure J — the standard one on $\mathbb{C}^4$ — tamed by ω and a *holomorphic* resolution $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$ with ω_i a standard Kähler form on it. Thus on $U_i \backslash V_i$ both forms tame the same complex structure and so we can take convex combinations of them.

This completes the "if" part of the Proof. For "only if", certainly if the C_i are symplectic then there is a two-form which is nontrivial on each. Any such nondegenerate form ω gives a 2-form on $X \backslash (\cup_i L_i)$, via the isomorphism (2.1). This fits into the exact sequence of the pair $(X, X\backslash(\cup_i L_i))$ (using the Thom isomorphism $H^3(X, X\backslash(\cup_i L_i)) \cong H^0(\cup_i L_i)$) as follows:

$$\begin{array}{ccccc} H^2(X\backslash(\cup_i L_i)) & \to & \bigoplus_i H^0(L_i) \cong \bigoplus_i \mathbb{R} & \to & H^3(X) \\ \omega & \mapsto & \bigoplus_i (\int_{C_i} \omega). & & \end{array}$$

Since the third map is (Poincaré dual to) the inclusion of the fundamental classes of the L_i into X, this gives the required good relation (2.10) $\sum (\int_{C_i} \omega)[L_i] = 0$. q.e.d.

Remarks. As we tend $N \to \infty$ above, $\omega + \widetilde{\sigma}/N$ is symplectic, and in the limit degenerates along each of the C_i (cf. the Remark after Theorem 2.7), giving us a two-form locally isomorphic to a pull-back from the node in $\mathbb{C}^4$ to a small resolution.

The proof shows that even if the conditions of the Theorem are not satisfied, we can always induce a distinguished homotopy class of almost complex structures on the surgered manifold. To do this, we choose some nondegenerate two-form ω_i near each $C_i \subset Y$ which extends the degenerate form ω, and then — via cut-off functions and the same tameness and convexity argument — extend to a non-closed nondegenerate global two-form. Such forms are in one-to-one correspondence with homotopy classes of almost complex structures.

Clearly the surgery we have described does not change the fundamental group of the manifold. Moreover, if we surger n Lagrangian

spheres L_i, then we increase the Euler characteristic by $2n$. It is then easy to deduce the following (which is well-known — see for instance [4]; we learnt it from [11]):

Theorem 2.11. *If the n L_i of Theorem 2.9 span an r dimensional subset of $H_3(X)$, then $b_3(Y) = b_3(X) - 2r$, and $b_2(Y) = b_2(X) + (n-r)$.*

Intuitively, for every 3-cycle we lose by degenerating the L_i, we lose another by Poincaré duality (H_3 is always even dimensional and has a symplectic basis). This "dual" 3-cycle L_i' intersects L_i and on the resolution becomes a 3-chain bounding the curve C_i; hence we lose $2r$ lots of H_3 and r lots of our new n 2-cycles C_i. (Dually the $n-r$ relations amongst the $[L_i]$ are given by 4-chains on X that become 4-cycles on Y as their boundaries have been collapsed; thus h_4 also increases by $(n-r)$.)

Since our surgery is an almost complex operation, we can also ask how the Chern classes of the almost complex structure are affected. c_1 is represented by the zero set of a transverse section of the canonical bundle $\Lambda^3_{\mathbb{C}} T^*X$. We can choose a standard nowhere-vanishing holomorphic section on $T^*S^3 \backslash \{\text{Zero section}\}$ in its holomorphic coordinates (2.1). This corresponds to a section on the resolution which extends across the $\mathbb{P}^1$ by Hartog's theorem; the extension is still non-vanishing since it is nonzero outside a codimension-two subvariety $\mathbb{P}^1$. Thus we can refine c_1 of both the smoothing X and resolution Y to lie in

$$H^2(X, \cup_i L_i) \cong H^2(Y, \cup_i C_i),$$

mapping to $H^2(X)$ and $H^2(Y)$ respectively. In this sense c_1 is preserved by the transition. Thus, in particular:

Lemma 2.12. (*Reverse*) *conifold transitions preserve the condition $c_1 = 0$.*

In six dimensions, a homotopy class of almost complex structures is completely determined by the first Chern class. (The second Chern class is then determined by the identity $c_2 = (c_1^2 - p_1)/2$ and the third Chern class is just the Euler characteristic.) All smooth six-manifolds with $\pi_1 = 0$ and $\mathrm{Tor}(H^*) = 0$ are almost complex, with the almost complex structures indexed by the integral lift c_1 of the Stiefel-Whitney class w_2 . The triple of integers $(c_1^3, c_1 c_2, c_3)$ is necessarily of the form $(2\alpha, 24\beta, 2\gamma)$ for suitable integers α, β, γ and all such triples are in fact realised by simply connected symplectic manifolds [14]. However, very little is known about the existence of symplectic manifolds with given

Chern classes, in particular with $c_1 = 0$.

In the next section we will give examples of the symplectic surgeries provided by Theorems 2.7 and 2.9. It has proved remarkably difficult, however, to prove definitely that the surgery does not preserve the subclass of Kähler manifolds. In the Calabi-Yau context, which, because of mirror symmetry, we would like to work, there is an obvious obstruction:

Lemma 2.13. *Let X be a simply connected symplectic six-manifold with $c_1(X) = 0$ and $b_3(X) = 0$. Then X is not homotopy equivalent to any Kähler manifold.*

The proof is that any simply connected Kähler manifold with $c_1 = 0$ has *holomorphically trivial* canonical bundle (since by Hodge theory, $H^{0,1} = 0$), hence has a nowhere zero holomorphic three-form Ω. This is automatically closed and nonzero in H^3, since $\int \Omega \wedge \overline{\Omega} > 0$. In order to obtain a manifold with $b_3 = 0$ from a conifold transition, one needs a collection of disjoint Lagrangians — satisfying a good relation — spanning $b_3/2$ dimensions in H_3 (this is the maximum possible, by Poincaré duality). Smoothly there is no obstruction to finding such spheres, but the situation in symplectic geometry is not clear, and in many examples (cf. the next section) the numerology and geometry seem to conspire precisely to make this impossible. Indeed, so many examples "just fail" that it is natural to wonder if there is some obstruction to finding such a collection of disjoint Lagrangian spheres; it is even natural to wonder if all symplectic Calabi-Yaus have $b_3 \geq 2$ just like Kähler Calabi-Yaus.

In this regard, it is worth pointing out that there are other obstructions to being Kähler which can *never* be violated by conifold transitions. For instance, Chern-Weil theory implies that for a Calabi-Yau n-fold the L^2-norm of the curvature tensor for the Ricci-flat metric is given by $c_2 \cdot \omega^{n-2}$, which must therefore be nonnegative.

Proposition 2.14. *Let X be a symplectic manifold obtained by conifold transitions on a Kähler Calabi-Yau 3-fold. Then $c_2(X) \cdot [\omega_X] > 0$.*

Proof. Via the surgery, the second Chern class changes by addition of the $\mathbb{P}^1$s we introduce in the resolution. (This follows from symmetry together with the local computation of Tian and Yau [36] who compute the effect on c_2 of a flop.) By construction our final symplectic form evaluates positively on these, so the surgery can only increase the value of $c_2 \cdot [\omega]$. q.e.d.

It is possible that every symplectic six-manifold with $c_1 = 0$ has $c_2 \cdot [\omega] \geq 0$ (even if they aren't all Kähler). For Calabi-Yaus with "large complex structure limit points", mirror symmetry gives a topological interpretation to this positivity: the Calabi-Yau should admit a fibration by Lagrangian tori, and the limiting locus of critical points of the fibration (as we tend towards the large complex structure limit point) should be codimension four and give a distinguished symplectic cycle representing c_2, cf. [34], [12].

Remarks. Li and Ruan [20] have studied the effect of reverse conifold transitions on quantum cohomology. The effect of the conifold transition (2.9) on QH^* is an interesting open question. Whereas the reverse conifold transition removes $\mathbb{P}^1$s and their Gromov-Witten contributions, so the conifold transition removes Lagrangian S^3s and should have a mirror effect on Joyce's invariant [17].

Salur [30] has considered deformations of (an appropriate modification of) special Lagrangian submanifolds in symplectic Calabi-Yaus. Her results accordingly apply to manifolds constructed from conifold transitions.

3. Assorted examples

In this section, we present various examples of the surgeries. To warm up we shall consider the situation in two complex dimensions, where the symplectic geometry of ordinary double points is already interesting.

(a) The question that shall motivate us is the following: in the minimal model programme, the (lack of) ampleness of the canonical class of a variety plays a fundamental role. Let (X, ω) be a symplectic manifold. When does the canonical class $K_X \in H^2(X; \mathbb{Z})$ itself contain symplectic forms?

For four-manifolds the canonical class is particularly decisive for the global geometry. A theorem of Liu [22] asserts that if X^4 contains a symplectic surface C with $K_X \cdot C < 0$ then in fact X is diffeomorphic to a del Pezzo surface, and indeed following work of Lalonde and McDuff [18], the symplectic form is isotopic to a standard Kähler form. In particular, X is Fano and $-K_X$ (hence also K_X) contains Kähler forms. In general, there are some obvious necessary conditions for K_X to contain symplectic forms: we must certainly have $(K_X)^2 > 0$. An observation going back to McDuff [24] shows that if $b_+(X) = 1$ then

this necessary condition is in fact sufficient. The proof, however, suggests no adaptation to the general case, and indeed there seems to be nothing special here about the canonical class: for minimal 4-manifolds with $b_+ = 1$, every class in H^2 of positive square contains symplectic forms [21]. This certainly fails in general (for intriguing examples see [37]); moreover, Taubes' work in Seiberg-Witten theory shows that if $b_+(X) > 1$ then for K_X to contain symplectic forms, X must contain no smooth -1-spheres.

Perhaps surprisingly, the constraints of positive square and minimality *are* sufficient in the integrable case. The following was observed independently by Catanese [3] (with a quite different proof).

Proposition 3.1. *If X is a minimal complex surface with $(K_X)^2 > 0$, then the canonical class K_X contains symplectic forms.*

Proof. The result is obvious for rational surfaces (where, however, the form may not be deformation equivalent to the usual Kähler form — think of $\mathbb{P}^2$). Using the classification of surfaces we can therefore assume X is a minimal complex surface of general type. A classical theorem [1] asserts that for $r \geq 5$ the morphism $|rK_X| : X \to \mathbb{P}^N$ defined by a multicanonical linear system is an embedding away from the union of all holomorphic -2-spheres, which are contracted to *rational* double point (or A-D-E) singularities. Suppose first all the -2-spheres are isolated, hence the singularities are nodes. Then there are local analytic co-ordinates z_i on projective space near any node such that the image of X is defined by $\{\sum_{i=1}^{3} z_i^2 = 0, z_j = 0 \; \forall j > 3\}$. By an argument of Seidel ([33], Lemma 1.7) there is an isotopy of Kähler forms on $\mathbb{P}^N$, starting with the Fubini-Study form and compactly supported near each node, which yields a form which in our analytic co-ordinates is exactly the standard form $\omega_{\mathbb{C}^N}$ in a small neighbourhood of the nodes. (Seidel's argument is local, so we adjust the form on a ball within the domain of definition of our co-ordinates; the ambient linear structure of $\mathbb{P}^N$ plays no role.) Since the isotopy is through Kähler forms they are all nondegenerate on the image of X, which is complex. Pulling back this form from projective space to X, we get a closed two-form η on X in the cohomology class K_X which is nondegenerate away from the isolated -2-spheres and conforms to the standard model near each one. That is, there is a neighbourhood U of each contracted sphere C such that

$\eta|_{U\setminus C} \cong \omega_{\mathbb{C}^3}|_{\{\sum z_i^2=0\}\setminus\{0\}}$. Using the fact that

$$\left(\left\{\sum_{j=1}^{3} z_j^2 = 0\right\} \Big\backslash \{0\}, \omega_{\mathbb{C}^3}\right) \cong \left(T^*S^2\backslash S^2, \omega_{can}\right). \tag{3.2}$$

we can extend the closed form η over U as a global symplectic form, making the two-sphere Lagrangian. Since the extended form vanishes on the two dimensional homology of T^*S^2 — indeed, the form is exact over U — it represents the same cohomology class as η, as required.

For the general case, consider the total space of a smoothing $\{f(z_1, z_2, z_3) = t\}_{t\in\mathbb{C}}$ of an ADE singularity $\{f(z_1, z_2, z_3) = 0\}$, equipped with the restriction of the standard Kähler form. By Seidel's result on isotopies of Kähler forms as above, the symplectic form on X induced from the Fubini-Study form is smoothly isomorphic to the standard form in an open neighbourhood of the contracted spheres, minus the spheres themselves. As in [33], there is a symplectic parallel transport on the fibres of the smoothing, which — restricting to a ray in the base starting at zero — shows that the complement of the ADE singularity in the zero-fibre is symplectomorphic to the complement of the vanishing cycles in a nearby fibre. This general fibre contains a tree of Lagrangian spheres; the contact boundary of such a domain is always ω-convex [7]. Gray's stability theorem for contact structures [9] shows that the boundary of the neighbourhood of the ADE chain in the central fibre is contactomorphic to this; the isomorphism of symplectic neighbourhoods coming from the parallel transport shows it is also ω-convex. A theorem of Etnyre [7] allows one to glue symplectic domains in this setting, so we can replace a neighbourhood of the ADE chain by a neighbourhood of the vanishing cycles in a nearby fibre. (Differentiably this is a trivial surgery - we remove a plumbed neighbourhood of a tree of spheres and then replace it by a diffeomorphism of the boundary which is isotopic to the identity.) We can therefore symplectically extend the two-form η pulled back from projective space with a global symplectic form η' for which all the spheres are Lagrangian, hence has unchanged cohomology class. q.e.d.

The canonical class contains Kähler forms only if the surface has no holomorphic -2-spheres, so this result is a purely symplectic phenomenon. It would be interesting to use Proposition 3.1 as an obstruction to integrability, for instance on a homotopy Kähler manifold with $\pm K_X$ the only Seiberg-Witten basic classes.

(b) We now return to the 3-fold case and give examples of Lagrangian S^3s with good relations (2.9), therefore admitting symplectic conifold transitions. As a simple local case, consider the Lagrangian $S^3 \subset \mathbb{P}^1 \times \mathbb{C}^2$ given by the product of the Hopf map and the unit norm inclusion. Here we have to take the symplectic structure on $\mathbb{P}^1$ given by *minus* that coming from symplectic reduction of the $S^3 \subset \mathbb{C}^2$. Alternatively, we can compose the Hopf map with the antipodal map, and use the symplectic structure compatible with the usual complex structure. Taking smaller S^3s in $\mathbb{C}^2$ (and so smaller symplectic forms on $\mathbb{P}^1$), and using the Darboux theorem to make any Kähler surface look locally symplectically like $\mathbb{C}^2$, we get:

Lemma 3.3. *Fix a Kähler surface S, and denote by ω_S the pullback of its Kähler form to $\mathbb{P}^1 \times S$. By taking ε sufficiently small, we can find arbitrarily many disjoint null-homologous Lagrangian three-spheres in $\mathbb{P}^1 \times S$ with symplectic structure $\omega_S + \varepsilon\omega_{\mathbb{P}^1}$.*

(We have only left to show that the spheres are null-homologous: in the original model the sphere bounded the 4-chain $\{([v], v) \in \mathbb{P}^1 \times \mathbb{C}^2 \ : \ 0 < |v| < 1\}$.)

So we can apply Theorem 2.9 to these examples to produce, for instance, three-folds with arbitrarily high b_2 and $b_3 = 0$ which are not obviously blowups of smooth 3-folds. In the $\mathbb{P}^1 \times \mathbb{P}^2$ case, it should be possible to show using standard projective 3-fold theory that the degeneration to a single node and resolution cannot be Kähler [5], but there are still some points to check.

(c) Though many of the above surgeries are surely not realisable within Kähler geometry, we have so far been unable to prove it in a particular case. If one could find a *homotopically* trivial Lagrangian S^3 bounding an embedded D^4, then non-Kähler manifolds would certainly result.

Lemma 3.4. *The symplectic conifold transition in a Lagrangian S^3 bounding an embedded D^4 would violate Hard Lefschetz.*

Proof. Recall that the Hard Lefschetz theorem for Kähler 3-folds [10] implies in particular that the intersection pairing

$$\cap [PD(\omega)] : H_4(X) \to H_2(X)$$

is an isomorphism. In particular, if there is some element $D \in H_4(X)$ for which $D \cap D' = 0 \in H_2$ for *every* $D' \in H_4$, then X cannot be

Kähler. In our case, the class D comes from the four-ball bounding the Lagrangian sphere.

The single homology relation $[L] = 0$ satisfies the conditions of the surgery Theorem 2.9; following the proof of that result, the symplectic structure on the resolution X is obtained by deforming with the Poincaré dual of a four-cycle which is the lift of the bounding topological four-ball which is transverse to the $\mathbb{P}^1$ at one point. Topologically, this lift is just an embedded four-sphere D inside the resolution. For any other four-cycle D', the intersection product $D \cap D' \in H_2(X)$ is geometrically represented by a two-cycle lying inside D, hence lies in the image of $H_2(D) \to H_2(X)$, but this is trivial since D is a sphere. Hence, the conifold transition of Z along L violates the Hard Lefschetz theorem and is not homotopy Kähler. q.e.d.

Unfortunately, we do not know of any examples of even homologically trivial Lagrangian three-spheres in Calabi-Yau 3-folds. (The known constructions of such spheres yield submanifolds whose Floer homology is probably not well-defined, whereas the Floer homology of any pair of Lagrangian three-spheres in a symplectic Calabi-Yau is well-defined by work of Fukaya, Oh, Ohta and Ono.) In this direction, however, we give a new construction of null-homologous Lagrangian 3-spheres in certain fibre products; these are therefore not simple $\mathbb{P}^1 \times S$ product examples as in Example (b).

(d) The idea of this construction is to represent S^3 as a family of tori T^2 over an interval, shrinking to circles at the end-points. Provided the circles that collapse at the two ends span the first homology of T^2, the closed three-cycle will be topologically a sphere, represented as a genus one Heegaard splitting. To obtain Lagrangian spheres this way is slightly more subtle, but one good case is where the T^2-fibres are themselves Lagrangian submanifolds in the fibres of a four-torus-fibred Kähler 3-fold. Such three-folds arise naturally from fibre products of elliptic surfaces, as in the work of Schoen [32].

Let $\pi_1 : S_1 \to \mathbb{P}^1$ and $\pi_2 : S_2 \to \mathbb{P}^1$ be elliptic fibrations with smooth generic fibres, only nodal singularities in fibres and with sections (and hence no multiple fibres). The fibre product $S_1 \times_{\mathbb{P}^1} S_2 \to \mathbb{P}^1$ is smooth except over points of $\mathbb{P}^1$ which are critical for both π_i, where at the points $(\text{Node}, \text{Node})$ there are ordinary double points, which therefore admit small resolutions. In particular, in the generic case, if we start with two rational elliptic surfaces each of which has 12 singular fibres, and no singular fibres are in common, the fibre product is a smooth

algebraic 3-fold, in fact Calabi-Yau [32]. Observe that if the original elliptic surfaces are equipped with Kähler forms, and if there are no common singular fibres, there is a natural Kähler form on the fibre product, restricted from $S_1 \times S_2$.

Fix points $a, b \in \mathbb{P}^1$ such that $a \in \mathrm{Crit}(\pi_1)$ and $b \in \mathrm{Crit}(\pi_2)$, and an arc γ in $\mathbb{P}^1$ from a to b (disjoint from all other critical values). Suppose for simplicity there is at most one node in each fibre. There is a symplectic parallel transport in each S_i, which gives rise to Lagrangian thimbles Δ_i $(i = 1, 2)$ over the arc γ such that in S_1 the thimble Δ_1 has boundary a smooth circle inside $(S_1)_b$ and passes through the node in the fibre $(S_1)_a$, whilst inside S_2 the thimble Δ_2 contains the node in $(S_2)_b$ and has boundary a circle inside $(S_2)_a$. Now consider the fibre-products of the thimbles (see Figure 1),

$$L = \Delta_1 \times_\gamma \Delta_2 = (\Delta_1 \times \Delta_2) \cap (S_1 \times_{\mathbb{P}^1} S_2) \ \subset \ S_1 \times S_2.$$

This is a family of two-tori over γ, one circle of which collapses at a and the other at b; moreover, the natural product Kähler form on $S_1 \times S_2$ restricts trivially to L and hence so does the Kähler form on the fibre product. A local computation shows that L is smooth; this is obvious away from the end-points of γ, and over the end-points the result follows from smoothness of the thimbles at their origins, which is well-known [33]. Hence we have constructed a Lagrangian three-sphere inside the 3-fold $S_1 \times_{\mathbb{P}^1} S_2$.

Taking suitable elliptic fibrations, one can again obtain many examples this way (and often the computation of intersection pairings can be reduced to counting intersections of arcs inside the base $\mathbb{P}^1$). There is an interesting special case, when the vanishing cycles for the original elliptic surface are homotopically trivial.

Lemma 3.5. *If the elliptic fibrations S_1 and S_2 have no common singular fibres and each have a homotopically trivial vanishing cycle, the (smooth Kähler) fibre product Z contains a homologically trivial Lagrangian three-sphere.*

Proof. In the above construction of the Lagrangian 3-sphere, γ_1 bounds a unique disc in each fibre; putting these together gives a 3-chain $D \subset S_1$, D^2-fibred over γ except at a where its fibrewise boundary collapses and the D^2 becomes an S^2 — the rational component of the singular fibre $(S_1)_a$. Then we have

$$\partial(D \times_\gamma \Delta_2) = \Delta_1 \times_\gamma \Delta_2 \ + \ S^2 \times (\Delta_2)_a \ = \ L \ + \ S^2 \times (\Delta_2)_a.$$

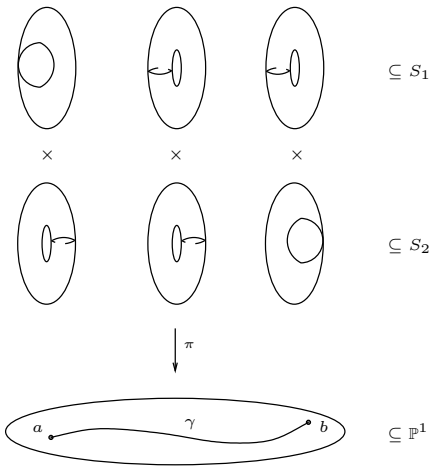

Figure 1: Fibred Lagrangian three-spheres.

Since the vanishing cycle $(\Delta_2)_a$ was also assumed to be null-homotopic, its product with the rational component S^2 of $(S_1)_a$ is null-homologous; thus L is also null-homologous. q.e.d.

It is straightforward to find suitable elliptic surfaces — if $\pi : S \to \mathbb{P}^1$ is not relatively minimal (so some fibres contain exceptional (-1)-spheres, i.e., are reducible with one component an S^2 meeting only one other at a single point), then the vanishing cycles associated to these nodes in the fibres are homotopically trivial. Taking a holomorphic automorphism ϕ of $\mathbb{P}^1$ that does not fix any of the critical points of E, the fibre product of π and $\phi \circ \pi$ contains a Lagrangian sphere as above. Note that if we blow down the (-1)-sphere in E, the image of the thimble is not Lagrangian with respect to any Kähler form downstairs.

Besides their intrinsic interest, such Lagrangians give interesting applications of the surgery, since they automatically satisfy a good relation (2.9).

(e) Although we cannot demonstrably produce non-Kähler symplectic Calabi-Yaus, we can at least produce a large collection of symplectic manifolds with $c_1 = 0$, so many that some *ought* to be non-Kähler; it is frustrating and intriguing that a year's work has not produced a proof. Here are some examples that start with the quintic hy-

persurface in $\mathbb{P}^5$. They show that it is possible to apply Theorem (2.9) in concrete cases, and to produce symplectic manifolds whose Betti numbers are thought not to be realised by Kähler Calabi-Yaus. Even if they turned out to be Kähler, they would give interesting new examples of rigid (i.e., no complex structure deformations, equivalent to $h^{2,1} = 0$ or $b_3 = 2$) Calabi-Yaus, many more than are thought to exist.

Proposition 3.6. *Symplectic Calabi-Yau manifolds exist with* $b_3 = 2$ *and any* $2 \leq b_2 \leq 25$.

Proof. [Sketch] Consider the hypersurfaces $Q_\lambda \subset \mathbb{P}^5$ defined by

$$\sum_{i=1}^{5} x_i^5 - \lambda \prod_{i=1}^{5} x_i = 0. \tag{3.7}$$

Each has an obvious $(\mathbb{Z}/5)^3$ projective symmetry group $\{(\alpha^{i_1}, \ldots, \alpha^{i_5}) : \sum_j i_j = 0 \bmod 5\}$. At $\lambda = 5$ this family includes Schoen's quintic [31] Q_5 with 125 nodes, the $(\mathbb{Z}/5)^3$-orbit of the node at $[1 : \cdots : 1]$. For $\lambda \in \mathbb{R}\backslash\{5\}$, Q_λ is smooth and has a Kähler structure inherited from $\mathbb{P}^4$. As symplectic manifolds these Q_λ are all isomorphic, and we call the general such manifold Q. To prove the Proposition, we exhibited explicit homology relations between subsets of the 125 Lagrangian vanishing cycles associated to the nodes in Q_5. To find these homology relations, we begin by constructing enough cycles to span $H_3(Q)$. These cycles will in fact be unions of real slices: we were heavily inspired by the calculations in Appendix A of the masterpiece [2].

Consider the open chain Δ^k in Q_λ give by taking $x_1, \ldots, x_4 \in (0, \infty)$, and x_5 a root (described, along with k, below) of Equation (3.7):

$$x_5^5 = -(x_1^5 + \cdots + x_4^5) + \lambda x_1 \ldots x_5. \tag{3.8}$$

For $\lambda = 0$ we choose the root $x_5 = e^{(2k-1)\pi i/5}(x_1^5 + \cdots + x_4^5)^{\frac{1}{5}}$, where the one-fifth power means the positive real one. For $\lambda \in (0, 5)$, it is shown in Appendix A of [2] that there is a continuous family of choices of x_5 compatible with this one; i.e., the roots of (3.8) do not come together as a double root for any given $x_1, \ldots, x_4$ for λ in this range. In fact, differentiating (3.8) shows that the equation for x_5 has a double root only when

$$5^5(x_1^5 + \cdots + x_4^5)^4 = 4^4(\lambda)^5(x_1 x_2 x_3 x_4)^5. \tag{3.9}$$

This has no real solutions for $\lambda < 5$ (as the geometric mean of the x_i^5 is no larger than the arithmetic mean). For $\lambda > 5$ [2] show that the

$k = 0, 1$ roots coincide, but we may choose one (for x_i real) by taking $\lambda \in \mathbb{R} + i\epsilon$ with ϵ small and positive (this rules out double roots by (3.9)), and then tend ϵ to zero. This defines Δ^k for all $\lambda \in [0, \infty)$.

Δ^k is naturally a 3-simplex with its four boundaries where one of the x_i ($i \leq 4$) becomes zero, six edges where two x_i are zero, and four vertices where only one x_i (and x_5) is nonzero. (In fact by scaling the x_i, $i = 1, \ldots, 4$ projectively so that they sum to 1, we get the standard 3-simplex $\{x_i > 0,\ \sum_i x_i = 1\}$ in $\mathbb{R}^4$.)

We now describe some similar cells that, along with $\Delta = \Delta^k$, will make up a simplicial complex in Q_λ ($\lambda \in (0, 5)$), again for a fixed $k \in \{0, 1, 2, 3, 4\}$. Picking $\alpha_i \in \{0, 1, \alpha := e^{2\pi i/5}\}$, denote by $\Delta_{\alpha_1\alpha_2\alpha_3\alpha_4} = \Delta^k_{\alpha_1\alpha_2\alpha_3\alpha_4}$ the set

$$x_i \in \alpha_i(0, \infty),\ i = 1, \ldots, 4, \tag{3.10}$$

with x_5 the kth root of (3.8) as above, with argument tending towards $e^{(2k-1)\pi i/5}$ as $\lambda \to 0$.

Thus Δ_{1111} is our 3-simplex $\Delta = \Delta^k$ above, and we get $2^4 = 16$ isomorphic copies by letting the α_i range through $\{1, \alpha\}$. Similarly, allowing one α_i to be zero we get the boundary simplices $x_i = 0$ of these 3-simplices; setting two α_is to zero gives the edges, three the vertices.

We set C^1 to be the simplicial complex $\bigcup \Delta_{\alpha_1\alpha_2\alpha_3 1}$, and $C^\alpha = \bigcup \Delta_{\alpha_1\alpha_2\alpha_3\alpha}$; these have common boundary $C^0 = \bigcup \Delta_{\alpha_1\alpha_2\alpha_3 0}$. (The four faces of any 3-simplex come from changing an α_i from an α or a 1 to 0, so 3-simplices meet along a common face if their three indices differ in only one place.) Finally we define $L^k = C^1 \cup_{C^0} C^\alpha$. We claim this is topologically a three-sphere. Let $\gamma \subset \mathbb{C}$ be the V-shaped union of the two halflines $[0, \infty) \cup \alpha[0, \infty)$. Then L^k is the image of $\gamma_1 \times \gamma_2 \times \gamma_3 \times \gamma_4 \cong \mathbb{R}^4$ in Q under the map $[\gamma_1 : \gamma_2 : \gamma_3 : \gamma_4 : x_5]$ where x_5 is the kth root of (3.8). But this map simply divides $\mathbb{R}^4$ by projective rescaling by $(0, \infty)$ acting radially (and we do not allow the origin as it doesn't define a projective coordinate), making the image $\mathbb{R}^4 \backslash \{0\}/(0, \infty) = S^3$.

At $\lambda = 0$, Δ^k is in fact *special Lagrangian* (as the fixed point set of the antiholomorphic involution $x_i \mapsto \bar{x}_i,\ x_5 \mapsto \alpha^{2k-1}\bar{x}_5$ on Q_0) so the L^k are piecewise smooth Lagrangians. Using the symmetry group $(\mathbb{Z}/5\mathbb{Z})^3$, we obtain 625 piecewise smooth Lagrangian three-spheres in Q. (Although we do not need the fact, we remark that any disjoint set can be *smoothed* as Lagrangians. To show this either glue in a near-Lagrangian local model along the edges and flow the result to make it

Lagrangian — the obstruction is just $[\omega]|_L \in H^2(L)$, and so zero — or tinker explicitly with the defining equations for the L^k.) We omit the proof of the following computational Lemma:

Lemma 3.11. *The* 625 *spheres described above span the entire* (204 *dimensional*) *third homology of the quintic.*

This can be proved by explicitly computing the intersection numbers (indeed the geometric intersections) of all the cycles with one another: this yields a vast matrix, whose rank we computed using MAPLE. Next, one can compute the geometric intersections between the cycles above and the vanishing cycles for the nodes in Schoen's quintic Q_5. Given this data, a computer search also yields good relations between disjoint sets of these vanishing cycles. In this way, we found disjoint sets of k spheres spanning 101 dimensions in homology, for each $102 \leq k \leq 125$, proving the Proposition. (However, computer searches suggest that no disjoint set of spheres — drawn from vanishing cycles or from the piecewise Lagrangian cycles constructed above — spans a 102-dimensional subspace in homology; this suggests we cannot achieve $b_3 = 0$ this way.) q.e.d.

Schoen's nodal quintic Q_5 [31] — which has the advantage of being rigid — is also the total space of an abelian surface fibration. (It is the relative Jacobian for a certain pencil of genus two curves on $\mathbb{P}^1 \times \mathbb{P}^1$; the base-points of the pencil give a distinguished model for the compactified Jacobian of the reducible singular fibres. The vanishing cycle for the node at $[1, 1, 1, 1, 1]$ in Schoen's example is, surprisingly, homologous to the sphere L^1 described above, as can also be checked by computing intersection numbers and using MAPLE.) However, the singular fibres seem too degenerate to find spheres that lift to the small resolution in this representation. Schoen has also classified which fibre products of rational elliptic surfaces yield rigid CY 3-folds [32]. In some cases, it is again possible to identify the nonzero 3-cycles on the rigid varieties; in general one can again laboriously check that none of these rigid fibre products contains a Lagrangian sphere of the fibred variety described above — the spheres one obtains on the nodal fibre products always lift to have boundary on the small resolutions.

(f) Finally, we mention that one could use Theorem 2.7 to symplectically smooth a Calabi-Yau (for instance) with node(s) which has no complex smoothing (i.e., the $\mathbb{P}^1$s in a small resolution satisfy no good relation [8]). Again, we have been unable to find an example where it

can be proved that the resulting manifold has no Kähler structure at all. It clearly has no Kähler degeneration back to the nodal variety; perhaps the symplectic degeneration over a disc can be completed over $\mathbb{P}^1$ to give a route to producing symplectic non-Kähler 8-manifolds.

4. Appendix: The local model

In this section we give the local model for smoothings and resolutions of nodes using homogeneous spaces and the like. We start with $\mathbb{R}^n$ and work in $\mathbb{R}^n \times (\mathbb{R}^n)^*$ with its natural symplectic structure. Define

$$M = \{(a,b) \in \mathbb{R}^n \times (\mathbb{R}^n)^* \,:\, \langle b, a\rangle = 0,\ a \neq 0 \neq b\}/\sim$$

where $\sim$ is the equivalence relation given by the orbits of the symplectic $(0,\infty)$-action $(a,b) \mapsto (\lambda a, \lambda^{-1} b)$. Thus M has a natural symplectic structure, also visible by mapping it as an (unramified) double cover to the coadjoint orbit

$$N = \{A \in \operatorname{End}\mathbb{R}^n \,:\, \operatorname{tr} A = 0,\ \operatorname{rank} A = 1\},$$

via $(a,b) \mapsto A = b \otimes a$. ($N$ is most obviously an adjoint orbit, but $\mathfrak{gl}(V) \subset V^* \otimes V$ is self-dual via the trace map.)

To see M as a complex variety we use the flat metric on $\mathbb{R}^n$ and its dual to find unique representatives in each $(0,\infty)$-orbit with $|a|^2 = |b|^2$, i.e., to write M as

$$\{(a,b) \in \mathbb{R}^n \times (\mathbb{R}^n)^* \setminus (0,0) \,:\, \langle b, a\rangle = 0,\ |a|^2 - |b|^2 = 0\}.$$

If we identify $\mathbb{R}^n \times (\mathbb{R}^n)^* \cong \mathbb{C}^n$ via $z_i = a_i + ib_i$, then $|a|^2 - |b|^2 = \sum(a_i^2 - b_i^2) = \operatorname{Re} \sum z_i^2$ and $\langle b, a\rangle = \sum a_i b_i = \operatorname{Im} \sum z_i^2/2i$, and we find M is the usual node minus the nodal point:

$$M = \left\{\sum z_i^2 = 0\right\} \subset \mathbb{C}^n \setminus \{0\}.$$

The symplectic and complex structures we have exhibited on M combine to give the Kähler structure inherited from the above embedding in $\mathbb{C}^n$; we will see this again via moment maps in the next section.

Writing $T^*S_a^{n-1} = \{(a,b) \in \mathbb{R}^n \times (\mathbb{R}^n)^* \,:\, |a| = 1,\ \langle b, a\rangle = 0\}$, and similarly for $T^*S_b^{n-1}$ the cotangent bundle of the dual sphere in $(\mathbb{R}^n)^*$, we have isomorphisms

$$T^*S_a^{n-1} \setminus S_a^{n-1} \xleftarrow{\sim} M \xrightarrow{\sim} T^*S_b^{n-1} \setminus S_a^{n-1}.$$

Here the first map is $(a,b) \mapsto (a/|a|\,,\, -|a|b)$ and the second $(a,b) \mapsto (b/|b|\,,\, -|b|a)$, and the maps are symplectomorphisms with respect to the canonical symplectic structures (cf. 2.1). Thus we see T^*S^{n-1} as a sort of symplectic resolution of the node (rather than as a smoothing); adding in the (Lagrangian) zero-section S^{n-1}_a or S^{n-1}_b at the node resolves it compatibly with the symplectic structure. These really are the smoothing; in fact in the obvious way they are

$$T^*S^{n-1}_a \cong \Big\{\sum z_i^2 = \epsilon\Big\} \qquad \text{and} \qquad T^*S^{n-1}_b \cong \Big\{\sum z_i^2 = -\epsilon\Big\}$$

where $\epsilon \in (0,\infty)$. Choosing different values of $\epsilon \in \mathbb{C}^*$ gives symplectic isotopies between these different smoothings, corresponding to different splittings of $\mathbb{C}^n$ into $\mathbb{R}^n \oplus (\mathbb{R}^n)^*$, twisting the standard one by $\sqrt{\epsilon}$. On looping ϵ once round 0 these isotopies have monodromy the symplectic Dehn twists of [33].

Alternatively we can form an *oriented real blow-up*[1] $\widehat{M}$ of $M \cup (0,0)$ (the double cover of the space of matrices $\{b \otimes a \,:\, \langle b,a\rangle = 0\}$ branched over the zero matrix), replacing the origin with its link

$$S(M) = \{(a,b) \,:\, |a| = 1 = |b|,\ \langle b,a\rangle = 0\}. \tag{4.1}$$

(I.e., $\widehat{M} := \{(a,b),(x,y) \in M \times S(M) \,:\, |a|x = a,\ |b|y = b\}$.) But $S(M)$ is the sphere bundle of $T^*S^{n-1}_a$ (and $T^*S^{n-1}_b$) so that $\widehat{M}$ is also the oriented blow up of T^*S^{n-1} in its zero-section S^{n-1}, with exceptional set an S^{n-2}-bundle over S^{n-1}.

So via the two induced different projections $S(M) \to S^{n-1}_a$ and $S(M) \to S^{n-1}_b$ we can blow down the blow-up $\widehat{M}$ via two different S^{n-2}-fibrations to get the two different symplectic resolutions/smoothings.

$$\begin{array}{ccccc} & & \widehat{M} & & \\ & \swarrow & & \searrow & \\ T^*S^{n-1}_a & & & & T^*S^{n-1}_b \\ & \searrow & & \swarrow & \\ & & \text{node.} & & \end{array} \tag{4.2}$$

(Compare this to the 2 small *complex* resolutions of the 3-fold node obtained by blowing down the blow-up via two different $\mathbb{P}^1$-fibrations, but notice in this real case the two resolutions are in fact isomorphic

[1]Here "oriented" means that we divide the normal directions only by positive real scalars.

as there is a whole family of blow downs of $\widehat{M}$ interpolating between the above two. Really this situation is more analogous to, and in fact (the double cover of) a real slice of, the Mukai flop $T^*\mathbb{P}^n \leftsquigarrow T^*(\mathbb{P}^n)^*$ [19].) It is important then that swapping S_a^{n-1} and S_b^{n-1} will *not induce the flop* on the (3-fold) resolution, as they are isotopic, even though the birational symplectomorphism $T^*S_a^3 \leftsquigarrow T^*S_b^3$ of (4.2) does not extend across the zero-sections. We shall see that the flop actually corresponds to changing orientation on S^3.

Alternatively we may blow down our master space $\widehat{M}$ in a different way to get the *complex* blow-up of the node. We now take $(a, b) \in \mathbb{R}^n \oplus \mathbb{R}^n$, then via the map $(a, b) \mapsto a \wedge b \in \Lambda^2\mathbb{R}^n$ we get a map $S(M) \to Gr^+$ to the Grassmannian of *oriented* 2-planes in $\mathbb{R}^n$. Gr^+ is naturally complex (thanks to Simon Donaldson for reminding us of this) by mapping such an oriented plane to the complex line $\mathbb{C}.(a+ib)$ in $\mathbb{C}^n$. Extending the metric bilinear form on $\mathbb{R}^n$ by linearity to a quadratic form on $\mathbb{C}^n$ we see as before (from $|a|^2 - |b|^2 = 0 = \langle a, b\rangle$) that the points of $\mathbb{P}^{n-1}$ (lines in $\mathbb{C}^n$) in the image of this map are the quadric $\sum z_i^2 = 0$, and we have exhibited the complex blow-up of the node — with exceptional divisor the quadric in $\mathbb{P}^{n-1}$ — as a blow down of $\widehat{M}$, dividing by the complex phase $U(1)$-fibres rotating a and b around each other in the exceptional locus $S(M)$.

The exceptional loci are all homogeneous spaces, and the maps we have defined can be described by the following diagram of maps between their exceptional sets; the varieties themselves are cones or bundles over these. The left-hand side is the symplectic smoothing/resolution side, birational to the complex resolution right-hand side:

(4.3)

$$\begin{array}{ccccc} & & \frac{\mathrm{SO}(n)}{\mathrm{SO}(n-2)} = S(M) & & \\ & \swarrow & & \searrow & \\ S^{n-1} = \frac{\mathrm{SO}(n)}{\mathrm{SO}(n-1)} & & & & \frac{\mathrm{SO}(n)}{\mathrm{SO}(2)\times\mathrm{SO}(n-2)} = Gr^+ \\ & \searrow & & \swarrow & \\ & & \text{node.} & & \end{array}$$

In the upper half of the diagram, the first arrow is an $\mathrm{SO}(n-1)/\mathrm{SO}(n-2) = S^{n-2}$-bundle, with choices (e.g., the different smoothings given by $T^*S_a^{n-1}$ and $T^*S_b^{n-1}$) given by the connected set of choices of embedding $\mathrm{SO}(n-2) \hookrightarrow \mathrm{SO}(n-1)$. The second arrow is an S^1-fibration, dividing out by complex phase.

The 3-fold case

In the special case $n = 4$ of interest to us, exceptional things happen on both sides. On the left-hand side $S(M) \cong S^3 \times S^2$ is a *trivial* S^{n-2} bundle, but in many different ways. The easiest way to represent this is via quaternionic geometry; let S^2 be the fixed set of complex structures J on $\mathbb{R}^4$ compatible with the flat metric and a fixed orientation. Then $S^3 \times S^2 \cong S(M)$ via $(a, J) \mapsto (a, b = Ja)$. The different splittings of $\mathbb{C}^4$ into $\mathbb{R}^4 \oplus \mathbb{R}^4$ give the different such trivialisations of $S(M)$, and so isotopies between them, e.g., between $S^3_a \times S^2$ and $S^3_b \times S^2$. Changing the orientation, however, changes things more dramatically; first changing the orientation on S^3, and secondly changing J. The induced map on $S^3 \times S^2$ is

$$(a, J \circ) \mapsto (a, \circ\, a^{-1}Ja), \tag{4.4}$$

where we are thinking of $a \in \mathbb{R}^4 \cong \mathbb{H}$ as a unit quaternion, J as a unit *imaginary* quaternion, and we note that post- (instead of pre-) multiplication by unit imaginary quaternions gives the complex structures of the opposite orientation. Thus the above is the right map, since in both cases b is $J \circ a = a \circ a^{-1}Ja$.

On the right-hand side this splitting of complex structures into two S^2s corresponds to the double cover $\mathrm{SO}(4) \to \mathrm{SO}(3) \times \mathrm{SO}(3)$, giving two extra projections

$$Gr^+ = \frac{\mathrm{SO}(4)}{\mathrm{SO}(2) \times \mathrm{SO}(2)} \longrightarrow \frac{\mathrm{SO}(3)}{\mathrm{SO}(2)} = S^2.$$

This fits the two *small* complex resolutions of the node into the diagram (4.3).

The easiest description of these maps, and the induced blow down from $\widehat{M}$, is to take an oriented plane (or a pair (a, b) of norm one with $\langle a, b\rangle = 0$) to the unique complex structure $J \in S^2$ on $\mathbb{R}^4$ compatible with the metric and orientation, and such that the plane is preserved by J (equivalently $Ja = b$). Then, as mentioned above, changing orientation changes $J\circ$ to $\circ\, a^{-1}Ja$ and swaps the S^2 factors in the corresponding isomorphism $Gr^+ \cong S^2 \times S^2$. We can also map $Gr^+ \to S^2 \times S^2$ via taking a plane to the unique J (resp. J') which preserves the plane and is compatible with the (resp. opposite) orientation, and thereby see that interchanging a and b interchanges J and J', since $Ja = b$ whilst $J'b = a$. (Alternatively we may use metric and orientation to write $\Lambda^2\mathbb{R}^4 = \Lambda^+ \oplus \Lambda^-$ and identify oriented planes with sums of unit

norm self-dual and anti-self-dual 2-forms in $S(\Lambda^+) \times S(\Lambda^-)$; changing orientation then swaps the two factors.)

So we see that *change of orientation on* $\mathbb{R}^4$ *corresponds to flopping the small resolution*. Indeed the choice of orientation or compatible oriented complex structure J on $\mathbb{R}^4$ can be related directly to the choice of resolution by using $(x_1 + iJx_1)$, etc. in the the choice of factorisation of $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0$ that produces the small resolution as a graph. Here we have chosen a real slice of $\mathbb{C}^4$ to talk about complex structures J on $\mathbb{R}^4$; different choices of real structure isotop J but do not change the S^2 it lies in, or the small resolution, just as they isotop the Lagrangian S^3 in the smoothing from, for instance, S^3_a to S^3_b.

On the smoothing side this change of orientation corresponds to changing the orientation of S^3, so this explains again how the signs of the λ_i in Theorem 2.9 can be changed (i.e., the orientations on the L_i can be changed) by flopping.

Finally then, we can describe the surgery that the conifold transition produces. We glue in an $S^3 \times D^3$ to $S^3 \times S^2$ to get the smoothing (and the many isotopic choices of product structure on $S^3 \times S^2$ that we have seen make the various smoothings isotopic), but glue in a $D^4 \times S^2$ to get either resolution. The gluings for the two resolutions differ via an involution of $S^3 \times S^2$, given by the composition of the map (4.4), and change of orientation on the S^3 (unit quaternions) and S^2 (unit imaginary quaternions). Notice (4.4) is nontrivial even to the S^2 as it restricts to the Hopf fibration on $S^3 \times \{*\}$; the diffeomorphism acts by the matrix $\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$ on $\pi_3(S^3 \times S^2) = \mathbb{Z}(\text{Degree}) \oplus \mathbb{Z}(\text{Hopf})$. In the symmetric co-ordinates (4.1) on $S(M)$, the map $(a, b) \mapsto (b, -a)$ corresponds to the monodromy around a half-circle in $\mathbb{C}^*$ (with square inducing the Dehn twist map), whilst the symmetry $(a, b) \mapsto (b, a)$ is the diffeomorphism coming from the flop.

Moment maps

Finally we show how our node and its smoothings, and the geometric structures on them, can be seen in a surprising way via moment maps. We can form S^{n-1} as the quotient of $\mathbb{R}^n$ by the dilation $\mathbb{R}$-action $a \mapsto e^{\lambda}a$ once we remove the fixed point $0 \in \mathbb{R}^n$. So we might try to form T^*S^{n-1} as a symplectic quotient of $T^*\mathbb{R}^n$ by the induced symplectic $\mathbb{R}$-action $(a, b) \mapsto (e^{\lambda}a, e^{-\lambda}b)$ on $\mathbb{R}^n \times (\mathbb{R}^n)^*$.

The derivative of this action gives a Hamiltonian vector field $(a, b) \mapsto$

$(a,-b)$ on $\mathbb{R}^n\times(\mathbb{R}^n)^*$ with Hamiltonian $h_1=\langle b,a\rangle$. Using the standard complex structure J on $\mathbb{R}^n\times(\mathbb{R}^n)^*\cong\mathbb{C}^n$ gives, unusually, another *Hamiltonian* vector field $(a,b)\mapsto J(a,-b)=(b,a)$ with Hamiltonian $h_2=\frac{1}{2}(|b|^2-|a|^2)$.

To form the symplectic quotient we fix a level set $h_1=\epsilon_1$ (which effectively divides by the second $\mathbb{R}$-action induced by h_2) and divide this by the first action of $\mathbb{R}$. Alternatively, to take this second quotient, we could instead just fix a level set of $h_2=\epsilon_2$. So, defining the complex moment map $h=2i(h_1+ih_2)=|a|^2-|b|^2+2i\langle b,a\rangle=\sum(a_j+ib_j)^2=\sum z_j^2$, and setting it equal to $\epsilon=2i(\epsilon_1+i\epsilon_2)$, we arrive at the quadric

$$\sum_j z_j^2=\epsilon,\qquad z_j=a_j+ib_j.$$

So the node ($\epsilon=0$) and its smoothings all arise in this way, with their canonical symplectic structures (as quotients) and different complex structures (depending on the level ϵ we picked) both restricted from the ambient Kähler structure on $\mathbb{C}^4$.

Even more strangely, these two $\mathbb{R}$-actions do not commute, so do not arise from a holomorphic $\mathbb{C}$-action (which is why there is *no canonical* complex structure on the quotient). This is more familiar in the context of hyperkähler quotients, where the complex group action does not complexify to give a "quaternionic" group action, but moment maps nonetheless exist. In fact our situation is the real slice of just such a situation [16], [19], and this is where our extra Hamiltonian, or moment map, comes from; the non-commutativity of the two $\mathbb{R}$-actions arises from the non-commutativity of the quaternions that have featured throughout this Appendix.

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