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THE YAMABE INVARIANT FOR NON-SIMPLY CONNECTED MANIFOLDS

BORIS BOTVINNIK & JONATHAN ROSENBERG

Abstract

The Yamabe invariant is an invariant of a closed smooth manifold defined using conformal geometry and the scalar curvature. Recently, Petean showed that the Yamabe invariant is nonnegative for all closed simply connected manifolds of dimension ≥ 5. We extend this to show that Yamabe invariant is nonnegative for all closed manifolds of dimension ≥ 5 with fundamental group of odd order having all Sylow subgroups abelian. The main new geometric input is a way of studying the Yamabe invariant on Toda brackets. A similar method of proof shows that all closed oriented manifolds of dimension ≥ 5 with non-spin universal cover, with finite fundamental group having all Sylow subgroups elementary abelian, admit metrics of positive scalar curvature, once one restricts to the "complement" of manifolds whose homology classes are "toral." The exceptional toral homology classes only exist in dimensions not exceeding the "rank" of the fundamental group, so this proves important cases of the Gromov-Lawson-Rosenberg Conjecture once the dimension is sufficiently large.

1. Introduction

The positive solution of the Yamabe problem [27] tells us that if M is a compact smooth manifold (without boundary), then every conformal class $\mathcal{C}$ of Riemannian metrics on M contains a metric (known as a Yamabe metric) of constant scalar curvature with the following special property. Its scalar curvature is the infimum of the scalar curvature s_g, taken over all metrics in $\mathcal{C}$ with constant scalar curvature and total volume 1. The value of this scalar curvature is called the Yamabe constant $Y(M, \mathcal{C})$ of $\mathcal{C}$. Equivalently, $Y(M, \mathcal{C})$ can be defined to be the minimum

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over metrics $g \in \mathcal{C}$ of the Einstein-Hilbert functional

$$I(g) = \frac{\int_M s_g \; d\,\mathrm{vol}_g}{\mathrm{vol}_g(M)^{\frac{n-2}{n}}}.$$

It is therefore natural to ask if there is a "best" Yamabe metric, and if so what its scalar curvature is. That motivates the following definition from [12]. The Yamabe invariant of M is defined by

$$Y(M) = \sup_{\mathcal{C}} Y(M, \mathcal{C}). \tag{1.1}$$

This supremum is not always attained, so the answer to the question about whether M has a "best" metric of constant scalar curvature might be "no." The best that is known is that there are singular metrics (with singularities at a finite number of points) which serve as the "best" approximation to an Einstein metric on M.

Nevertheless, $Y(M)$ is a diffeomorphism invariant of M. It also turns out that $Y(M) > 0$ if and only if M admits a metric of positive scalar curvature, a much-studied condition ([7], [8], [9], [22] [30], [31], [24], [25], [4]). However, $Y(M) = 0$ is possible even when M admits no scalar-flat metric.

In dimension 2, Gauss-Bonnet quickly shows that $Y(M) = 4\pi\chi(M)$. In dimension 4, $Y(M)$ can be positive, 0, or negative, and a lot is known about it from Seiberg-Witten theory ([16] and [19]). Similarly, there is a conjectural connection between $Y(M)$ and "geometrization" when $\dim M = 3$ (see for instance [2]). But even when $\dim M = 3$, and especially when $\dim M > 4$, it is not yet known if there are **any** manifolds with $Y(M) < 0$. (The obvious candidates for such manifolds are hyperbolic manifolds, but for all we know they could have vanishing Yamabe invariant.) In fact, Petean [20] has proved that $Y(M) \geq 0$ for any simply connected manifold of dimension at least 5.

In this paper we study the Yamabe invariant for manifolds with finite fundamental groups. Our first main result is the following.

Theorem 1.1. *Let M be a closed, connected, oriented manifold with finite fundamental group π, $\dim M \geq 5$. Suppose all Sylow subgroups of π are abelian. Assume either that M is spin and the order of π is odd, or else that the universal cover of M is non-spin. Then $Y(M) \geq 0$.*

The proof of this result is somewhat involved. First of all, we use surgery tools (developed in the study of positive scalar curvature) to

reduce the assertion of Theorem 1.1 to special situations. In particular, we show that it is enough to study the case when π is a finite abelian p-group. The central objects to understand here are the bordism groups $\Omega_n(B\pi)$ and $\Omega_n^{\mathrm{Spin}}(B\pi)$, and the proof amounts to the fact that all elements of these bordism groups may be represented by manifolds with nonnegative Yamabe invariant. A computation of these groups is quite hard, and the actual answer is known only for elementary abelian groups of odd order and few other cases (see [10]). Instead we use the Künneth formula to build manifolds with nonnegative Yamabe invariant to represent generators of these bordism groups. There are two types of "building blocks": tensor products (which are realized by direct products of manifolds) and torsion products (which geometrically are just Toda brackets).

We recall that Toda bracket $\langle M, P, L\rangle$ is defined when $M \times P = \partial V$ and $P \times L = \partial U$. Then the manifold

$$W = V \times L \cup_{M\times P\times L} M \times U$$

represents the Toda bracket $\langle M, P, L\rangle$.

As usual, to prove new geometric results we have to employ some *new geometric techniques*. Roughly, we show (under some restrictions) that if $Y(M)$ and $Y(L)$ are ≥ 0 (resp., > 0), then $Y(W) \geq 0$ (resp., > 0). We prove this by analytical means using elementary differential geometry.

Our second main result is the following.

Theorem 1.2. *Let M be a closed, connected, oriented manifold with finite fundamental group π. Suppose all Sylow subgroups of π are elementary abelian of rank $\leq r$. Assume that the universal cover of M is non-spin and that $\dim M > \max(4, r)$. Then M has a metric of positive scalar curvature.*

To put these results in context, it's worth recalling what is known about positive scalar curvature for manifolds with finite fundamental group. For such manifolds (of dimension ≥ 5) whose universal cover is non-spin, *there are no known obstructions to positive scalar curvature.* For spin manifolds of dimension ≥ 5 with finite fundamental group, the only known obstructions to positive scalar curvature come from the index theory of the Dirac operator ([24], [25]), and it is known that "stably" these are the only obstructions [24]. In fact in [22], it was conjectured (on the basis of extremely spotty evidence) that the index theory of the Dirac operator provides the only obstructions to positive

scalar curvature on manifolds of dimension ≥ 5 with finite fundamental group. This conjecture has sometimes been called the Gromov-Lawson-Rosenberg Conjecture. However, the "stable" theorem by itself does not actually answer the question of whether any *particular* manifold with vanishing Dirac obstructions admits a metric of positive scalar curvature. It *is* known [4] that for spin manifolds of dimension ≥ 5 with finite fundamental group with periodic cohomology, the Dirac obstructions are the only obstructions to positive scalar curvature. A similar theorem was proved by Schultz [29], and independently by Botvinnik and Gilkey [3], for spin manifolds of dimension ≥ 5 with fundamental group $\mathbb{Z}/p \times \mathbb{Z}/p$, p an odd prime. But very little was previously known about positive scalar curvature for manifolds with elementary abelian fundamental group of rank > 2. The proof of Theorem 1.2 is based on a reduction to the results of [3], again using Toda brackets.

The outline of the paper is as follows. Section 2 recalls the surgery and bordism theorems necessary for attacking the problems. Section 3 contains our basic geometric results on Toda brackets. Section 4 puts together the topological and geometrical tools to prove Theorem 1.1 and related results, and Section 5 proves Theorem 1.2 and related results.

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2. Basic topological reduction tools

To warm up, we recall the following result of Petean for simply connected manifolds:

Theorem 2.1 ([20])**.** *If M^n is a connected, simply connected closed manifold of dimension $n \geq 5$, then $Y(M) \geq 0$.*

The proof of this fact is based on the following *surgery theorem*:

Theorem 2.2 (Petean, Yun [21])**.** *If M is a closed manifold with connected components M_i, and if another closed connected manifold M' can be obtained from M by surgeries in codimension ≥ 3, and if $Y(M_i) \geq 0$ for each i, then $Y(M') \geq 0$.*

Proof. This is really three theorems in one. If $Y(M_i) > 0$ for all

i, then M admits a metric of positive scalar curvature, hence so does M', by the surgery theorem of Gromov-Lawson and Schoen-Yau ([8] and [28]—some of the details are carefully redone in Theorem 3.1 of [25]), and so $Y(M') > 0$. If M is disconnected and $Y(M_i) = 0$ for some components and $Y(M_j) > 0$ for other components, then we may first replace M by the connected sum of its components, which has $Y \geq 0$ by iterated application of Case (b) of [21], Theorem 1. (See also [12].) This reduces us to the case where M is connected. If M is connected and $Y(M) \leq 0$, then the Corollary to Theorem 1 of [21] says $Y(M') \geq Y(M)$, so if $Y(M)$ is exactly 0, $Y(M') \geq 0$. q.e.d.

In this paper we will discuss what can be learned about the Yamabe invariant for non-simply connected manifolds, using Theorem 2.2.

Many of the basic facts about manifolds of positive scalar curvature, which are proved using the surgery theorem of Gromov-Lawson and Schoen-Yau, have obvious counterparts for manifolds with nonnegative Yamabe invariant, obtained by substituting Theorem 2.2 in the proof. The proofs are almost identical to those in the positive scalar curvature case, so while we will give complete statements of the results, we will be brief when it comes to details of the proofs.

First we need to convert the Surgery Theorem, Theorem 2.2, to a Bordism Theorem. We repeat some definitions from [24] and [25]:

Definition 2.3. Let $B \to B\mathrm{O}$ be a fibration. A *B-structure* on a manifold is defined to be a lifting of the (classifying map of the) stable normal bundle to a map into B. Then one has bordism groups Ω_n^B of manifolds with B-structures, defined in the usual way. (For instance, if $B = B\mathrm{Spin}$, mapping as usual to $B\mathrm{O}$, then $\Omega_n^B = \Omega_n^{\mathrm{Spin}}$.) We note that given a connected closed manifold M, there is a choice of such a B for which M has a B-structure and the map $M \to B$ is a 2-equivalence.

Examples 2.4. The following special cases show that many of the classical bordism theories arise via this construction.

1. If M is a spin manifold, choose $B = B\pi \times B\mathrm{Spin}$, where $\pi = \pi_1(M)$, and let $B \to B\mathrm{O}$ be the projection onto the second factor composed with the map $B\mathrm{Spin} \to B\mathrm{O}$ induced by $\mathrm{Spin} \to O$. Map M to the first factor by means of the classifying map for the universal cover, and to the second factor by means of the spin structure. The map $M \to B$ is a 2-equivalence since it induces an isomorphism on π_1 and $\pi_2(B) = 0$. The associated bordism theory is $\Omega_*^{\mathrm{Spin}}(B\pi)$.

2. If M is oriented and the universal cover $\widetilde{M}$ of M is non-spin, choose $B = B\pi \times B\mathrm{SO}$, where $\pi = \pi_1(M)$, and let $B = B\mathrm{SO} \to B\mathrm{O}$ be the obvious map. Map M to the first factor by means of the classifying map for the universal cover, and to the second factor by means of the orientation. The map $M \to B$ is a 2-equivalence since it induces an isomorphism on π_1 and $\pi_2(B) \cong \pi_2(B\mathrm{SO}) \cong \pi_1(\mathrm{SO}) \cong \mathbb{Z}/2$, with the map $\pi_2(M) \to \pi_2(B)$ corresponding to $w_2(\widetilde{M})$. The associated bordism theory is $\Omega_*(B\pi)$.

3. If M is not orientable and the universal cover of M is non-spin, let $\pi = \pi_1(M)$, and let B be defined by the homotopy pull-back diagram

$$\begin{array}{ccc} B & \longrightarrow & B\pi \\ \downarrow & & \downarrow{\scriptstyle w_1} \\ B\mathrm{O} & \xrightarrow{w_1} & \mathbb{RP}^\infty, \end{array}$$

where the maps labeled w_1 are defined by the first Stiefel-Whitney class. Note that $B\mathrm{O}$ has fundamental group $\mathbb{Z}/2$ and that $w_1\colon B\mathrm{O} \to \mathbb{RP}^\infty$ induces an isomorphism on π_1, so that B has fundamental group π. The map $B \to B\mathrm{O}$ can be taken to be a fibration with fiber $B\pi'$, where $\pi' = \ker w_1$ is the fundamental group of the oriented double cover of M. Then the maps of M to $B\pi$ by means of the classifying map for the universal cover and to $B\mathrm{O}$ by means of the classifying map for the stable normal bundle define a map from M to B which is a 2-equivalence for the same reason as in the last example. We will denote the associated bordism theory by $\Omega_*(B\pi; w_1)$; it is a "twisted version" of oriented bordism. In the special case where π splits as $\pi' \times \mathbb{Z}/2$, with $\pi' = \ker(w_1\colon \pi \to \mathbb{Z}/2)$, then B becomes simply $B\pi' \times B\mathrm{O}$, and the associated bordism theory is $\mathfrak{N}_*(B\pi')$. In general, $\Omega_*(B\pi; w_1)$ is more complicated to describe, though the following theorem often tells as much as one needs to know about it.

Theorem 2.5 (Kurazono [13]). *In Example* 2.4(3), *if* $w\colon \pi \to \mathbb{Z}/2$ *is surjective, there is an Atiyah-Hirzebruch spectral sequence*

$$H_s(B\pi, \Omega_t \otimes \mathbb{Z}_w) \Rightarrow \Omega_{s+t}(B\pi; w),$$

where $\mathbb{Z}_w$ *denotes the local coefficient system on* $B\pi$ *locally isomorphic to* $\mathbb{Z}$, *with the twist given by the map* $w\colon \pi \to \mathbb{Z}/2 \cong \mathrm{Aut}(\mathbb{Z})$.

The simply connected cases of the positive scalar curvature analogue of the following theorem were proved in [8]; the general case of the positive scalar curvature analogue, with this formulation, is in [24] and [25].

Theorem 2.6 (Bordism Theorem)**.** *Let M^n be a connected B-manifold with $n = \dim M \geq 5$, and assume that the map $M \to B$ is a 2-equivalence. Then $Y(M) \geq 0$ if and only if the B-bordism class of M lies in the subgroup of Ω_n^B generated by B-manifolds with nonnegative Yamabe invariant.*

Sketch of proof. Let N be a B-manifold B-bordant to M. The hypotheses combine (via the method of proof of the s-Cobordism Theorem) to show that M can be obtained from N by surgeries in codimension ≥ 3. Then if each component of N has nonnegative Yamabe invariant, one can apply Theorem 2.2 to conclude that the same is true for M. This does it since addition in Ω_n^B comes from disjoint union and additive inverses correspond to reversal of orientation, etc., which doesn't affect the Yamabe invariant of the underlying manifold. q.e.d.

Fortunately for applications, one can do better than this. For simplicity, we restrict attention to the three cases discussed in Examples 2.4.

Theorem 2.7 (Jung, Stolz)**.** *Let M^n be a compact connected manifold with $n = \dim M \geq 5$.*

1. *If, as in Example* 2.4(1)*, M is spin with fundamental group π, then $Y(M) \geq 0$ if and only if the class of $M \to B\pi$ in $ko_n(B\pi)$ lies in the subgroup $ko_n^{\geq 0}(B\pi)$ generated by classes of $M' \to B\pi$ with M' a spin manifold with nonnegative Yamabe invariant, and $M' \to B\pi$ a map (not necessarily an isomorphism on π_1). Here ko_* is the homology theory corresponding to the connective real K-theory spectrum.*

2. *If, as in Example* 2.4(2)*, M is oriented with fundamental group π, and the universal cover of M is* ***not*** *spin, then $Y(M) \geq 0$ if and only if the class of $M \to B\pi$ in $H_n(B\pi, \mathbb{Z})$ lies in the subgroup $H_n^{\geq 0}(B\pi, \mathbb{Z})$ generated by classes of $M' \to B\pi$ with M' an oriented manifold with nonnegative Yamabe invariant, and $M' \to B\pi$ a map (not necessarily an isomorphism on π_1).*

3. *If, as in Example* 2.4(3)*, M is non-orientable with fundamental group π and first Stiefel-Whitney class $w\colon \pi \to \mathbb{Z}/2$, and if*

the universal cover of M is ***not*** *spin, then $Y(M) \geq 0$ if and only if the class of $M \to B\pi$ in $H_n(B\pi, \mathbb{Z}_w)$ lies in the subgroup $H_n^{\geq 0}(B\pi, \mathbb{Z}_w)$ generated by classes of $M' \to B\pi$, with M' a manifold with nonnegative Yamabe invariant, $M' \to B\pi$ a map, and $w_1(M')$ factoring through w.*

Sketch of proof. It was proved by Jung and Stolz (see [24], [25], and [26]) that the kernel of the map $\Omega_n^{\mathrm{Spin}}(B\pi) \to ko_n(B\pi)$ in Case 1, the kernel of the map $\Omega_n(B\pi) \to H_n(B\pi, \mathbb{Z})$ in Case 2, and the kernel of the map $\Omega_n(B\pi; w) \to H_n(B\pi, \mathbb{Z}_w)$ (the edge homomorphism of the spectral sequence of Theorem 2.5) in Case 3 are represented by manifolds with positive scalar curvature. Thus the result immediately follows from Theorem 2.6. q.e.d.

This is now enough machinery to deal with "easy" torsion-free fundamental groups:

Theorem 2.8. *Let M^n be a closed connected n-manifold with a fundamental group π which is either free abelian or of homological dimension ≤ 4. (This includes the fundamental groups of aspherical 2-manifolds, 3-manifolds, and 4-manifolds.) Assume either that M is spin or that its universal cover is non-spin. In the spin case, also assume that the Atiyah-Hirzebruch spectral sequence $H_p(B\pi, ko_q) \Rightarrow ko_*(B\pi)$ collapses. (This is automatic if π is of homological dimension ≤ 3.) Then if $n \geq 5$, M has nonnegative Yamabe invariant.*

Proof. By Theorem 2.7, it's enough to show that for each of these groups π, $H_n^{\geq 0}(B\pi, \mathbb{Z}_w)$ exhausts $H_n(B\pi, \mathbb{Z}_w)$ for each $w\colon \pi \to \mathbb{Z}/2$, trivial or not, and $ko_n^{\geq 0}(B\pi)$ exhausts $ko_n(B\pi)$ for $n \geq 5$. The oriented non-spin case is easy, since for π free abelian and any n, $H_n(B\pi, \mathbb{Z})$ is generated additively by the classes of tori, which carry flat metrics and thus have Yamabe invariant zero, whereas if π has homological dimension ≤ 4, $H_n(B\pi, \mathbb{Z})$ vanishes for $n \geq 5$. In the non-oriented case, if π has homological dimension ≤ 4, it is still true that $H_n(B\pi, \mathbb{Z}_w)$ vanishes for $n \geq 5$, for any twist w. If π is free abelian and w is nonzero, then we can always split π as $\pi' \times \mathbb{Z}$, where w is trivial on π' and nontrivial on the $\mathbb{Z}$ factor. Since $H_*(B\mathbb{Z}, \mathbb{Z}_w)$ is nonzero only in degree 0 when $w \neq 0$, $H_*(B\pi, \mathbb{Z}_w)$ splits by the Künneth Theorem as $H_*(B\pi', \mathbb{Z}) \otimes H_0(B\mathbb{Z}, \mathbb{Z}_w)$. Since $H_*(B\pi', \mathbb{Z})$ is generated by tori, $H_*(B\pi, \mathbb{Z}_w)$ is generated by products with a torus factor, and the non-oriented result follows by Proposition 3.2. So consider the spin case. When π is free abelian, the Atiyah-Hirzebruch spectral sequence col-

lapses and

$$ko_n(B\pi) \cong \bigoplus_{p+q=n} H_p(B\pi, \mathbb{Z}) \otimes ko_q.$$

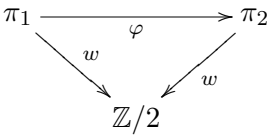
Thus this group is generated by the classes of $f\colon T^p \times N^q \to B\pi$, where the map f factors through T^p. Since, as pointed out in [20], ko_* is generated by the classes of manifolds of nonnegative Yamabe invariant, we have the desired result. The other cases are similar but easier. q.e.d.

Most of this paper will now deal with the opposite extreme, the case where $\pi_1(M)$ is finite. In this case, the following results reduce us to the case where the fundamental group is a p-group.

Lemma 2.9. *Suppose M^n is a closed connected manifold with $Y(M) \geq 0$, and suppose $\widetilde{M}$ is a finite covering of M. Then $Y(\widetilde{M}) \geq 0$.*

Proof. Let m be the number of sheets of the covering $\widetilde{M} \to M$. By assumption, given $\varepsilon > 0$, we can choose a conformal class $\mathcal{C}$ on M with $Y(M, \mathcal{C}) \geq -\varepsilon$. That means there is a metric g on M with unit volume and constant scalar curvature $s \geq -\varepsilon$. Lift the metric g up to $\widetilde{M}$. That gives a metric on $\widetilde{M}$ with volume m and scalar curvature $s \geq -\varepsilon$. Rescaling, we get a metric on $\widetilde{M}$ with unit volume and scalar curvature $\geq -m^{-\frac{2}{n}}\varepsilon$. This being true for all $\varepsilon > 0$, it follows that $Y(\widetilde{M}) \geq 0$. q.e.d.

Proposition 2.10. *If π_1 and π_2 are groups and if $\varphi\colon \pi_1 \to \pi_2$ is a group homomorphism, then φ sends $H_n^{\geq 0}(B\pi_1, \mathbb{Z})$ to $H_n^{\geq 0}(B\pi_2, \mathbb{Z})$ and $ko_n^{\geq 0}(B\pi_1)$ to $ko_n^{\geq 0}(B\pi_2)$. It also sends $H_n^{\geq 0}(B\pi_1, \mathbb{Z}_w)$ to $H_n^{\geq 0}(B\pi_2, \mathbb{Z}_w)$ when the diagram*



commutes. If π_1 is a subgroup of π_2 of finite index, then the transfer map on H_n or ko_n sends $H_n^{\geq 0}(B\pi_2, \mathbb{Z})$ to $H_n^{\geq 0}(B\pi_1, \mathbb{Z})$, $H_n^{\geq 0}(B\pi_2, \mathbb{Z}_w)$ to $H_n^{\geq 0}(B\pi_1, \mathbb{Z}_{w|_{\pi_1}})$, and $ko_n^{\geq 0}(B\pi_2)$ to $ko_n^{\geq 0}(B\pi_1)$.

Proof. The first statement is obvious from the definitions in Theorem 2.7. The second statement follows from Lemma 2.9, since the transfer is realized geometrically via coverings. q.e.d.

Theorem 2.11 (Kwasik, Schultz [14]). *Let M^n be a closed connected n-manifold with finite fundamental group π. Assume either that M is spin or that its universal cover is non-spin. For each prime p, let $i_p\colon \pi_p \hookrightarrow \pi$ be the inclusion of a Sylow p-subgroup of π, and let*

$$\begin{aligned} t_p\colon H_n(B\pi,\mathbb{Z}) &\to H_n(B\pi_p,\mathbb{Z}),\\ t_p\colon H_n(B\pi,\mathbb{Z}_w) &\to H_n(B\pi_p,\mathbb{Z}_{w\circ i_p}),\\ t_p\colon ko_n(B\pi) &\to ko_n(B\pi_p) \end{aligned}$$

be the transfer maps. Then M has nonnegative Yamabe invariant if and only if $t_p([M])$ lies in the subgroup $H_n^{\geq 0}(B\pi_p,\mathbb{Z})$ in the oriented non-spin case, or in $ko_n^{\geq 0}(B\pi)$ in the spin case, for each p dividing the order of π. In the non-orientable non-spin case, M has nonnegative Yamabe invariant if and only if $t_2([M])$ lies in $H_n^{\geq 0}(B\pi_p,\mathbb{Z})$ for all odd primes dividing $|\pi|$ and in $H_n^{\geq 0}(B\pi_2,\mathbb{Z}_{w\circ i_2})$ (with respect to the restriction of the twist w).

Proof. The proof is almost word-for-word as in [14], but we review the argument. The "only if" statement is contained in Proposition 2.10. As for the "if" statement, let $A = \widetilde{H}_n(B\pi,\mathbb{Z})$, $\widetilde{H}_n(B\pi,\mathbb{Z}_w)$, or $\widetilde{ko}_n(B\pi)$, and let B be the subgroup $\widetilde{H}_n^{\geq 0}(B\pi,\mathbb{Z})$, $\widetilde{H}_n^{\geq 0}(B\pi,\mathbb{Z}_w)$, or $\widetilde{ko}_n^{\geq 0}(B\pi)$. Similarly let $A_p = \widetilde{H}_n(B\pi_p,\mathbb{Z})$, $\widetilde{H}_n(B\pi_p,\mathbb{Z}_{w\circ t_p})$, or $\widetilde{ko}_n(B\pi_p)$, and let B_p be the subgroup $\widetilde{H}_n^{\geq 0}(B\pi_p,\mathbb{Z})$, $\widetilde{H}_n^{\geq 0}(B\pi_p,\mathbb{Z}_{w\circ t_p})$, or $\widetilde{ko}_n^{\geq 0}(B\pi_p)$. (We can work with reduced homology since $H_*(\mathrm{pt}) = H_*^{\geq 0}(\mathrm{pt})$ and $ko_*(\mathrm{pt}) = ko_*^{\geq 0}(\mathrm{pt})$.) Note that A is a finite group and B is a subgroup; we are trying to show that an element $[M]$ of A lies in B if $t_p([M]) \in B_p$ for all p. Now $\alpha_p = i_p \circ t_p$ is an endomorphism of A which is an isomorphism on $A_{(p)}$, since $[\pi : \pi_p]$ is a unit modulo p. If $t_p([M]) \in B_p$ for all p, then $\alpha_p([M]) \in i_p(B_p) \subseteq B$ for all p, by Proposition 2.10. So that means the image of $[M]$ in $A_{(p)}$ lies in $B_{(p)}$ for all p, and thus $[M]$ lies in B. q.e.d.

Theorem 2.12 (Kwasik, Schultz [14]). *Let π be a finite group, and let*

$$e\colon \Omega^\infty\Sigma^\infty B\pi_+ \to \Omega^\infty\Sigma^\infty B\pi_+$$

be an idempotent in the stable homotopy category (giving a stable splitting of $B\pi$). Then e maps $H_n^{\geq 0}(B\pi,\mathbb{Z})$, $H_n^{\geq 0}(B\pi,\mathbb{Z}_w)$, and $ko_n^{\geq 0}(B\pi)$ into themselves.

Sketch of proof. As pointed out in [14], the proof of the Segal Conjecture implies that the stable splittings of $B\pi$ are essentially linear

combinations of products of transfer maps and maps induced by group homomorphisms, so the result then follows from Proposition 2.10. q.e.d.

3. Analytic tools

In this section we present a number of analytic results that can be used to study the classes of manifolds with positive scalar curvature or with nonnegative Yamabe invariant. First we need a basic characterization of manifolds in the latter class.

Proposition 3.1. *Let M^n be a closed n-manifold. Then:*

(i) *If M does not admit a metric of positive scalar curvature, then*

$$Y(M) = -\inf_g \left(\int_M |s_g|^{\frac{n}{2}} d\mathrm{vol}_g \right)^{\frac{2}{n}}. \tag{3.1}$$

Here the infimum is taken over all Riemannian metrics g on M, and s_g denotes the scalar curvature of g.

(ii) *Suppose that for each $\varepsilon > 0$, there exists a metric g on M with volume 1 and $|s_g| < \varepsilon$. Then $Y(M) \geq 0$. The converse is true if $n \geq 3$ or if $Y(M) = 0$.*

Proof. Statement (i) is Proposition 1 in [16]. As for (ii), suppose the condition is satisfied. If $Y(M) > 0$, then we have nothing to prove, and if not, (i) shows that $Y(M) \geq 0$. In the converse direction, suppose $Y(M) \geq 0$. If $n \geq 3$ and if $Y(M) > 0$, then by a theorem of Kazdan and Warner [11], M admits a metric g with $s_g \equiv 0$, and obviously we may rescale g to have volume 1 without changing this condition. If, on the other hand, $Y(M) = 0$, that means, by definition of the Yamabe invariant (recall Equation (1.1)), that for all $\varepsilon > 0$, there exists a metric g on M with volume 1 and $s_g \leq 0$ constant and $> -\varepsilon$. So again the condition of (ii) is satisfied. q.e.d.

Another basic fact is the following:

Proposition 3.2. *Suppose M^m and N^n are closed manifolds, $n = \dim N \geq 1$, and $Y(N) \geq 0$. Then $Y(M \times N) \geq 0$.*

Proof. If $Y(N) > 0$, then N admits a metric of positive scalar curvature and so does $M \times N$, so $Y(M \times N) > 0$. If $Y(N) = 0$, then by Proposition 3.1, given $\varepsilon > 0$, there exists a metric g_ε on N with

volume 1 and $|s_g| < \varepsilon$. Choose any metric g' on M with volume 1. If we give $M \times N$ the product metric $g_\varepsilon \times tg'$ (where tg' means g' rescaled by multiplying all distances by t), then this product metric has scalar curvature $t^{-2}s_{g'} + s_{g_\varepsilon}$ and volume t^m. So the integral in (3.1) (with $M \times N$ in place of M) is

$$\iint \left|t^{-2}s_{g'} + s_{g_\varepsilon}\right|^{\frac{n+m}{2}} d\mathrm{vol}_{tg'} d\mathrm{vol}_\varepsilon \leq t^m \left|Ct^{-2} + \varepsilon\right|^{\frac{n+m}{2}}, \tag{3.2}$$

for some constant C (the maximum of $|s_{g'}|$ over M) independent of t and ε. So the idea is to take t large, and then given t, to take ε of order t^{-2}. In Equation (3.2), we see that the integral on the left-hand side is bounded by a constant times

$$t^m \left(t^{-2}\right)^{\frac{n+m}{2}} = t^{-n},$$

which goes to zero as $t \to \infty$. Thus by Proposition 3.1, the result follows. q.e.d.

Next, we discuss the extension of the minimal hypersurface technique of [28] to the study of nonnegative Yamabe invariant. Suppose M^n is a closed manifold with metric g, and suppose H^{n-1} is a stable minimal hypersurface in M. In [28], it was shown that if $s_g > 0$, then $Y(H, [\bar{g}]) > 0$, where $\bar{g}$ denotes the induced metric on H and $[\bar{g}]$ is its conformal class. In particular, there is a metric in $[\bar{g}]$ with positive scalar curvature, and this can be used to rule out positive scalar curvature metrics on many non-simply connected manifolds. Now it is not true that just because $Y(M) \geq 0$, then $Y(H) \geq 0$, since by Proposition 3.2, we can get a counterexample by taking $M = H \times S^1$ and $Y(H) < 0$ (say with $n - 1 = 2$ or 4). However, the same estimates used in the proof of Theorem in [28] show that if $s_g \geq K$, where K is a constant, then because the second variation of the $(n-1)$-dimensional volume of H is positive, one has

$$\int_H \frac{(\bar{s} - K)\phi^2}{2} + \int_H |\nabla\phi|^2 > 0 \tag{3.3}$$

for all functions $\phi \in C^\infty(H)$ not vanishing identically. (Here $\bar{s}$, the scalar curvature of H, and ∇ are to be computed with respect to the induced metric $\bar{g}$.) Assume $n > 3$ and consider the "conformal Laplacian"

$$L_H = \frac{4(n-2)}{n-3}\Delta_H + \bar{s}$$

of H, where Δ_H is the usual (nonnegative) Laplacian. (Recall that the dimension of H is $n-1$, not n.) Then for ϕ as above we have

$$\begin{aligned}(3.4)\quad &\frac{1}{2}\langle L_H\phi,\,\phi\rangle \\ &= \frac{2(n-2)}{n-3}\int_H |\nabla\phi|^2 + \int_H \frac{\bar{s}\phi^2}{2} \\ &= \frac{n-1}{n-3}\int_H |\nabla\phi|^2 + \left(\int_H \frac{(\bar{s}-K)\phi^2}{2} + \int_H |\nabla\phi|^2\right) + \frac{K}{2}\int_H \phi^2 \\ &> \frac{K}{2}\int_H \phi^2.\end{aligned}$$

Note the use of Equation (3.3) at the last step. This implies that if K is close to 0, then the conformal Laplacian L_H is not too negative, and thus $Y(H,\bar{g})$ is not too negative, provided that the $(n-1)$-dimensional volume of H is not too large.

If $n = 3$, things are even easier: we instead take $\phi \equiv 1$ in Equation (3.3) and apply Gauss-Bonnet. These arguments thus prove the following two results:

Theorem 3.3. *Let M^n be a closed manifold with metric g, and suppose H^{n-1} is a stable minimal hypersurface in M. Also suppose that the metric g is scalar-flat. Then $Y(H) \geq 0$.*

Proof. Immediate from the above estimates. q.e.d.

Theorem 3.4. *Let M^2 be a closed oriented surface of genus $g > 1$, and let $N^3 = S^1 \times M^2$. Then $Y(N) = 0$ by Proposition* 3.2. *(It cannot be strictly positive, by* [9]*, Theorem* 8.1*, for example.) Thus there is a sequence g_i of metrics on N^3 with volume* 1 *and constant scalar curvatures s_i, with the scalar curvatures tending to* 0 *as $i \to \infty$. On the other hand, for any such sequence of metrics,* $\operatorname{diam}(N, g_i) \to \infty$.

Proof. Choose the metrics g_i as in the statement of the theorem. Choose minimal submanifolds M_i which are absolutely area-minimizing in the homology class $[M^2] \in H_2(S^1 \times M^2, \mathbb{Z})$ for the metric g_i. By inequality (3.3) with $\phi \equiv 1$,

$$(3.5)\qquad \liminf_{i\to\infty} \int_{M_i} (\bar{s}_i - s_i)\, d\mathrm{vol}_{\bar{g}_i} \geq 0.$$

On the other hand, each M_i must be a surface of genus > 1, since it represents a nontrivial homology class in the infinite cyclic cover $\mathbb{R}\times M^2$,

while each mapping of a sphere into this space is null-homotopic and each mapping of a torus into this space factors through a circle (since each abelian subgroup of $\pi_1(M)$ is cyclic) and is thus trivial in H_2. So by Gauss-Bonnet, $\int_{M_i} \bar{s}_i d\mathrm{vol}_{\bar{g}_i} \leq -4\pi$. Comparing this with Equation (3.5), we see the area of M_i with respect to $\bar{g}_i$ must tend to ∞ as $i \to \infty$, while the average value of $\bar{s}_i$ must go to 0, and in particular, $\mathrm{diam}(M_i, \bar{g}_i) \to \infty$. This in turn means $\mathrm{diam}\, g_i \to \infty$, since otherwise we could choose representatives for the homology class $[M]$ in (N, g_i) with bounded diameters, a contradiction. q.e.d.

The next two results are the most significant in this paper; they will be used in the next section to deal with "Toda brackets," among the most intractable of bordism classes.

Theorem 3.5. *Let M_0 and M_1 be closed manifolds, not necessarily connected, that admit metrics of positive scalar curvature. Suppose $M_0 = \partial W_0$ and $M_1 = \partial W_1$ for some compact manifolds with boundary, W_0 and W_1. Form a new manifold*

$$M = (W_0 \times M_1) \cup_{M_0 \times M_1} (M_0 \times W_1)$$

of dimension $n_0 + n_1 + 1$, where n_0 and n_1 are the dimensions of M_0 and M_1. Then M admits a metric of positive scalar curvature.

Proof. We start by choosing metrics of positive scalar curvature, g_0 and g_1, on M_0 and M_1, respectively. Extend them to metrics $\bar{g}_0$ and $\bar{g}_1$ on W_0 and W_1, which are product metrics in neighborhoods of the boundaries. The trick is to write M as a union of four pieces (not two) as follows:

$$M = (W_0 \times M_1) \cup_{M_0 \times M_1} (T_0 \times M_1) \cup_{M_0 \times M_1} (M_0 \times T_1) \cup_{M_0 \times M_1} (M_0 \times W_1),$$

where the "tubes" T_0 and T_1 are (as smooth manifolds) $M_0 \times I$ and $M_1 \times I$, respectively. Call the pieces here A_0, $T_0 \times M_1$, $M_0 \times T_1$, and A_1 in that order. Since g_0 and g_1 have positive scalar curvature, we can choose (very small) constants $t_0 > 0$ and $t_1 > 0$ so that the metric $\bar{g}_0 \times t_1 g_1$ on A_0 and the metric $t_0 g_0 \times \bar{g}_1$ on A_1 have positive scalar curvature. Now all we have to do is choose the metric g_{T_0} on T_0 to interpolate between $t_0 g_0$ and g_0 and the metric g_{T_1} on T_1 to interpolate between $t_1 g_1$ and g_1. If the tubes T_0 and T_1 are "very long," it is possible to do this so that T_0 and T_1 have positive scalar curvature, by

the "Isotopy implies concordance" theorem, [8], Lemma 3. (In fact, in this case, one can write down an explicit warped product metric that does the trick.) Then all the metrics fit together to give a metric of positive scalar curvature on M. q.e.d.

The next theorem is quite similar, but considerably more delicate.

Theorem 3.6. *Let M_0 and M_1 be closed manifolds, not necessarily connected, each with nonnegative Yamabe invariant. (When M_i is disconnected, we mean that each component is required to have nonnegative Yamabe invariant.) Suppose $M_0 = \partial W_0$ and $M_1 = \partial W_1$ for some compact manifolds with boundary, W_0 and W_1. Form a new manifold*

$$M = (W_0 \times M_1) \cup_{M_0 \times M_1} (M_0 \times W_1)$$

of dimension $n_0 + n_1 + 1$, where n_0 and n_1 are the dimensions of M_0 and M_1. Then, excluding the case where $Y(M_0) = 0$, $n_1 = 2$, and $Y(M_1) > 0$, it follows that $Y(M) \geq 0$.

Proof. We follow the same approach as in the proof of Theorem 3.5. If $Y(M_0)$ and $Y(M_1)$ are both strictly positive, we're done by Theorem 3.5, so we may assume at least one of M_0 and M_1 has $Y = 0$. Then we're excluding the case where the other manifold is a disjoint union of copies of S^2 or $\mathbb{RP}^2$, so by Proposition 3.1, we may assume both manifolds have metrics of unit volume which are almost scalar-flat. By Proposition 3.1, it is enough to show that M has a metric for which the integral in (3.1) is as small as one likes. We will estimate the integral separately over the four pieces of M (as in the last proof) and add the results. Fix $\varepsilon > 0$ and choose metric g_0 and g_1 on M_0 and M_1, respectively, each with volume 1 and with small constant scalar curvatures, s_0 and s_1, respectively, with $|s_0|$, $|s_1| < \varepsilon$. Extend g_0 and g_1 to metrics $\bar{g}_0$ and $\bar{g}_1$ on W_0 and W_1, which are product metrics in neighborhoods of the boundaries. Then the scalar curvature of the metric $\bar{g}_0 \times t_1 g_1$ on A_0 is $s_{\bar{g}_0} + t_1^{-2} s_1$ and the scalar curvature of the metric $t_0 g_0 \times \bar{g}_1$ on A_1 is $s_{\bar{g}_1} + t_0^{-2} s_0$. (The constants t_0 and t_1 will be chosen later.) Furthermore, the volumes of these metrics are $\mathrm{vol}(\bar{g}_0) \times t_1^{n_1}$ for A_0 and $\mathrm{vol}(\bar{g}_1) \times t_0^{n_0}$ for A_1. Letting t_0 and t_1 go to 0, we see there are constants $c_0 > 0$ and

$c_1 > 0$ with

(3.6)
$$\iint\limits_{A_0} \left| s_{\bar{g}_0} + t_1^{-2} s_1 \right|^{\frac{n_0+n_1+1}{2}} d\mathrm{vol}_{\bar{g}_0} d\mathrm{vol}_{t_1 g_1} \leq c_0 t_1^{-(n_0+n_1+1)} \varepsilon^{\frac{n_0+n_1+1}{2}} t_1^{n_1},$$

(3.7)
$$\iint\limits_{A_1} \left| s_{\bar{g}_1} + t_0^{-2} s_0 \right|^{\frac{n_0+n_1+1}{2}} d\mathrm{vol}_{t_0 g_0} d\mathrm{vol}_{\bar{g}_1} \leq c_1 t_0^{-(n_0+n_1+1)} \varepsilon^{\frac{n_0+n_1+1}{2}} t_0^{n_0}.$$

The right-hand sides of (3.6) and (3.7) can be rewritten as

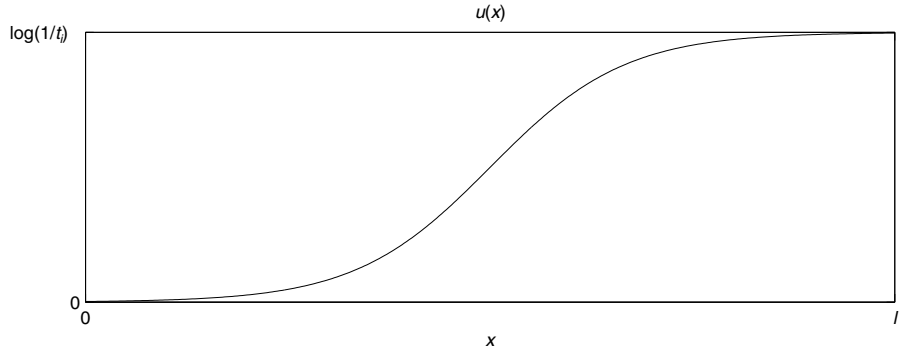
$$c_0 t_1^{-(n_0+1)} \varepsilon^{\frac{n_0+n_1+1}{2}} = c_0 \varepsilon^{n_1/2} \left(\frac{\varepsilon}{t_1^2} \right)^{\frac{n_0+1}{2}}$$

and

$$c_1 t_0^{-(n_1+1)} \varepsilon^{\frac{n_0+n_1+1}{2}} = c_1 \varepsilon^{n_0/2} \left(\frac{\varepsilon}{t_0^2} \right)^{\frac{n_1+1}{2}},$$

respectively.

Next we need to deal with the tubes T_0 and T_1. We give these warped product metrics of the form $f_i(x) g_i \times g_{\mathbb{R}}$, $i = 0, 1$, where $g_{\mathbb{R}}$ is the standard metric on the line, and x is the parameter along the length of the tube. The function f_i will be chosen to interpolate between 0 and t_i. If we write $f_i = \exp(-u_i)$, we need to choose u_i as in the following picture, so that the graph has vanishing first and second derivatives at both ends:



Here l, to be taken large, is the length of the tube. Since $t_i < 1$ and $\mathrm{vol}(g_i) = 1$, the volume of T_i will be bounded by l, as will the volume

of $T_0 \times M_1$ or $M_0 \times T_1$, when we take the product with the metric g_1 on M_1 or g_0 on M_0. The scalar curvature of T_i is given by Equation (7.35) on p. 157 of [9], which gives:

$$\frac{1}{f_i^2} s_i - \frac{n_i(n_i-1)}{f_i^2}(f_i')^2 - \frac{2n_i}{f_i} f_i''.$$

Since $f = \exp(-u_i)$, $\frac{f_i'}{f_i} = -u_i'$ and $\frac{f_i''}{f_i} = (u_i')^2 - u_i''$. Now we can choose u_i so that u_i' is bounded by a constant times $\frac{\log(1/t_i)}{l}$ and u_i'' is bounded by a constant times $\frac{\log(1/t_i)}{l^2}$. Thus the scalar curvature of T_i is bounded in absolute value by

$$\frac{\varepsilon}{t_i^2} + d_i \frac{(\log(1/t_i))^2}{l^2}$$

for some constant d_i. Thus the integrals over $T_0 \times M_1$ and $M_0 \times T_1$ give:

$$\iint\limits_{T_0 \times M_1} |s_{T_0} + s_1|^{\frac{n_0+n_1+1}{2}} \, d\mathrm{vol}_{T_0} d\mathrm{vol}_{g_1} \tag{3.8}$$

$$\leq l \left| \varepsilon + \frac{\varepsilon}{t_0^2} + d_0 \frac{(\log(1/t_0))^2}{l^2} \right|^{\frac{n_0+n_1+1}{2}},$$

$$\iint\limits_{M_0 \times T_1} |s_{g_0} + s_{T_1}|^{\frac{n_0+n_1+1}{2}} \, d\mathrm{vol}_{g_0} d\mathrm{vol}_{T_1} \tag{3.9}$$

$$\leq l \left| \varepsilon + \frac{\varepsilon}{t_1^2} + d_1 \frac{(\log(1/t_1))^2}{l^2} \right|^{\frac{n_0+n_1+1}{2}}.$$

Now all we have to do is choose the parameters t_0, ε, and l to make all of (3.6), (3.7), (3.8), and (3.9) simultaneously small. We do this as follows. First choose t_0 and t_1 very small. Then choose l large enough so that the terms

$$l \left| \frac{(\log(1/t_i))^2}{l^2} \right|^{\frac{n_0+n_1+1}{2}} = \frac{(\log(1/t_i))^{n_0+n_1+1}}{l^{n_0+n_1}}$$

are small. Then finally choose ε/t_i^2 extremely small so that

$$l \left(\frac{\varepsilon}{t_i^2} \right)^{\frac{n_0+n_1+1}{2}}$$

is also small. That does it. q.e.d.

4. Applications to non-negativity of the Yamabe invariant

We're now ready for the first main result of this paper.

Theorem 4.1. *Let M^n be a closed, connected, orientable n-manifold with abelian fundamental group, with non-spin universal cover, and with $n \geq 5$. Then M has nonnegative Yamabe invariant.*

Proof. By Theorem 2.7, it's enough to show that $H_*^{\geq 0}(B\pi_1(M), \mathbb{Z})$ exhausts the image in $H_*(B\pi_1(M), \mathbb{Z})$ of $\Omega_*(B\pi_1(M))$. Write $\pi_1(M)$ as $\mathbb{Z}^k \times \pi$, with π finite abelian. Since the homology of a free abelian group is torsion free, the Künneth Theorem gives

$$H_n(B\pi_1(M), \mathbb{Z}) \cong \bigoplus_{p+q=n} H_p(B\mathbb{Z}^k, \mathbb{Z}) \otimes H_q(B\pi, \mathbb{Z}),$$

and so the homology of $B\pi_1(M)$ is generated by classes of products of tori with homology classes of $B\pi$. So by Proposition 3.2, we only have to show that $H_*^{\geq 0}(B\pi, \mathbb{Z})$ exhausts the image in $H_*(B\pi, \mathbb{Z})$ of $\Omega_*(B\pi_1(M))$. In other words, we are reduced to the case of *finite* abelian groups. By Theorem 2.11, we can further assume that $\pi_1(M)$ is a finite abelian p-group for some prime p. We will come back to finite abelian p-groups after a short digression. q.e.d.

Lemma 4.2. *Let π be a cyclic group of prime power order p^k. Then each class in $H_n(B\pi, \mathbb{Z})$ is represented by an oriented manifold with nonnegative Yamabe invariant, and if $n > 1$, by an oriented manifold with positive scalar curvature.*

Proof. Note that $H_{2n+1}(B\pi, \mathbb{Z})$ is cyclic of order p^k, with a generator represented by the lens space $S^{2n+1}/\pi \to B\pi$, and $H_{2n}(B\pi, \mathbb{Z})$ vanishes for $n > 0$. Since the lens space has positive scalar curvature except in the exceptional case $n = 0$, when it has a flat metric, the statement is immediate. q.e.d.

Proof of Theorem 4.1, *continued.* Recall that we have already reduced to the case where the fundamental group π of M is a finite abelian p-group, hence a finite product of cyclic p-groups. We have to deal with the Tor terms in the Künneth Theorem, and also consider the possibility that the Thom map (sending the class of an oriented manifold to its homology fundamental class) $\Omega_*(B\pi) \to H_*(B\pi, \mathbb{Z})$ may not be surjective, and may not be split onto its image. Thus the argument will require some care. We prove the theorem by induction on the rank (the number of cyclic factors in a product decomposition) of π. If the rank is 1, π

is cyclic and we are done by Lemma 4.2. So assume the result is true for p-groups of smaller rank, and write $\pi = \pi' \times \mathbb{Z}/p^k$, where we may assume that p^k is less than or equal to the order of every cyclic factor of π', and thus less than or equal to the order of every cyclic factor of the homology of $B\pi'$. First assume that $p = 2$. This case is easier because $M\mathrm{SO}$ localized at 2 is a direct sum of Eilenberg-Mac Lane spectra (see [18] and [33], or [23] for a review of the literature), and thus $H_*(B\pi, \mathbb{Z})$ can be identified with a direct summand in $\Omega_*(B\pi)$, and similarly for π'. By inductive hypothesis, each cyclic factor (say of order 2^s, $s \geq k$) in $H_j(B\pi', \mathbb{Z})$ is generated by the class of a manifold $M' \to B\pi'$, where M' has nonnegative Yamabe invariant. If $n - j$ is odd, then we get a corresponding tensor term in the Künneth formula for the homology of $B\pi$, and it is represented by a product of M' with either S^1 or a lens space, and so it is represented by a manifold with nonnegative Yamabe invariant. If $n - j \geq 2$ is even, there is a contribution to $H_n(B\pi, \mathbb{Z})$ of the form

$$\mathrm{Tor}_{\mathbb{Z}}\big(H_j(B\pi', \mathbb{Z}),\, H_{n-j-1}(B\mathbb{Z}/2^k, \mathbb{Z})\big) = \mathrm{Tor}_{\mathbb{Z}}(\mathbb{Z}/2^s, \mathbb{Z}/2^k),$$

which we need to represent by a manifold of nonnegative Yamabe invariant. Since $\mathrm{Tor}^1_{\mathbb{Z}}$ is left exact and $\mathrm{Tor}_{\mathbb{Z}}(\mathbb{Z}/2^s, \mathbb{Z}/2^k)$ is cyclic of order 2^k, the map

$$\mathrm{Tor}_{\mathbb{Z}}(\mathbb{Z}/2^k, \mathbb{Z}/2^k) \to \mathrm{Tor}_{\mathbb{Z}}(\mathbb{Z}/2^s, \mathbb{Z}/2^k)$$

induced by the inclusion $\mathbb{Z}/2^k \hookrightarrow \mathbb{Z}/2^s$ is an isomorphism, so without loss of generality, we may replace M' by something representing a multiple of its homology class, and assume $s = k$. Choose $M'' \to B\mathbb{Z}/2^k$, with M'' either S^1 or a lens space, of dimension $n - j - 1$, generating $H_{n-j-1}(B\mathbb{Z}/2^k, \mathbb{Z})$. We may assume the bordism classes of $M' \to B\pi'$ and $M'' \to B\mathbb{Z}/2^k$ both have order 2^k. Then their Tor product in the homology of $B\pi$ may be represented by the cobordism Massey product $\langle M', 2^k, M''\rangle$ (see [1]), or in other words, by a Toda bracket construction as in Theorem 3.6. More precisely, choose W_0 bounding $2^k M'$ over $B\pi'$ and W_1 bounding $2^k M''$ over $B\mathbb{Z}/2^k$, and glue together $W_0 \times M''$ and $M' \times W_1$ along their common boundary. By Theorem 3.6, the resulting manifold is represented by a manifold with nonnegative Yamabe invariant. (Note that the exceptional case of that theorem never arises.) This completes the inductive step.

Now consider the case where p is odd. In this case, it's important to point out that the inductive hypothesis is simply that the *image* of

$\Omega_*(B\pi') \to H_*(B\pi', \mathbb{Z})$ is represented by manifolds with nonnegative Yamabe invariant, as this map is not usually surjective. However, in this case we have one additional tool in our arsenal, namely Landweber's Künneth Theorem for oriented bordism [15]. More precisely, we apply Theorem A of [15], which applies since $\widetilde{H}_*(B\mathbb{Z}/p^k, \mathbb{Z})$ consists entirely of odd torsion and the Atiyah-Hirzebruch spectral sequence $\widetilde{H}_*(B\mathbb{Z}/p^k, \Omega_*) \Rightarrow \widetilde{\Omega}_*(B\mathbb{Z}/p^k)$ collapses for dimensional reasons. (Ω_* localized at p is free over $\mathbb{Z}_{(p)}$ and concentrated in degrees divisible by 4, and $\widetilde{H}_*\big(B\mathbb{Z}/p^k, \mathbb{Z}_{(p)}\big)$ is nonzero only in odd degrees.) Note also that the proof of Landweber's Theorem shows that $\Omega_*(B\mathbb{Z}/p^k)_{(p)}$ has homological dimension 1 over $\Omega_*(\mathrm{pt})_{(p)}$. Now observe that we have a commutative diagram with exact rows:

(4.1)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega_*(B\pi')\otimes_{\Omega_*}\Omega_*(B\mathbb{Z}/p^k) & \longrightarrow & \Omega_*(B\pi) & \longrightarrow & \mathrm{Tor}^1_{\Omega_*}\big(\Omega_*(B\pi'),\Omega_*(B\mathbb{Z}/p^k)\big) & \longrightarrow & 0 \\
\| & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \| \\
0 & \longrightarrow & H_*(B\pi')\otimes_{\mathbb{Z}}H_*(B\mathbb{Z}/p^k) & \longrightarrow & H_*(B\pi) & \longrightarrow & \mathrm{Tor}_{\mathbb{Z}}\big(H_*(B\pi'),H_*(B\mathbb{Z}/p^k)\big) & \longrightarrow & 0,
\end{array}$$

in which the vertical arrows are induced by the natural transformation $\Omega_* \to H_*$. Note that the map $\Omega_*(B\mathbb{Z}/p^k) \to H_*(B\mathbb{Z}/p^k)$ is surjective, and denote the image of $\Omega_*(B\pi') \to H_*(B\pi')$ by $RH_*(B\pi')$ (for "representable homology"). The image of α is then obviously $RH_*(B\pi') \otimes_{\mathbb{Z}} H_*(B\mathbb{Z}/p^k)$; classes here are represented by products of manifolds of nonnegative Yamabe invariant (because of the inductive hypothesis), so these have nonnegative Yamabe invariant. The image of β is by definition $RH_*(B\pi)$, whereas the image of γ is evidently contained in $\mathrm{Tor}_{\mathbb{Z}}\big(RH_*(B\pi'), H_*(B\mathbb{Z}/p^k)\big)$. We need to examine the structure of $\mathrm{Tor}^1_{\Omega_*}\big(\Omega_*(B\pi'), \Omega_*(B\mathbb{Z}/p^k)\big)$. For this we use an explicit free Ω_*-resolution of $\widetilde{\Omega}_*(B\mathbb{Z}/p^k)$ found in [6, §50], at least in the case $k = 1$; this resolution is also a consequence of [15, Lemma 5.3]. Namely, if z_{2j-1} is the class of the standard lens space $L^{2j-1}(p^k; 1, \dots, 1)$ in $\widetilde{\Omega}_*(B\mathbb{Z}/p^k)$, then the z_{2j-1} generate $\widetilde{\Omega}_*(B\mathbb{Z}/p^k)$ as an Ω_*-module, and satisfy relations of the form

$$p^k z_{2j-1} + x_4^j z_{2j-5} + x_8^j z_{2j-9} + \cdots = 0,$$

where $x_{4\ell}^j \in \Omega_{4\ell}$. Then there is an exact sequence of graded Ω_*-modules

$$0 \to \bigoplus_{j=1}^{\infty} \Omega_* \epsilon_{2j-1} \xrightarrow{\partial} \bigoplus_{j=1}^{\infty} \Omega_* \delta_{2j-1} \xrightarrow{\varepsilon} \widetilde{\Omega}_*(B\mathbb{Z}/p^k) \to 0,$$

with the ϵ_{2j-1} and δ_{2j-1} generators (of degree $2j-1$) of graded free modules, and with $\varepsilon(\delta_{2j-1}) = z_{2j-1}$, and with

$$\partial(\epsilon_{2j-1}) = p^k\delta_{2j-1} + x_4^j\delta_{2j-5} + x_8^j\delta_{2j-9} + \cdots .$$

From this we can read off $\mathrm{Tor}^1_{\Omega_*}\big(\Omega_*(B\pi'), \Omega_*(B\mathbb{Z}/p^k)\big)$; in total degree 2ℓ, it can be identified with

$$\begin{aligned}(4.2)\quad &\ker\left(\bigoplus_{j=1}^{\ell}\Omega_{2(\ell-j)+1}(B\pi')\epsilon_{2j-1} \xrightarrow{\mathrm{id}\otimes\partial} \bigoplus_{j=1}^{\ell}\Omega_{2(\ell-j)+1}(B\pi')\delta_{2j-1}\right)\\ &\cong \left\{Y \in \bigoplus_{j=1}^{\ell}\Omega_{2j-1}(B\pi') : AY = 0\right\},\end{aligned}$$

where (indexing elements of $\Omega_{2j-1}(B\pi')$ by their degrees) the lower-triangular system $AY = 0$ has the explicit form

$$\begin{pmatrix} p^k & 0 & 0 & \dots \\ * & p^k & 0 & \dots \\ * & * & p^k & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} y_1 \\ y_5 \\ y_9 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \quad \begin{pmatrix} p^k & 0 & 0 & \dots \\ * & p^k & 0 & \dots \\ * & * & p^k & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} y_3 \\ y_7 \\ y_{11} \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix},$$

the $*$'s indicating elements of the torsion-free part of Ω_*. Under the map $\Omega_* \to H_*$, all the $x_{4\ell}^j$'s go to zero, so the lower-triangular matrix A maps to the scalar matrix p^kI.

Now we can write down generators for the Tor term in the upper right in (4.1). In the language of [1, §5.10], they are "matrix cobordism Massey products" $\langle Y, A^t, Z\rangle$, where Y is a row vector in $\Omega_*(B\pi')$ (the transpose of the column vector in (4.2)) and Z is a column vector in $\Omega_*(B\mathbb{Z}/p^k)$. We can explicitly construct a representative for such a class as a Toda bracket. To give a concrete example, consider the generator of $\mathrm{Tor}^1_{\Omega_*}(\Omega_*(B\mathbb{Z}/3), \Omega_*(B\mathbb{Z}/3))$ in total degree 6 (thus contributing to $\Omega_7(B\mathbb{Z}/3 \times B\mathbb{Z}/3)$) corresponding to the relation $x_4z_1 + 3z_5 = 0$. The classes z_1 and z_5 in $\Omega_*(B\mathbb{Z}/3)$ are represented by lens spaces $L^1 = S^1$ and L^5, and there is a simply connected 4-manifold X^4 representing x_4. Choose N^2 (a "pair of pants") bounding $3L^1$ over $B\mathbb{Z}/3$ and a bordism W^6 between $-X^4 \times L^1$ and $3L^5$ over $B\mathbb{Z}/3$. Then our Toda bracket is constructed in four pieces by gluing first $L^5 \times N^2$, then $W^6 \times L^1$, then $L^1 \times W^6$, and then $N^2 \times L^5$. (See Figure 1.)

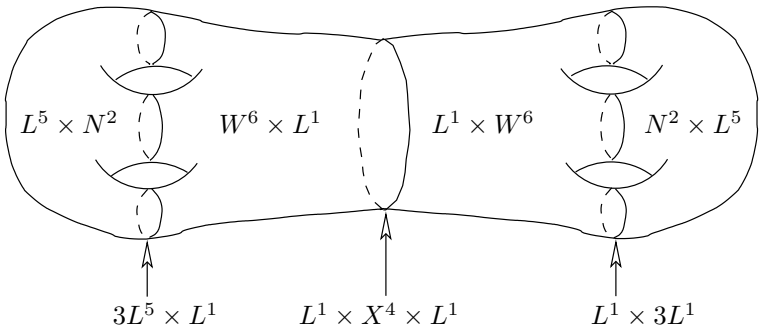


Figure 1: Construction of a matrix Toda bracket.

In the notation of [1], this is the matrix Toda bracket

$$\left\langle \begin{pmatrix} z_1 & z_5 \end{pmatrix}, \begin{pmatrix} 3 & x_4 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} z_5 \\ z_1 \end{pmatrix} \right\rangle.$$

The general case works essentially the same way — the matrix Toda bracket is assembled out of products, glued together along their boundaries, where at least one of the factors is either a generator of $\Omega_*(B\pi')$, which by inductive hypothesis has nonnegative Yamabe invariant, or else a lens space generator of $\Omega_*(B\mathbb{Z}/p^k)$, which also has nonnegative Yamabe invariant. Now invoke Theorem 3.6. Again, the exceptional case of that Theorem never arises in our context. So this shows that the image of γ is represented by manifolds of nonnegative Yamabe invariant. Chasing diagram (4.1) now shows that $RH_*(B\pi)$ is represented by manifolds with nonnegative Yamabe invariant, which completes the inductive step for the case p odd. q.e.d.

Corollary 4.3. *Let M^n be a closed, connected, oriented n-manifold with finite fundamental group π, with non-spin universal cover, and with $n \geq 5$. Also assume all Sylow subgroups of π are abelian. Then M has nonnegative Yamabe invariant.*

Proof. This is immediate from Theorem 4.1 and Theorem 2.11. q.e.d.

In the odd order case, we can carry this over to the spin case as well:

Theorem 4.4. *Let M^n be a closed, connected, spin n-manifold with finite fundamental group π of odd order, and with $n \geq 5$. Also assume*

all Sylow subgroups of π are abelian. Then M has nonnegative Yamabe invariant.

Proof. By Petean's theorem [20], this is true when π is trivial. As before, it's enough to consider the case of an abelian p-group, p odd. But for π of odd order, the natural map $\widetilde{\Omega^{\mathrm{Spin}}}_*(B\pi) \to \widetilde{\Omega}_*(B\pi)$ is an isomorphism, since the "forget spin structure" map of spectra $M\mathrm{Spin} \to M\mathrm{SO}$ is an equivalence after localizing at p (see [18]). We prove the result by induction on the rank of π. Thus write $\pi = \pi' \times \mathbb{Z}/p^k$, and assume by inductive hypothesis that the theorem is true for π'. Since $\widetilde{\Omega^{\mathrm{Spin}}}_*(B\pi') \to \widetilde{\Omega}_*(B\pi')$ and $\widetilde{\Omega^{\mathrm{Spin}}}_*(B\mathbb{Z}/p^k) \to \widetilde{\Omega}_*(B\mathbb{Z}/p^k)$ are isomorphisms, we have by Landweber's Theorem [15] an exact sequence

$$0 \to \Omega_*^{\mathrm{Spin}}(B\pi') \otimes_{\Omega_*^{\mathrm{Spin}}} \Omega_*^{\mathrm{Spin}}(B\mathbb{Z}/p^k) \to \Omega_*^{\mathrm{Spin}}(B\pi) \to \mathrm{Tor}^1_{\Omega_*^{\mathrm{Spin}}}\big(\Omega_*^{\mathrm{Spin}}(B\pi'), \Omega_*^{\mathrm{Spin}}(B\mathbb{Z}/p^k)\big) \to 0. \tag{4.3}$$

By inductive hypothesis, each element of $\Omega_s^{\mathrm{Spin}}(B\pi')$ is represented by a map $M' \to B\pi'$, with M' a spin s-manifold with nonnegative Yamabe invariant, and similarly each element of $\Omega_t^{\mathrm{Spin}}(B\mathbb{Z}/p^k)$ is represented by a map $M'' \to B\mathbb{Z}/p^k$, with M'' a spin t-manifold with nonnegative Yamabe invariant. Then $[M' \to B\pi'] \otimes [M'' \to B\mathbb{Z}/p^k]$ in the tensor term on the left side of (4.3) is represented by $M' \times M'' \to B\pi$, which has nonnegative Yamabe invariant. Furthermore, the Tor term $\mathrm{Tor}^1_{\Omega_*^{\mathrm{Spin}}}([M' \to B\pi'], [M'' \to B\mathbb{Z}/p^k])$ on the right of (4.3) is generated by matrix Toda brackets, just as in the proof of Theorem 4.1. As before, it follows from Theorem 3.6 that these Toda brackets have nonnegative Yamabe invariant, and this completes the inductive step. q.e.d.

5. Applications to positive scalar curvature

It turns out that the method of proof of Theorem 4.1, if we replace Theorem 3.6 by Theorem 3.5, gives substantial results on the positive scalar curvature problem for manifolds with finite abelian fundamental group for which the universal cover is non-spin, since all of the homology generators constructed above have positive scalar curvature by Theorem 3.5, *except* for those involving Toda brackets and products of one-dimensional homology classes. We proceed to formalize this as follows:

Definition 5.1. Let π be a finitely generated abelian group. Call a class in $H_n(B\pi, \mathbb{Z})$ *toral* if it is represented by a map $T^n \to B\pi$. Note that any such map is determined up to homotopy by the associated map $\mathbb{Z}^n \to \pi$ on fundamental groups, which we may assume without loss of generality to have image of rank $n \leq r$, where $r = \operatorname{rank} \pi$, that is, the minimal number of cyclic factors when we write π as a direct sum of cyclic groups, so toral classes only exist in degrees $n \leq r$. Let $H_n^{\mathrm{toral}}(B\pi, \mathbb{Z}) \subseteq H_n(B\pi, \mathbb{Z})$ be the subgroup generated by the toral classes, and call this the *toral subgroup*.

Proposition 5.2. *Let π be an elementary abelian p-group of rank r, that is, $(\mathbb{Z}/p)^r$. Then for all $n \geq 1$, $H_n(B\pi, \mathbb{Z})$ is also elementary abelian, of rank equal to*

$$\sum_{j=1}^{n} (-1)^{n-j} \binom{j+r-1}{r-1}.$$

The toral subgroup $H_n^{\mathrm{toral}}(B\pi, \mathbb{Z})$ is a direct summand in $H_n(B\pi, \mathbb{Z})$, of rank the binomial coefficient $\binom{r}{n}$. (Note that this vanishes for $n > r$.)

Proof. The homology groups $H_n(B\mathbb{Z}/p, \mathbb{Z})$ vanish for $n > 0$ even and are $\mathbb{Z}/p$ for n odd. So by iterated applications of the Künneth Theorem, all integral homology groups of π (other than H_0, which is $\mathbb{Z}$), are elementary abelian p-groups. Consider the Poincaré series

$$P(r,t) = 1 + \sum_{n=1}^{\infty} t^n \dim_{\mathbb{Z}/p} H_n(B(\mathbb{Z}/p)^r, \mathbb{Z}).$$

Then

$$P(1,t) = 1 + t + t^3 + t^5 + \cdots = 1 + \frac{t}{1-t^2} = \frac{1+t-t^2}{1-t^2}. \tag{5.1}$$

The Künneth Theorem gives the recursion relation

$$P(r+1,t) = P(r,t)P(1,t) + t\,(P(r,t)-1)\,(P(1,t)-1)\,, \tag{5.2}$$

where the first term comes from the "tensor terms" and the second term comes from the "Tor terms." Putting together Equations (5.1) and (5.2) yields by induction on r the formula

$$P(r,t) = \frac{1+t(1-t)^r}{(1-t)^r(1+t)} = \frac{t}{1+t} + \frac{1}{(1-t)^r(1+t)}.$$

For $n \geq 1$, the coefficient of t^n in this expression is

$$(-1)^{n+1} + \sum_{j=0}^{n} (-1)^{n-j} \binom{j+r-1}{r-1},$$

which is the expression in the statement of the Proposition. On the other hand, the toral subgroup is generated just by the products of distinct generators of H_1, so in degree n, we have $\binom{r}{n}$ possibilities.

q.e.d.

Definition 5.3. For any space X, we denote by $RH_*(X)$ the image of the Thom map $\Omega_*(X) \to H_*(X, \mathbb{Z})$, and call it the *representable homology*. (This already made an appearance in the proof of Theorem 4.1.) Note that RH_* is a homotopy functor (in fact, $RH_s(X) = E^\infty_{s,0}$ in the bordism spectral sequence $H_s(X, \Omega_t) \Rightarrow \Omega_*(X)$), though not a homology theory. By Lemma 4.2, $RH_*(B\pi) = H_*(B\pi, \mathbb{Z})$ when π is a cyclic group. Clearly $H_*^{\text{toral}}(B\pi, \mathbb{Z}) \subseteq RH_*(B\pi)$ when H_*^{toral} is defined as in Definition 5.1.

The following fact, which is somewhat surprising, will be our key technical tool:

Proposition 5.4. *Let π be an elementary abelian p-group of rank 2, where p is an odd prime. Then $RH_{\text{odd}}(B\pi)$ is generated (as an abelian group) by the images of $RH_*(B\sigma)$, as σ runs over the cyclic subgroups of π.*

Proof. This is proved in [3], using explicit calculations of the eta-invariants of lens spaces. q.e.d.

The parallel to this when $p = 2$ is the following:

Proposition 5.5. *Let $\pi = (\mathbb{Z}/2) \times (\mathbb{Z}/2)$. Then $H_{2k-1}(B\pi, \mathbb{Z})$ is generated (as an abelian group) by manifolds of positive scalar curvature mapping to $B\pi$, for all $k > 1$.*

Proof. Recall that MSO localized at 2 is an Eilenberg-Mac Lane spectrum, so the map $\Omega_*(B\pi) \to H_*(B\pi, \mathbb{Z})$ is split surjective. In particular, $RH_*(B\pi) = H_*(B\pi)$. By the Künneth Theorem, $H_{2k-1}(B\pi, \mathbb{Z})$ is a vector space over $\mathbb{F}_2$ with basis consisting of $z_{2k-1} \otimes 1$, $1 \otimes z_{2k-1}$, and the homology Toda brackets $\langle z_{2j-1}, 2, z_{2k-2j-1} \rangle$, $1 \leq j \leq k-1$. Here z_{2j-1} is the class of $\mathbb{RP}^{2j-1} \hookrightarrow \mathbb{RP}^\infty$ in $H_{2j-1}(B\mathbb{Z}/2, \mathbb{Z})$. Clearly the classes $z_{2k-1} \otimes 1$ and $1 \otimes z_{2k-1}$ have positive scalar curvature when

$k > 1$. The classes $\langle z_{2j-1}, 2, z_{2k-2j-1}\rangle$ have positive scalar curvature when $1 < j < k-1$, by Theorem 3.5. So it remains to deal with Toda brackets involving z_1. There is one special case: $H_3(B\pi, \mathbb{Z})$ turns out to be spanned by $z_3 \otimes 1$ and $1 \otimes z_3$ together with the class of the diagonal embedding Δ of $\mathbb{RP}^3$ in $\mathbb{RP}^\infty$, as one can see by observing that

$$\Delta^*\colon\ H^*(B\pi, \mathbb{Z}/2) = \mathbb{F}_2[u,v] \to H^*(\mathbb{RP}^3, \mathbb{Z}/2) = \mathbb{F}_2[x]/(x^4)$$

sends both of the generators u and v to x, so that $\Delta_*([\mathbb{RP}^3])$ pairs nontrivially with both u^2v and v^2u, and thus can't be in the span of the classes $z_3 \otimes 1$ and $1 \otimes z_3$. In fact $\langle z_1, 2, z_1\rangle = z_3 \otimes 1 + 1 \otimes z_3 + \Delta_*(z_3)$. So it remains to deal with Toda brackets $\langle z_1, 2, z_{2j-1}\rangle$. (The case of $\langle z_{2j-1}, 2, z_1\rangle$, $1 < j < k-1$ is the same by symmetry.) For this we need the fact that S^1 and $\mathbb{RP}^{2j-1}$ have orientation-reversing diffeomorphisms commuting up to homotopy with the nontrivial maps $S^1 \to \mathbb{RP}^\infty$ or $\mathbb{RP}^{2j-1} \to \mathbb{RP}^\infty$. Thus $S^1 \amalg S^1 \cong S^1 \amalg (-S^1)$ bounds a cylinder $S^1 \times I$ (over $\mathbb{RP}^\infty$), and similarly $\mathbb{RP}^{2j-1} \amalg \mathbb{RP}^{2j-1} \cong \mathbb{RP}^{2j-1} \amalg (-\mathbb{RP}^{2j-1})$ bounds the cylinder $\mathbb{RP}^{2j-1} \times I$ (over $\mathbb{RP}^\infty$). So $\langle z_1, 2, z_{2j-1}\rangle$ can be constructed by gluing together $(S^1 \times I) \times \mathbb{RP}^{2j-1}$ and $S^1 \times (\mathbb{RP}^{2j-1} \times I)$. Since the standard metric on $\mathbb{RP}^{2j-1}$ has positive scalar curvature for $j > 1$ and the standard metric on $S^1 \times I$ is flat, it is clear that we obtain a metric of positive scalar curvature on the Toda bracket. q.e.d.

Before proceeding further, we would like to remind the reader of some known results on $\Omega_*(B\pi)$, where π is an elementary abelian p-group of rank r with p an odd prime. Recall that

$$(\Omega_*)_{(p)} \cong \mathbb{Z}_{(p)}[x_4, x_8, \dots, x_{4k}, \dots]$$

with $|x_{4k}| = 4k$. Let $P = \mathbb{Z}_{(p)}[\ x_{4i} \mid 2i \neq p^\ell - 1\ ]$, and $M(P)$ denote the corresponding generalized Moore spectrum (just a wedge of shifted sphere spectra, localized at p). Then there is a splitting of the spectrum

(5.3)
$$M\mathrm{SO}_{(p)} = BP \wedge M(P) \cong \bigvee_{\text{multiindices } i} \Sigma^{4|i|} BP, \quad \text{where } 2i_k \neq p^\ell - 1.$$

Here $BP_* = \mathbb{Z}_{(p)}[v_1, \dots, v_k, \dots]$, $|v_j| = 2(p^j - 1)$. In particular, we have a BP_*-module isomorphism $(\Omega_*)_{(p)} \cong BP_* \otimes P$. Our goal is to describe the representable homology $RH_*(B\pi)$, which is the image of the map $\Omega_*(B\pi) \to H_*(B\pi; \mathbb{Z})$. Since π is an elementary abelian p-group with p odd, it is enough to work in the p-local category; thus we will

work with the group $RH_*(B\pi)_{(p)} = \operatorname{Im}\big(\Omega_*(B\pi)_{(p)} \to H_*(B\pi;\mathbb{Z}_{(p)})\big)$. Clearly $RH_0(B\pi)_{(p)} = RH_0(B\pi)\otimes\mathbb{Z}_{(p)}$, and $RH_j(B\pi)_{(p)} = RH_j(B\pi)$ for $q>0$. Let $\varphi^0\colon M\mathrm{SO}_{(p)} \to BP$ be the projection on the first BP from (5.3). Let $h_*\colon BP_*(B\pi) \to H_*(B\pi;\mathbb{Z}_{(p)})$ be the standard homomorphism. Then the map $\Omega_*(B\pi)_{(p)} \to H_*(B\pi;\mathbb{Z}_{(p)})$ is decomposed as

$$\Omega_*(B\pi)_{(p)} \xrightarrow{\varphi^0_*} BP_*(B\pi) \xrightarrow{h_*} H_*(B\pi;\mathbb{Z}_{(p)}).$$

Thus it is enough to work with $BP_*(B\pi)$. We need the following results on the structure of $BP_*(B\mathbb{Z}/p)$, see, say [10]. We have that $BP^*(\mathbb{CP}^\infty) = BP^*[[x]]$, where $x\in BP^2(\mathbb{CP}^\infty)$ is the first Chern class. Here we identify $BP^{-*} = BP_*$. Start with the standard fibration

$$B\mathbb{Z}/p \longrightarrow \mathbb{CP}^\infty \xrightarrow{p} \mathbb{CP}^\infty,$$

where the map $p\colon \mathbb{CP}^\infty = K(\mathbb{Z},2) \to K(\mathbb{Z},2) = \mathbb{CP}^\infty$ induces multiplication by p on π_2. Then $p^* : BP^*(\mathbb{CP}^\infty) \to BP^*(\mathbb{CP}^\infty)$ is given as

$$p^*(x) = [p](x) = \sum_{i\geq 0} a_i x^{2i+1}, \qquad a_i \in BP_{4i}.$$

(We have re-indexed from the way things are written in [10], since if p is odd, then BP_* is concentrated in degrees divisible by 4.) In particular, $BP^*(B\mathbb{Z}/p) \cong BP^*[[x]]/[p](x)$. Let $I_n = (p, v_1, \dots, v_{n-1})$. The coefficients a_i satisfy:

- $a_i \in I_n$ for $i < (p^n-1)/2$;
- $a_{(p^n-1)/2} \equiv v_n \bmod I_n$, in particular, $a_0 = p$, and $a_i = 0$ for $0 < i < (p-1)/2$, and $a_{(p-1)/2} = v_1$.

The BP_*-module $\widetilde{BP}_*(B\mathbb{Z}/p)$ is generated by elements

$$z_{2m-1} \in BP_{2m-1}(B\mathbb{Z}/p)$$

(represented by the standard lens spaces), subject to the relations:

$$R_m = \sum_{i=0}^{m-1} a_i z_{m-4i} = 0. \tag{5.4}$$

Now we are ready to prove the following result.

Theorem 5.6. *Let π be an elementary abelian p-group of rank r, where p is an odd prime. Then $RH_*(B\pi)$ is generated (as an abelian group) by elements $x_1 \otimes \cdots \otimes x_j \in H_*(B\sigma_1) \otimes \cdots \otimes H_*(B\sigma_j)$, with $\sigma_1 \times \cdots \times \sigma_j \hookrightarrow \pi$, $j \leq r$, and with each σ_i a cyclic p-group. (However, the embedding $\sigma_1 \times \cdots \times \sigma_j \hookrightarrow \pi$ is not necessarily the "standard" one.)*

Proof. We prove this by induction on the rank r. When $r = 1$, the statement is trivially true, and when $r = 2$, this is Proposition 5.4. Now assume the result for smaller values of r, and write $\pi = \pi' \times \mathbb{Z}/p$, where π' has rank $r-1$. We use the BP-version of the diagram (4.1):

(5.5)

$$\begin{array}{ccccccccc} 0 & \to & BP_*(B\pi')\otimes_{BP_*} BP_*(B\mathbb{Z}/p) & \to & BP_*(B\pi) & \to & \mathrm{Tor}^1_{BP_*}\big(BP_*(B\pi'),BP_*(B\mathbb{Z}/p)\big) & \to & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & H_*(B\pi')\otimes_{\mathbb{Z}} H_*(B\mathbb{Z}/p) & \longrightarrow & H_*(B\pi) & \longrightarrow & \mathrm{Tor}_{\mathbb{Z}}\big(H_*(B\pi'),H_*(B\mathbb{Z}/p)\big) & \longrightarrow & 0, \end{array}$$

The image of α is taken care of by inductive hypothesis. Also by inductive hypothesis, the image of γ is generated by the images of

$$\mathrm{Tor}^1_{BP_*}\big(BP_*(B\sigma_1) \otimes_{BP_*} BP_*(B\sigma_2) \otimes_{BP_*} \ldots \otimes_{BP_*} BP_*(B\sigma_j),\, BP_*(B\mathbb{Z}/p)\big) \tag{5.6}$$

for the various subgroups $\sigma_1 \times \ldots \sigma_j \hookrightarrow \pi'$. The image of (5.6) is contained in a copy of $RH_*\left(B(\mathbb{Z}/p)^{r-1}\right)$ if $j < r-1$, which is also covered by the inductive hypothesis. So we may assume $j = r-1$. To simplify notation, write H_* for $\widetilde{H}_*(B\mathbb{Z}/p)$ and (as in [10]) N_* for $\widetilde{BP}_*(B\mathbb{Z}/p)$. Thus we are reduced to studying the image of the map

(5.7)

$$\gamma\colon\ \mathrm{Tor}^1_{BP_*}\big(\overbrace{N_* \otimes_{BP_*} \cdots \otimes_{BP_*} N_*}^{r-1},\, N_*\big) \to \mathrm{Tor}_{\mathbb{Z}}\big(\overbrace{H_* \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} H_*}^{r-1},\, H_*\big).$$

The BP_*-module on the left in (5.7) is computed in [10, Theorem 4.1]; this subquotient of $BP_*(B\pi)$ is a direct sum

$$\mathrm{Tor}^1_{BP_*}\big((N_*)^{\otimes^{r-1}},\, N_*\big) \cong \bigoplus_{j=1}^{p^{r-1}-1} (N_*)^{\otimes^{r-1}} y_{2j}$$

of copies of $\big(N_*\big)^{\otimes^{r-1}}$, shifted up in degrees by $2j$, $0 < j < p^{r-1}$. Similarly, since H_* is a direct sum of cyclic groups of degree p in odd degrees

and $(H_*)^{\otimes^{r-1}}$ is an elementary abelian p-group, the $\mathbb{Z}$-module on the right in (5.7) is a subquotient of $H_*(B\pi)$ of the form $\bigoplus_{j=1}^{\infty}(H_*)^{\otimes^{r-1}}y_{2j}$, a direct sum of copies of $\left(H_*\right)^{\otimes^{r-1}}$, shifted up in degrees by $2j$, $0 < j < \infty$. Since N_* surjects onto H_* under the map $BP_*(B\mathbb{Z}/p) \to H_*(B\mathbb{Z}/p;\, \mathbb{Z})$, we see that the image of the map (5.7) is precisely a direct sum of copies of $\left(H_*\right)^{\otimes^{r-1}}$, shifted up in degrees by $2j$, $0 < j < p^{r-1}$. We need to show that this graded group is generated by products of lens spaces.

Let $I = (i_1, \ldots, i_{r-1})$ be a multi-index with all the i_k's odd, and let $z_I = z_{i_1} \otimes \cdots \otimes z_{i_{r-1}}$ be the corresponding product of lens spaces (or the element of $\left(N_*\right)^{\otimes^{r-1}}$ or $\left(H_*\right)^{\otimes^{r-1}}$ represented by this product). Then we need to show that $z_I y_{2j} \in RH_*(B\pi)$ is represented by linear combinations of products of lens spaces, for $0 < j < p^{r-1}$. The element $z_I y_{2j} \in RH_*(B\pi)$ is represented by the homology Toda bracket $\langle z_I,\, p,\, z_{2j-1}\rangle$. So if z_I has order p in $\left(N_*\right)^{\otimes^{r-1}}$ and z_{2j-1} has order p in N_*, we can lift the homology Toda bracket to a BP-Toda bracket of the same form. Suppose $j = 1$ or 2 and some i_k in I is $= 1$ or 3. Then the lens spaces z_{2j-1} and z_{i_k} have order p in N_*, and so z_I also has order p in $BP_*(B\pi')$; hence we have (in BP_*)

$$
\begin{aligned}
(5.8)\qquad & \langle z_I,\, p,\, z_{2j-1}\rangle \\
& = \langle z_{i_1} \otimes \cdots \otimes z_{i_{k-1}} \otimes z_{i_k} \otimes z_{i_{k+1}} \otimes \cdots \otimes z_{i_{r-1}},\, p,\, z_{2j-1}\rangle \\
& = \pm z_{i_1} \otimes \cdots \otimes z_{i_{k-1}} \otimes \langle z_{i_k},\, p,\, z_{2j-1}\rangle \otimes z_{i_{k+1}} \otimes \cdots \otimes z_{i_{r-1}}
\end{aligned}
$$

by the product property of Toda brackets, [1, 2.1, axiom 3]. The bracket in the middle is of rank 2, so we can apply Proposition 5.4 to it, and so our element is represented by a linear combination of products of lens spaces. This takes care of the case where $j = 1$ or 2 and some i_k in I is $= 1$ or 3. In general, we are interested only in the image of γ. Notice that the elements of $\mathrm{Tor}^1_{BP_*}\left((N_*)^{\otimes^{r-1}},\, N_*\right)$ are represented by BP_*-matrix Toda brackets. Since the BP_*-module N_* is given in terms of the generators z_m and the relations R_m, it is enough to consider the Toda brackets of the form:

$$
\left\langle \begin{pmatrix} z_{i_k} & z_{i_k-4} & \ldots \end{pmatrix},\, A,\, \begin{pmatrix} \vdots \\ z_{2j-5} \\ z_{2j-1} \end{pmatrix} \right\rangle
$$

and

$$\left\langle z_{i_1} \otimes \cdots \otimes z_{i_{k-1}} \otimes \left(z_{i_k},\, z_{i_k-4}, \dots\right) \otimes z_{i_{k+1}} \otimes \cdots \otimes z_{i_{r-1}},\, A, \begin{pmatrix} \vdots \\ z_{2j-5} \\ z_{2j-1} \end{pmatrix} \right\rangle,$$

where A is the matrix

$$\begin{pmatrix} p & 0 & 0 & 0 & \dots \\ a_1 & p & 0 & 0 & \dots \\ a_2 & a_1 & p & 0 & \dots \\ a_3 & a_2 & a_1 & p & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

If we now replace z_{i_k} in (5.8) by the row vector $\begin{pmatrix} z_{i_k} & z_{i_k-4} & \dots \end{pmatrix}$, and replace p in (5.8) by A and z_{2j-1} on the right of (5.8) by $\begin{pmatrix} \vdots \\ z_{2j-5} \\ z_{2j-1} \end{pmatrix}$, then such matrix Toda brackets generate the desired subgroup of $RH_*(B\pi)$, and the argument goes through just as before. q.e.d.

Definition 5.7. Let π be a finite elementary abelian p-group, say $(\mathbb{Z}/p)^r$, and let H_* denote homology with $\mathbb{Z}$ coefficients. We define $RH_*^{\text{atoral}}(B\pi)$, the *atoral quotient* of representable homology, to be the quotient group $RH_*(B\pi)/H_*^{\text{toral}}(B\pi)$.

Theorem 5.8. *Let π be an elementary abelian p-group, and let $n \geq 5$. Let H_* denote homology with $\mathbb{Z}$ coefficients. Then every class in $RH_*^{\text{atoral}}(B\pi)$ is represented by an oriented manifold of positive scalar curvature. In particular, if $n > \operatorname{rank}\pi$, then **every** orientable n-manifold with fundamental group π and with non-spin universal cover has a metric of positive scalar curvature.*

Proof. First consider the case where p is odd. We apply Theorem 5.6. This reduces us to the case of classes of the form $x_1 \otimes \cdots \otimes x_j \in H_*(B\sigma_1) \otimes \cdots \otimes H_*(B\sigma_j)$, with each x_i represented by either S^1 or a lens space. The class is toral if all the x_i's are 1-dimensional. The remaining products have positive scalar curvature and project onto the atoral quotient.

Finally consider the case where $p = 2$. In this case all homology classes are representable, and we argue by induction on $\operatorname{rank}\pi$. To start the induction, Lemma 4.2 takes care of the case where $\operatorname{rank}\pi = 1$.

Proposition 5.5 takes care of the case where $\operatorname{rank}\pi = 2$. So assume $\operatorname{rank}\pi > 2$ and write $\pi = \pi' \times (\mathbb{Z}/2) = \pi'' \times (\mathbb{Z}/2) \times (\mathbb{Z}/2)$ We may assume by inductive hypothesis that the theorem is true for π' and π''. The Künneth Theorem gives a short exact sequence

$$0 \to H_*(B\pi') \otimes_{\mathbb{Z}} H_*(B\mathbb{Z}/2) \to H_*(B\pi) \to \operatorname{Tor}_{\mathbb{Z}}\big(H_*(B\pi'), H_*(B\mathbb{Z}/2)\big) \to 0,$$

(and similarly for $H_*(B\pi')$ in terms of $H_*(B\pi'')$), and each element of the Tor term is represented by Toda brackets. Also, $H_*^{\text{toral}}(B\pi)$ is contained in the image of $H_*(B\pi') \otimes H_*(B\mathbb{Z}/2)$. The group $H_*(B\pi') \otimes H_*(B\mathbb{Z}/2)$ is spanned by product classes $z \otimes z_{2k-1}$, which are toral exactly when z is toral and $k = 1$, and which are obviously represented by manifolds of positive scalar curvature otherwise.

Thus it remains to show that every element of

$$\operatorname{Tor}\big(H_*(B\pi'),\, H_*(B\mathbb{Z}/2)\big)$$

is represented by a manifold of positive scalar curvature. Consider a Toda bracket $\langle z, 2, z_{2k-1}\rangle$. We may further reduce to the case where either $z = z'' \otimes z_{2j-1}$ or $z = \langle z'', 2, z_{2j-1}\rangle$, with $z'' \in H_*(B\pi'')$. There are several cases. If $z = z'' \otimes z_{2j-1}$, then

$$\langle z'' \otimes z_{2j-1}, 2, z_{2k-1}\rangle = z'' \otimes \langle z_{2j-1}, 2, z_{2k-1}\rangle,$$

and we may apply Proposition 5.5 to the second factor, which is always represented by a manifold of positive scalar curvature.

It remains to deal with an iterated Toda bracket

$$\big\langle \langle z'', 2, z_{2j-1}\rangle, 2, z_{2k-1}\big\rangle.$$

In this case we need the associativity formula [1, 2.1.6]:

$$\big\langle \langle z'', 2, z_{2j-1}\rangle, 2, z_{2k-1}\big\rangle + \big\langle z'', \langle 2, z_{2j-1}, 2\rangle, z_{2k-1}\big\rangle + \big\langle z'', 2, \langle z_{2j-1}, 2, z_{2k-1}\rangle\big\rangle = 0.$$

The bracket $\langle 2, z_{2j-1}, 2\rangle$ in the middle term lies in $H_{2j}(B\mathbb{Z}/2) = 0$, so the middle term can be eliminated. In the last term, we can apply Proposition 5.5 to conclude that $\langle z_{2j-1}, 2, z_{2k-1}\rangle$ is represented by a manifold of positive scalar curvature. If z'' has positive scalar curvature, we are done by Theorem 3.5. But if z'' is toral, we may split off a z_1 factor and reduce to a lower-rank case. So in either event, we are done by induction. q.e.d.

In the case of odd order elementary abelian groups, it is not so hard to prove the analogue of Theorem 5.8 for the spin case, largely because of the fact that for p odd, $M\mathrm{Spin} \to M\mathrm{SO}$ is a p-local equivalence. The details are carried out in the sequel paper [5].

Problem 5.9. Are toral homology classes (for an elementary abelian p-group) represented by manifolds of positive scalar curvature? We suspect not, but we know of no way to approach this question.

Problem 5.10. Is Theorem 4.4 true without the odd order assumption? We presume so, but the proof would necessarily be much more complicated, since computing $ko_*(B\pi)$ for a 2-group is quite difficult.

Problem 5.11. Is Theorem 5.8 true for arbitrary abelian p-groups? Again we suspect so, but the necessary calculations are difficult.

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University of Oregon
Eugene, OR 97403-1222

University of Maryland
College Park, MD 20742-4015