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VARIATIONALLY COMPLETE ACTIONS ON COMPACT SYMMETRIC SPACES

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Abstract

We prove that an isometric action of a compact Lie group on a compact symmetric space is variationally complete if and only if it is hyperpolar.

1. Introduction

The main result of this paper is the following theorem:

Theorem. *An isometric action of a compact Lie group on a compact symmetric space is variationally complete if and only if it is hyperpolar.*

Variationally complete actions were introduced by Bott in [1] (see also [2]). Let G be a compact Lie group acting on a complete Riemannian manifold M by isometries. A geodesic γ in M is called *G-transversal* if it is orthogonal to the G-orbit through $\gamma(t)$ for every t. One can show that a geodesic γ is G-transversal if there is a point t_0 such that $\dot\gamma(t_0)$ is orthogonal to $G\gamma(t_0)$. A Jacobi field along a geodesic in M is called *G-transversal* if it is the variational vector field of a variation through G-transversal geodesics. The action of G on M is called *variationally complete* if every G-transversal Jacobi field J in M that is tangent to the G-orbits at two different parameter values is the restriction of a Killing field on M induced by the G-action. It is proved in [2] on p. 974 that instead of requiring tangency at two different points

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in the definition of variational completeness it is equivalent to require tangency at one point and vanishing at another point.

Conlon considered in [3] actions of a Lie group G on a complete Riemannian manifold M with the property that there is a connected, complete submanifold Σ of M that meets all orbits of G in such a way that the intersections between Σ and the orbits of G are all orthogonal. Such a submanifold is called a *section* and an action admitting a section is called *polar*. Notice that neither do we assume as Conlon in [3] that Σ is closed nor do we assume that it is properly embedded as is usually required in the recent literature on the subject. It is easy to see that a section Σ is totally geodesic in M. An action admitting a section that is flat in the induced metric is called *hyperpolar*. Conlon proved in [3] that a hyperpolar action of a compact Lie group on a complete Riemannian manifold is variationally complete. Notice that he does not use in his proof that the sections of the action are closed. His result therefore implies one direction of the main theorem of this paper.

In the case of Euclidean spaces, hyperpolar representations were classified by Dadok in [4]. As a consequence of his classification he obtained that a hyperpolar representation of a compact Lie group is orbit equivalent to the isotropy representation of a symmetric space. (We recall that two isometric actions are said to be *orbit equivalent* if there is an isometry between the action spaces under which the orbits of the two actions correspond.)

On the other hand, we classified variationally complete representations in [6]. As a consequence of our classification we obtained that a variationally complete representation of a compact Lie group is hyperpolar. Di Scala and Olmos gave in [5] a very short, simple proof of this result. It follows that a representation is hyperpolar if and only if it is variationally complete.

Next we discuss the case of compact symmetric spaces. There are two important classes of examples of hyperpolar actions on them. It is clear that a cohomogeneity one action of a compact Lie group on a compact symmetric space is hyperpolar. Hermann constructed in [9] a class of examples of variationally complete actions on compact symmetric spaces which in fact turned out to be hyperpolar. Namely, if K_1 and K_2 are two symmetric subgroups of the same compact Lie group G, then the action of K_1 on G/K_2 is hyperpolar and so is the action of $K_1 \times K_2$ on G. Kollross [10] classified hyperpolar actions on compact *irreducible* symmetric spaces. It follows from his classification that, in the irreducible case, all hyperpolar actions belong to either one of these

classes.

Finally, we add that the concepts of hyperpolarity and variational completeness admit natural extensions in the context of proper Fredholm actions of Hilbert-Lie groups on Hilbert spaces (see [11] and [12]). We shall make use of them in the course of our proof.

2. The proof of the theorem

Conlon proved in [3] that hyperpolar actions are variationally complete as we pointed out in the introduction. In this section we will prove the converse of his result for actions on compact symmetric spaces. Our strategy is as follows. First we reduce to the case of a symmetric space of compact type. Next we lift the action on the symmetric space to a variationally complete action of a path group on a Hilbert space. Then we use an argument similar to that in [5] to show that the lifted action is hyperpolar. Finally, it is easy to see that a section for the action on the Hilbert space induces a flat section for the original action.

Let M be a compact Riemannian symmetric space. We identify M with the coset space G/K, where G is the connected component of the group of all isometries of M and K is the isotropy subgroup of a chosen base point. Let H be a compact Lie group acting by isometries on M. An action of a group is variationally complete (resp. hyperpolar) if and only if the same is true for the restriction of that action to the connected component of the group. This follows from the definition in the case of variationally complete actions and is easy in the case of hyperpolar actions, see Proposition 2.4 in [8]. We can thus assume that H is a connected, closed subgroup of G.

Let $\hat{M}$ be a compact Riemannian covering space of M which splits as a product of a torus T^k and a symmetric space N of compact type. Let $\hat{G}$ be the connected component of the group of all isometries of $\hat{M}$. Then $\hat{G}$ is a covering group of G and we let $\hat{H}$ denote the subgroup of $\hat{G}$ which is the connected component of the inverse image of H. Notice that the action of $\hat{H}$ on $\hat{M}$ is variationally complete if the one of H on M is so. In this case, the restriction σ_2 of the action of $\hat{H}$ to N is clearly still variationally complete. We will prove below that a variationally complete action on a symmetric space of compact type is hyperpolar. Since the restriction σ_1 of the action of $\hat{H}$ to T^k is orbit equivalent to the action of a torus on T^k and therefore hyperpolar, it follows that the product action $\sigma_1 \times \sigma_2$ on $T^k \times N = \hat{M}$ is hyperpolar. The action of

$\hat{H}$ on M is therefore also hyperpolar since it is orbit equivalent to the product action of $\sigma_1 \times \sigma_2$ on $\hat{M}$, which can be seen by arguments similar to those in the proof of Theorem 4 (ii) in [4] or of Proposition 3.4 (d) in [6]. Now we can simply project a section of the $\hat{H}$-action in $\hat{M}$ down to M to see that the action of H on M is hyperpolar.

It follows from the discussion in the previous paragraph that from now on we can restrict to the case where M is of compact type. In this case the group G is semisimple. There is an involution σ of G such that K is open in the fixed point set of σ. Now the Lie algebra of G decomposes into the ± 1-eigenspaces of $d\sigma$, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, where $\mathfrak{k}$ is the Lie algebra of K, and the Riemannian metric in M is the G-invariant metric induced by some Ad_G-invariant inner product on $\mathfrak{g}$. The group H is a closed subgroup of G so that it acts on M by left translations, and $H \times K$ acts on G by $(h,k) \cdot g = hgk^{-1}$, where $h \in H$, $k \in K$ and $g \in G$. Clearly, the projection $\pi : G \to G/K$ is an equivariant Riemannian submersion.

Lemma 1. *If the action of H on M is variationally complete, then the action of $H \times K$ on G is also variationally complete.*

Proof. Let γ be an $H \times K$-transversal geodesic in G defined on $[0,1]$ and let J be an $H \times K$-transversal Jacobi field along γ such that $J(0) = 0$ and $J(1)$ is tangent to the $H \times K$-orbit through $\gamma(1)$. We must show that J is the restriction along γ of an $H \times K$-Killing field.

We first identify the tangent and normal spaces to the $H \times K$-orbits in G. Let $\mathfrak{h}$ be the Lie algebra of H. Then $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ and we denote by $\mathfrak{h}^\perp$ its orthogonal complement in $\mathfrak{g}$ with respect to the Ad_G-invariant inner product on $\mathfrak{g}$. Note that $[\mathfrak{h}, \mathfrak{h}^\perp] \subset \mathfrak{h}^\perp$ and that $\mathfrak{k}$ and $\mathfrak{p}$ are also mutually orthogonal in $\mathfrak{g}$. Given $a \in G$, the tangent space to $H \cdot a \cdot K$ at the point a is $\mathfrak{h} \cdot a + a \cdot \mathfrak{k}$. Therefore the left translates under a^{-1} of the tangent and normal spaces of $H \cdot a \cdot K$ at a are respectively $\mathfrak{l}^a := \mathfrak{k} + \mathrm{Ad}_{a^{-1}}\mathfrak{h}$ and $\mathfrak{m}^a := \mathfrak{p} \cap \mathrm{Ad}_{a^{-1}}\mathfrak{h}^\perp$. It will be convenient to set $\mathfrak{q}^a := \mathfrak{k} \cap \mathrm{Ad}_{a^{-1}}\mathfrak{h}$.

Let $\gamma(0) = a \in G$. Then $\gamma(t) = ae^{tX}$ for $X \in \mathfrak{m}^a$. Since $J(0) = 0$, setting $\gamma_\alpha(t) = ae^{t(X+\alpha Y)}$, where $Y \in \mathfrak{m}^a$ and $Y = a^{-1} \cdot J'(0)$, defines a variation of $\gamma = \gamma_0$ through $H \times K$-transversal geodesics which induces J. Since $\pi : G \to G/K$ is an equivariant Riemannian submersion and each γ_α is a horizontal curve with respect to π, we have that $\{\pi\gamma_\alpha\}$ is a variation of $\bar{\gamma} = \pi\gamma$ through H-transversal geodesics in M. Moreover, the associated H-transversal Jacobi field $\bar{J}$ along $\bar{\gamma}$ satisfies $\bar{J}(0) = 0$, $\bar{J}'(0) = a \cdot \pi_* Y$, and $\bar{J}(1)$ is tangent to the H-orbit through $\pi(ae^X)$. It

follows by variational completeness of the action of H on M that $\bar{J}$ is the restriction along $\bar{\gamma}$ of an H-Killing field on M, namely $\bar{J}(t) = Z \cdot \bar{\gamma}(t)$ for some $Z \in \mathfrak{h}$, which yields

$$\begin{aligned}\bar{J}(t) &= \frac{\partial}{\partial \alpha}\Big|_{\alpha=0} e^{\alpha Z}\pi(ae^{tX}) \\ &= \frac{\partial}{\partial \alpha}\Big|_{\alpha=0} ae^{tX}\pi(e^{-tX}a^{-1}e^{\alpha Z}ae^{tX}) \\ &= ae^{tX}\cdot \pi_* \mathrm{Ad}_{(ae^{tX})^{-1}} Z.\end{aligned}$$

This equation gives $\bar{J}(0) = a \cdot \pi_* \mathrm{Ad}_{a^{-1}} Z$. Since we already know that $\bar{J}(0) = 0$, we get that $\pi_* \mathrm{Ad}_{a^{-1}} Z = 0$, and then $\mathrm{Ad}_{a^{-1}} Z \in \mathfrak{k}$. Hence $\mathrm{Ad}_{a^{-1}} Z \in \mathfrak{q}^a$.

Since $X \in \mathfrak{p}$, the left translations by e^{tX} define isometries of the symmetric space M which form a one-parameter group of transvections, that is, each one of them induces parallel transport along the geodesic $s \mapsto \pi(e^{sX})$. Denote by ∇ the Levi-Civita connection of M. Then we have

$$\begin{aligned}a^{-1}\cdot \bar{J}'(0) &= a^{-1}\cdot \nabla_{a\cdot\pi_* X}\bar{J} \\ &= \nabla_{\pi_* X}(a^{-1}\cdot \bar{J}) \\ &= \frac{d}{dt}\Big|_{t=0} \pi_* \mathrm{Ad}_{(ae^{tX})^{-1}} Z \\ &= \pi_*[\mathrm{Ad}_{a^{-1}} Z, X].\end{aligned}$$

We deduce that $\bar{J}'(0) = a\cdot\pi_*[\mathrm{Ad}_{a^{-1}} Z, X]$. But we already know that $\bar{J}'(0) = a\cdot\pi_* Y$. Since $Y \in \mathfrak{m}^a \subset \mathfrak{p}$ and $[\mathrm{Ad}_{a^{-1}} Z, X] \in [\mathfrak{q}^a, \mathfrak{m}^a] \subset \mathfrak{m}^a \subset \mathfrak{p}$, this implies $Y = [\mathrm{Ad}_{a^{-1}} Z, X]$.

Finally, consider the $H \times K$-Killing field in G given by

$$U \cdot y = \frac{\partial}{\partial \alpha}\Big|_{\alpha=0} e^{\alpha Z} y e^{-\alpha \mathrm{Ad}_{a^{-1}} Z}.$$

Then

$$U\cdot\gamma(t) = \frac{\partial}{\partial \alpha}\Big|_{\alpha=0} e^{\alpha Z} a e^{tX} a^{-1} e^{-\alpha Z} a.$$

Therefore $U\cdot\gamma(0) = 0$ and

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0} U\cdot\gamma(t) &= \frac{\partial}{\partial \alpha}\Big|_{\alpha=0}\frac{\partial}{\partial t}\Big|_{t=0} e^{\alpha Z} a e^{tX} a^{-1} e^{-\alpha Z} a \\ &= \left(\frac{\partial}{\partial \alpha}\Big|_{\alpha=0} \mathrm{Ad}_{e^{\alpha Z} a} X\right)\cdot a \\ &= [Z, \mathrm{Ad}_a X]\cdot a.\end{aligned}$$

Hence $a^{-1}\cdot \frac{d}{dt}\big|_{t=0} U\cdot\gamma(t) = \mathrm{Ad}_{a^{-1}}[Z,\mathrm{Ad}_a X] = Y$. It follows that $J(t) = U\cdot\gamma(t)$. q.e.d.

Let $V = L^2([0,1];\mathfrak{g})$ denote the Hilbert space of L^2-integrable paths $u:[0,1]\to\mathfrak{g}$, and let $\hat{G} = H^1([0,1];G)$ denote the Hilbert-Lie group of H^1-paths in G parametrized on $[0,1]$ (the elements of $\hat{G}$ are the absolutely continuous paths $g:[0,1]\to G$ whose derivative is square integrable). We have that $\hat{G}$ acts by affine isometries on V: $g*u = gug^{-1} - g'g^{-1}$, where $g\in\hat{G}$ and $u\in V$. Let $\mathcal{P}(G, H\times K)$ denote the subgroup of all paths $g\in\hat{G}$ such that $(g(0),g(1))\in H\times K$. Let $\varphi: V\to G$ be the parallel transport map defined by $\varphi(u) = h(1)$, where $h\in\hat{G}$ is the unique solution of $h^{-1}h' = u$, $h(0) = 1$. Then it is known that (see [11, 12, 13]):

(a) the action of $\hat{G}$ on V is proper and Fredholm;

(b) $\varphi(g*u) = g(0)\varphi(u)g(1)^{-1}$, where $g\in\hat{G}$ and $u\in V$;

(c) $\mathcal{P}(G,H\times K)*u = \varphi^{-1}((H\times K)\cdot\varphi(u))$;

(d) the action of $\mathcal{P}(G,1\times G)$ on V is simply transitive;

(e) $\varphi: V\to G$ is a Riemannian submersion and a principal $\Omega_1(G) := \mathcal{P}(G,1\times 1)$-bundle;

(f) the horizontal distribution $\mathcal{H}$ of φ is given by $\mathcal{H}(u) = \{\mathrm{Ad}_g\hat{Y} : Y\in\mathfrak{g}\}$, where $\hat{Y}$ denotes the constant path with value Y and g is the unique element of $\mathcal{P}(G,1\times G)$ which satisfies $u = g*\hat{0}$;

(g) the tangent space to the orbit through $\hat{Y}$ at $\hat{Y}$ is

$$\{[\xi,\hat{Y}] - \xi' :\ \xi\in H^1([0,1],\mathfrak{g}),\ \xi(0)\in\mathfrak{h},\ \xi(1)\in\mathfrak{k}\}.$$

Lemma 2. *If the action of $H\times K$ on G is variationally complete, then the action of $\mathcal{P}(G,H\times K)$ on V is also variationally complete.*

Proof. Let γ be a transversal geodesic in V defined on $[0,1]$ and let J be a transversal Jacobi field along γ such that $J(0) = 0$ and $J(1)$ is tangent to the orbit through $\gamma(1)$. We must show that J is the restriction along γ of a Killing field induced by the $\mathcal{P}(G,H\times K)$-action.

Let $\gamma(0) = u\in V$. Since $\mathcal{P}(G,H\times K)\supset\Omega_1(G)$, we have that the normal space to the orbit through u is contained in the horizontal subspace $\mathcal{H}(u)$. Let g be the unique element of $\mathcal{P}(G,1\times G)$ which satisfies $u = g*\hat{0}$. Now $\gamma(t) = u + t\mathrm{Ad}_g\hat{X}$ for some $X\in\mathfrak{g}$. Since

$J(0) = 0$, setting $\gamma_\alpha(t) = u + t\mathrm{Ad}_g(\hat{X} + \alpha\hat{Y}) = g * t(\widehat{X + \alpha Y})$, where $J'(0) = \mathrm{Ad}_g\hat{Y}$ for some $Y \in \mathfrak{g}$, defines a variation of $\gamma = \gamma_0$ through transversal geodesics which induces J. Since $\varphi : V \to G$ is an equivariant Riemannian submersion with respect to the homomorphism $h \in \mathcal{P}(G, H \times K) \mapsto (h(0), h(1)) \in H \times K$, and each γ_α is a horizontal curve with respect to φ, we have that $\{\varphi\gamma_\alpha\}$ is a variation of $\bar{\gamma} = \varphi\gamma$ through transversal geodesics in G. Moreover, the associated transversal Jacobi field $\bar{J}$ along $\bar{\gamma}$ satisfies $\bar{J}(0) = 0$, $\bar{J}'(0) = d\varphi_u(\mathrm{Ad}_g\hat{Y}) = Y \cdot a$, where $a = g(1)^{-1} = \varphi(u)$, and $\bar{J}(1)$ is tangent to the orbit through $\bar{\gamma}(1)$. It follows by variational completeness of the action of $H \times K$ on G that $\bar{J}$ is the restriction along $\bar{\gamma}$ of an $H \times K$-Killing field on G, namely $\bar{J}(t) = Z \cdot \bar{\gamma}(t) - \bar{\gamma}(t) \cdot W$ for some $Z \in \mathfrak{h}$, $W \in \mathfrak{k}$.

Note that $\bar{\gamma}(t) = e^{tX}a$. Therefore we can write $\bar{J}(t) = e^{tX} \cdot (\mathrm{Ad}_{e^{-tX}}Z - \mathrm{Ad}_a W) \cdot a$. It follows that $0 = \bar{J}(0) = (Z - \mathrm{Ad}_a W) \cdot a$, so that $W = \mathrm{Ad}_{a^{-1}}Z \in \mathfrak{k} \cap \mathrm{Ad}_{a^{-1}}\mathfrak{h} = \mathfrak{q}^a$ and $\bar{J}(t) = e^{tX} \cdot (\mathrm{Ad}_{e^{-tX}}Z - Z) \cdot a$. Then we have that $Y \cdot a = \bar{J}'(0) = [Z, X] \cdot a$, which gives $Y = [Z, X]$.

Finally, consider the one-parameter subgroup $\{g_\alpha\}$ of $\mathcal{P}(G, H \times K)$ given by $g_\alpha = ge^{\alpha Z}g^{-1}$. Note that $g_\alpha(0) = e^{\alpha Z} \in H$ and $g_\alpha(1) = a^{-1}e^{\alpha Z}a \in K$, so that g_α is well defined. We next show that the Killing field induced by $\{g_\alpha\}$ coincides with J along γ. It suffices to compare their initial values. Since $g_\alpha * \gamma = (ge^{\alpha Z}) * t\hat{X}$, we have

$$ge^{\alpha Z} * \hat{0} = -(ge^{\alpha Z})'(ge^{\alpha Z})^{-1} = -g'g^{-1} = g * \hat{0} = u,$$

and

$$\begin{aligned}
\frac{\partial}{\partial t}\Big|_{t=0}\frac{\partial}{\partial \alpha}\Big|_{\alpha=0} ge^{\alpha Z} * t\hat{X} &= \frac{\partial}{\partial \alpha}\Big|_{\alpha=0}\frac{\partial}{\partial t}\Big|_{t=0} t\mathrm{Ad}_{ge^{\alpha Z}}\hat{X} + u \\
&= \frac{\partial}{\partial \alpha}\Big|_{\alpha=0}\mathrm{Ad}_{ge^{\alpha Z}}\hat{X} \\
&= \mathrm{Ad}_g\widehat{[Z, X]} \\
&= \mathrm{Ad}_g\hat{Y}.
\end{aligned}$$

This completes the proof. q.e.d.

Lemma 3. *If the action of $\mathcal{P}(G, H \times K)$ on V is variationally complete, then it is hyperpolar.*

Proof. Let N_0 in V be an orbit which is the preimage of a principal orbit of $H \times K$ in G and γ a geodesic starting orthogonally to N_0 in p, i.e., $\xi = \gamma'(0)$ is in the normal space $\nu_p(N_0)$. Let N_1 be the orbit

through $q = \gamma(1)$ and assume that also N_1 is the preimage of a principal orbit in G. It follows from variational completeness that $\gamma(1)$ is not a focal point of N_0 along γ. We will prove that the tangent spaces of N_0 at $\gamma(0)$ and of N_1 at $\gamma(1)$ coincide if considered to be affine subspaces of V. It follows that the normal spaces of N_0 at $\gamma(0)$ and of N_1 at $\gamma(1)$ coincide which implies that the action is hyperpolar.

Let $E_p \subset T_pN_0$ be the direct sum of the eigenspaces of the Weingarten operator $A^{N_0}_\xi$ corresponding to the nonvanishing eigenvalues and define a corresponding subspace $E_q \subset T_qN_1$ with respect to the Weingarten map $A^{N_1}_\xi$. Similarly let Z_p be the zero eigenspace in T_pN_0 and Z_q the zero eigenspace in T_qN_1. We will first prove that $E_p = E_q$ using an argument from [5]. Then we will prove that $Z_p = Z_q$. It will follow that $T_pN_0 = T_qN_1$ finishing the proof.

Let $X \in E_p$ be an eigenvector corresponding to the nonvanishing eigenvalue λ of $A^{N_0}_\xi$. Then $J(t) = (1 - \lambda t)X$ is a transversal Jacobi field along γ that is tangent to the orbit N_0 and vanishes in $t = \frac{1}{\lambda}$. Variational completeness now implies that this Jacobi field is induced by the action. Therefore $J(t)$ is tangent to the orbit through $\gamma(t)$ for every t. Since $\gamma(1)$ is not a focal point of N_0 along γ, we have that $1 - \lambda \neq 0$ and then X lies in T_qN_1. It is now clear that X is an eigenvector of $A^{N_1}_\xi$ corresponding to a nonzero eigenvalue. This proves that $E_p \subset E_q$. Analogous arguments imply $E_q \subset E_p$ proving $E_p = E_q$.

It is left to prove that $Z_p = Z_q$. Let $X \in Z_p$. Let $\{\psi_\alpha\}$ be a one-parameter subgroup of $\mathcal{P}(G, H \times K)$ such that

$$X = \frac{d}{d\alpha}\Big|_{\alpha=0} \psi_\alpha p.$$

The variation $\{\psi_\alpha \gamma\}$ induces a Jacobi field J along γ with $J(0) = X$. We first show that $J(t) = X + t\eta$ where $\eta \in \nu_p(N_0)$. In fact

$$J'(0) = \frac{d}{d\alpha}\Big|_{\alpha=0} \psi_{\alpha *}\xi = -A_\xi X + \nabla^\perp_{\frac{d}{d\alpha}} \psi_{\alpha *}\xi \,|_{\alpha=0} = \nabla^\perp_{\frac{d}{d\alpha}} \psi_{\alpha *}\xi \,|_{\alpha=0}$$

which shows that $J'(0)$ lies in $\nu_p(N_0)$ as we wanted to prove.

Our next goal is to show that $\eta = 0$. Assume that $\eta \neq 0$. We know that $X + \eta \in T_qN_1$. Then $\eta \notin \nu_q(N_1)$ since $\langle X + \eta, \eta \rangle = ||\eta||^2 \neq 0$. This implies that $X + \eta \notin Z_q$ since otherwise $J'(1) = \eta$ would be a normal vector. Let $Y \in E_q$ be the image of $X + \eta$ under the orthogonal projection along Z_q into E_q. We have that $Y \neq 0$ or, which is the same, $\langle Y, X + \eta \rangle \neq 0$. We know that $\langle Y, X \rangle = 0$ since $E_p = E_q$. We also have

that $\langle Y, \eta \rangle = 0$ since Y is a tangent vector of N_0 in p and η is a normal vector. Hence we also get $\langle Y, X + \eta \rangle = 0$ which is a contradiction. This proves that $\eta = 0$. We therefore have that $J(t) = X$ which implies that $X \in T_q N_1$ and it is clear that $X \in Z_q$. It follows that $Z_p \subset Z_q$. The proof that $Z_q \subset Z_p$ is analogous. Hence $Z_p = Z_q$. q.e.d.

With the following lemma we finish the proof of the theorem in the introduction.

Lemma 4. *If the action of $\mathcal{P}(G, H \times K)$ on V is hyperpolar, then the action of H on M is also hyperpolar.*

Proof. Let Σ be a section of the action of $\mathcal{P}(G, H \times K)$ on V and let A be the image of Σ under φ in G. We will show that A is a flat section of the action of $H \times K$ on G. Since Σ is horizontal with respect to the Riemannian submersion φ, we have that $\varphi|_\Sigma : \Sigma \to G$ is an isometric immersion. Property (c) before Lemma 2 implies that $d\varphi_u(T_u\Sigma)$ is perpendicular to the $H \times K$-orbit through $\varphi(u)$ in G for all $u \in V$. It follows that A is a submanifold in G which meets all orbits perpendicularly. Moreover A is flat since $\varphi|_\Sigma$ is an isometric immersion. This finishes the proof that the action of $H \times K$ on G is hyperpolar. Similar arguments show that $\pi(A)$ is a flat section of the action of H on M. q.e.d.

It is interesting to remark that it follows from the results in Section 2 of [7] that, in the case where the metric in G is induced from the Killing form of $\mathfrak{g}$, we have that A is a torus, and hence A and $\pi(A)$ are *properly embedded* sections. In the case where the metric in G is not induced from the Killing form of $\mathfrak{g}$, it is easy to construct examples of hyperpolar actions with sections that are not properly embedded. One such example is the diagonal action of $\mathbf{SO}(3)$ on the product of spheres $S^2(1) \times S^2(R)$ where R^2 is irrational. (This follows because $\gamma(t) = \big((\cos t, \sin t, 0), \big(R \sin \frac{t}{R^2}, R \cos \frac{t}{R^2}, 0\big)\big)$ is a geodesic normal to the orbits.) Here is another example. Let the symmetric space M be a torus, and let T^k be a proper subtorus of M that we think of as a Lie group. The orbits of T^k are the cosets of T^k, and the image Σ under the exponential map of the normal space of T^k in M at the identity element is clearly a section. If M is not a rational torus, the subtorus T^k can be chosen so that Σ is not properly embedded, see Section 2 in [7].

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