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HOMOGENEITY OF EQUIFOCAL SUBMANIFOLDS

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Abstract

Equifocal submanifolds are an extension of the notion of isoparametric submanifolds in Euclidean spaces to symmetric spaces and consequently they share many of the properties well-known for their isoparametric relatives. An important step in understanding isoparametric submanifolds was Thorbergsson's proof of their homogeneity in codimension at least two which in particular solved the classification problem in this case. In this paper we prove the analogous result for equifocal submanifolds using the generalization of Thorbergsson's result to infinite dimensions due to Heintze and Liu.

1. Introduction

A closed submanifold M in a Riemannian symmetric space S is called *equifocal* (cf. [15]) if its normal bundle νM is globally flat and abelian and the focal data (i.e., the focal directions and distances) are invariant under parallel translation in the normal bundle.

In euclidean space $S = \mathbb{R}^n$, equifocal submanifolds are the same as isoparametric ones. Together with their focal manifolds they form a large class of interesting spaces which among others contains all flag manifolds and most symmetric spaces. The study of these submanifolds has a long history and goes back to Levi-Civita and E. Cartan, cf. [17] for a detailed historical account. Those of codimension ≥ 3 (if irreducible and substantial) are homogeneous submanifolds as has been shown by Thorbergsson [16]. But homogeneous isoparametric submanifolds are well-known: they are the principal orbits of certain orthogonal representations called *polar* which are closely related to symmetric

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spaces. For codimension 2, however, Thorbergsson's theorem fails and the classification is still open (while for codimension 1 there are only the spheres).

In [15], Terng and Thorbergsson also posed the problem of extending the result on homogeneity of isoparametric submanifolds in a suitable way to equifocal ones in simply connected, compact symmetric spaces. One result in this direction was obtained by M. Brück (cf. [1]) who could prove that an equifocal submanifold is homogeneous if one of its focal submanifolds consists of a single point. In the present paper, we provide an answer to the question of Terng and Thorbergsson:

Theorem. *A complete connected irreducible equifocal submanifold of codimension ≥ 2 in a simply connected compact symmetric space is homogeneous.*

As in the euclidean case, the homogeneous equifocal submanifolds of a symmetric space $S = G/K$ are well-known: They are orbits of so called *hyperpolar* isometric group actions on S which have recently been classified by Kollross [8]; essentially, the irreducible ones with codimension ≥ 2 are the principal orbits of subgroups $H \subset G$ such that G/H is a symmetric space as well.

The proof is based on a method due to Terng and Thorbergsson (cf. [15]): There is a natural Riemannian submersion $\pi : V \to S = G/K$, where V is the Hilbert space of L^2-curves in the Lie algebra $\mathfrak{g}$ of G. If $\overline{M} \subset S$ is equifocal, its preimage or lift $M = \pi^{-1}(\overline{M}) \subset V$ is an isoparametric submanifold of V with equal codimension. The theory of isoparametric submanifolds in euclidean space has been extended to Hilbert space; in particular is has been shown by Heintze and Liu [5] that the irreducible ones of codimension ≥ 2 are homogeneous. The main contribution of the present paper is to show that homogeneity of $M \subset V$ implies homogeneity of $\overline{M} \subset G/K$.

To obtain this result we have to show that sufficiently many isometries of M preserve the fibration associated to the submersion π. In fact, through a study of the Lie algebra $\mathcal{K}^M$ of Killing fields tangent to a lift M we will see that under slight restrictions on M all one parameter groups of isometries have this property. This turns out to be true for arbitrary lifts M, not only for those of equifocal submanifolds.

On V there is an isometric action of the group $H^1([0,1], G)$ of H^1-differentiable paths in G which preserves the fibration. First, we enlarge the class of the Killing fields induced by this action by algebraically

similar ones and in particular the parallel ones. We will see that within this bigger class of Killing fields only those induced by the $H^1([0,1],G)$-action can be tangent to a lift and in particular that there are no parallel Killing fields in $\mathcal{K}^M$. In a second step we construct, given a Killing field tangent to a lift, a sequence converging within $\mathcal{K}^M$ in a weak sense to a parallel field. A closer study of this convergence shows that this limit will be nonzero (which would contradict part one) unless the initial field is induced by the action of $H^1([0,1],G)$.

The following section contains some preliminary material, in particular about the submersion π, as well as remarks on notation and subtleties due to the infinite dimensional context. Moreover, we solve a first part of the problem by showing that homogeneity of a submanifold of $S = G/K$ is equivalent to the homogeneity of its lift under the natural submersion $G \to G/K$. In the subsequent sections we procede with the investigation of the Killing fields tangent to a lift as sketched above.

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2. Preliminaries and notation

Throughout this paper, let G be a Lie group, $\mathfrak{g}$ denote its Lie algebra and $S = G/K$ be a simply connected compact symmetric space. We can assume that G is a simply connected compact semisimple Lie group equipped with the biinvariant metric induced by the Killing form. Terng and Thorbergsson (cf. [15]) introduced as an essential tool in the study of submanifolds in symmetric spaces the following Riemannian submersion $V = L^2([0,1],\mathfrak{g}) \to G$. For proofs of all the facts stated below consult [15] or [7].

Let $H^1([0,1],G)$ denote the group of weakly differentiable paths in G whose weak derivative g' is square integrable (with the group structure given by pointwise multiplication of paths) and ΩG the subgroup of loops based at the identity of G. The group $H^1([0,1],G)$ acts isometrically on the Hilbert space $L^2([0,1],\mathfrak{g})$ via gauge transformations

$$g * u = gug^{-1} - g'g^{-1}.$$

We will frequently denote the isometry induced by a path g simply by g as well. Restricted to $P_eG = \{g \in H^1([0,1],G) \mid g(0) = e\}$, this action

induces a map

$$P_eG \longrightarrow L^2([0,1],\mathfrak{g}), \quad g \mapsto g^{-1} * 0 = g^{-1}g',$$

which turns out to be an isometry. Composed with the endpoint map $P_eG \to G, g \mapsto g(1)$ its inverse yields the desired Riemannian submersion

$$\begin{aligned} \pi : L^2([0,1],\mathfrak{g}) &\longrightarrow P_e(G) \longrightarrow G \\ u = g^{-1} * 0 &\longmapsto g \longmapsto g(1). \end{aligned}$$

From this description we see that the fibers of π are precisely the orbits of ΩG under the above action. Denoting the associated horizontal and vertical distributions on $L^2([0,1],\mathfrak{g})$ by $\mathcal{H}$ and $\mathcal{V}$, we see that

$$\begin{aligned} \mathcal{V}_0 &= \left\{ u \in L^2([0,1],\mathfrak{g}) \,\middle|\, \int_0^1 u\,dt = 0 \right\} \text{ and thus} \\ \mathcal{H}_0 &= \{u \in L^2([0,1],\mathfrak{g}) \,|\, u \equiv \text{ const}\}. \end{aligned}$$

We will therefore frequently not distinguish between $\mathcal{H}_0$ and $\mathfrak{g}$. The projection $L^2([0,1],\mathfrak{g}) \to \mathcal{H}_0$ is given by $u^{\mathcal{H}_0} = \int_0^1 u\,dt$. Also, a simple calculation shows that the derivative of the isometry $P_eG \to L^2([0,1],\mathfrak{g})$ yields an identification of $T_e\Omega G = \{u \in H^1([0,1],\mathfrak{g}) \,|\, u(0) = u(1) = 0\}$ and $\mathcal{V}_0$ given by $u \mapsto u'$.

The isometries induced by $H^1([0,1],G)$ preserve the fibration associated to π; more precisely, we have the relation

$$\pi(g * u) = g(0)\ \pi(u)\ g^{-1}(1),$$

for $g \in H^1([0,1],G)$. Conversely, this also shows that isometries of G can be lifted to $L^2([0,1],\mathfrak{g})$: if $\varphi : G \to G$ is given by $\varphi(x) = g_0\, x\, g_1^{-1}$ and g any curve joining g_0 and g_1 then the isometry of $L^2([0,1],\mathfrak{g})$ induced by g covers φ:

$$\begin{array}{ccc} L^2([0,1],\mathfrak{g}) & \xrightarrow{g*.} & L^2([0,1],\mathfrak{g}) \\ \pi\downarrow & & \downarrow\pi \\ G & \xrightarrow{\varphi} & G. \end{array}$$

As in finite dimensions there is the usual relation between skew adjoint operators and isometries on a Hilbert space given by the exponential map, which also in this context is a local diffeomorphism near

the origin (cf. [9]). Thus it is possible to obtain information about one parameter groups of isometries via the investigation of Killing fields. However, it seems to be unknown whether the group of extrinsic isometries of a submanifold in Hilbert space (i. e. the group of isometries on the Hilbert space preserving the submanifold) is a Lie group as well. This problem can be circumvented in our situation since in their proof of homogeneity of isoparametric submanifolds in Hilbert space Heintze and Liu construct one parameter groups of isometries.

The Killing fields induced by the action of $H^1([0,1],G)$ on $L^2([0,1],\mathfrak{g})$ are given by

$$K_v x = [v,x] - v',$$

where $v \in H^1([0,1],\mathfrak{g})$ and $'$ denotes weak derivative: If g_s is a one parameter family in $H^1([0,1],G)$ with $g_0 \equiv e$ then the derivative of $g_s * x = \mathrm{Ad}_{g_s} x - g_s' g_s^{-1}$ with respect to s evaluated at $s = 0$ yields precisely K_v for $v(t) = \frac{\partial}{\partial s}\Big|_{s=0} g_s(t)$.

We will use the following notation: The natural projection for the symmetric space S will be denoted by $\overline{\pi} : G \to G/K = S$. The symbol $\overline{M}$ will always refer to a submanifold of S; a star as in M^* will indicate the corresponding lift to G via $\overline{\pi}$ and the lift to the Hilbert space $L^2([0,1],\mathfrak{g})$ will be called M. Thus we have the following picture:

$$\begin{array}{ccccc} V & \xrightarrow{\pi} & G & \xrightarrow{\overline{\pi}} & G/K \\ \cup & & \cup & & \cup \\ M & & M^* & & \overline{M} \end{array}$$

where $M^* = \overline{\pi}^{-1}(\overline{M})$, $M = \pi^{-1}(M^*)$ and $\pi, \overline{\pi}$ are the natural submersions.

It is proven in [15] that a submanifold $\overline{M}$ is equifocal in G/K if and only if its lift M^* to G is equifocal which in turn is equivalent to the fact that the corresponding lift M to $L^2([0,1],\mathfrak{g})$ is isoparametric.

We may moreover assume that all manifolds under consideration are irreducible, i. e. do not split as a product: for isoparametric as well as equifocal submanifolds such a splitting can be recognized from the associated Weyl group (cf. [12], [3]) and these groups coincide for $\overline{M}$, M^* and M (cf. [15]). In particular, we can assume that $\overline{M}$ does not contain a trivial factor, i. e. it does not split as a product containing a whole factor of S.

We now consider the situation of the submersion $G \to G/K$ starting with a simple algebraic fact.

Lemma 1. *Let G be a group and $H \subset G \times G$ a subgroup. Let $H_i \subset G$, $i = 1, 2$ be the projection of H onto the first resp. second factor of $G \times G$ and $\overline{H}_1 = \{g \in G \,|\, (g, e) \in H\}$, $\overline{H}_2 = \{g \in G \,|\, (e, g) \in H\}$. Then $\overline{H}_i$ is a normal subgroup of H_i.*

Theorem 2. *Let G/K be a simply connected symmetric space of compact type, (G, K) a symmetric pair and $\pi : G \to G/K$ the canonical projection. If $\overline{M}$ is a submanifold of G/K s.th. $M^* = \overline{\pi}^{-1}(\overline{M})$ is extrinsically homogenous in G then so is $\overline{M}$. The converse holds trivially.*

Proof. Let $H \subset G \times G$ be the connected component of $\mathrm{I}(M^*)$, the group of extrinsic isometries of M^*. Obviously we have $e \times K \subset H$ and thus $K \subset \overline{H}_2 \subset H_2 \subset G$.

Since K is a maximal subgroup of G it follows that $\overline{H}_2 = G$ or $\overline{H}_2 = K$ (if necessary consider only the connected components). In the first case we have $e \times G \subset H$ and H is transitive on G, so $M^* = G$ and $\overline{M} = G/K$. Otherwise we see that since $K = \overline{H}_2$ is not a normal subgroup of G the above lemma excludes $H_2 = G$ and so we obtain $H_2 = K$. Thus $H \subset G \times K$ and so the isometries of M^* preserve the fibration $G \to G/K$. q.e.d.

3. Horizontal projection of Killing fields

As mentioned above, we want to show that all Killing fields tangent to the lift M of an irreducible submanifold $M^* \subset G$ are induced by the $H^1([0, 1], G)$-action. In this section, we will prove this for the class of Killing fields of the form

$$(*) \qquad Kx = [v, x] - b,$$

for some $v, b \in L^2([0, 1], \mathfrak{g})$. Such a vector field is induced by the action of $H^1([0, 1], G)$ iff $v \in H^1([0, 1], \mathfrak{g})$ and $b = v'$ and our aim is to show that only in this case it can be tangent to a lift. Note that M^* is not assumed to be equifocal.

Theorem 3. *Let $M = \pi^{-1}(M^*)$ be the lift of a submanifold $M^* \subset G$. If M^* does not contain a trivial factor of G then every Killing field of the form $(*)$ tangent to M is induced by the action of $H^1([0, 1], G)$.*

Informally, the idea is to see towards which adjacent fibers the points of the standard fiber $\pi^{-1}(e)$ will be moved by a Killing field K, i.e., to consider the horizontal projection of K along this fiber. More explicitely, this horizontal projection can be described by the map

$$\Pi_K : \Omega G \longrightarrow \mathfrak{g} \cong \mathcal{H}_0, \quad \Pi_K(g) = (g_*K)_0{}^{\mathcal{H}_0} = \int_0^1 \mathrm{Ad}_g\, K(g^{-1} * 0)\, dt.$$

Here, the term g_*K denotes push forward of the vector field K by the isometry induced by g. Thus, we push forward K from all points of the standard fiber to the origin and take its horizontal projection to $\mathcal{H}_0$.

Clearly, since the isometries induced by ΩG preserve lifts, if K is tangent to $M = \pi^{-1}(M^*)$ then so is g_*K and thus $\operatorname{Im}\Pi_K$ is contained in the horizontal part of the tangent space T_0M which we identify with T_eM^*

$$\operatorname{Im}\Pi_K \subset T_eM^* \cong T_0M \cap \mathcal{H}_0.$$

We state some more facts about the map Π:

(i) The assignment $K \mapsto \Pi_K$ is linear.

(ii) The horizontal projection is constant for Killing fields induced by the action of $H^1([0,1], G)$, since the asscociated one-parameter group of isometries preserves the fibration.

(iii) $\Pi_K(hg) = \Pi_{g_*K}(h)$, in particular $(d\Pi_K)_g\ (dR_g)_e = (d\Pi_{g_*K})_e$, where R_g denotes right-multiplication with g.

(iv) If $Kx = [v, x] - b$ as above, then $(d\Pi_K)_e\, u = \int_0^1 [v - \widetilde{b}, u']\, dt$ for all $u \in T_e\Omega G$.

Here and below, the symbol $\widetilde{\ }$ or $\widetilde{\ \ }$ denotes integration, e.g., $\widetilde{b}(t) = \int_0^t b(\tau)d\tau$. To check (iv), we can assume $b = 0$ by subtracting $\overline{K}x = \widetilde{[b, x]} - b$ which, by (i) and (ii) above, changes Π_K only by a constant. But then, if g_s is a curve in ΩG with $g_0 \equiv e$

$$\begin{aligned}
\frac{\partial}{\partial s}\Big|_{s=0} \Pi_K(g_s) &= \int_0^1 \frac{\partial}{\partial s}\Big|_{s=0} \mathrm{Ad}_{g_s}[v, g_s^{-1} * 0]\, dt \\
&= \int_0^1 \left[v, \frac{\partial}{\partial s}\Big|_{s=0} (g_s^{-1} * 0)\right]\, dt \\
&= \int_0^1 \left[v, \frac{\partial g_s'}{\partial s}\Big|_{s=0}\right]\, dt.
\end{aligned}$$

For the moment, we assume G to be simple and $Kx = [v, x] - b$. In this situation, we will find a surprising dichotomy: either the projection Π_K is constant (which is the case if and only if K is induced by the action of $H^1([0,1], G)$) or its image is not contained in any proper affine subspace of $\mathfrak{g}$.

Proposition 4. *If G is simple and K as above, then $\operatorname{Im}\Pi_K$ is not contained in any proper affine subspace of $\mathfrak{g}$ unless $v \in H^1([0,1], \mathfrak{g})$ and $b = v'$.*

As an immediate corollary we obtain the above theorem for simple G: If $\dim M^* < \dim G$ then in view of the above dichotomy the relation $\operatorname{span}(\operatorname{Im}\Pi_K) \subset T_e M^* \neq \mathfrak{g}$ implies $\Pi_K \equiv \text{const}$ and so K must be induced by the $H^1([0,1], G)$-action.

Proof. As in the calculation above, subtracting an $H^1([0,1], G)$-Killing field, we may restrict to the case $Kx = [v, x]$. Then

$$\begin{aligned}(g_* K)x &= \operatorname{Ad}_g [v, g^{-1} * x] \\ &= [\operatorname{Ad}_g v, x] - [\operatorname{Ad}_g v, g * 0],\end{aligned}$$

since $\operatorname{Ad}_g(g^{-1} * x) = g * (g^{-1} * x) + g' g^{-1} = x - g * 0$ and we obtain

$$(d\Pi_{g_* K})_e \, u = \int_0^1 [w, u'] \, dt \quad \text{with} \quad w = w(g) = \operatorname{Ad}_g v - [\widetilde{\operatorname{Ad}_g v, g * 0}].$$

Now, if $\operatorname{Im}\Pi_K$ is contained in some affine subspace of $\mathfrak{g}$ then there is a nonzero $X \in \mathfrak{g}$ such that $\langle (d\Pi_K)_g \; u_g, X\rangle = 0$ for all $g \in \Omega G$ and $u_g \in T_g \Omega G$ or equivalently

$$\langle (d\Pi_{g_* K})_e \, u, X\rangle = 0 \quad \text{for all } g \in \Omega G \text{ and } u \in T_e \Omega G.$$

To put it another way, identifying $u \in T_e \Omega G$ with $\mathfrak{u} = u' \in \mathcal{V}_0$ we obtain

$$\begin{aligned}0 &= \left\langle \int_0^1 [w, \mathfrak{u}] \, dt, X \right\rangle \\ &= -\int_0^1 \langle \mathfrak{u}, [w, X]\rangle \, dt \\ &= -\langle \mathfrak{u}, [w, X]\rangle_{L^2} \quad \text{for all } g \in \Omega G \text{ and } \mathfrak{u} \in \mathcal{V}_0\end{aligned}$$

and thus

$$[\operatorname{Ad}_g v - [\widetilde{\operatorname{Ad}_g v, g * 0}], X] \in \mathcal{H}_0 = \mathfrak{g} \quad \text{for all } g \in \Omega G.$$

Choosing $Y \in \mathfrak{g}$ such that $Z := [X, Y] \neq 0$ and taking inner products with Y, we see that for all $g \in \Omega G$

$$\langle \mathrm{Ad}_g\, v - \widetilde{[\mathrm{Ad}_g\, v, g * 0]}, Z\rangle = \text{const} \quad \text{as a function of } t \in [0,1], \text{ or}$$
$$\langle v, \mathrm{Ad}_g^{-1}\, Z\rangle = \langle \widetilde{[\mathrm{Ad}_g\, v, g * 0]}, Z\rangle + \text{const}.$$

Since the nontrivial adjoint orbits of G are full, $\mathfrak{g}$ has a basis of the form $\mathrm{Ad}_{h_i} Z$ for suitable $h_i \in G$. Now consider loops g_i such that $g_i|_{(\varepsilon,1-\varepsilon)} = h_i$ for some $\varepsilon > 0$. Since $g_i * 0\,|_{(\varepsilon,1-\varepsilon)} = 0$ the first term on the right-hand side above is constant on $(\varepsilon, 1-\varepsilon)$ and thus so is v.

q.e.d.

Thus, for simple Lie groups a large class of Killing fields can be seen not to be tangent to any lift by considering their horizontal projection along a single fiber. In the semisimple case the situation is a bit more involved, since a trivial factor cannot be recognized by investigating only one tangent space. However, from the algebraic form of the Killing fields under consideration we see that their horizontal projection splits: if $G = G_1 \times \cdots \times G_k$ with G_i simple and $\mathfrak{g}_i$ the Lie algebra of G_i then

$$\mathrm{Im}\,\Pi_K = \mathrm{Im}\,\Pi_{K_1} \times \cdots \times \mathrm{Im}\,\Pi_{K_n} \subset \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_k,$$

where $K_i(x) = [v_i, x] - b_i$ and v_i, b_i denote the projection to $\mathfrak{g}_i$ of v, b and $\Pi_{K_i} : G_i \to \mathfrak{g}_i$ is the corresponding horizontal projection. So again in the situation where M is the lift of $M^* \subset G$ and $Kx = [v, x] - b$ tangent to M we see from the argument in the simple case that T_eM^* contains all the ideals $\mathfrak{g}_i$ such that $b_i \neq v_i'$. Let $\mathfrak{g}_K$ denote the complement of these, i. e. the maximal ideal in $\mathfrak{g}$ for which the projection $(v - \widetilde{b})^{\mathfrak{g}_K}$ is constant.

The following proposition yields Theorem 3 in the general case.

Proposition 5. *For every $x \in M^*$ we have $(\mathfrak{g}_K)^\perp x \subset T_xM^*$. If M^* does not contain a trivial factor, then $\mathfrak{g}_K = \mathfrak{g}$ and thus $b = v'$.*

Proof. We noted above that $(\mathfrak{g}_K)^\perp \subset T_eM^*$. Now let $x \in M^*$ and $g(t) = \exp(tX)$ be a geodesic from e to x. Then $g * M$ is the lift of M^*x^{-1} and g_*K is tangent to $g * M$. So we obtain the relation $(\mathfrak{g}_{g_*K})^\perp \subset T_e(M^*x^{-1}) = (T_xM^*)x^{-1}$. We will prove that $\mathfrak{g}_{g_*K} = \mathfrak{g}_K$.

To see this recall that

$$(g_*K)x = [\mathrm{Ad}_g\, v, x] - [\mathrm{Ad}_g\, v, g * 0] - \mathrm{Ad}_g\, b.$$

Thus to prove that $\mathfrak{g}_K \subset \mathfrak{g}_{g_*K}$ we have to consider the projection onto $\mathfrak{g}_K$ of

$$\begin{aligned}&\mathrm{Ad}_g\, v - \widetilde{[\mathrm{Ad}_g\, v, g * 0]} - \widetilde{\mathrm{Ad}_g\, b}\\ &= \mathrm{Ad}_g(v - \widetilde{b}) - \widetilde{[\mathrm{Ad}_g\, v, g * 0]} - \widetilde{\mathrm{Ad}_g\, b} + \mathrm{Ad}_g\, \widetilde{b}\end{aligned}$$

Observe that this projection is (weakly) differentiable since $(v - \widetilde{b})^{\mathfrak{g}_K} = \text{const}$. Then deriving the above projection onto $\mathfrak{g}_K$ we obtain using $g * 0 = -X$

$$\mathrm{Ad}_g[X, v - \widetilde{b}] - [\mathrm{Ad}_g\, v, -X] - \mathrm{Ad}_g\, b + \mathrm{Ad}_g[X, \widetilde{b}] + \mathrm{Ad}_g\, b \equiv 0.$$

This proves $\mathfrak{g}_K \subset \mathfrak{g}_{g_*K}$ and the converse inclusion is obtained by exchanging the roles of K and g_*K.

The second claim and thus Theorem 3 now follows from Lemma 3.4 in [3], which states that a submanifold of G has a trivial factor if and only if there is an ideal $\mathfrak{k} \subset \mathfrak{g}$ such that $\mathfrak{k}x \subset T_xM^*$ for all $x \in M^*$.

q.e.d.

4. Constructing parallel Killing fields

In this section we finish the proof of our main theorem by proving that only the $H^1([0,1], G)$-Killing fields can be tangent to a lift containing no trivial factor. By the results of the previous section, it remains to show that the linear part has the required form. So we have to prove the following:

Theorem 6. *If $M^* \subset G$ does not contain a trivial factor and $Kx = Ax + b$ is a Killing field tangent to $M = \pi^{-1}(M^*)$ then $Ax = [v, x]$ for some $v \in L^2([0,1], \mathfrak{g})$.*

The strategy for the proof is as follows: First, note that by Theorem 3 in particular there are no parallel Killing fields in $\mathcal{K}^M$. We will associate to a Killing field $K \in \mathcal{K}^M$ and a suitable direction $v \in V$ a sequence $K_n \in \mathcal{K}^M$ which converges to a parallel Killing field K_∞ in a weak sense. This limit is still contained in $\mathcal{K}^M$, contradicting Theorem 3 unless K_∞ is zero. By a closer examination of this process we will see that thereto K must be contained in the class of Killing fields we considered in the previous section, and the proof is complete.

The suitable topology on the space $\mathcal{K}$ is the one induced by the norm topology on $\mathrm{End}(V)$ and the weak topology on V. Abusing notation, convergence in this topology will be called *weak convergence*, denoted by "$\rightharpoonup$". It is easy to see that $\mathcal{K}^M$ is closed in this weaker topology: $K_n \rightharpoonup K$ clearly implies $K_n x \rightharpoonup Kx$ (weak convergence in V) for every $x \in V$ and since the tangent spaces of M are closed, in particular wealky closed, $K_n x \in T_x M$ for all n implies $Kx \in T_x M$.

We now turn to the construction mentioned above. Consider points $v = fX \in V$, where either:

- $\int_0^1 f dt = 0$, or
- $f \equiv$ const and v is the initial velocity of a closed geodesic in G of period one.

Since by the above assumptions $g = \exp(\widetilde{f}X)$, where $\widetilde{f}(t) = \int_0^t f d\tau$, is a loop and we have $v = g^{-1} * 0$ such points lie on the fiber $\pi^{-1}(e)$. Clearly, the same is true for any integer multiple of v and we find sequences of points in $\pi^{-1}(e)$ lying on a line and going to infinity. Now, given $K \in \mathcal{K}^M$, a fixed direction $v = fX$ as above and loops $g_n = \exp(n\widetilde{f}X)$ consider the sequence

$$K_n := g_{n*} K / n.$$

If $Kx = Ax + b$ then the linear and constant part of K_n are given by

$$\begin{aligned} A_n &= \mathrm{Ad}_{g_n} A \, \mathrm{Ad}_{g_n}^{-1} / n \qquad \text{and} \\ b_n &= \mathrm{Ad}_{g_n}(A(g_n^{-1} * 0) + b)/n \\ &= \mathrm{Ad}_{g_n}(A(fX) + b/n). \end{aligned}$$

Since ΩG acts isometrically on M clearly $\mathcal{K}^M$ is invariant under the action of ΩG by push forward and thus $K_n \in \mathcal{K}^M$. Now, the above formulae show that $A_n \to 0$ and $\|b_n\| \leq$ const. Thus K_n has a subsequence converging wealky to a parallel Killing field $K_\infty \in \mathcal{K}^M$. By the results of previous section this is impossible unless $K_\infty = 0$.

The condition $K_n \rightharpoonup 0$ for all v as above yields an important restriction on the linear part of K:

Proposition 7. *If $v = fX$ as above then $K_n \rightharpoonup 0$ implies $A(fX) \subset \mathrm{Im}\,\mathrm{ad}_X$.*

Proof. Let $g_n := \exp(n\widetilde{f}X)$ and decompose $A(fX) = v_1 + v_2$ with $v_1 \subset \ker \mathrm{ad}_X$ and $v_2 \subset \mathrm{Im}\,\mathrm{ad}_X$. Then the constant part of $K_n = {g_n}_* K/n$ is given by

$$\begin{aligned} b_n &= \mathrm{Ad}_{g_n}(v_1 + v_2 + b/n) \\ &= v_1 + \mathrm{Ad}_{g_n}(v_2 + b/n) \end{aligned}$$

since Ad_{g_n} acts trivially on (curves in) $\ker \mathrm{ad}_X$.

Thus $\langle v_1, b_n\rangle_{L^2} = \|v_1\|^2_{L^2} + \langle v_1, b_n\rangle_{L^2}/n \to \|v_1\|^2_{L^2}$ and $K_n \rightharpoonup 0$ implies $v_1 = 0$. q.e.d.

In other words, we see that the linear part of tangent Killing fields must have the form $A(fX) = [v(fX), X]$ for some $v(fX) \in L^2([0,1], \mathfrak{g})$, if f and X are chosen suitably. In fact, in the case where $f \equiv$ const, say one, we can drop the assumption that X is the initial velocity of a closed geodesic of period one: first, scaling X doesn't affect the above form. Now, given any $Y \in \mathfrak{g}$, we can find an approximating sequence X_i in some abelian subalgebra of $\mathfrak{g}$ and corresponding to closed geodesics (of any period). We may assume that $\ker \mathrm{ad}_{X_i} = \ker \mathrm{ad}_Y$ such that $[v(X_i), X_i] = AX_i \longrightarrow AX$ implies convergence of $v(X_i)$ modulo $\ker \mathrm{ad}_Y$ which is sufficient for our purposes.

Our aim is now to show that $v(fX) = fv_0$ for some fixed $v_0 \in L^2([0,1], \mathfrak{g})$. To this end, we first prove that $v(fX)$ can be chosen independently of X:

Proposition 8. *Let K be a Killing field in $\mathcal{K}^M$ and f, X as above. Then there is a (uniquely determined) operator*

$$v : L^2([0,1]) \to L^2([0,1], \mathfrak{g})$$

such that

$$K(fX) = [v(f), X] - b.$$

The proof is based on the following lemma:

Lemma 9. *If $A : \mathfrak{g} \to \mathfrak{g}$ is a linear map of the form $Ax = [v(x), x]$ for some map $v : \mathfrak{g} \to \mathfrak{g}$ then $A \in \mathrm{ad}(\mathfrak{g})$, i.e., there is a fixed $v \in \mathfrak{g}$ such that $Ax = [v, x]$.*

Proof. Clearly A is skew symmetric and tangent to the adjoint orbits in $\mathfrak{g}$; so the associated one-parameter group e^{tA} leaves the orbits invariant.

Now if G is simple and of rank ≥ 2 it is known from Dadok (cf. [2]) that $\mathrm{Ad}(G)$ is the maximal subgroup of $\mathrm{SO}(\mathfrak{g})$ acting with these orbits, thus $e^{tA} \in \mathrm{Ad}(G)$ and so $A \in \mathrm{ad}(\mathfrak{g})$. For $\mathfrak{g} = \mathfrak{su}(2)$ the result is trivial since $\mathrm{ad}(\mathfrak{su}(2)) = \mathfrak{so}(3)$.

In the semisimple case we can restrict A to the simple ideals of $\mathfrak{g}$ and the claim follows. q.e.d.

Proof of Proposition 8. We know that $A(fX) = [v(fX), X]$ for f, X as required.

First, we treat the case $\int_0^1 f dt = 0$, where we have no condition on X. Then the result follows immediately applying the lemma to the map $X \mapsto A(fX)(t)$ for fixed f and $t \in [0,1]$. (The expression $A(fX)(t)$ of course only makes sense for representatives of the L^2-curve $A(fX)$. However, it is easy to see that the resulting $v(f) \in L^2([0,1], \mathfrak{g})$ does not depend on this choice.)

For the case where $f \equiv \text{const}$, say one, and X is the initial velocity of a closed geodesic of periode one we note that any

Uniqueness, linearity and continuity follow since $\mathfrak{g}$ is semisimple.

q.e.d.

We are now in a position to finish the proof of the main theorem by showing that the operator $v : L^2([0,1]) \to L^2([0,1], \mathfrak{g})$ is of the form $v(f) = fv_0$ for some $v_0 \in L^2([0,1], \mathfrak{g})$ and thus $A(fX) = [v_0, fX]$ which by the previuos section implies the result.

Proof of the main theorem.

Note that the form $A(fX) = [v(f), X]$ is invariant not only under the action of ΩG but also of $H^1([0,1], G)$: if $K \in \mathcal{K}^M$ and $g \in H^1([0,1], G)$ then $g_* K$ is tangent to $g * M$, the lift of $g(0)\, M^*\, g(1)^{-1}$, and Proposition 8 applies.

To finish the proof we show that $v(f) = fv(1)$ using the action of $H^1([0,1], G)$. Comparing the linear parts of K and $\overline{K} = g_*^{-1} K$ yields

$$(**) \qquad \mathrm{Ad}_g \, \overline{A} = A \, \mathrm{Ad}_g$$

where

$$\overline{A}(fX) = [\overline{v}(f), X] \text{ for some } \overline{v} : L^2([0,1]) \to L^2([0,1], \mathfrak{g}).$$

It is now convenient to consider the complexified situation. So let $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{g}^{\mathbb{C}}$ and

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{t} + \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

be a complex root space decomposition, i.e., $[H, X_\alpha] = \alpha(H)\, X_\alpha$ for $H \in \mathfrak{t}$ and $X_\alpha \in \mathfrak{g}_\alpha$ for a set of roots Δ.

Fix $n \in \mathbb{Z}$, a root $\alpha \in \Delta$ and a path $g(t) = \exp(tH)$ with $H \in \mathfrak{t}$, $\alpha(H) = 2\pi n$. Comparing both sides of $(**)$ yields for $X \in \mathfrak{t}$

$$[\mathrm{Ad}_g\, \overline{v}(1), X] = [v(1), X], \text{ since } \mathrm{Ad}_g\,|_{\mathfrak{t}} = \mathrm{id}$$

and for $X_\alpha \in \mathfrak{g}_\alpha$

$$[\mathrm{Ad}_g\, \overline{v}(1), e^{2\pi i n t} X_\alpha] = [v(e^{2\pi i n t}), X_\alpha], \text{ since } \mathrm{Ad}_g(X_\alpha) = e^{2\pi i n t} X_\alpha.$$

Thus we obtain

$$\begin{aligned} \mathrm{Ad}_g\, \overline{v}(1) &= v(1) \bmod \mathfrak{t} \\ e^{2\pi i n t}\, \mathrm{Ad}_g\, \overline{v}(1) &= v(e^{2\pi i n t}) \bmod \mathcal{C}(\mathfrak{g}_\alpha) \end{aligned}$$

(i.e., agree up to a curve in $\mathfrak{t}$ resp. $\mathcal{C}(\mathfrak{g}_\alpha)$), where $\mathcal{C}(\mathfrak{g}_\alpha)$ denotes the centralizer of $\mathfrak{g}_\alpha$. Combining the two relations yields

$$e^{2\pi i n t} v(1) - v(e^{2\pi i n t}) =: d \subset \mathfrak{t} + \mathcal{C}(\mathfrak{g}_\alpha)$$

and it remains to show $d = 0$.

Now varying $\alpha \in \Delta$ and using $\bigcap_\alpha \mathcal{C}(\mathfrak{g}_\alpha) = 0$ (since $\mathfrak{g}$ is semisimple) shows $d \subset \mathfrak{t}$, and, considering other maximal abelian subalgebras $\mathfrak{t}$, the same argument proves $e^{2\pi i n t} v(1) = v(e^{2\pi i n t})$. q.e.d.

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