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COMPACT KÄHLER MANIFOLDS WITH NONPOSITIVE BISECTIONAL CURVATURE

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Abstract

In this article, we prove that for any compact Kähler manifold M^n with real analytic metric and nonpositive bisectional curvature, there exists a finite cover M' of M such that M' is a holomorphic and metric fiber bundle over a compact Kähler manifold N with nonpositive bisectional curvature and $c_1(N) < 0$, and the fiber is a flat complex torus. This partially confirms a conjecture of Yau.

1. Introduction and statement of results

A major component of the attempt to generalize the classical uniformization theorem for Riemann surfaces to higher dimensions is to understand complex manifolds which admit complete Kähler metrics whose bisectional curvature keeps a sign. In this context, bisectional curvature is the appropriate curvature to use because it is tied to the complex structure and, unlike holomorphic (sectional) curvature which is the other curvature defined in terms of the complex structure, it contains enough information to make deeper questions meaningful. By comparison, assumptions on the sectional curvature are often too strong. (Nevertheless, we note that many open questions related to sectional curvature remain unanswered. A simple example is whether a compact Kähler manifold with nonpositive sectional curvature, which is of general type, must be Kobayashi hyperbolic. So far this has been confirmed only for dimension 2 ([47]) by using a result of Lu and Yau [28]).)

If we restrict ourselves to compact manifolds, those with nonnegative bisectional curvature are well understood after the works of Frankel

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[19], Howard-Smyth-Wu [21], [37], Mori [33], Siu-Yau [35], and Mok [29]. The final result says that, given a compact Kähler manifold M^n with nonnegative bisectional curvature, there exists a finite covering manifold M' of M such that the Albanese map $\pi : M' \to T^q$ of M' is a surjective holomorphic fiber bundle, whose fiber has $c_1 > 0$ and is in fact biholomorphic to a product of irreducible compact Hermitian symmetric spaces.

On the other hand, the current state of knowledge in the nonpositive case is much less satisfactory. Perhaps the following example will serve to illustrate this point. In Yau's famous problem set [44], Question 35 asks whether a simply-connected, complete Kähler manifold with negative bisectional curvature must be Stein. A weaker question is whether a compact Kähler manifold with negative bisectional curvature must have infinite fundamental group. Bun Wong observed that these questions can be reduced to the 2-dimensional case, and he also constructed examples of simply-connected algebraic surfaces with ample cotangent bundle ([36]). Contrast this with the same question using sectional rather than bisectional curvature: the well-known Cartan-Hadamard Theorem states that any complete Riemannian manifold with nonpositive sectional curvature is a $K(\pi, 1)$ space, and this theorem has been responsible for the vast recent development in the theory of nonpositively curved Riemannian manifolds, beginning with the classical works of Bishop-O'Neill [5], Yau [42], Lawson-Yau [27], Eberlein-O'Neill [15] in the late 60's and the 70's, leading to the explosion in the 80's and 90's. The paper [14] gives an excellent discussion and references on the subject.

As in the Riemannian case, it is often very important to understand the difference between the negatively curved case and the nonpositively curved (but not quasi-negatively curved) case. The former tends to be "hyperbolic" in some suitable sense, while the latter usually possesses certain rigidity properties.

For compact Kähler manifolds with nonpositive bisectional curvature, the first structure question one can ask is the following conjecture of Yau which we learned from his seminar at Harvard:

Conjecture (Yau). *Let M^n be a compact Kähler manifold with nonpositive bisectional curvature. Then there exists a finite cover M' of M such that M' is a holomorphic and metric fiber bundle over a compact Kähler manifold N with nonpositive bisectional curvature and $c_1(N) < 0$, and the fiber is a (flat) complex torus.*

Recall that the fiber bundle is called a *metric* bundle, if for any $p \in N$, there is some neighborhood $p \in U \subseteq N$ such that the bundle over U is isometric to the product of the fiber with U.

This conjecture can be considered as the dual of the splitting theorem in [21] in the nonnegative bisectional curvature case. It also has a generalization to the numerically effective case (see §4), which is dual to the reduction theorem of Demailly-Peternell-Schneider [8].

The main purpose of this article is to confirm the above conjecture of Yau under the additional assumption that the metric of M^n is real analytic:

Theorem E. *Let M^n be a compact Kähler manifold with real analytic metric and nonpositive bisectional curvature. Then there exists a finite cover M' of M such that M' is a holomorphic and metric fiber bundle over a compact Kähler manifold N with nonpositive bisectional curvature and $c_1(N) < 0$, and the fiber is a flat complex torus.*

As noted in [10], the finite cover M' here can be chosen so that it (and any further cover of it) is diffeomorphic to the product of the fiber and the base. However, there are M such that any finite cover of M is not biholomorphic or isometric to the product. The first such examples were constructed by Lawson and Yau [27].

As an immediate consequence of Theorem E, we have the following:

Corollary F. *Let M^n be as in Theorem* E *above. Then its Kodaira dimension is equal to its* Ricci rank*, i.e., the maximum of the rank of its Ricci tensor at each point.*

Previously, this was known when M^n is assumed to have nonpositive sectional curvature ([49]).

We believe that Yau's conjecture is true in its full generality. The presence of the real analyticity assumption in Theorem E is, in our minds, merely the reflection of a technical failure that we could not overcome at this point in time.

A key step in the proof of Theorem E is to show that the totally geodesic foliation $\mathcal{L}$ defined by the kernel of the Riemannian curvature tensor is actually a holomorphic foliation (Theorem A in §2). This result should be of independent interest, as the foliation technique in complex geometry often requires this holomorphicity (cf. [30]).

Theorem A enables one to propagate the so-called conullity operators and the Ricci tensor along a leaf of $\mathcal{L}$, and estimate their growth rates. Under the assumption that the curvature is bounded (e.g., when

the manifold is compact to begin with), those conullity operators must all vanish, with the result that, at least locally, $\mathcal{L}$ is left invariant by the holonomy group. In the real analytic case, this would mean that the universal covering space $\widetilde{M}$ of the manifold M in Theorem E has a Euclidean de Rham factor of dimension $(n-r)$, where r is the Ricci rank of M.

Another key step in the proof of Theorem E is to show that the leaves of $\mathcal{L}$ in $\widetilde{M}$ actually close up, e.g., their images in M are compact. This is parallel to the situation in the Eberlein theory on Euclidean de Rham factors for nonpositively curved Riemannian manifolds ([10]–[13]). Here one must show that the projection of the deck transformation group onto the non-Euclidean de Rham factors is discrete (cf. Claim 1 in §4). To this end, we make use of the partial stability of the tangent bundle of M (which boils down to the maximum principle). Although the techniques in accomplishing this key step (Claim 1) are totally different from the Riemannian case, the geometric flavor is remarkably similar. In fact, the Eberlein theory has been the guiding spirit in our quest here.

Some of the results of this article were announced in [40] and [49].

2. Holomorphicity of the Ricci kernel foliation

We first fix the notation and terminology. The basic reference is [50].

Let (M^n, g) be a Kähler manifold with nonpositive bisectional curvature. Denote by ρ the Ricci $(1,1)$-form. It is negative semi-definite everywhere on M. Denote by r the maximum of the (complex) rank of ρ, and by $U \subseteq M$ the open subset where ρ has rank r. We say that M has *degenerate Ricci* if $r < n$. That is, if the Ricci tensor is nowhere negative definite. In this case, denote by $\mathcal{L}$ the distribution in U given by the kernel of ρ.

By linear algebra, the nonpositivity of the bisectional curvature implies that $X \in \mathcal{L}$ if and only if $R(X, *, *, *) \equiv 0$, where R is the curvature tensor (see [40]). So $\mathcal{L}$ is the kernel of the curvature tensor. Thus it is a foliation, whose leaves are totally geodesic, flat complex submanifolds of U. By a theorem of Ferus [16], each leaf of $\mathcal{L}$ is complete if M is complete.

We shall refer to r as the *Ricci rank* and $\mathcal{L}$ the *Ricci kernel foliation* of M^n.

In general, the complex foliation $\mathcal{L}$ may not be holomorphic, i.e.,

even though the leaves of $\mathcal{L}$ are complex submanifolds, they may not vary holomorphically from leaf to leaf. More precisely, the leaves of $\mathcal{L}$ being complex submanifolds means that locally, there are $n-r$ C^∞ vector fields $V_1, \ldots, V_{n-r}$ of type $(1,0)$ so that they are linearly independent and span the fibers of $\mathcal{L}$ at each point. Then by definition, $\mathcal{L}$ is *holomorphic* if the $\{V_i\}$ can be chosen to be holomorphic vector fields. When M^n is complete, Theorem A following says that $\mathcal{L}$ is always a holomorphic foliation. In case the leaves of $\mathcal{L}$ are one dimensional, Theorem A is implied by (the completeness of the leaves and) a result due to Burns (Theorem 3.1 and Corollary 3.2 in [4]), where he studied the more general Monge-Ampere foliations. When M has nonpositive sectional curvature, Theorem A was proved in [48] (in this case, $\mathcal{L}$ is left invariant by the holonomy group). The proof there also implies the case where the leaves of $\mathcal{L}$ are $(n-1)$ dimensional.

Theorem A. *If (M^n, g) is a complete Kähler manifold with nonpositive bisectional curvature, then its Ricci kernel foliation $\mathcal{L}$ is a holomorphic foliation.*

Proof. Since the proof is somewhat long, we divide it into two parts. In the first part, we use the completeness of the leaves of $\mathcal{L}$ and equations arising from the geometry of the situation to obtain Condition $(\star)$ below which must be satisfied by conullity operators of the form C_{aT+bJT}, and to find a basis relative to which C_{aT+bJT} assumes a particularly simple form. It will turn out that $\mathcal{L}$ is holomorphic iff the sub-matrix B in the matrix of C_{aT+bJT} below is zero (see (1)). In the second half of the proof, we use certain differential equations along a leaf of $\mathcal{L}$ to show that, if B is not zero, then Condition $(\star)$ would be violated. The contradiction then completes the proof. q.e.d.

We begin by setting things up the same way as in the proof of Theorem 3 in [48], with minor modifications to suit our situation here. Let us denote by $\mathcal{L}^\perp$ the distribution (of type $(1,0)$ tangent vectors) in U representing the orthogonal complement of $\mathcal{L}$. Also denote by $\mathcal{F}$ and $\mathcal{F}^\perp$ the underlying real distribution of $\mathcal{L}$ or $\mathcal{L}^\perp$. $\mathcal{F}$ is of real rank $2r$.

Recall that the *conullity operator* of a totally geodesic foliation $\mathcal{F}$ in a Riemannian manifold is defined by (cf. [2], [7])

$$C_T(X) = -(\nabla_X \widetilde{T})^\perp$$

where T and X are tangent vectors in $\mathcal{F}$ and $\mathcal{F}^\perp$, respectively, and $\widetilde{T}$ is a local vector field in $\mathcal{F}$ extending T. Here $Y^\perp$ stands for the $\mathcal{F}^\perp$-component of Y. These operators are well-defined tensors, and satisfy

the Riccati type equations

$$\nabla_T C_S = C_S \circ C_T - C_{\nabla_T S} - \{R(T,\cdot)S\}^{\perp}$$

for any two vector fields T and S in $\mathcal{F}$. In our case, the curvature term vanishes, and if we choose S to be parallel in each leaf of $\mathcal{F}$, then the above equation becomes

$$\nabla_T C_S = C_S \circ C_T.$$

In particular, along any geodesic $\gamma(t)$ contained in a leaf F of $\mathcal{F}$, one has the Riccati equation:

$$\nabla_T C_T = (C_T)^2$$

where $T = \gamma'(t)$. Since each F is complete by Ferus' theorem [16], the Riccati equation has the following consequence: *at each point of U, C_T for any $T \in \mathcal{F}$ cannot have nonzero real eigenvalues* (cf. [40]; this is an observation implicit in [2] and explicitly pointed out by [7]).

For $T \in \mathcal{F}$, extend C_T linearly over $\mathbf{C}$ to the complexification

$$\mathcal{F}^{\perp} \otimes \mathbf{C} = \mathcal{L}^{\perp} \oplus \overline{\mathcal{L}^{\perp}}.$$

Choose a local frame $\{e_i, \overline{e_i}\}_{i=1}^{r}$ such that each $e_i \in \mathcal{L}^{\perp}$. Write $C_T(e_i) = \sum_j A_{ij} e_j + B_{\bar{i}j}\overline{e_j}$, then the matrix of C_T relative to the basis $\{e, \overline{e}\}$ is

$$C_T = \begin{bmatrix} A & \overline{B} \\ B & \overline{A} \end{bmatrix}, \qquad \text{and} \quad C_{JT} = JC_T = \sqrt{-1}\begin{bmatrix} A & -\overline{B} \\ B & -\overline{A} \end{bmatrix}$$

where J is the almost complex structure of M. For any real numbers a and b, write $\lambda = a + \sqrt{-1}b$, we have

$$C_{aT+bJT} = \begin{bmatrix} \lambda A & \overline{\lambda B} \\ \lambda B & \overline{\lambda A} \end{bmatrix}. \tag{1}$$

Therefore, by the same reasoning using the Riccati equation, we have the following:

Condition ($\star$)**.** For any $\lambda \in \mathbf{C}$, the above $(2r \times 2r)$ matrix C_{aT+bJT} in A, B has no nonzero real eigenvalue at any point of U.

This condition will be the focus of our discussion below.

Note that A and B represents the holomorphic and anti-holomorphic part of the conullity operator C_T. By a simple argument (see [40]), $\mathcal{L}$

being holomorphic is equivalent to $B = 0$ for any $T \in \mathcal{F}_x$ at any $x \in U$. Our goal is therefore to show that $B = 0$.

(As a side remark, when $r = 1$, it is easy to see that Condition $(\star)$ already implies that $A = B = 0$, so $\mathcal{L}$ is holomorphic (and the fibers of $\mathcal{L}$ are locally invariant subspaces of the holonomy group) in this case. In particular, if M^n is a complete complex submanifold of $\mathbf{C}^N$ with Ricci rank (which equals the Gauss rank) 1, then it must be a cylinder, this is known as the Abe's cylinder theorem ([1]), which is the complex version of the classical Hartman-Nirenberg cylinder theorem ([20]).)

Fix any $x \in U$ and $T \in \mathcal{F}_x$. In order to show that $B = 0$ at the point x, first we will find a a basis $\{e_i\}_{i=1}^r$ of $V = \mathcal{L}_x^\perp$ relative to which the matrices A and B look particularly simple. To this end, we need to make use of the interaction between A, B and the curvature tensor.

Let us take a local tangent frame $\{e_i, e_\alpha\}$ of M^n of type $(1,0)$ near x, such that each $e_i \in \mathcal{L}^\perp$, each $e_\alpha \in \mathcal{L}$, and $T = e_n + \overline{e_n}$. We will fix the range of indices as follows.

$$1 \le i, j, \cdots \le r \ ; \quad r+1 \le \alpha, \beta, \cdots \le n.$$

Denote by θ, Θ the matrices of the connection and the curvature under the frame e. We have

$$d\varphi = \varphi \wedge \theta, \quad d\theta = \Theta + \theta \wedge \theta$$

where $\varphi = (\varphi_1, \ldots, \varphi_n)$ is the row vector of the coframe dual to e. We have

$$\theta_{\alpha i} = -\sum_k (A^\alpha_{ki} \varphi_k + B^\alpha_{\bar{k}i} \overline{\varphi_k})$$

where A^α, B^α are the matrices which appear in the matrix of C_{T^α} for $T^\alpha = e_\alpha + \overline{e_\alpha}$ as before. Here and below we are only interested in the case $\alpha = n$, so the superscript α will be omitted in the following.

Write $\Omega_{a\bar{b}} = \sum_{c=1}^n \Theta_{ac} g(e_c, e_b)$. Then

$$\Omega_{a\bar{b}} = \sum_{c,d=1}^n R_{a\bar{b}c\bar{d}} \ \varphi_c \wedge \overline{\varphi_d}$$

where $R_{a\bar{b}c\bar{d}} = R(e_a, \overline{e_b}, e_c, \overline{e_d})$ are the components of the curvature tensor with respect to the frame e. We have $\Omega_{\alpha\bar{*}} = \Omega_{*\bar{\alpha}} = 0$ so the second Bianchi identity implies that

$$\sum_{j=1}^r \theta_{\alpha j} \wedge \Omega_{j\bar{k}} = 0$$

for any α and k. That is,

$$\sum_{i,j} A^{\alpha}_{ij}\varphi_i \wedge \Omega_{j\bar{k}} = 0, \qquad \sum_{i,j} B^{\alpha}_{\bar{i}\bar{j}}\overline{\varphi_i} \wedge \Omega_{j\bar{k}} = 0.$$

Specialize to $\alpha = n$, the above identities are equivalent to

$$\sum_j (A_{ij}R_{j\bar{k}p\bar{l}} - A_{pj}R_{j\bar{k}i\bar{l}}) = 0 \tag{2}$$

$$\sum_j (B_{\bar{i}j}R_{j\bar{k}p\bar{l}} - B_{\bar{l}j}R_{j\bar{k}p\bar{i}}) = 0 \tag{3}$$

for any i, p, k, l. Let $H = \{H_{a\bar{b}}\}$ denote the matrix of the Ricci tensor with respect to the basis $\{e, \overline{e}\}$. Using the symmetries of the curvature tensor, we get:

$$\begin{aligned}(BH)_{\bar{i}\bar{k}} &= \sum_j B_{\bar{i}j}H_{j\bar{k}} = \sum_{i,l} B_{\bar{i}j}R_{j\bar{k}l\bar{l}} = \sum_{j,l} B_{\bar{i}j}R_{j\bar{l}l\bar{k}} \\ &= \sum_{j,l} B_{\bar{k}j}R_{j\bar{l}l\bar{i}} \qquad \text{(by (3))} \\ &= \sum_{j,l} B_{\bar{k}j}R_{j\bar{i}l\bar{l}} = \sum_j B_{\bar{k}j}H_{j\bar{i}} = ({}^tH{}^tB)_{\bar{i}\bar{k}}.\end{aligned}$$

Hence,

$$BH = {}^t(BH).$$

Since H is negative definite on $V = \mathcal{L}_x^{\perp}$, there will be basis of V under which $H = -I$. With respect to this basis, the matrix B becomes a complex symmetric matrix. It is well-known that for any given complex symmetric matrix B, there exists a unitary matrix Q such that ${}^tQBQ$ is diagonal, and with nonnegative diagonal entries (cf. [6], p. 28). Writing P for ${}^t\overline{Q}$, we have proved that $\overline{P}BP^{-1}$ is a diagonal matrix with nonnegative diagonal entries.

Define a new basis $\{\epsilon_i\}$ in V using P above, so that $\epsilon_i = \sum_j P_{ij}e_j$. Then with respect to $\{\epsilon_i\}$, the matrix of C_{aT+bJT} in (1) is now

$$C_{aT+bJT} = \begin{bmatrix} \lambda A^{\epsilon} & \overline{\lambda B^{\epsilon}} \\ \lambda B^{\epsilon} & \overline{\lambda A^{\epsilon}} \end{bmatrix} \tag{4}$$

where

$$A^{\epsilon} = PAP^{-1}, \qquad B^{\epsilon} = \overline{P}BP^{-1}.$$

So the matrix of C_{aT+bJT} with respect to $\{\epsilon_i\}$ will have the property that the two off-diagonal blocks are diagonal matrices with nonnegative diagonal entries. From now on, we may assume for the sake of notational simplicity that $\{\epsilon_i\}$ is just $\{e_i\}$ so that with respect to $\{e_i\}$, $B = \operatorname{diag}\{b_1, \ldots, b_r\}$, where $b_1 \geq \cdots \geq b_r \geq 0$. Denote by

$$V = V_1 \oplus \cdots \oplus V_s$$

the decomposition of V into eigen-spaces corresponding to *distinct* eigenvalues $b_1 > b_2 > \cdots > b_s$.

For any $1 \leq a < b \leq s$, and any $e_i \in V_a$, $e_l \in V_b$, we claim that $A_{il} = 0$. To see this, we use Equation (3) and the fact that $B_{\bar{i}j} = b_i \delta_{ij}$ for all i, j to get

$$b_i R_{il\bar{p}\bar{k}} = b_l R_{l\bar{i}p\bar{k}}.$$

For any vector $Z = \sum_k Z_k e_k$ in V, if we multiply the preceding equation by $Z_p \overline{Z_k}$ and sum over p and k, we obtain

$$b_i R(e_i, \overline{e_l}, Z, \overline{Z}) \;=\; b_l R(e_l, \overline{e_i}, Z, \overline{Z}).$$

Noting that $R(e_l, \overline{e_i}, Z, \overline{Z})$ is the complex conjugate of $R(e_i, \overline{e_l}, Z, \overline{Z})$, and that b_i and b_l are distinct real numbers, taking the absolute value of both sides gives $R(e_i, \overline{e_l}, Z, \overline{Z}) = 0$ for any $Z \in V$ and for any $i \neq l$. If now p and k are any two integers among $1, \ldots, r$, then

$$R(e_i, \overline{e_l}, e_p + z e_k, \overline{e_p + z e_k}) = 0$$

for any complex number z. Expanding by linearity and taking derivative with respect to $\overline{z}$ then give

$$R_{il\bar{p}\bar{k}} = 0 \qquad \text{for all } p, k, \text{ and } i \neq l.$$

Applying this to Equation (2) still with $i \neq l$, we have $\sum_j A_{ij} R_{j\bar{l}p\bar{k}} = 0$ for all p, k. Setting $p = k$ and summing over p lead to $\sum_j A_{ij} H_{j\bar{l}} = 0$, where $H_{j\bar{l}}$ are the components of the Ricci tensor as above. Again, H being negative definite on V, we finally get $A_{il} = 0$ for $i \neq l$.

We have proved that A is block diagonal with respect to $\{e\}$. Therefore in order to check Condition $(\star)$, we may simply check it on a subspace V_a $(1 \leq a \leq s)$, in which $B = hI$ for some $h \geq 0$. This concludes our discussion of the first half of the proof.

In the second half, we will show that if the constant h in $B = hI$ is positive, then Condition $(\star)$ would be violated. Therefore B must be 0, and Theorem A is proved.

With the chosen point x as above, let $\gamma(t)$, $t \in \mathbf{R}$ be the geodesic with $\gamma(0) = x$ and $\gamma'(0) = T$. We now parallel translate the frame $\{e_i, e_\alpha\}$ of V parallelly along γ to $\{e_i(t), e_\alpha(t)\}$ at $\gamma(t)$. The parallel translate of T is of course just $\gamma'(t)$, which will continue to be denoted simply by T since there is no danger of confusion. Now along γ, we have the analogue of (1) for the conullity operator $C_{aT+bJT}(t)$ at each $\gamma(t)$ with respect to the parallel frame $\{e(t)\}$:

$$C_{aT+bJT}(t) = \begin{bmatrix} \lambda A(t) & \overline{\lambda B(t)} \\ \lambda B(t) & \overline{\lambda A(t)} \end{bmatrix}$$

where $\lambda = a + \sqrt{-1}b$ and a, b are real. Then for each t, we can find, as before, a matrix $P_{ij}(t)$ and a new frame $\{\epsilon_i(t)\}$, so that $\epsilon_i(t) = \sum_j P_{ij}(t)e_j(t)$ and $A^\epsilon(t)$ and $B^\epsilon(t)$, the analogues of the matrices in (4), are block diagonal, with the diagonal blocks of $B^\epsilon(t)$ being (nonnegative) scalar multiples of the identity matrix. Clearly, our previous argument can be done in a smooth way so that $P_{ij}(t)$ is a smooth matrix function of t and consequently $\{\epsilon_i(t)\}$ is a smooth frame along γ.

Since $e(t)$ is parallel along γ, the Riccati equation satisfied by $C_{aT+bJT}(t)$ has the consequence that

$$A'(t) = A(t)^2 + \overline{B(t)}B(t), \qquad B'(t) = B(t)A(t) + \overline{A(t)}B(t)$$

where the derivatives are with respect to t. In the (rather extensive) computations to follow, we will omit any reference to t in the interest of notational economy. Then because $A^\epsilon = PAP^{-1}$ and $B^\epsilon = \overline{P}BP^{-1}$ (see the equations below (4)), the above equations become

$$(A^\epsilon)' = (A^\epsilon)^2 + \overline{B^\epsilon}B^\epsilon + SA^\epsilon - A^\epsilon S$$
$$(B^\epsilon)' = B^\epsilon A^\epsilon + \overline{A^\epsilon}B^\epsilon + \overline{S}B^\epsilon - B^\epsilon S,$$

where $S = P'P^{-1}$. Now assume that $B \neq 0$. Denote by A_1, B_1 and S_1 the first diagonal block of A^ϵ, B^ϵ and S. We have $B_1 = hI$ and $h(0) > 0$ (we note explicitly that h now denotes a function of t). The first diagonal block of the above equations says that

$$A_1' = A_1^2 + h^2 I + S_1 A_1 - A_1 S_1$$
$$h'I = h(A_1 + \overline{A_1}) + h(\overline{S_1} - S_1).$$

By looking at the real and imaginary parts, the second equation says that

$$h'I = h(A_1 + \overline{A_1}), \qquad h(\overline{S_1} - S_1) = 0.$$

The first equation and the uniqueness theorem for first order ODE imply that h must be nowhere zero (because $h(0) > 0$). Thus $h(t) > 0$ for all $t \in \mathbf{R}$. By the second equation, S_1 is real.

Define the function

$$f = \frac{1}{2} \log h.$$

Then the equation $h'I = h(A_1 + \overline{A_1})$ implies that we can decompose A_1 into real and imaginary parts as $A_1 = f'I + ihD$ for a real matrix D. Substituting this expression of A_1 into $A_1' = A_1^2 + h^2 I + S_1 A_1 - A_1 S_1$ and equating the real and imaginary parts of both sides, we have

$$D^2 = \left(1 + \frac{f'^2 - f''}{h^2}\right) I$$
$$D' = Q_1 D - D Q_1.$$

Taking the derivative of D^2, and making use of the formula for D', we get

$$\left(1 + \frac{f'^2 - f''}{h^2}\right)' = 0.$$

Thus

$$\frac{f'^2 - f''}{h^2} = c$$

for some constant c. We claim that $c \geq 0$.

To prove this claim, rewrite the equation as $f'^2 - f'' = ce^{4f}$, or

$$y'' = cy^{-3}, \quad \text{where} \quad y = e^{-f} = 1/\sqrt{h} > 0.$$

Either $c = 0$ or $c \neq 0$. Suppose first that $c = 0$. In that case, $y'' = 0$ and $y(t) = c_1 t + c_2$. Since $y > 0$ on the real line, we must have $c_1 = 0$ and $c_2 > 0$. Thus $c = 0$ implies that f and h are constant and $D^2 = I$. Suppose $c \neq 0$. Then multiplying both sides of $y'' = cy^{-3}$ by y' and integrating, we obtain

$$(y')^2 = \frac{-c}{y^2} + c_1,$$

so that

$$yy' = \pm\sqrt{-c + c_1 y^2}$$

Since $c \neq 0$, y is not a constant, thus $c_1 \neq 0$. By solving the above equation, we get

$$y^2 = \frac{1}{c_1}[c + (c_2 \pm c_1 t)^2]$$

for some constant c_2. In order to have $y > 0$ for all t, it is necessary to have $c_1 > 0$ and $c > 0$. Therefore we have proved the claim that $c \geq 0$.

To summarize up to this point, we have proved that

$$A_1 = f'I + ihD \quad \text{and} \quad D^2 = (1+c)I \tag{5}$$

for some constant $c \geq 0$. To obtain a contradiction, we next claim that for each t, there exists a complex number $\lambda(t)$ such that 1 is an eigenvalue of the matrix

$$\begin{bmatrix} \lambda(t)A_1(t) & \overline{\lambda(t)}h(t)I \\ \lambda(t)h(t)I & \overline{\lambda(t)A_1(t)} \end{bmatrix}.$$

This of course will contradict Condition ($\star$).

Now fix a t and, with this understood, we will suppress any reference to t in the following argument. The preceding claim is seen by a direct computation to be equivalent to the assertion that 0 is an eigenvalue of

$$(I - \overline{\lambda A_1})(I - \lambda A_1) - |\lambda|^2 h^2 I.$$

Substituting the values of A_1 and D in (5) into this expression, we get $\psi I + 2\Im(\lambda)hD$, where $\Im$ denotes the imaginary part of a complex number and

$$\psi = 1 + (ch^2 + f'^2)|\lambda|^2 - 2\mathrm{Re}(\lambda)f'.$$

Since $D^2 = (1+c)I$ by (5), D will have an eigenvalue $\epsilon\sqrt{1+c}$, with $\epsilon = 1$ or -1. Thus $\psi I + 2\Im(\lambda)hD$ has 0 as an eigenvalue iff there exists $\lambda \in \mathbf{C}$ such that

$$\psi + 2\Im(\lambda)h\epsilon\sqrt{1+c} = 0.$$

We now show that such is the case.

Write $\lambda = su - i\epsilon u$, where s, u are real, then the preceding equation in ψ becomes a quadratic equation in u,

$$\alpha u^2 - 2\beta u + 1 = 0, \tag{6}$$

with

$$\alpha = (1+s^2)(ch^2 + f'^2) > 0, \qquad \beta = sf' + h\sqrt{1+c}.$$

First, suppose $c = 0$. We know that $f' = 0$ and therefore $\alpha = 0$, so Equation (6) becomes $-2hu + 1 = 0$, which has the obvious solution

$u = 1/(2h)$. Now let us assume that $c > 0$. If we let $s = \frac{f'\sqrt{1+c}}{ch}$, then $\beta > 0$, and

$$\begin{aligned}\beta^2 - \alpha &= (sf' + h\sqrt{1+c})^2 - (1+s^2)(ch^2 + f'^2) \\ &= h^2 - f'^2 + 2sf'h\sqrt{1+c} - s^2ch^2 \\ &= h^2 + \frac{f'^2}{c} \;>\; 0.\end{aligned}$$

So the quadratic Equation (6) in u will have real solutions. This concludes the proof of Theorem A. q.e.d.

3. Local splitting when curvature is bounded

In this section, we will show that if M^n in Theorem A also has bounded curvature, then the nearby leaves of $\mathcal{L}$ are all parallel to each other, in a sense to be made precise.

Recall that a function f on a complete Riemannian manifold M is said to have *sub-k growth,* if

$$\lim_{i\to\infty} \frac{|f(x_i)|}{d^k(x_i, x_0)} = 0$$

for any sequence $\{x_i\}$ in M with the distance $d(x_i, x_0)$ going to infinity. Here x_0 is a fixed point. If $k = 2$, then it is more common to say that f has *sub-quadratic growth.*

We say that the leaves of $\mathcal{L}$ *vary parallelly* if, within each connected component U_a of U (the open set where the Ricci form has maximum rank r), parallel translation from one point of M to another maps $\mathcal{L}$ onto itself. We would at times express this fact more informally by saying that *the leaves of $\mathcal{L}$ in U_a are parallel to each other.* From the definition of the conullity operator, this is the same as saying that, within U_a, all the conullity operators of $\mathcal{L}$ vanish. In terms of the holonomy group of U_a, this is also equivalent to the fact that each fiber $\mathcal{L}_p$ is an invariant subspace of the (restricted) holonomy group of U_a. By the de Rham decomposition theorem, each point of U_a would have a neighborhood which splits holomorphically and isometrically as $L \times Y^r$ where L is flat and Y has dimension r if the leaves of $\mathcal{L}$ vary parallelly (cf. the proof of the de Rham decomposition theorem for Kähler manifolds on p. 172 of [25]). The theorem we are after is then the following:

Theorem B. *Let M^n be a complete Kähler manifold with nonpositive bisectional curvature, with Ricci rank $r < n$, such that its scalar curvature s has sub-quadratic growth. Then the leaves of the Ricci kernel foliation $\mathcal{L}$ vary parallelly.*

Proof. Let $l \geq 0$ be the smallest integer such that $C_T^{l+1} = 0$ for all conullity operator C_T. The idea of the proof is to show that the scalar curvature s will grow roughly like d^{2l}. In particular, if $l \geq 1$, then s grows at least like d^2 and is therefore not sub-quadratic, contradiction. Therefore $l = 0$ and all conullity operators vanish. The leaves of $\mathcal{L}$ are thus parallel to each other within each component of U.

Fix a point $x \in U$ and a real tangent direction T at x such that $T \in \mathcal{L}$ and $C_T^l \neq 0$. Write $v = \frac{1}{2}(T - \sqrt{-1}JT)$, and denote by $f : \mathbf{C} \to \mathcal{L}_x$ (where by abuse of notation we are now using $\mathcal{L}_x$ to denote the *leaf* of $\mathcal{L}$ passing through x) the holomorphic isometric immersion so that $f(0) = x$ and $f'(0) = v$. Let $\{e_i\}_{i=1}^r$ be a basis of $\mathcal{L}_x^\perp$. Since the bundle $f^*\mathcal{L}^\perp$ is flat, we can extend v and $\{e_i\}$ parallelly along $\mathbf{C}$, and have $\nabla_v e_i = \nabla_{\overline{v}} e_i = 0$. We will use t to denote the standard holomorphic coordinate of $\mathbf{C}$. Because $R_{v***} = 0$, the second Bianchi identity implies that

$$\nabla_v R_{XYZW} = R_{(C_vX)YZW} + R_{X(C_vY)ZW}$$

if X, Y, Z, W are parallel along $f(\mathbf{C})$. So the Ricci tensor H satisfies

$$\nabla_t H_{i\bar{j}} = \sum_{k=1}^r A_{ik} H_{k\bar{j}}, \qquad \nabla_{\bar{t}} H_{i\bar{j}} = \sum_{k=1}^r H_{i\bar{k}} \overline{A_{jk}}$$

(where we continue to observe the convention on the range of indices: $1 \leq i, j, \ldots, \leq r$) and $\nabla_t A = A^2$, $\nabla_{\bar{t}} A = 0$. Here $C_v(e_i) = A_{ij} e_j$. Clearly,

$$A = (I - tA_0)^{-1} A_0, \qquad H = (I - tA_0)^{-1} H_0 (I - \bar{t} A_0^*)^{-1}$$

are solutions (A_0^* denotes the conjugate transpose of A_0). Note that along any geodesic starting from 0, the uniqueness of the solution H given above is guaranteed by that of the first order system of ODEs. Thus on $\mathbf{C}$, we have

$$H = \sum_{i,j=0}^{l} t^i \bar{t}^j A_0^i H_0 (A_0^*)^j.$$

Since the matrix $A_0^l H_0 (A_0^l)^*$ is nonpositive, and $A_0^l \neq 0$, $H_0 < 0$, we know that the trace of $A_0^l H_0 (A_0^l)^*$ is negative. So the scalar curvature s, which is the trace of H, has leading term $c|t|^{2l}$ with $c < 0$. Since $|t|$ dominates (a constant multiple of) the distance function d (from x) in M, this means that s grows like d^{2l}. As previously mentioned, if $l \geq 1$, this would contradict the assumption of sub-quadratic growth on s. q.e.d.

Corollary C. *If M^n is a compact Kähler manifold with nonpositive bisectional curvature which has Ricci rank $r < n$, then the open set U in which the Ricci tensor has maximum rank r in the universal cover $\widetilde{M}$ of M is, locally, holomorphically isometric to $L_a \times Y_a$, where L_a is a complete flat Kähler manifold, and Y_a is a Kähler manifold with nonpositive bisectional curvature and negative Ricci curvature.*

In particular, if the metric of M is real analytic, then $\widetilde{M} = \mathbf{C}^{n-r} \times Y^r$ with Y a simply-connected, complete Kähler manifold with nonpositive bisectional curvature and quasi-negative Ricci curvature.

Recall that the term *quasi-negative* for a tensor means that the tensor is nonpositive everywhere and negative definite somewhere. The last assertion about an M with a real analytic metric follows from Nijenhuis' theorem about real analytic Riemannian manifolds ([24], Theorem 10.8 on p. 101). Indeed, the leaves of $\mathcal{L}$ within each component of U being parallel to each other, the local holonomy groups of M leave $\mathcal{L}$ invariant. But the real analyticity of the metric implies that the holonomy group of M itself coincides with the local holonomy groups, by Nijenhuis' theorem, so $\mathcal{L}$ is globally invariant under the holonomy group. By the de Rham decomposition theorem, the assertion follows.

As a special case, if $M^n \subseteq \mathbf{C}^{n+p}$ is a complex submanifold, equipped with the restriction of a complex Euclidean metric, then M^n has nonpositive bisectional curvature, and its Ricci rank r is equal to its Gauss rank, i.e., the dimension of the image of the Gauss map, which sends a point $x \in M$ to the tangent space $T_x M$, considered as a point in the Grassmannian $\mathrm{Gr}_{\mathbf{C}}(n, n+p)$. So as a special case of Theorem B, we have the following:

Corollary D. *Let $M^n \subseteq \mathbf{C}^{n+p}$ be a complex submanifold equipped with the restriction of an Euclidean metric. If M^n is complete, with Gauss rank $r < n$, and the scalar curvature of M has sub-quadratic growth, then M^n is a cylinder: $M^n = \mathbf{C}^{n-r} \times N^r$, where $N^r \subseteq \mathbf{C}^{n-r+p}$ (with Gauss rank r).*

Of course the immersed case goes as well. For Euclidean submanifold $M^n \subseteq \mathbf{C}^{n+p}$ with Gauss rank $r < n$, the Ricci kernel foliation $\mathcal{L}$ is just the Gauss foliation, i.e., the kernel of the differential of the Gauss map. So $\mathcal{L}$ is automatically a holomorphic foliation (even without the completeness assumption). So the proof of Corollary D actually does not need Theorem A, and is thus much easier.

Corollary D may be considered to be a cylinder theorem. Other types of cylinder theorems and related topics are discussed in [17], [38] and [39].

4. A splitting theorem for compact manifolds

Our goal of this section is to prove the following theorem, which partially confirms a conjecture of Yau (see §1) and is the main result of this article.

Theorem E. *Let M^n be a compact Kähler manifold with real analytic metric and nonpositive bisectional curvature. Denote by r its Ricci rank. Then there exists a finite covering M' of M, such that $q : M' \to N^r$ is a holomorphic fiber bundle over a compact Kähler manifold N^r with nonpositive bisectional curvature and $c_1(N) < 0$, while the fiber of q is a complex $(n-r)$ torus T.*

Furthermore, M' is diffeomorphic to $T \times N$, and q is a metric bundle, i.e., $\forall\ x \in N$, there exists a small neighborhood $x \in V \subseteq N$ such that $q^{-1}(V)$ is isometric to $T \times V$.

Proof. When $r = n$, the canonical line bundle K_M of M is nef and big. If K_M is not ample, then the result of Kawamata [22] would imply that M contains a rational curve. But this is impossible since M has nonpositive bisectional curvature. Therefore, K_M must be ample, that is, $c_1(M) < 0$.

In the following, let us assume that $r < n$. By Corollary C, we know that the universal covering space $\widetilde{M}$ of M is holomorphically isometric to $\mathbf{C}^{n-r} \times Y^r$, where Y^r is a simply-connected, complete Kähler manifold with nonpositive bisectional curvature, and the Ricci tensor of Y^r is negative definite somewhere. Denote by Γ the deck transformation group, and by I_1, I_2 the group of holomorphic isometries of $\mathbf{C}^{n-r}$ and Y^r, respectively.

Since any $f \in \Gamma$ has to preserve the product structure, it is in the form $f = (f_1, f_2)$, where $f_i \in I_i$, $i = 1, 2$. Denote by $p_i : \Gamma \to I_i$ the

projection map, and by $\Gamma_i = p_i(\Gamma)$ the image groups for $i = 1, 2$. The main point in the proof of Theorem E is to show that:

Claim 1. The group Γ_2 is discrete.

After Claim 1 is established, the rest of the argument will follow the proof of the main theorem in [10]. For completeness and for the convenience of the reader, we give an outline of this argument as well, but the credit here is due to Eberlein. Let us assume Claim 1 and finish the proof first. Following Eberlein [10], we claim that:

Claim 2. There exists a finite index subgroup $\Gamma' \subseteq \Gamma$ such that Γ'_2 acts freely on Y, and Γ'_1 contains translations only. Here $\Gamma'_i = p_i(\Gamma')$, $i = 1, 2$.

To see this, notice that by Claim 1, we have a compact complex analytic space $Z = Y/\Gamma_2$ and a surjective holomorphic map $M \to Z$ induced by the projection map $\widetilde{M} \to Y$. Denote by $K \times \{1\} \subseteq \Gamma$ the kernel of $p_2 : \Gamma \to \Gamma_2$, then $K \subseteq \Gamma_1$ is a discrete group of isometry on $\mathbf{C}^{n-r}$, with compact quotient because the fibers of $M \to Z$ are compact.

Let $K_0 \subseteq K$ be the intersection $K \cap A$ with the translation group A of $\mathbf{C}^{n-r}$, then $K_0 \cong \mathbf{Z}^{2n-2r}$, and $T = \mathbf{C}^{n-r}/K_0$ is a complex torus. Since both K and A are normalized by Γ_1, so is K_0. That is, for any $f_1 \in \Gamma_1$ with $f_1(x) = Bx + a$, the unitary matrix B must preserve the lattice K_0. So all these B form a finite group Ψ.

If we replace Γ by the kernel of the composition $\Gamma \to \Gamma_1 \to \Psi$, which has finite index in Γ, then we may assume that Γ_1 contains translations only. For the sake of simplicity, we will continue to denote this subgroup by Γ. But keep in mind that it may be a finite index subgroup in the original Γ.

Consider the homomorphism $\rho : \Gamma_2 \to T$, defined by $\rho(f_2) = [f_1]$. Here we identified the translation group A with $\mathbf{C}^{n-r}$ itself. Then the image $\rho(\Gamma_2)$ is a finitely generated abelian group. Its torsion part is therefore a finite group. Write Γ' for the kernel of the map from Γ to the torsion part of $\rho(\Gamma_2)$, then Γ' has finite index in Γ, and clearly Γ'_1 still contains translations only, and $\rho(\Gamma'_2)$ is torsion free. It is easy to see that the latter means that Γ'_2 acts freely on Y. So Claim 2 is proved.

By Claim 2, we have a finite covering $M' = \widetilde{M}/\Gamma'$ of M, and a holomorphic surjection $q : M' \to N$ induced by the projection of $\widetilde{M}$ to Y. Here $N = Y/\Gamma'_2$ is a compact Kähler manifold. q makes M' a holomorphic fiber bundle over N with the fiber being a complex torus

T. M is also isometrically a (flat torus) fiber bundle over N.

The proofs of Lemma 2 and Lemma 4 of [10] can be carried over without any modification, so M' is diffeomorphic to $T\times N$. To do this, one first lifts ρ to a homomorphism $\widetilde{\rho}:\Gamma_2'\to A$, which is possible because the image of ρ is torsion free, then by the fact that A is contractible, one can use a triangulation of N to construct a smooth map $F:Y\to \mathbf{C}^{n-r}$ such that $F(f_2(y))=F(y)+\widetilde{\rho}(f_2)$ for any $f_2\in\Gamma_2'$ and any $y\in Y$. So the diffeomorphism

$$([x],y)\mapsto([x]+F(y),y)$$

of $T\times Y$ descends to a diffeomorphism between $T\times N$ and M'. We refer the readers to Eberlein's paper [10] and the references therein for the details of the post-Claim 1 arguments.

Now let us focus on Claim 1. We will use an idea of Nadel [34], in which he proved that the automorphism group of the universal covering space of a projective manifold with $c_1<0$ must be semi-simple. Here our M does not have $c_1<0$, but the splitting result in Corollary C will enable the stability argument to go through.

Following Eberlein ([12], p. 215), let us consider the group G which is the closure of $\Gamma\cdot A$ in the isometry group of $\widetilde{M}$. Here as before, A is the translation group of $\mathbf{C}^{n-r}$. Since A is a normal abelian subgroup of the Lie group G, the identity component G^0 of G is solvable by Zassenhaus' Lemma ([3], p. 149). So $p_2(G^0)$ is a solvable group where, as usual, p_2 is the projection into the group of holomorphic isometries of Y.

If $p_2(G^0)=\{1\}$, then it is not hard to see that Γ_2 must be discrete (see for instance p. 25 of [10]). So let us assume that $p_2(G^0)\neq\{1\}$ and will derive a contradiction. Let A^* be the last nonidentity subgroup in the derived series for $p_2(G^0)$. Then A^* is abelian, and is normalized by Γ_2 since $p_2(G^0)$ is so.

Denote by V the Lie algebra of $A^*\subseteq \mathrm{Aut}(\widetilde{M})$. Then V is a nontrivial, Γ-invariant, finite dimensional complex vector space. Take a basis $v_1,\dots,v_k$ of V, and view them as holomorphic vector fields on $\widetilde{M}$. Then by Theorem 3.1 of [34], they cannot be linearly dependent over the field of meromorphic functions on $\widetilde{M}$. That is, at a generic point $p\in\widetilde{M}$, $v_1(p),\dots,v_k(p)$ are linearly independent over $\mathbf{C}$.

Note that we can restrict our considerations to $\{x\}\times Y$ for a generic x, and use the fact that Y has negative Ricci curvature near p. Each v_i is tangent to the Y directions, so Theorem 3.1 of [34] applies.

Since A^* is normalized by Γ_2 thus Γ, V will be Γ-invariant, that is, we have a representation

$$\rho : \Gamma \to GL(V).$$

Let E be the flat holomorphic vector bundle over M given by the representation ρ, that is, E is the quotient of $\widetilde{M} \times V$ by Γ acting as

$$f(p, v) = (f(p), \rho(f)(v))$$

for any $f \in \Gamma$. Denote by $T_2 \subseteq T_M$ the subbundle of the holomorphic tangent bundle of M corresponding to the Y factor. The map

$$(p, v) \mapsto v_p$$

descends to a holomorphic map $\varphi : E \to T_2$. At a generic point $p \in M$, the vectors $v_1(p), \ldots, v_k(p)$ are linearly independent, so φ is injective at p.

Consider the determinant line bundle $\Lambda^k E$ on M. It has $c_1(\Lambda^k E) = 0$ in the de Rham cohomology group since it comes from a representation of $\pi_1(M)$. That is, if h is any Hermitian metric on it, its curvature form Θ_h, which is a pure imaginary closed $(1,1)$ form on M, would be d-exact. Since M is compact Kähler, by the $\partial\overline{\partial}$-lemma, we know that

$$\Theta_h = \partial\overline{\partial} f$$

for some smooth real function f. So the Hermitian metric $h_0 = e^f h$ on the line bundle would have zero curvature: $\Theta_{h_0} \equiv 0$.

Take the k-th wedge product of φ, we get a holomorphic sheaf map $\Lambda^k\varphi : \Lambda^k E \to \Lambda^k T_2$. This means that the bundle $F := (\Lambda^k E)^{-1} \otimes \Lambda^k T_2$ has a nontrivial global holomorphic section $\sigma \in H^0(M, F)$.

Let us equip $\Lambda^k E$ with the metric h_0, and equip $\Lambda^k T_2$ with the induced metric from the original Kähler metric on M. This way F becomes a Hermitian vector bundle. It is easy to see that it has nonpositive curvature in the sense that $\Theta_{w\overline{w}}$ is a nonpositive $(1,1)$ form for any section w of F. Furthermore, at a generic point $p \in M$, there exists a vector $w \in F_p$ and tangent direction $z \in T_p M$ such that $\Theta_{w\overline{w}}(z, \overline{z}) < 0$. The last part is caused by the negativity of the Ricci tensor in the T_2 directions at a generic point $p \in M$.

So if we apply Hopf's maximum principle and the standard Bochner identity to the Laplacian of the function $|\sigma|^2$ on M, the curvature condition would imply that σ is parallel, and is actually identically zero.

This of course will be a contradiction. Here we used the fact that, the curvature of F is given by

$$\Theta^F_{\sigma\overline{\sigma}} = \Theta_{e_1\overline{e_1}} + \cdots + \Theta_{e_k\overline{e_k}}$$

where Θ is the curvature form of $T_2 \subseteq T_M$, $\sigma = \tau \otimes e$, with τ a local section of $(\Lambda^k E)^{-1}$ and $e = e_1 \wedge \cdots \wedge e_k$ is a local section of $\Lambda^k T_2$.

This type of simple-minded application of the Bochner technique was first made in [26]. It can also be found on p. 52 of [23] (cf. Theorem 1.9 and 1.10). This completes the proof of Claim 1, and thus of Theorem E. q.e.d.

Remark. First of all, note that in Theorem E, any finite cover of M' is also a holomorphic and metric fiber bundle with flat torus fiber over a compact Kähler manifold with $c_1 < 0$.

Secondly, by a theorem of Sid Frankel [18], any projective manifold N with $c_1(N) < 0$ admits a finite cover in the form $N_1 \times N_2$, where N_1 is a locally Hermitian symmetric space, and the automorphism group of the universal cover of N_2 is discrete. (Of course N_1 or N_2 does not have to appear). So one can state a combination of this with Theorem E, which we will omit here.

The third point is that the M' in Theorem E (or any other finite cover of M) may not be biholomorphically a product (of T with N), even though it is diffeomorphic to the product space. This will happen when the image of ρ is infinite. The first such construction (where N is a compact Riemann surface of genus $g \geq 2$) is given by Lawson and Yau in [27].

Finally, for the product M of two compact complex manifolds, the space of all Kähler metrics on M with nonpositive bisectional curvature is characterized by the product of the corresponding spaces for the factors, extended by the shift of global holomorphic 1-forms (cf. [46] for more details). When M is a holomorphic fiber bundle or fibration, similar characterizations occur (cf. the paper by Paul Yang [41]).

It is difficult to imagine that the real analyticity assumption in Theorem E is anything but a technical one, although we do not know how to remove it at this point and time. In other words, we believe that Yau's conjecture stated in §1 should hold true in its full generality. In fact, in light of Demailly's notion of nefness for holomorphic vector bundles over a compact Kähler manifold, and the reduction theorem of Demailly, Peternell and Schneider [8], we propose the following conjecture which is a slight generalization of Yau's:

Generalized Yau's Conjecture. *Let M^n be a compact Kähler manifold with nef cotangent bundle in the sense of Demailly. Denote by κ its Kodaira dimension. Then there exists a finite covering M' of M, such that M' is a holomorphic fibration without singular fibers over a projective manifold N^κ of dimension κ, each fiber is a complex torus, and $c_1(N) < 0$.*

Let us assume that this conjecture is true. If M^n is a compact Kähler manifold with nonpositive bisectional curvature and Kodaira dimension κ, then M has nef cotangent bundle. The truth of the conjecture now implies that there exists a holomorphic fibration $f : M' \to N^\kappa$ where M' is a finite cover of M, $c_1(N) < 0$, and each fiber of f is a nonsingular complex torus. Since M' also has nonpositive bisectional curvature (with the pull back metric from M), any complex torus embedded in M' must be flat and totally geodesic. By Yang's result [41], f is a metric bundle, that is, for any $p \in N$, there exists a small neighborhood $p \in U \subseteq N$ such that $f^{-1}(U)$ is isometric to a product $T \times V$ where T is a fiber of f. We can define a metric on U by identifying it with V. It is not hard to see that this will patch up nicely to give a Kähler metric on N which has nonpositive bisectional curvature. Since $c_1(N) < 0$, the Ricci tensor of N has to be negative definite somewhere, thus the Ricci rank of M' (hence of M) is equal to κ.

Thus, an affirmative answer to the above conjecture implies Yau's conjecture. This justifies its name.

So far, very little is known about the Generalized Yau's Conjecture except in the low dimensional cases (e.g., when n is 2 or 3), where the classification theory could kick in to help. In [45], Zhang had some interesting results on related topics. He also formulated the following conjecture: for a projective manifold with nef cotangent bundle, the canonical line bundle K is ample if and only if $\chi(K) > 0$. In [45], he was able to confirm this for $n \leq 4$.

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