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# *D*-EQUIVALENCE AND *K*-EQUIVALENCE

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## Abstract

Let $X$ and $Y$ be smooth projective varieties over $\mathbb{C}$. They are called *$D$-equivalent* if their derived categories of bounded complexes of coherent sheaves are equivalent as triangulated categories, and *$K$-equivalent* if they are birationally equivalent and the pull-backs of their canonical divisors to a common resolution coincide. We expect that the two equivalences coincide for birationally equivalent varieties. We shall provide a partial answer to the above problem in this paper.

## 1. Introduction

Let $X$ be a smooth projective variety. We denote by $D(X) = D^b(\mathrm{Coh}(X))$ the derived category of bounded complexes of coherent sheaves on $X$ (in §6, we shall consider a generalization where $X$ has singularities). It is known that $D(X)$ has a structure of a triangulated category.

**Definition 1.1.** Let $X$ and $Y$ be smooth projective varieties. They are called *$D$-equivalent* if their derived categories $D(X)$ and $D(Y)$ of bounded complexes of coherent sheaves are equivalent as triangulated categories, i.e., there exists an equivalence of categories $\Phi : D(X) \to D(Y)$ which commutes with the translations and sends any distinguished triangle to a distinguished triangle. They are called *$K$-equivalent* if they are birationally equivalent and if there exists a smooth projective variety $Z$ with birational morphisms $f : Z \to X$ and $g : Z \to Y$ such that the pull-backs of the canonical divisors are linearly equivalent: $f^*K_X \sim g^*K_Y$.

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We shall consider the following conjecture which predicts that the $D$ and $K$-equivalences coincide for birationally equivalent varieties.

**Conjecture 1.2.** *Let $X$ and $Y$ be birationally equivalent smooth projective varieties. Then the following are equivalent.*

(1) *There exists an equivalence of triangulated categories $D(X) \cong D(Y)$.*

(2) *There exists a smooth projective variety $Z$ and birational morphisms $f : Z \to X$ and $g : Z \to Y$ such that $f^*K_X \sim g^*K_Y$.*

The category of coherent sheaves $\mathrm{Coh}(X)$ reflects the biregular geometry of $X$, but we expect that the derived category $D(X)$ captures more essential properties such as its birational geometry.

A derived category is a purely algebraic object. But one can sometimes recover the geometry from it:

**Theorem 1.3** ([2])**.** *Let $X$ be a smooth projective variety. Assume that $K_X$ or $-K_X$ is ample.*

(1) *Let $Y$ be another smooth projective variety. Assume that there exists an equivalence of categories $\Phi : D(X) \to D(Y)$ which commutes with the translations. Then there is an isomorphism $\phi : X \to Y$.*

(2) *The group of isomorphism classes of exact autoequivalences of $D(X)$ is isomorphic to the semi-direct product of $\mathrm{Aut}(X)$ and $\mathrm{Pic}(X) \oplus \mathbb{Z}$.*

We shall prove a generalization of Bondal-Orlov's theorem in this paper:

**Theorem 1.4** (= Theorem 2.3)**.** *Let $X$ and $Y$ be smooth projective varieties. Assume that the bounded derived categories of coherent sheaves on them are equivalent as triangulated categories: $D(X) \cong D(Y)$. Then the following hold:*

(0) $\dim X = \dim Y$. *Let $n$ be the common dimension.*

(1) *If $K_X$ (resp. $-K_X$) is nef, then $K_Y$ (resp. $-K_Y$) is also nef, and an equality on the numerical Kodaira dimension $\nu(X) = \nu(K_Y)$ (resp. $\nu(X, -K_X) = \nu(Y, -K_Y)$) holds.*

(2) *If $\kappa(X) = n$, i.e., $X$ is of general type, or if $\kappa(X, -K_X) = n$, then $X$ and $Y$ are birationally equivalent. Moreover, there exist birational morphisms $f : Z \to X$ and $g : Z \to Y$ from a smooth projective variety $Z$ such that $f^*K_X \sim g^*K_Y$.*

We also consider the following conjecture:

**Conjecture 1.5.** *For a given smooth projective variety $X$, there exist only finitely many smooth projective varieties $Y$ up to isomorphisms such that $D(Y)$ is equivalent to $D(X)$ as a triangulated category.*

We shall give an affirmative answer for surfaces in §3 by extending a result of Bridgeland and Maciocia [7]:

**Theorem 1.6** (= Theorems 3.1 and 3.2)**.** *Let $X$ be a smooth projective surface. Then there exist at most finitely many smooth projective surfaces $Y$ up to isomorphism such that the derived categories $D(X)$ and $D(Y)$ are equivalent as triangulated categories. Moreover, if $X$ contains a $(-1)$-curve but is not isomorphic to a relatively minimal elliptic rational surface, then any such $Y$ is isomorphic to $X$.*

The above conjecture can be regarded as a generalization of the conjecture which predicts that there exist only finitely many minimal models up to isomorphisms in a fixed birational equivalence class ([11]). Note that we do not assume the minimality of $X$ in Conjecture 1.5.

We consider the reverse direction from $K$-equivalence to $D$-equivalence in the latter half of the paper. We collect some facts from minimal model theory in §4, and we calculate some examples in arbitrary dimension in §5. In the case of dimension 3, we have a complete answer even for the case of singular varieties:

**Theorem 1.7** (= Theorems 4.6 and 6.5)**.** *Let $X$ and $Y$ be normal projective varieties of dimension 3 having only $\mathbb{Q}$-factorial terminal singularities, and let $\mathcal{X}$ and $\mathcal{Y}$ be their canonical covering stacks. Assume that $X$ and $Y$ are $K$-equivalent. Then the bounded derived categories of coherent orbifold sheaves $D(\mathcal{X})$ and $D(\mathcal{Y})$ are equivalent as triangulated categories.*

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## 2. From $D$-equivalence to $K$-equivalence

We need the concept of Fourier-Mukai transformation:

**Definition 2.1.** Let $X$ and $Y$ be smooth projective varieties, and let $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ be projections. For an object $e \in D(X \times Y)$, we define an *integral functor* $\Phi^e_{X\to Y} : D(X) \to D(Y)$ by

$$\Phi^e_{X\to Y}(a) = p_{2*}(p_1^*(a) \otimes e)$$

for $a \in D(X)$, where $p_1^*$ and $\otimes$ are the right derived functors and $p_{2*}$ is the left derived functor. An integral functor is said to be a *Fourier-Mukai transformation* if it is an equivalence.

The following theorem by Orlov is fundamental for the proof of Theorem 2.3.

**Theorem 2.2** ([19])**.** *Let $\Phi : D(X) \to D(Y)$ be a functor of bounded derived categories of coherent sheaves which commutes with the translations and sends any distinguished triangle to a distinguished triangle. Assume that $\Phi$ is fully faithful and has a right adjoint. Then there exists an object $e \in D(X \times Y)$ such that $\Phi$ is isomorphic to the integral functor $\Phi^e_{X\to Y}$. Moreover, $e$ is uniquely determined up to isomorphism.*

The following theorem guarantees that the $D$-equivalence implies the $K$-equivalence at least for general type varieties.

**Theorem 2.3.** *Let $X$ and $Y$ be smooth projective varieties. Assume that the bounded derived categories of coherent sheaves on them are equivalent as triangulated categories: $D(X) \cong D(Y)$. Then the following hold:*

(0) $\dim X = \dim Y$. *Let $n$ be the common dimension.*

(1) *If $K_X$ (resp. $-K_X$) is nef, then $K_Y$ (resp. $-K_Y$) is also nef, and an equality on the numerical Kodaira dimension $\nu(X) = \nu(Y)$ (resp. $\nu(X, -K_X) = \nu(Y, -K_Y)$) holds.*

(2) *If $\kappa(X) = n$, i.e., $X$ is of general type, or if $\kappa(X, -K_X) = n$, then $X$ and $Y$ are birationally equivalent. Moreover, there exist birational morphisms $f : Z \to X$ and $g : Z \to Y$ from a smooth projective variety $Z$ such that $f^*K_X \sim g^*K_Y$.*

*Proof.* By Theorem 2.2, there exists an object $e \in D(X \times Y)$ such that $\Phi^e_{X\to Y} : D(X) \to D(Y)$ is an equivalence. Let

$$e^\vee = R\mathrm{Hom}_{\mathcal{O}_{X\times Y}}(e, \mathcal{O}_{X\times Y}),$$

the the derived dual object. By the Grothendieck duality, the right and left adjoint functors of $\Phi = \Phi^e_{X\to Y}$ are given by $\Phi^{e^\vee\otimes p_1^*\omega_X[\dim X]}_{Y\to X}$ and $\Phi^{e^\vee\otimes p_2^*\omega_Y[\dim Y]}_{Y\to X}$.

Since $\Phi$ is an equivalence, the right and left adjoint functors of $\Phi = \Phi^e_{X\to Y}$ are isomorphic. By Theorem 2.2 again, we have an isomorphism of objects

$$e^\vee \otimes p_1^*\omega_X[\dim X] \cong e^\vee \otimes p_2^*\omega_Y[\dim Y].$$

It follows immediately that $\dim X = \dim Y$.

Let $H^i(e^\vee)$ be the cohomology sheaves, $\Gamma$ the union of the supports of the $H^i(e^\vee)$ for all $i$, $\Gamma = \bigcup_j Z_j$ the decomposition to irreducible components, and let $\nu_j : \widetilde{Z}_j \to Z_j$ be the normalizations. We take a $Z_j$ and assume that it is an irreducible component of the support of $H^i(e^\vee)$. By taking the determinant of both sides of the isomorphism

$$\nu_j^*(H^i(e^\vee) \otimes p_1^*\omega_X) \cong \nu_j^*(H^i(e^\vee) \otimes p_2^*\omega_Y)$$

we obtain

$$\nu_j^* p_1^*\omega_X^{\otimes m_j} \cong \nu_j^* p_2^*\omega_Y^{\otimes m_j}$$

where $m_j$ is the rank of $\nu_j^* H^i(e^\vee)$.

(1) Since $\Phi^e_{X\to Y}$ is an equivalence, the projections $p_1|_\Gamma : \Gamma \to X$ and $p_2|_\Gamma : \Gamma \to Y$ are surjective. Let $Z_1$ be an irreducible component of $\Gamma$ which dominates $Y$. If $K_X$ is nef, then $m_1\nu_1^* p_1^* K_X \sim m_1\nu_1^* p_2^* K_Y$ is also nef, hence so is $K_Y$. We have also $\nu(X) \geq \nu(\widetilde{Z}_1, \nu_1^* p_2^* K_Y) = \nu(Y)$, thus $\nu(X) = \nu(Y)$. The case where $-K_X$ is nef is proved similarly.

(2) If $\kappa(X) = n$, then there exist an ample $\mathbb{Q}$-divisor $A$ and an effective $\mathbb{Q}$-divisor $B$ on $X$ such that $K_X \sim_{\mathbb{Q}} A+B$ by Kodaira's lemma. Let $Z_1$ be an irreducible component of $\Gamma$ which dominates $X$. Then the projection $p_2|_{Z_1} : Z_1 \to Y$ is quasi-finite on $Z_1 \backslash p_1^{-1}(\mathrm{Supp}(B))$. Indeed, if there exists a curve $C$ which is contained in $Z_1 \cap p_2^{-1}(y)$ for a point $y \in Y$ but not entirely in $p_1^{-1}(\mathrm{Supp}(B))$, then we have $(p_2^* K_Y \cdot C) = 0$ while $(p_1^* K_X \cdot C) \geq (p_1^* A \cdot C) > 0$, a contradiction. Since $\dim X = \dim Y = n$, it follows that $\dim Z_1 = n$ and $Z_1$ also dominates $Y$.

We claim that the set $\Gamma \cap p_1^{-1}(x)$ consisits of 1 point for a general point $x \in X$. Indeed, the previous argument showed already that $\Gamma \cap p_1^{-1}(x)$ is a finite set. If it is not connected, then the natural map $\mathrm{Hom}_{D(X)}(\mathcal{O}_x, \mathcal{O}_x) \to \mathrm{Hom}_{D(Y)}(\Phi(\mathcal{O}_x), \Phi(\mathcal{O}_x))$ is not surjective, a contradiction. Therefore, $Z_1$ is a graph of a birational map. If we take $Z$ to be any resolution of $Z_1$, then the conclusion holds.

The case where $\kappa(X, -K_X) = n$ is proved similarly. q.e.d.

**Remark 2.4.**

(0) The differential geometric picture of the above proof is that the kernel object $e$ of the Fourier-Mukai transformation cannot spread itself if the Ricci curvature is non-vanishing.

(1) In the case where $K_X$ or $-K_X$ is ample, we can also reprove Theorem 1.3 (2) by a similar argument as above.

Indeed, if we take $B = 0$, then $Z_1$ becomes a graph of an isomorphism, say $h$. Now $e$ can be considered as a complex of sheaves on $X$ so that we have $\Phi(\mathcal{O}_x) \cong h(e \otimes_{\mathcal{O}_X} \mathcal{O}_x)$ for any $x \in X$, where the tensor product is taken in $D(X)$. Since

$$\mathrm{Hom}^p_{D(X)}(\Phi(\mathcal{O}_x), \Phi(\mathcal{O}_x)) = 0$$

for any $p < 0$, it follows that there exists an integer $i_0$ such that $e[i_0]$ is a sheaf. Since

$$\mathrm{Hom}_{D(X)}(\Phi(\mathcal{O}_x), \Phi(\mathcal{O}_x)) = \mathbb{C}$$

$e[i_0]$ is invertible.

We note that we did not assume in Theorem 1.3 that $\Phi$ sends any distinguished triangle to a distinguished triangle.

(2) We can extend Theorem 1.3 (2) to the case where $X$ admits quotient singularities if $K_X$ generates the local class group at any point as in [12]. Namely, let $\mathcal{X}$ be the smooth stack which lies naturally above $X$ and let $D(\mathcal{X}) = D^b(\mathrm{Coh}(\mathcal{X}))$ be the derived category of bounded complexes of coherent sheaves on $\mathcal{X}$ (see §6). Then $\mathrm{Auteq}(D(\mathcal{X}))$ is isomorphic to the semi-direct product of $\mathrm{Aut}(X)$ and $\mathrm{Pic}(\mathcal{X}) \oplus \mathbb{Z}$. The proof is the same as in [2].

On the other hand, if $K_X$ does not generate the local class group, then the group of autoequivalences is much larger. For example, if $Y$ is a smooth projective minimal surface of general type and $X$ is its canonical model, then $D(\mathcal{X})$ is equivalent to $D(Y)$. If $C$ is an exceptional curve

of the resolution $Y \to X$, then $\mathcal{O}_C(-1)$ is a 2-spherical object in $D(Y)$ and generates an autoequivalence of infinite order ([21], see also §4).

(3) If $\nu(X) = \nu(Y) = 0$ in Theorem 2.3 (1), then $K_X \sim 0$ if and only if $K_Y \sim 0$ because $\Phi$ commutes with the Serre functors. More generally, it is known that the orders of the canonical divisors coincide ([7] Lemma 2.1).

## 3. Fourier-Mukai partners of surfaces

We have a complete picture of $D$ and $K$-equivalences for surfaces. We start with the case of minimal surfaces:

**Theorem 3.1** ([7])**.** *Let $X$ be a smooth projective surface. Assume that there is no $(-1)$-curve on $X$. Then there exist at most finitely many smooth projective surfaces $Y$ such that the derived categories $D(X)$ and $D(Y)$ are equivalent as triangulated categories.*

We note that there are Fourier-Mukai partners which are not birationally equivalent in the case of abelian or K3 or elliptic surfaces ([16], [17], [19], [20], [7], [10]). It is rather surprising that the existence of a $(-1)$-curve reduces the symmetry drastically:

**Theorem 3.2.** *Let $X$ be a smooth projective surface. Assume that there exists a $(-1)$-curve on $X$. Then there exist at most finitely many smooth projective surfaces $Y$ such that the derived categories $D(X)$ and $D(Y)$ are equivalent as triangulated categories. Moreover, if $X$ is not isomorphic to a relatively minimal elliptic rational surface, then any such $Y$ is isomorphic to $X$.*

*Proof.* We use the notation of the proof of Theorem 2.3. Let $C$ be a $(-1)$-curve and $\Gamma_C = p_1^{-1}(C) \cap \Gamma$. Since $-K_X|_C$ is ample, the projection $p_2|_{\Gamma_C} : \Gamma_C \to Y$ is a finite morphism. We have two possibilities that $\dim \Gamma_C = 1$ or 2.

Assume first that $\dim \Gamma_C = 1$. We take an irreducible component $Z_1$ of $\Gamma$ which dominates $X$, and let $Z_{1,C} = p_1^{-1}(C) \cap Z_1$ and $C' = p_2(Z_{1,C})$. We know that $\dim Z_{1,C} = \dim C' = 1$. It follows that $\dim Z_1 = 2$ and the projection $p_1|_{Z_1} : Z_1 \to X$ is generically finite, hence a birational morphism as in the proof of Theorem 2.3.

If $Z_1$ dominates $Y$, then the other projection $p_2|_{Z_1} : Z_1 \to Y$ is also birational, and $X$ and $Y$ are $K$-equivalent through $Z_1$. Hence $X$ and $Y$ are isomorphic (cf. Lemma 4.2).

Otherwise, we have $p_2(Z_1) = C'$. There exists an open dense subset $U \subset X$ such that $p_1$ induces an isomorphism $p_1^{-1}(U) \cap \Gamma = p_1^{-1}(U) \cap Z_1 \to U$. Take two distinct points $x_1, x_2 \in U$ which correspond to the same point $y \in C'$, i.e., $y = p_2(p_1^{-1}(x_1) \cap \Gamma) = p_2(p_1^{-1}(x_2) \cap \Gamma)$. Then both $\Phi(\mathcal{O}_{x_1})$ and $\Phi(\mathcal{O}_{x_2})$ are supported at $y$, hence $\mathrm{Hom}^p_{D(Y)}(\Phi(\mathcal{O}_{x_1}),\Phi(\mathcal{O}_{x_2}))$ $\neq 0$ for some $p$, a contradiction.

Assume next that $\dim \Gamma_C = 2$. Then $p_2|_{\Gamma_C} : \Gamma_C \to Y$ is dominant. Since $(K_X \cdot C) < 0$, we deduce that $-K_Y$ is nef and $\nu(Y, -K_Y) = 1$. Hence $-K_X$ is also nef and $\nu(X, -K_X) = 1$ by Theorem 2.3. By the classification of surfaces, such a surface is isomorphic to either a minimal elliptic ruled surface or a rational surface with Euler number 12. Since $X$ has a $(-1)$-curve, $X$ is a rational surface. By [7] Proposition 2.3, $Y$ is also a rational surface.

We have the possibilities that $\dim \Gamma = 2$ or 3. If $\dim \Gamma = 2$, then we obtain our result as before. If $\dim \Gamma = 3$, then $X$ and $Y$ are dominated by families of curves whose intersection numbers with the canonical divisors vanish. Thus $X$ and $Y$ are relatively minimal rational elliptic surfaces. By [7] Proposition 4.4, we obtain our result. Here we note that the proof there works also for relatively minimal elliptic surfaces of negative Kodaira dimension. q.e.d.

We can extend some of the above argument to higher dimensional case:

**Proposition 3.3.** *Let $X$ and $Y$ be smooth projective varieties. Assume that $\kappa(X) \geq 0$ but $K_X$ is not nef, and that there is an extremal contraction morphism $\phi : X \to W$ which contracts a prime divisor $D$ to a point. Assume that the derived categories $D(X)$ and $D(Y)$ are equivalent as triangulated categories. Then $X$ and $Y$ are birational and $K$-equivalent.*

*Proof.* We use the notation of the proof of Theorem 2.3. The proof is similar to that of Theorem 3.2.

Let $\Gamma_D = p_1^{-1}(D) \cap \Gamma$. If $\dim \Gamma_D = n - 1$ for $n = \dim X$, then there exists an irreducible component $Z_1$ of $\Gamma$ of dimension $n$ which dominates $X$. Then it follows that $X$ and $Y$ are birational and $K$-equivalent as in the proof of Theorem 3.2.

Assume that $\dim \Gamma_D \geq n$. Since $-K_X|_D$ is ample, the projection $p_2|_{\Gamma_D} : \Gamma_D \to Y$ is a finite morphism. Hence $\dim \Gamma_D = n$, and $-K_Y$ is nef with $\nu(Y, -K_Y) = n - 1$, a contradiction to $\kappa(X)$. q.e.d.

**Remark 3.4.** We cannot expect similar statements for other types

of contractions. For example, let $A$ be an abelian surface, $\hat{A}$ its dual, and $S$ a smooth projective surface which contains a $(-1)$-curve. Let $X = A \times S$ and $Y = \hat{A} \times S$. Then $X$ has a divisorial contraction, $D(X) \cong D(Y)$, but $X$ and $Y$ are not birational in general.

## 4. Flops and minimal models

We consider normal varieties which are not necessarily smooth in this section.

**Definition 4.1.** Let $X$ and $Y$ be normal quasiprojective varieties whose canonical divisors are $\mathbb{Q}$-Cartier divisors. A birational map $\alpha : X \dashrightarrow Y$ is said to be *crepant* if there exists a smooth quasiprojective variety $Z$ with birational projective morphisms $f : Z \to X$ and $g : Z \to Y$ such that $\alpha \circ f = g$ and $f^*K_X \sim_{\mathbb{Q}} g^*K_Y$.

**Lemma 4.2.** *Let $\alpha : X \dashrightarrow Y$ be a crepant birational map between quasiprojective varieties with only terminal singularities. Then $\alpha$ is an isomorphism in codimension 1; i.e., there exist closed subvarieties $E \subset X$ and $F \subset Y$ of codimension at least 2 such that $\alpha$ induces an isomorphism $X \setminus E \cong Y \setminus F$.*

*Proof.* Since $X$ has only terminal singularities, a prime divisor $D$ on $Z$ is mapped by $f$ to a subvariety of codimension at least 2 on $X$ if and only if it appears in the relative canonical divisor $K_{Z/X} = K_Z - f^*K_X$ as an irreducible component. Since a similar statement holds for $g$, our assertion follows from the equality $K_{Z/X} = K_{Z/Y}$. q.e.d.

**Definition 4.3.** A projective variety $X$ with only canonical singularities is called *minimal* if $K_X$ is nef.

The minimality of a variety is characterized by the minimality of its canonical divisor:

**Lemma 4.4.** *Let $X$ and $Y$ be normal projective varieties whose canonical divisors are $\mathbb{Q}$-Cartier divisors. Assume that $X$ and $Y$ are birationally equivalent, $X$ has only canonical singularities and that $K_X$ is nef. Then the inequality $K_X \leq K_Y$ holds in the following sense: Let $Z$ any smooth projective variety with projective birational morphisms $f : Z \to X$ and $g : Z \to Y$. Then there exists a positive integer $m$ such that $m(g^*K_Y - f^*K_X)$ is linearly equivalent to an effective divisor. In particular, any birational map between minimal varieties is crepant.*

*Proof.* We write $f^*K_X + A = g^*K_Y + B$, where $A$ and $B$ are effective divisors without common irreducible components. Since $X$ has only canonical singularities, we may assume that codim $g(\mathrm{Supp}(B)) \geq 2$.

Assuming that $B \neq 0$, we shall derive a contradiction. Let $H$ and $M$ be very ample divisors on $Y$ and $Z$, respectively, and let $n = \dim Y$ and $d = \dim g(\mathrm{Supp}(B))$. We consider a generic surface section

$$S = g^*H_1 \cap \cdots \cap g^*H_d \cap M_1 \cap \cdots \cap M_{n-d-2}$$

for $H_i \in |H|$ and $M_j \in |M|$. By the Hodge index theorem, we have $(g^*H^d \cdot M^{n-d-2} \cdot B^2) < 0$, while $(g^*H^d \cdot M^{n-d-2} \cdot B \cdot (f^*K_X + A - g^*K_Y)) \geq 0$ because $K_X$ is nef and $(g^*H^d \cdot g^*K_Y \cdot M^{n-d-2} \cdot B) = 0$, a contradiction. q.e.d.

We consider a special kind of crepant birational maps called flops:

**Definition 4.5.** Let $X$ and $Y$ be quasiprojective varieties with only canonical singularities, and $D$ a $\mathbb{Q}$-Cartier divisor on $X$. A birational map $\alpha : X \dashrightarrow Y$ is said to be a *$D$-flop*, or simply a *flop*, if there exist a normal quasiprojective variety $W$ and crepant birational projective morphisms $\phi : X \to W$ and $\psi : Y \to W$ which satisfy the following conditions:

(1) $\phi = \psi \circ \alpha$.

(2) $\phi$ and $\psi$ are isomorphisms in codimension 1.

(3) $D$ is $\phi$-ample, and for any $\mathbb{Q}$-Cartier divisor $A$ on $X$, there exist a $\mathbb{Q}$-Cartier divisor $A_0$ on $W$ and a rational number $r$ such that $A \sim_{\mathbb{Q}} \phi^*A_0 + rD$.

(4) Let $D'$ be the strict transform of $D$ on $Y$. Then $-D'$ is $\psi$-ample, and for any $\mathbb{Q}$-Cartier divisor $B$ on $Y$, there exist a $\mathbb{Q}$-Cartier divisor $B_0$ on $W$ and a rational number $r'$ such that $B \sim_{\mathbb{Q}} \psi^*B_0 + r'D'$.

We can define flops of complex analytic spaces instead of quasiprojective varieties in a similar way. In this case, $X$ and $Y$ are complex analytic spaces which are relatively projective over a complex analytic space $W$.

Any crepant birational map between projective varieties with only $\mathbb{Q}$-factorial terminal singularities is expected to be decomposed into a sequence of flops:

**Theorem 4.6.** *Let $\alpha : X \dashrightarrow Y$ be a crepant birational map between projective varieties of dimension 3 with only $\mathbb{Q}$-factorial terminal singularities. Then $\alpha$ is decomposed into a sequence of flops.*

*Proof.* We may assume that the subvariety $E$ of Lemma 4.2 is purely 1-dimensional. We may also assume that any irreducible component of $E$ is the image of a curve on $Z$ which is mapped to a point on $Y$. Since $\alpha$ is crepant, we have $K_X|_E \sim_{\mathbb{Q}} 0$. Let $H$ be an ample Cartier divisor on $Y$ such that $H - K_Y$ is still ample, and let $H'$ be its strict transform on $X$. By construction, any curve $C$ such that $(H' \cdot C) \leq 0$ is contained in $E$. We run the minimal model program with respect to $K_X + \epsilon H'$, where $\epsilon$ is a small positive number, for only those extremal rays on which $H'$ is nonpositive. Then the associated extremal curves are contained in $E$, so we obtain an $H'$-flop. We denote the result after the flop again by the same letters such as $X, E$ and $H'$. After a finite flops, we have no more extremal rays on which $H'$ is nonpositive. Then $H'$ becomes nef and big. Since $H'$ is ample outside $E$, $H' - K_X$ is also nef and big, By the base point free theorem, we obtain a birational morphism $X \to Y$, which should be an isomorphism. q.e.d.

## 5. From $K$-equivalence to $D$-equivalence

The following is a special case of the implication from (2) to (1) in Conjecture 1.2:

**Conjecture 5.1.** *Let $X$ and $Y$ be smooth projective varieties and $\alpha : X \to W \leftarrow Y$ a flop. Then there exists an equivalence of triangulated categories $\Phi : D(X) \to D(Y)$.*

The examples in this section suggest that the integral functor $\Phi^e_{X \to Y}$ for the structure sheaf $e = \mathcal{O}_{X \times_W Y}$ of the subscheme $X \times_W Y \subset X \times Y$ might work.

We consider the following two examples of flops in this section:

**Example 5.2.**

(1) A *standard flop.* Let $X$ be a smooth projective variety of dimension $2m + 1$ for some positive integer $m$, and $E$ a subvariety of $X$. Assume that $E \cong \mathbb{P}^m$, and $N_{E/X} \cong \mathcal{O}_{\mathbb{P}^m}(-1)^{m+1}$. Let $f : Z \to X$ be the blowing-up with center $E$. Then the exceptional divisor $G$ is isomorphic to $\mathbb{P}^m \times \mathbb{P}^m$ and can be blown-down to another direction, so that we obtain a birational morphism $g : Z \to Y$ and a subvariety

$F = g(G) \cong \mathbb{P}^m$. There is a projective variety $W$ with contraction morphisms $\phi : X \to W$ and $\psi : Y \to W$ whose exceptional loci are $E$ and $F$, respectively, and such that $w_0 = \phi(E) = \psi(F)$ is the only singular point of $W$. Then $\alpha = g \circ f^{-1} = \psi^{-1} \circ \phi$ is a flop.

(2) *Mukai's flop.* Let $W_0$ be a generic hypersurface section of $W$ in (1) through the singular point $w_0$. Let $X_0 = \phi^{-1}(W_0)$, $Y_0 = \psi^{-1}(W_0)$, $\phi_0 = \phi|_{X_0}$, and $\psi_0 = \psi|_{Y_0}$. Then $X_0$ and $Y_0$ are smooth, and $\alpha_0 = \psi_0^{-1} \circ \phi_0$ is a flop. The inverse image $\widetilde{Z}_0 = f^{-1}(X_0) = g^{-1}(Y_0)$ is reducible with 2 irreducible components $G$ and $Z_0$, where $Z_0$ is smooth. The restrictions $f_0 = f|_{Z_0}$ and $g_0 = g|_{Z_0}$ are again birational morphisms, and $\alpha_0 = g_0 \circ f_0^{-1}$. We set $G_0 = G \cap Z_0$. Then $f_0(G_0) = E$ and $g_0(G_0) = F$.

We need the following concepts:

**Definition 5.3.** A set $\Omega$ of objects of $D(X)$ is said to a *spanning class* if the following hold for any $a \in D(X)$.

(1) $\mathrm{Hom}^p(a, \omega) = 0$ for all $p \in \mathbb{Z}$ and all $\omega \in \Omega$ implies that $a \cong 0$

(2) $\mathrm{Hom}^p(\omega, a) = 0$ for all $p \in \mathbb{Z}$ and all $\omega \in \Omega$ implies that $a \cong 0$.

For example, the set of point sheaves $\{\mathcal{O}_P\}$ for a smooth projective variety is a spanning class ([3] Example 2.2).

**Definition 5.4.** A *Serre functor* $S_X : D(X) \to D(X)$ is an autoequivalence of triangulated categories which induces bifunctorial isomorphisms

$$\mathrm{Hom}_{D(X)}(a, b) \to \mathrm{Hom}_{D(X)}(b, S_X(a))^*$$

for $a, b \in D(X)$.

If a Serre functor exists, then it is unique up to isomorphisms. If $X$ is smooth and projective, then $S_X(a) = a \otimes \omega_X[\dim X]$ is a Serre functor.

In order to prove that a functor $\Phi : D(X) \to D(Y)$ to be fully faithful, it is sufficient to check it for the spanning class ([3] Theorem 2.3):

$$\Phi : \mathrm{Hom}^p(\omega_1, \omega_2) \cong \mathrm{Hom}^p(\Phi(\omega_1), \Phi(\omega_2))$$

for all $p \in \mathbb{Z}$ and all $\omega_1, \omega_2 \in \Omega$. Moreover, by [5] Theorem 2.3, provided that $\Phi = \Phi^e_{X \to Y}$ is fully faithful, it is an equivalence if and only if it commutes with the Serre functor. Theorefore, in order to prove our

conjecture, we may consider locally over an analytic neighborhood of a point of $W$ and replace the given flop by any other flop which is analytically isomorphic to the original one. If $\Phi$ is proved to be fully faithful, then it is automatically an equivalence in our case.

**Proposition 5.5** ([1]). *In Example* 5.2 (1), *$Z$ is isomorphic to the fiber product $X \times_W Y$ which is a closed subscheme of $X \times Y$, and the functor*

$$g_* f^* = \Phi_{X \to Y}^{\mathcal{O}_Z} : D(X) \to D(Y)$$

*is an equivalence of triangulated categories.*

*Proof.* We may replace $X$, $Y$ and $Z$ by the total space of the vector bundles $N_{E/X}$, $N_{F/Y}$ and $N_{G/Z}$, respectively. We denote by $\mathcal{O}_X(k)$, $\mathcal{O}_Y(l)$ and $\mathcal{O}_Z(k,l)$ the pull-backs of $\mathcal{O}_E(k)$, $\mathcal{O}_F(l)$ and $\mathcal{O}_G(k,l)$, respectively. The set of objects

$$\{\mathcal{O}_X(-k) \in D(X) | k = 0, 1, \ldots, m\}$$

spans $D(X)$. Since $K_{Z/X} \sim mG$, we have

$$\mathcal{O}_X(-k) \xrightarrow{f^*} \mathcal{O}_Z(-k,0) \cong \mathcal{O}_Z(0,k)(kG) \xrightarrow{g_*} \mathcal{O}_Y(k).$$

We have

$$\mathrm{Hom}^p(\mathcal{O}_X(-k_1), \mathcal{O}_X(-k_1)) \cong \mathrm{Hom}^p(\mathcal{O}_Y(k_1), \mathcal{O}_Y(k_1)) \cong 0$$

for $p \neq 0$ and $k_1, k_2 = 0, 1, \ldots, m$ by the vanishing theorem, and

$$\Phi_{X \to Y}^{\mathcal{O}_Z} : \mathrm{Hom}(\mathcal{O}_X(-k_1), \mathcal{O}_X(-k_1)) \cong \mathrm{Hom}(\mathcal{O}_Y(k_1), \mathcal{O}_Y(k_1))$$

because $X$ and $Y$ are isomorphic in codimension 1. Therefore, $\Phi_{X \to Y}^{\mathcal{O}_Z}$ is an equivalence by the remarks preceding to the proposition. q.e.d.

**Lemma 5.6** ([8]). *Let $\pi_X : X \to S$ and $\pi_Y : Y \to S$ be smooth projective morphisms from smooth quasiprojective varieties to a smooth quasiprojective curve. Let $s_0 \in S$ be a point, and let $X_0 = \pi_X^{-1}(s_0)$ and $Y_0 = \pi_X^{-1}(s_0)$ be fibers. Let $i_{X_0} : X_0 \to X$, $i_{Y_0} : Y_0 \to Y$ and $i_{X \times_S Y} : X \times_S Y \to X \times Y$ be the embeddings. Let $e \in D(X \times_S Y)$ be an object, and let $e_0 = e \otimes \mathcal{O}_{X_0 \times Y_0}$ and $e' = i_{X \times_S Y *}(e)$. Then there is an isomorphism of functors from $D(X_0)$ to $D(Y)$:*

$$i_{Y_0 *} \circ \Phi_{X_0 \to Y_0}^{e_0} \cong \Phi_{X \to Y}^{e'} \circ i_{X_0 *}.$$

*Proof.* Let $i_{X_0\times Y_0} : X_0 \times Y_0 \to X \times_S Y$ be the embedding. Let $p_1 : X \times_S Y \to X$, $p_2 : X \times_S Y \to Y$, $p_{1,0} : X_0 \times Y_0 \to X_0$ and $p_{2,0} : X_0 \times Y_0 \to Y_0$ be projections. For $a \in D(X_0)$, we have

$$\begin{aligned} i_{Y_0*} \circ \Phi^{e_0}_{X_0\to Y_0}(a) &\cong i_{Y_0*}p_{2,0*}(p_{1,0}^*(a) \otimes e_0) \cong p_{2*}i_{X_0\times Y_0*}(p_{1,0}^*(a) \otimes e_0) \\ &\cong p_{2*}(i_{X_0\times Y_0*}p_{1,0}^*(a) \otimes e) \cong p_{2*}(p_1^* i_{X_0*}(a) \otimes e) \\ &\cong \Phi^{e'}_{X\to Y} \circ i_{X_0*}(a). \end{aligned}$$

q.e.d.

**Corollary 5.7.** *In Example* 5.2 (2), *the functor*

$$\Phi^{\mathcal{O}_{\tilde{Z}_0}}_{X_0\to Y_0} : D(X_0) \to D(Y_0)$$

*is an equivalence of triangulated categories.*

*Proof.* Since $Z = X \times_W Y$ is a subscheme of $X \times_S Y$, we have the following isomorphisms:

$$i_{X_0*}\Phi^{\mathcal{O}_{\tilde{Z}_0}(mG)}_{Y_0\to X_0}\Phi^{\mathcal{O}_{\tilde{Z}_0}}_{X_0\to Y_0} \cong \Phi^{\mathcal{O}_Z(mG)}_{Y\to X}\Phi^{\mathcal{O}_Z}_{X\to Y} i_{X_0*} \cong i_{X_0*}.$$

For any $a \in D(X_0)$, let $b \in D(X_0)$ be the cone of the natural morphism

$$a \to \Phi^{\mathcal{O}_{\tilde{Z}_0}(mG)}_{Y_0\to X_0}\Phi^{\mathcal{O}_{\tilde{Z}_0}}_{X_0\to Y_0}(a).$$

Then $i_{X_0*}(b) \cong 0$, hence $b \cong 0$. q.e.d.

The following concept is useful for constructing autoequivalences of derived categories.

**Definition 5.8** ([21])**.** An object $s \in D(X)$ is called *$n$-spherical* if

$$\mathrm{Hom}^p_{D(X)}(s,s) \cong \begin{cases} \mathbb{C} & \text{if } p = 0, n \\ 0 & \text{otherwise.} \end{cases}$$

The *twisting functors* $T_s, T'_s : D(X) \to D(X)$ are defined such that the following triangles are distinguished:

$$\begin{aligned} R\mathrm{Hom}_X(s,a) \otimes s \to a \to T_s(a) \to R\mathrm{Hom}_X(s,a) \otimes s[1] \\ T'_s(a) \to a \to R\mathrm{Hom}_{\mathbb{C}}(R\mathrm{Hom}_X(a,s), s) \to T'_s(a)[1] \end{aligned}$$

where $R\mathrm{Hom}_X$ denotes the derived global Hom. If $s$ is $n$-spherical for $n = \dim X$, then $T_s$ and $T'_s$ are equivalences and $T_s \circ T'_s \cong \mathrm{Id}_{D(X)}$.

**Example 5.9.**

(1) $\mathcal{O}_E$ in Example 5.2 (1) is a $(2m+1)$-sherical object. Indeed, since $N_{E/X} \cong \mathcal{O}_E(-1)^{m+1}$, we have

$$\mathcal{E}xt^p_{\mathcal{O}_{X_0}}(\mathcal{O}_E, \mathcal{O}_E) \cong \bigwedge^p (\mathcal{O}_E(-1)^{m+1}).$$

Hence

$$\mathrm{Hom}^p_{D(X)}(\mathcal{O}_E, \mathcal{O}_E) \cong \begin{cases} \mathbb{C} & \text{if } p = 0, 2m+1 \\ 0 & \text{otherwise.} \end{cases}$$

(2) $\mathcal{O}_E$ in Example 5.2 (2) is *not* a $2m$-sherical object. Indeed, since $N_{E/X_0} \cong \Omega^1_E$, we have

$$\mathcal{E}xt^p_{\mathcal{O}_{X_0}}(\mathcal{O}_E, \mathcal{O}_E) \cong \Omega^p_E.$$

Hence

$$\mathrm{Hom}^p_{D(X_0)}(\mathcal{O}_E, \mathcal{O}_E) \cong \begin{cases} \mathbb{C} & \text{if } p = 0, 2, \ldots, 2m \\ 0 & \text{otherwise.} \end{cases}$$

There is some relationship between the flops and the twistings.

**Example 5.10.** If $m = 1$ in Example 5.2 (1), then there are isomorphisms

$$\Phi^{\mathcal{O}_Z}_{Y\to X} \circ \Phi^{\mathcal{O}_Z}_{X\to Y}(\mathcal{O}_X(-k)) \cong T'_{\mathcal{O}_E(-1)}(\mathcal{O}_X(-k))$$

for $k = 0, 1$.

Indeed, we have

$$\begin{array}{ccccc} \mathcal{O}_X & \xrightarrow{\Phi^{\mathcal{O}_Z}_{X\to Y}} & \mathcal{O}_Y & \xrightarrow{\Phi^{\mathcal{O}_Z}_{Y\to X}} & \mathcal{O}_X \\ \mathcal{O}_X(-1) & \xrightarrow{\Phi^{\mathcal{O}_Z}_{X\to Y}} & \mathcal{O}_Y(1) & \xrightarrow{\Phi^{\mathcal{O}_Z}_{Y\to X}} & \mathcal{I}_E(-1) \end{array}$$

where $\mathcal{I}_E$ is the ideal sheaf of $E$ in $X$. On the other hand,

$$R\mathrm{Hom}_X(\mathcal{O}_X, \mathcal{O}_E(-1)) = 0, \quad R\mathrm{Hom}_X(\mathcal{O}_X(-1), \mathcal{O}_E(-1)) = \mathbb{C}$$

hence

$$T'_{\mathcal{O}_E(-1)}(\mathcal{O}_X) = \mathcal{O}_X, \quad T'_{\mathcal{O}_E(-1)}(\mathcal{O}_X(-1)) = \mathcal{I}_E(-1).$$

**Example 5.11.** In Example 5.2 (2), there are isomorphisms

$$\Phi^{\mathcal{O}_{Z_0}}_{X_0\to Y_0}\circ(\Phi^{\mathcal{O}_{\tilde{Z}_0}}_{X_0\to Y_0})^{-1}(\mathcal{O}_{Y_0}(k))\cong T_{\mathcal{O}_F(-1)}(\mathcal{O}_{Y_0}(k))$$

for $k=0,1,\dots,m$.

Indeed, since

$$\begin{aligned}&\mathrm{Hom}^p_{D(Y_0)}(\mathcal{O}_F(-1),\mathcal{O}_{Y_0}(k))\\ &\cong \mathrm{Hom}_{D(Y_0)}(\mathcal{O}_{Y_0}(k),\mathcal{O}_F(-1)[2m-p])^*\\ &\cong\begin{cases}0 & \text{if } k=0,\dots,m-1\\ 0 & \text{if } k=m \text{ and } p\neq m\\ \mathbb{C} & \text{if } k=m \text{ and } p=m.\end{cases}\end{aligned}$$

we have

$$T_{\mathcal{O}_F(-1)}(\mathcal{O}_{Y_0}(k))\cong\mathcal{O}_{Y_0}(k)$$

for $k=0,1,\dots,m-1$, and

$$\mathcal{O}_F(-1)[-m]\to\mathcal{O}_{Y_0}(m)\to T_{\mathcal{O}_F(-1)}(\mathcal{O}_{Y_0}(m))\to\mathcal{O}_F(-1)[-m+1]$$

is a distinguished triangle, where the first arrow is nontrivial.

On the other hand, we have an exact sequence

$$0\to\mathcal{O}_X\to\mathcal{O}_X\to\mathcal{O}_{X_0}\to 0$$

where the first arrow is the multiplication by an equation of $W_0\subset W$. Hence

$$\Phi^{\mathcal{O}_{\tilde{Z}_0}}_{X_0\to Y_0}(\mathcal{O}_{X_0}(-k))\cong\mathcal{O}_{Y_0}(k)$$

for $k=0,1,\dots,m$. If $k=0,1,\dots,m-1$, then we also have

$$\mathcal{O}_{X_0}(-k)\xrightarrow{f_0^*}\mathcal{O}_{Z_0}(-k,0)\cong\mathcal{O}_{Z_0}(0,k)(kG_0)\xrightarrow{g_{0*}}\mathcal{O}_{Y_0}(k)$$

because $K_{Z_0/Y_0}=(m-1)G_0$.

For $k=m$, we have an exact sequence

$$0\to\mathcal{O}_{Z_0}(0,m)((m-1)G_0)\to\mathcal{O}_{Z_0}(0,m)(mG_0)\to\omega_{G_0}(0,m)\to 0.$$

Since $g_{0*}(\omega_{G_0})\cong\omega_F[-m+1]\cong\mathcal{O}_F(-m-1)[-m+1]$, we obtain a distinguished triangle

$$\mathcal{O}_F(-1)[-m]\to\mathcal{O}_{Y_0}(m)\to g_{0*}f_0^*(\mathcal{O}_{X_0}(-m))\to\mathcal{O}_F(-1)[-m+1].$$

We claim that the first arrow is nontrivial as an element of

$$\mathrm{Hom}_{D(Y_0)}(\mathcal{O}_F(-1)[-m], \mathcal{O}_{Y_0}(m)) \cong \mathbb{C}.$$

Indeed, if not, then we would have

$$\begin{aligned}&\mathrm{Hom}_{D(Y_0)}(\mathcal{O}_F(-1)[-m], \mathcal{O}_{Y_0}(m))\\ &\cong \mathrm{Hom}^1_{D(Y_0)}(g_{0*}f_0^*(\mathcal{O}_{X_0}(-m)), \mathcal{O}_{Y_0}(m))\end{aligned}$$

but

$$\begin{aligned}&\mathrm{Hom}^1_{D(Y_0)}(g_{0*}f_0^*(\mathcal{O}_{X_0}(-m)), \mathcal{O}_{Y_0}(m))\\ &\cong \mathrm{Hom}^1_{D(Z_0)}(f_0^*(\mathcal{O}_{X_0}(-m)), g_0^!\mathcal{O}_{Y_0}(m))\\ &\cong \mathrm{Hom}^1_{D(Z_0)}(\mathcal{O}_{Z_0}(0,m)(mG_0), \mathcal{O}_{Z_0}(0,m)((m-1)G_0))\\ &\cong H^1(Z_0, \mathcal{O}_{Z_0}(-G_0)) \cong 0\end{aligned}$$

a contradiction. Therefore, we have the desired isomorphisms.

**Proposition 5.12.** *In Example* 5.2 (2), *if* $m \geq 2$, *then the functor*

$$g_{0*}f_0^* = \Phi^{\mathcal{O}_{Z_0}}_{X_0 \to Y_0} : D(X_0) \to D(Y_0)$$

*is not an equivalence.*

*Proof.* Let us write $\Phi = \Phi^{\mathcal{O}_{Z_0}}_{X_0 \to Y_0}$ and $a = \mathcal{O}_{X_0}(-m)$. We consider a spectral sequence

$$E_2^{p,q} = \bigoplus_{i \in \mathbb{Z}} \mathrm{Ext}^p(H^i(\Phi(a)), H^{q+i}(\Phi(a))) \Rightarrow \mathrm{Hom}^{p+q}_{D(Y_0)}(\Phi(a), \Phi(a))$$

given by the last line of [24] 4.6.10. We have

$$H^q(\Phi(a)) \cong \begin{cases} \mathcal{O}_{Y_0}(m) & \text{if } q = 0 \\ \mathcal{O}_F(-1) & \text{if } q = m-1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathrm{Ext}^p(\mathcal{O}_{Y_0}(m), \mathcal{O}_{Y_0}(m)) \cong \mathrm{Ext}^p(\mathcal{O}_{X_0}(-m), \mathcal{O}_{X_0}(-m)) = 0 \text{ for } p \neq 0$$

$$\mathrm{Ext}^p(\mathcal{O}_{Y_0}(m), \mathcal{O}_F(-1)) \cong \begin{cases} \mathbb{C} & \text{if } p = m \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{Ext}^p(\mathcal{O}_F(-1), \mathcal{O}_{Y_0}(m)) \cong \begin{cases} \mathbb{C} & \text{if } p = m \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{Ext}^p(\mathcal{O}_F(-1), \mathcal{O}_F(-1)) \cong \begin{cases} \mathbb{C} & \text{if } p = 0, 2, \dots, 2m \\ 0 & \text{otherwise.} \end{cases}$$

Then the terms $E_2^{p,0}$ for $p = 2, \dots, 2m-2$ survive, hence $\mathrm{Hom}^p_{D(X_0)}(a, a)$ and $\mathrm{Hom}^p_{D(Y_0)}(\Phi(a), \Phi(a))$ are not isomorphic for these $p$. q.e.d.

**Remark 5.13.** After this paper was written, Jan Wierzba informed us that Corollary 5.7 and Proposition 5.12 were already proved by Namikawa [18], though the proofs are different. Combining with a result in [9] or [26] (see also [14]), we obtain the implication from (2) to (1) in Conjecture 1.2 in the case of symplectic projective manifolds of dimension 4.

## 6. Flops of terminal 3-folds

We shall deal with singular verieties in this section.

The smoothness of the given varieties is an important assumption for the study of derived categories. For example, any coherent sheaf on a smooth projective variety has a finite locally free resolution, hence the Serre functor exists.

We can compare our situation with the deformation theory of maps from curves to varieties. The latter is not applicable to singular varieties because the smoothness assumption is essential for a good obstruction theory. However it provides deep results such as the theory of rationally connected varieties.

We can still deal with singular varieties as if they are smooth in some cases:

(1) If $X$ is a variety with only quotient singularities, then we consider a smooth stack $\mathcal{X}$ above $X$ as a natural substitute (cf. [12]).

(2) If $X$ has only hypersurface singularities, then we embed $X$ into a smooth variety by deformations (cf. [8]).

(3) If $X$ is a normal crossing variety, then we replace $X$ by its smooth hypercovering (cf. [15]).

We consider a mixture of (1) and (2) in this section.

**Definition 6.1.** Let $X$ be a normal quasiprojective variety such that the canonical divisor $K_X$ is a $\mathbb{Q}$-Cartier divisor. Each point $x \in X$ has an open neighborhood $U_x$ such that $m_x K_X$ is a principal Cartier divisor on $U_x$ for a minimum positive integer $m_x$. The *canonical covering* $\pi_x : \widetilde{U}_x \to U_x$ is a finite morphism of degree $m_x$ from a normal variety which is etale in codimension 1 and such that $K_{\widetilde{U}_x}$ is a Cartier divisor. The canonical coverings are etale locally uniquely determined, thus we can define the *canonical covering stack* $\mathcal{X}$ as the stack above $X$ given by the collection of canonical coverings $\pi_x : \widetilde{U}_x \to U_x$.

We denote by $D(\mathcal{X}) = D^b(\mathrm{Coh}(\mathcal{X}))$ the derived category of bounded complexes of coherent orbifold sheaves on $\mathcal{X}$ (cf. [12]).

The following was suggested by Burt Totaro.

**Proposition 6.2.** *Let $X$ be a normal projective variety such that the canonical divisor $K_X$ is a $\mathbb{Q}$-Cartier divisor. Then there exists an embedding*

$$\phi : X \to \mathbb{P}(a_1, \ldots, a_N)$$

*to a weighted projective space such that the stack structure on $X$ induced from the natural smooth stack structure of $\mathbb{P}(a_1, \ldots, a_N)$ coincides with the one defined by the canonical coverings.*

*Proof.* Let $H$ be an ample Cartier divisor such that $K_X + H$ is still ample as a $\mathbb{Q}$-Cartier divisor. The ring $R = \bigoplus_{m=0}^{\infty} H^0(X, m(K_X + H))$ is a finitely generated algebra over $\mathbb{C}$. Let $x_1, \ldots, x_N$ be a set of homogeneous generators of $R$ of degree $a_1, \ldots, a_N$. Then we obtain an embedding of $X$ to a weighted projective space

$$\phi : X \to \mathbb{P}(a_1, \ldots, a_N).$$

Since $K_X + H$ is ample, g.c.d.$(a_1, \ldots, a_N) = 1$.

We claim that

$$\text{g.c.d.}(a_1, \ldots, \check{a_i}, \ldots, a_N) = 1$$

for any $i = 1, \dots, N$, i.e., the sequence of integers $(a_1, \dots, a_N)$ is *well-formed*. Indeed, suppose that $(a_2, \dots, a_N) = c \neq 1$. Let $m$ be a sufficiently large integer which is not divisible by $c$, and consider an exact sequence

$$0 \to \mathcal{O}_X((m - a_1)(K_X + H)) \to \mathcal{O}_X(m(K_X + H)) \to \mathcal{F}_m \to 0$$

given by the multiplication by $x_1$, where $\mathcal{F}_m$ is a sheaf on $X_1 = \operatorname{div}(x_1)$. By assumption, we have $H^0(X, (m - a_1)(K_X + H)) \cong H^0(X, m(K_X + H))$, while $H^0(X_1, \mathcal{F}_m) \neq 0$ and $H^1(X, (m-a_1)(K_X+H)) = 0$ for large $m$, a contradiction.

Let us fix a point $p \in X$. Then there exists a homogeneous coodinate, say $x_1$, such that $x_1(p) \neq 0$. We have a commutative diagram

$$\begin{array}{ccccc} U & \xrightarrow{\phi} & U_{x_1} & \xrightarrow{\subset} & \mathbb{P}(a_1, \dots, a_N) \\ {\scriptstyle \pi_U}\uparrow & & \uparrow{\scriptstyle \pi_1} & & \\ \widetilde{U} & \xrightarrow{\widetilde{\phi}} & \widetilde{U}_{x_1} & & \end{array}$$

where $U$ is a small open neighborhood of $p$, $U_{x_1}$ is the open subset of $\mathbb{P}(a_1, \dots, a_N)$ defined by $x_1 \neq 0$, $\pi_U : \widetilde{U} \to U$ is a canonical covering, and $\pi_1 : \widetilde{U}_{x_1} \to U_{x_1}$ is the natural covering from an affine space with coordinates

$$x_2 x_1^{-a_2/a_1}, \dots, x_N x_1^{-a_N/a_1}.$$

Note that both $\pi_U$ and $\pi_1$ are etale in codimension 1.

Since $x_1(p) \neq 0$, we may choose a branch of $x_1^{1/a_1}$ on sufficiently small $U$. Then $\phi$ can be lifted to a morphism $\widetilde{\phi} : \widetilde{U} \to \widetilde{U}_{x_1}$ which we can check to be etale. Therefore, the two stack structures coincide. q.e.d.

**Remark 6.3.**

(1) By the proposition, any coherent orbifold sheaf on the canonical covering stack $\mathcal{X}$ has a surjection from a locally free orbifold sheaf on $\mathcal{X}$. But the Serre functor for the category $D(\mathcal{X})$ does not exist in general.

(2) Totaro ([22]) proved the following resolution theorem: on a smooth orbifold whose coarse moduli space is a separated scheme, any coherent orbifold sheaf has a finite resolution by locally free orbifold sheaves.

We still have a good spanning class for terminal 3-folds:

**Lemma 6.4.** *Let $X$ be a normal projective variety of dimension 3 with only terminal singularities, $m_x$ the index of $K_X$ at $x \in X$, and $\mathcal{X}$ the canonical covering stack of $X$. Then the set $\{\mathcal{O}_x(iK_X) | x \in X, 0 \leq i < m_x\}$ is a spanning class of $D(\mathcal{X})$.*

*Proof* (cf. [3] Example 2.2 and [8] Lemma 3.4). Let $a$ be a nonzero object of $D(\mathcal{X})$. Take a point $x_0$ in the support of $a$, and let $q_0$ be the maximal value of $q$ such that $H^q(a)_{x_0} \neq 0$. Then there exists an integer $i_0$ such that $\mathrm{Hom}(H^{q_0}(a), \mathcal{O}_{x_0}(i_0K_X)) \neq 0$. Then

$$\mathrm{Hom}^{-q_0}_{D(X)}(a, \mathcal{O}_{x_0}(i_0K_X)) \neq 0.$$

If the support of $a$ is not contained in the singular locus of $X$, then we take the above point $x_0$ from the smooth locus of $X$. By the Serre duality, we have $\mathrm{Hom}^{n+q_0}(\mathcal{O}_{x_0}, a) \neq 0$, where $n = \dim X$. Otherwise, let $q_1$ be the minimal value of $q$ such that $H^q(a)_{x_0} \neq 0$. Since $X$ has only isolated singularities, there exists an integer $i_1$ such that $\mathrm{Hom}(\mathcal{O}_{x_0}(i_1K_X), H^{q_1}(a)) \neq 0$. Hence $\mathrm{Hom}^{q_1}_{D(X)}(\mathcal{O}_{x_0}(i_1K_X), a) \neq 0$.

q.e.d.

**Theorem 6.5.** *Let $X$ and $Y$ be normal quasiprojective varieties of dimension 3 with only $\mathbb{Q}$-factorial terminal singularities,*

$$X \xrightarrow{\phi} W \xleftarrow{\psi} Y$$

*a flop, and $\mathcal{X}$ and $\mathcal{Y}$ the canonical covering stacks above $X$ and $Y$, respectively. Then the bounded derived categories of coherent orbifold sheaves $D(\mathcal{X})$ and $D(\mathcal{Y})$ are equivalent as triangulated categories.*

*Proof.* The assertion is already proved in the case where $K_X$ is a Cartier divisor by Bridgeland [4] and Chen [8] (see also [23]). Indeed, it is proved that the structure sheaf $\mathcal{O}_Z$ of the fiber product $Z = X \times_W Y$ is quasi-isomorphic to a finite complex of sheaves on $X \times Y$ flat over $X$ so that the integral functor $\Phi^{\mathcal{O}_Z}_{X \to Y} : D(X) \to D(Y)$ is defined and is an equivalence ([8] Lemma 2.1 and Proposition 4.2).

We shall give a new simpler proof, which is based on [23] §4.1, that $\Phi = \Phi^{\mathcal{O}_Z}_{X \to Y} : D(X) \to D(Y)$ is an equivalence in the case where $K_X$ is a Cartier divisor. We may assume that $W$ is a hypersurface singularity of multiplicity 2. Thus $W$ has an involution $\sigma$ such that $W/\langle\sigma\rangle$ is smooth. We may take $Y = X$ and $\psi = \sigma \circ \phi$.

First we prove that $\Phi(\mathcal{O}_X) \cong \mathcal{O}_Y$. Indeed, for any closed point $y \in Y$, the scheme theoretic fiber $g^{-1}(y)$ is isomorphic to the fiber

$f^{-1}(\psi(y))$. We have $H^0(\mathcal{O}_{f^{-1}(\psi(y))}) \cong \mathbb{C}$, hence the natural homomorphism $\mathcal{O}_Y \to R^0 g_* \mathcal{O}_Z$ is an isomorphism.

Any subscheme of $Z$ which is mapped by $g$ to an infinitesimal subscheme $\bar{y}$ of $Y$ supported at $y$ is isomorphic to a subscheme of the product $\overline{C} \times \bar{y}$ for a subscheme $\overline{C}$ of $X$ which is mapped by $\phi$ to an infinitesimal subscheme $\overline{w}$ of $W$ supported at $\phi(y)$. Since $R^1\phi_*\mathcal{O}_X = 0$, it follows that $R^1 g_* \mathcal{O}_Z = 0$.

Let $C_j$ $(j = 1, \dots, t)$ be the exceptional curves of $\phi$, and $L_i$ $(i = 1, \dots, t)$ invertible sheaves on $X$ such that $(L_i \cdot C_j) = \delta_{ij}$. Then $L_i$ are generated by global sections for all $i$. We note that $R^1\phi_* L_i^*$ may not necessarily vanish.

According to [23] §4.1, we construct locally free sheaves $M_i$ and $N_i$ on $X$ by the following exact sequences:

$$0 \to \mathcal{O}_X^{r_i} \to M_i \to L_i \to 0$$
$$0 \to N_i \to \mathcal{O}_X^{s_i} \to L_i \to 0$$

for some integers $r_i, s_i$ such that we have the vanishing higher direct image sheaves $R^1\phi_* M_i^* = 0$ and $R^1\phi_* N_i = 0$. By [23] Proposition 4.1.2, if we take $r_i$ and $s_i$ to be the minimal possible integers under the vanishing conditions, then we have

$$\phi_* N_i \cong \sigma_* \phi_* M_i$$

where we note that $\sigma_* L_i \cong L_i^*$. By construction, $M_i$ and $N_i^*$ are generated by global sections.

It follows that $R^1 g_* f^* N_i = 0$ from $R^1\phi_* N_i = 0$ as before. We consider an exact sequence

$$0 \to g_* f^* N_i \to \mathcal{O}_Y^{s_i} \to g_* f^* L_i \to 0.$$

Since there is a non-natural injection $g_* f^* L_i \to g_* \mathcal{O}_Z$, the sheaf $g_* f^* L_i$ is torsion free. Hence $g_* f^* N_i$ is a reflexive sheaf. Since $\psi_* g_* f^* N_i \cong \phi_* N_i \cong \psi_* M_i$, we conclude that $\Phi(N_i) \cong M_i$.

The set of sheaves $\Omega = \{\mathcal{O}_X, N_1, \dots, N_t\}$ is a spanning class of $D(X)$. $\omega$ is locally free, $\omega^*$ is generated by global sections and $R^1\phi_*\omega = 0$ for any $\omega \in \Omega$. Hence

$$\mathrm{Hom}^p_{D(X)}(\omega_1, \omega_2) = 0$$

for $p > 0$ and $\omega_1, \omega_2 \in \Omega$. Similarly we have

$$\mathrm{Hom}^p_{D(Y)}(\Phi(\omega_1), \Phi(\omega_2)) = 0.$$

Since $X$ and $Y$ are isomorphic in codimension 1, we have

$$\mathrm{Hom}_X(\omega_1, \omega_2) \cong \mathrm{Hom}_Y(\Phi(\omega_1), \Phi(\omega_2)).$$

Therefore, we have proved that $\Phi$ is an equivalence in the case where $K_X$ is a Cartier divisor.

Now we consider the general case. Let $\mathcal{W}$ be the canonical covering stack of $W$. Let $w \in W$ be a point, $W_w$ its small neighborhood on which $m_w K_W$ is a principal Cartier divisor, and $\pi_w : \widetilde{W}_w \to W_w$ a canonical covering. Then $m_w K_X$ and $m_w K_Y$ are also principal Cartier divisors on $X_w = \phi^{-1}(W_w)$ and $Y_w = \psi^{-1}(W_w)$, respectively, and we have corresponding canonical coverings $\pi_X : \widetilde{X}_w \to X_w$ and $\pi_Y : \widetilde{Y}_w \to Y_w$. Thus there are morphisms of stacks $\phi : \mathcal{X} \to \mathcal{W}$ and $\psi : \mathcal{Y} \to \mathcal{W}$. Let

$$\mathcal{Z} = \mathcal{X} \times_{\mathcal{W}} \mathcal{Y}$$

be the fiber product as a stack. Then it is a stack above $Z = X \times_W Y$ where local coverings are given by

$$\widetilde{Z}_w = \widetilde{X}_w \times_{\widetilde{W}_w} \widetilde{Y}_w \to Z_w = X_w \times_{W_w} Y_w.$$

Let $\mathfrak{f} : \mathcal{Z} \to \mathcal{X}$ and $\mathfrak{g} : \mathcal{Z} \to \mathcal{Y}$ be the induced morphisms.

We claim that the functor

$$\mathfrak{g}_* \mathfrak{f}^* : D(\mathcal{X}) \to D(\mathcal{Y})$$

is defined and is an equivalence. Indeed, over an open subset $W_w$, we know already that the integral functor

$$\Phi^{\mathcal{O}_{\widetilde{Z}_w}}_{\widetilde{X}_w \to \widetilde{Y}_w} : D(\widetilde{X}_w) \to D(\widetilde{Y}_w)$$

is an equivalence. Let $\mathcal{X}_w = \mathcal{X}|_{X_w} = [\widetilde{X}_w/G]$, $\mathcal{Y}_w = \mathcal{Y}|_{Y_w} = [\widetilde{Y}_w/G]$, $\mathcal{Z}_w = \mathcal{Z}|_{Z_w} = [\widetilde{Z}_w/G]$, $\mathfrak{f}_w = \mathfrak{f}|_{\mathcal{Z}_w}$ and $\mathfrak{g}_w = \mathfrak{g}|_{\mathcal{Z}_w}$. The Galois group $G = \mathbb{Z}/m_w$ acts equivariantly so that we have $D(\widetilde{X}_w)^G \cong D(\mathcal{X}_w)$ and $D(\widetilde{Y}_w)^G \cong D(\mathcal{X}_w)$ (cf. [7]). Hence we have a well-defined equivalence

$$\mathfrak{g}_{w*} \mathfrak{f}_w^* : D(\mathcal{X}_w) \to D(\mathcal{Y}_w).$$

By Lemma 6.4, we conclude the proof. q.e.d.

**Remark 6.6.** We note that the equivalence $\Phi = \mathfrak{g}_*\mathfrak{f}^* : D(\mathcal{X}) \to D(\mathcal{Y})$ does not induce an equivalence $D(X) \to D(Y)$ of usual derived categories for singular varieties. Indeed, we can construct a similar example as in [12] Example 5.1. There is a skyscraper sheaf $a \in D(\mathcal{X})$ supported over a non-Gorenstein singular point of $X$ such that $\pi_{X*}(a) = 0$ in $D(X)$, but its image $\Phi(a) \in D(\mathcal{Y})$ has a 1-dimensional support so that $\pi_{Y*}(\Phi(a)) \neq 0$ in $D(Y)$.

## References

[1] A.I. Bondal & D.O. Orlov, *Semiorthogonal decompositions for algebraic varieties*, alg-geom/9506012.

[2] A.I. Bondal & D.O. Orlov, *Reconstruction of a variety from the derived category and groups of autoequivalences*, alg-geom/9712029, Compositio Math. **125** (2001) 327–344, MR 2001m:18014.

[3] T. Bridgeland, *Equivalences of triangulated categories and Fourier-Mukai transforms*, math.AG/9809114, Bull. London Math. Soc. **31** (1999) 25–34, MR 99k:18014, Zbl 0937.18012.

[4] T. Bridgeland, *Flops and derived categories*, math.AG/0009053, Invent. Math. **147** (2002) 613–632, MR 1 893 007.

[5] T. Bridgeland, A. King & M. Reid, *Mukai implies McKay: The McKay correspondence as an equivalence of derived categories*, math.AG/9908027, J. Amer. Math. Soc. **14** (2001) 535–554.

[6] T. Bridgeland & A. Maciocia, *Fourier-Mukai transforms for quotient varieties*, math.AG/9811101.

[7] T. Bridgeland & A. Maciocia, *Complex surfaces with equivalent derived categories*, Math. Z. **236**(**4**) (2001) 677–697, MR 2002a:14017.

[8] J.-C. Chen, *Flops and equivalences of derived categories for threefolds with only Gorenstein singularities*, math.AG/0202005.

[9] K. Cho, Y. Miyaoka & N.I. Shepherd-Barron, *Characterizations of projective spaces and applications*.

[10] S. Hosono, B.H. Lian, K. Oguiso & S.-T. Yau, *Counting Fourier-Mukai partners and deformations*, math.AG/0202014.

[11] Y. Kawamata, *On the cone of divisors of Calabi-Yau fiber spaces*, alg-geom/9701006, Internat. J. Math. **8** (1997) 665–687, MR 98g:14043, Zbl 0931.14022.

[12] Y. Kawamata. *Francia's flip and derived categories*, math.AG/0111041.

[13] Y. Kawamata, K. Matsuda & K. Matsuki, *Introduction to the minimal model problem*, in 'Algebraic Geometry Sendai 1985', Advanced Studies in Pure Math. **10** (1987), Kinokuniya and North-Holland, 283–360, MR 89e:14015, Zbl 0672.14006.

[14] S. Kebekus *Characterizing the projective space after Cho, Miyaoka and Shepherd-Barron*, math.AG/0107069, in 'Complex Geometry — Collection of Papers Dedicated to Hans Grauert', Springer-Verlag, 2002, 147–155, MR 1 922 103.

[15] V.A. Lunts, *Coherent sheaves on configuration schemes*, math.AG/0012061, J. Algebra **244** (2001) 379–406, MR 2002k:14024.

[16] S. Mukai, *Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves*, Nagoya Math. J. **81** (1981) 153–175, MR 82f:14036, Zbl 0446.14019.

[17] S. Mukai. *On the moduli space of bundles on $K3$ surfaces*, I, Vector Bundles on Algebraic Varieties (Bombay, 1984), 341–413, Tata Inst. Fund. Res. Stud. Math., **11**, Tata Inst. Fund. Res., Bombay, 1987, MR 88i:14036, Zbl 0674.14023.

[18] Y. Namikawa, *Mukai flops and derived categories*, math.AG/0203287.

[19] D. Orlov, *Equivalences of derived categories and $K3$ surfaces*, alg-geom/9606006, J. Math. Sci. (New York) **84**(**5**) (1997) 1361–1381, MR 99a:14054, Zbl 0938.14019.

[20] D. Orlov, *On equivalences of derived categories of coherent sheaves on abelian varieties*, alg-geom/9712017.

[21] P. Seidel & R.P. Thomas, *Braid group actions on derived categories of coherent sheaves*, math.AG/0001043, Duke Math. Jour. **108** (2001) 37–108, MR 2002e:14030.

[22] B. Totaro, *The resolution property for schemes and stacks*, to appear in J. Reine Angew. Math.

[23] M. Van den Bergh, *Three-dimensional flops and non-commutative rings*, a preliminary manuscript.

[24] J.-L. Verdier, *Des catégories dérivées des catégories abéliennes*, Astérisque **239** (1996), MR 98c:18007.

[25] C.-L. Wang, *$K$-equivalence in birational geometry and characterizations of complex elliptic genera*, math.AG/0103063.

[26] J. Wierzba & J.A. Wisniewski, *Small Contractions of Symplectic 4-folds*, math.AG/0201028.

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