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GENERICITY OF GEODESIC FLOWS WITH POSITIVE TOPOLOGICAL ENTROPY ON S^2

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Abstract

We show that the set of C^∞ Riemannian metrics on S^2 or $\mathbb{R}\mathrm{P}^2$ whose geodesic flow has positive topological entropy is open and dense in the C^2 topology. The proof is partially based on an analogue of Franks' lemma for geodesic flows on surfaces.

To the memory of Michel Herman

1. Introduction

Let M be a closed surface endowed with a C^∞ Riemannian metric g and let ϕ_t^g be the geodesic flow of g. One of the most important dynamical invariants that one can associate to ϕ_t^g to roughly measure its orbit structure complexity is the *topological entropy,* which we shall denote by $h_{\mathrm{top}}(g)$. The first question one asks about $h_{\mathrm{top}}(g)$ is whether it vanishes or not. If $h_{\mathrm{top}}(g) > 0$ a well-known result of A. Katok [25] states that the dynamics of ϕ_t^g presents transverse homoclinic intersections and as a consequence the number of periodic hyperbolic geodesics grows exponentially with length. Moreover, other conclusions of a more geometrical nature can be drawn. Given p and q in M and $T > 0$, define $n_T(p, q)$ as the number of geodesic arcs joining p and q with length $\leq T$.

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R. Mañé showed in [35] that

$$\lim_{T\to\infty} \frac{1}{T} \log \int_{M\times M} n_T(p,q)\; dp\, dq = h_{\mathrm{top}}(g),$$

and therefore if $h_{\mathrm{top}}(g) > 0$, we have that on average the number of arcs between two points grows exponentially with length. Even better, K. Burns and G.P. Paternain showed in [13] that there exists a set of positive area in M such that for any pair of points p and q in that set, $n_T(p,q)$ grows exponentially with exponent $h_{\mathrm{top}}(g)$.

When the Euler characteristic of M is negative a result of E.I. Dinaburg [15] ensures that $h_{\mathrm{top}}(g) > 0$ for *any* metric g. Moreover, Katok in [26] showed that $h_{\mathrm{top}}(g)$ is greater than or equal to the topological entropy of a metric of constant negative curvature and the same area as g, with equality if and only if g itself has constant curvature. Therefore one is left with the problem of describing the behavior of the functional $g \mapsto h_{\mathrm{top}}(g)$ on the two-sphere (projective space) and the two-torus (Klein bottle). It is well-known that these surfaces admit various completely integrable metrics with zero topological entropy: flat surfaces, surfaces of revolution, ellipsoids and Poisson spheres. On the other hand V. Donnay [17] and Petroll [41] showed how to perturb a homoclinic or heteroclinic connection to create transverse intersections. Applying these type of perturbations to the case of an ellipsoid with three distinct axes one obtains convex surfaces with positive topological entropy. Examples of these type were first given by G. Knieper and H. Weiss in [30]. Explicit real analytic convex metrics arising from rigid body dynamics were given by Paternain in [39].

We would like to point out that Katok's theorem mentioned above about the existence of transverse homoclinic intersections when the topological entropy is positive, together with the structural stability of horseshoes implies that the set of C^∞ metrics for which $h_{\mathrm{top}}(g) > 0$ is *open* in the C^r topology for all $2 \leq r \leq \infty$. Therefore, the relevant question about topological entropy for surfaces with nonnegative Euler characteristic is the following: when is the set of C^∞ metrics with positive topological entropy dense?

Let us recall that a Riemannian metric is said to be *bumpy* if all closed geodesics are nondegenerate, that is, if the linearized Poincaré map of every closed geodesic does not admit a root of unity as an eigenvalue. An important tool for proving generic properties of geodesic flows is the bumpy metric theorem which asserts that the set of C^r bumpy metrics is a residual subset of the set of all C^r metrics endowed with

the C^r topology for all $2 \leq r \leq \infty$. The bumpy metric theorem is traditionally attributed to R. Abraham [1], but see also Anosov's paper [3]. Recall that a closed geodesic is said to be *hyperbolic* if its linearized Poincaré map has no eigenvalue of norm one and it is said to be *elliptic* if the eigenvalues of its linearized Poincaré map all have norm one but are not roots of unity. For a surface with a bumpy metric the closed geodesics are all elliptic or hyperbolic.

Let us recall some facts about heteroclinic orbits. Let SM be the unit sphere bundle of (M, g). Given two hyperbolic periodic orbits γ, η, of the geodesic flow ϕ_t, a *heteroclinic orbit* from γ to η is an orbit $\phi_{\mathbb{R}}(\theta)$ such that

$$\lim_{t\to-\infty} d(\phi_t(\theta), \gamma) = 0 \quad \text{and} \quad \lim_{t\to+\infty} d(\phi_t(\theta), \eta) = 0.$$

The orbit $\phi_{\mathbb{R}}(\theta)$ is said to be *homoclinic* to γ if $\eta = \gamma$. The *weak stable* and *weak unstable* manifolds of the hyperbolic periodic orbit γ are defined as

$$W^s(\gamma) := \Big\{\theta \in SM \;\Big|\; \lim_{t\to+\infty} d\big(\phi_t(\theta), \gamma\big) = 0\Big\},$$
$$W^u(\gamma) := \Big\{\theta \in SM \;\Big|\; \lim_{t\to-\infty} d\big(\phi_t(\theta), \gamma\big) = 0\Big\}.$$

The sets $W^s(\gamma)$ and $W^u(\gamma)$ are n-dimensional ϕ_t-invariant immersed submanifolds of the unit sphere bundle, where $n = \dim M$. Then a heteroclinic orbit is an orbit in the intersection $W^u(\gamma) \cap W^s(\eta)$. If $W^u(\gamma)$ and $W^s(\eta)$ are transversal at $\phi_{\mathbb{R}}(\theta)$ we say that the heteroclinic orbit is transverse. A standard argument in dynamical systems (see [27, §6.5.d] for diffeomorphisms) shows that if a flow contains a transversal homoclinic orbit then it has positive topological entropy.[1] (Note that for geodesic flows the closed orbits never reduce to fixed points.) Moreover, if there is a loop $\gamma_0, \ldots, \gamma_N = \gamma_0$ of orbits such that for each i, $0 \leq i \leq N-1$, there is a transverse intersection of $W^u(\gamma_i)$ with $W^s(\gamma_{i+1})$, then γ_0 has a homoclinic orbit and, in particular, ϕ_t has positive entropy.

For the case of the two-torus classical results of G.A. Hedlund [22] and H.M. Morse [37] ensure that for a bumpy metric there are always heteroclinic geodesics. In fact, minimal periodic geodesics are always hyperbolic (for bumpy surfaces) and if we choose in $\mathbb{R}^2$ a strip bounded by two periodic minimal geodesics c^+ and c^- such that it does not contain other periodic minimal geodesics, then there exist minimal geodesics c

[1] In fact it contains a hyperbolic basic set (see below).

and c^* such that c is α-asymptotic to c^- and ω-asymptotic to c^+ and viceversa for c^* (cf. [4, Theorem 6.8]). If these heteroclinic connections are not transverse, they can be perturbed using Donnay's theorem to easily obtain C^r density of metrics with positive topological entropy for all $2 \leq r \leq \infty$, for the two-torus. Similar arguments can be used for the Klein bottle. Clearly no argument like the one just described can be applied to surfaces with no real homology.

One of the main goals of this paper is to show:

Theorem A. *The set of C^∞ Riemannian metrics g on S^2 or $\mathbb{R}\mathrm{P}^2$ for which $h_{\mathrm{top}}(g) > 0$ is dense in the C^2 topology.*

Corollary B. *The set of C^∞ Riemannian metrics g on S^2 or $\mathbb{R}\mathrm{P}^2$ for which $h_{\mathrm{top}}(g) > 0$ is open and dense in the C^2 topology. In particular, if g belongs to this open and dense set, then the number of hyperbolic prime closed geodesics of length $\leq T$ grows exponentially with T.*

From the previous discussion and the theorem we obtain the following corollary which answers a question that Detlef Gromoll posed to the second author in 1988.

Corollary C. *There exists a C^2 open and dense set $\mathcal{U}$ of C^∞ metrics on S^2 such that for any $g \in \mathcal{U}$ there exists a set G of positive g-area such that for any p and q in G we have*

$$\lim_{T\to\infty} \frac{1}{T} \log n_T(p,q) = h_{\mathrm{top}}(g) > 0.$$

The last corollary is sharp in the sense that the sets G with positive g-area cannot be taken to have full area. In [12], Burns and Paternain constructed an open set of C^∞ metrics on S^2 for which there exists a positive area set $U \subset M$, such that for all $(p,q) \in U \times U$,

$$\limsup_{T\to\infty} \frac{1}{T} \log n_T(p,q) < h_{\mathrm{top}}.$$

It seems quite reasonable to conjecture that on any closed manifold the set of C^∞ Riemannian metrics whose geodesic flow exhibits a transverse homoclinic intersection is open and dense in the C^r-topology for any r with $2 \leq r \leq \infty$. Besides its intrinsic interest there is another motivation for looking at this conjecture. Quite recently, A. Delshams, R. de la Llave and T. Seara proved the existence of orbits of

unbounded energy (Arnold diffusion type of phenomenon) for perturbations of geodesic flows with a transverse homoclinic intersection by generic quasiperiodic potentials on any closed manifold [16]. Hopefully our methods here can be further developed to the point where they yield Theorem A for any closed manifold.

Let us describe the main elements that go into the proof of Theorem A. This will clarify at the same time why we can only obtain density in the C^2 topology. An important tool for proving generic properties for geodesic flows is a local perturbation result of W. Klingenberg and F. Takens [29]. We recall its precise statement in Section 2. We shall also see in Section 2 that the bumpy metric theorem, together with Klingenberg-Takens and a new perturbation lemma (cf. Lemma 2.6) imply the analogue of the Kupka-Smale theorem for geodesic flows. Namely, that C^r-generic Riemannian metrics on a manifold of any dimension have closed geodesics whose Poincaré maps have generic $(r-1)$-jets and the heteroclinic intersections of their hyperbolic closed geodesics are transversal.

If there exists an elliptic closed geodesic, using the Kupka-Smale theorem we can approximate our metric by one such that the Poincaré map of the elliptic closed geodesic becomes a generic exact twist map in a small neighborhood of the elliptic fixed point. Then a result of Le Calvez [31] implies that the twist map has positive topological entropy and therefore a metric of class C^4 with a nonhyperbolic closed geodesic can be approximated by one that has positive topological entropy. Details of this argument are given in Section 3. Now we are faced with the following question: how can we proceed if all the closed geodesics are hyperbolic and this situation persists in a neighborhood?

It is not known if the two-sphere (or projective space) admits a metric all of whose closed geodesics are hyperbolic. A fortiori, it is not known if this can happen for an open set of metrics (see [6] for a thorough discussion about the existence of a nonhyperbolic closed geodesic).

Let M be a closed surface and let $\mathcal{R}^1(M)$ be the set of C^r Riemannian metrics on M, $r \geq 4$, all of whose closed geodesics are hyperbolic, endowed with the C^2 topology and let $\mathcal{F}^1(M) = \operatorname{int}\big(\mathcal{R}^1(M)\big)$ be the interior of $\mathcal{R}^1(M)$ in the C^2 topology. Given a metric g let $\operatorname{Per}(g)$ be the union of the hyperbolic (prime) periodic orbits of g.

Using Mañé's techniques on dominated splittings in his celebrated paper [34] and an analogue of Franks' lemma for geodesic flows we will show:

Theorem D. *If $g \in \mathcal{F}^1(M)$, then the closure $\overline{\mathrm{Per}(g)}$ is a hyperbolic set.*

Theorem D together with results of N. Hingston and H.-B Rademacher (cf. [24, 45, 44]), will show (cf. Section 5):

Theorem 1.1. *If a C^4 metric on a closed surface cannot be C^2-approximated by one having an elliptic periodic orbit, then it has a nontrivial hyperbolic basic set.*

This theorem together with the previous discussion will allow us to prove Theorem A.

A *hyperbolic set* of a flow f^t (without fixed points) is a compact invariant subset Λ such that there is a splitting of the tangent bundle of the phase space $T_\Lambda \mathcal{N} = E^s \oplus E^u \oplus E^c$ which is invariant under the differential of f^t: $df^t(E^{s,u}) = E^{s,u}$, E^c is spanned by the flow direction and there exist $0 < \lambda < 1$ and $N > 0$ such that

$$\left\|df^N|_{E^s}\right\| < \lambda^N, \qquad \left\|df^{-N}|_{E^u}\right\| < \lambda^N.$$

A hyperbolic set Λ is said to be *locally maximal* if there exists an open neighborhood U of Λ such that

$$\Lambda = \bigcap_{t\in\mathbb{R}} f^t(U).$$

A hyperbolic *basic set* is a locally maximal hyperbolic set which has a dense orbit. It is said to be *nontrivial* if it is not a single closed orbit.

It is well-known that nontrivial hyperbolic basic sets have positive topological entropy [8]. Moreover, the dynamics on such a set can be modelled on suspensions of topological Markov chains (see [9, 10]). Also, the exponential growth rate of the number of periodic orbits in the basic set is given by the topological entropy ([8]):

$$h_{\mathrm{top}}(f^t|_\Lambda) = \lim_{T\to+\infty} \frac{1}{T} \log \#\{\gamma \,|\, \gamma \text{ periodic orbit of period } \leq T\}.$$

(For the case of diffeomorphisms all these facts can be found in [27, Chapter 18].)

Mañé's theory on dominated splittings is based on Theorem 5.1 below about families of periodic sequences of linear maps: if when perturbing each linear map of such a family, the return linear maps remain

hyperbolic, then their stable and unstable subspaces satisfy a uniform bound

$$\left\|T^N|_{E^s}\right\| \cdot \left\|T^{-N}|_{E^u}\right\| < \lambda_1 < 1, \tag{1}$$

for a fixed iterate N (eventually smaller than the periods), where T is the differential of our dynamical system. A splitting satisfying the uniform bound (1) is called a *dominated splitting*. The uniform bound (1) implies the continuity of the splitting, i.e., a dominated splitting on an invariant subset A of a dynamical system extends continuously to the closure $\overline{A}$.

The family of (symplectic) linear maps in our situation will be the following. Consider a periodic orbit γ of the geodesic flow and cut it into segments of length $\tau(\gamma)$ between ρ and 2ρ for some $\rho > 0$ which is less than the injectivity radius . Construct normal local transversal sections passing through the cutting points. Our family will be given by the set of all linearized Poincaré maps between consecutive sections (cf. proof of Proposition 5.5).

In order to apply Theorem 5.1, we first have to change "linear map" to "symplectic linear map" (cf. Lemma 5.4). Then we have to be able to modify independently each linearized Poincaré map of time τ on the periodic orbits, covering a neighborhood of fixed radius of the original linearized Poincaré map. This is done with the analogue of Franks' lemma for geodesic flows (cf. Section 4). Thus we obtain a dominated splitting on the closure of the set of C^2 persistently hyperbolic closed geodesics.

In Contreras [14] it is shown that a dominated invariant splitting $E \oplus F$ on a non-wandering ($\Omega(\Lambda) = \Lambda$) compact invariant set Λ of a symplectic diffeomorphism is hyperbolic, provided that E and F have the same dimension. This is also proved in Ruggiero [48], when the subspaces E and F are assumed to be lagrangian.

A result of N. Hingston [24] (cf. also Rademacher [24, 45, 44]) states that if all the periodic orbits of a metric in S^2 are hyperbolic, then they are infinite in number. Assuming that they are C^2-persistently hyperbolic, the theory above and Smale's spectral decomposition theorem imply that their closure contains a nontrivial basic set. (Alternatively, we could have also used for the 2-sphere the stronger results of Franks and Bangert [5, 19] which assert that any metric on S^2 has infinitely many geometrically distinct closed geodesics.)

Unfortunately, Mañé's techniques only work in the C^2 topology and that is why in Theorem A we can prove density of positive topological

entropy on the two-sphere or projective space only for the C^2 topology. We remark that the lack of a closing lemma for geodesic flows prevent us from concluding that the geodesic flow of a metric near g is Anosov as one would expect.

At this point it seems important to remark that if instead of considering Riemannian metrics we were considering Finsler metrics or hamiltonians, then Theorem A would have been a corollary of well-known results for hamiltonians (cf. [38, 46, 47, 51]). However, as is well-known, perturbation results within the set of Riemannian metrics are much harder, basically due to the fact that when we change the metric in a neighborhood of a point of the manifold we affect all the geodesics leaving from those points; in other words, even if the size of our neighborhood in the manifold is small, the effect of the perturbation in the unit sphere bundle could be large. This is the main reason why the closing lemma is not known for geodesic flows (cf. [43]), even though there is a closing lemma for Finsler metrics.

Another remark concerns the degree of differentiablity of our metrics. Theorem A holds if instead of requiring our metrics g to be C^∞ we require them to be C^r for $r \geq 2$. Given a C^2 metric g_0, we can approximate it by a C^∞ metric g_1 in the C^2 topology. Afterwards we C^2-approximate g_1 by a C^∞ metric g_2 with a basic set. Then the structural stability theorem works for an open C^2-neighborhood of g_2 of C^2 metrics. We need g_1 to be at least C^4 in three places: in Franks' Lemma 4.1; in the proof of Theorem 1.1; and to make the Poincaré map of an elliptic closed geodesic a twist map. Observe that we actually find a hyperbolic basic set and not just $h_{\mathrm{top}}(g) > 0$. Katok's theorem, which is based on Pesin theory, requires the Riemannian metrics to be of class at least $C^{2+\alpha}$. This restriction is overcome in our case by the use of the structural stability theorem. On the other hand for Corollary C a C^∞ hypothesis on the metrics is essential because, as in Mañé's formula [35], Yomdin's theorem [56] is used.

Related Work. There is an unpublished preprint by H. Weiss [54] that proves that within the set of positively curved 1/4-pinched metrics, those with positive topological entropy are C^r-dense. Weiss uses a result of G. Thorbergsson [52] which asserts that any positively curved 1/4-pinched metric on S^2 has a nonhyperbolic closed geodesic and similar arguments to the ones we give in Section 3, although the Kupka-Smale theorem for geodesic flows is not proven.

Michel Herman gave a wonderful lecture at IMPA [23] in which he

outlined a proof of the following theorem: within the set of C^∞ positively curved metrics on S^2 those with an elliptic closed geodesic are C^2-generic. Among other tools, he used an analogue of Franks' lemma just like the one we prove in the present paper. As a matter of fact, he not only pointed out a mistake in a draft of the manuscript we gave him, but he also explained to us how to solve the self-intersection problem that appears in the proof. This paper is dedicated to his memory.

It is worth mentioning that Herman's motivation was a claim by H. Poincaré [42] that said that any convex surface has a nonhyperbolic closed geodesic without self-intersections. This claim was proved wrong by A.I. Grjuntal [20].

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2. Bumpy metrics and the Kupka-Smale theorem

In this section M is a closed manifold of dimension n. We begin by recalling some elementary facts. Let ϕ_t^g be the geodesic flow of the Riemannian metric g acting on SM, the unit sphere bundle of M. Let $\pi : SM \to M$ be the canonical projection. Non-trivial closed geodesics on M are in one-to-one correspondence to the periodic orbits of ϕ_t^g. For a closed orbit $\gamma = \{\phi_t^g(z) : \ t \in [0, a]\}$ of period $a > 0$ we can define the *Poincaré map* $\mathcal{P}_g(\Sigma, \gamma)$ as follows: one can choose a local hypersurface Σ in SM through v and transversal to γ such that there are open neighborhoods Σ_0, Σ_a of v in Σ and a differentiable mapping $\delta : \Sigma_0 \to \mathbb{R}$ with $\delta(v) = a$ such that the map $\mathcal{P}_g(\Sigma, \gamma) : \Sigma_0 \to \Sigma_a$ given by

$$u \mapsto \phi^g_{\delta(u)}(u),$$

is a diffeomorphism.

Given a closed geodesic $c : \mathbb{R}/\mathbb{Z} \to M$, all iterates $c^m : \mathbb{R}/\mathbb{Z} \to M$; $c^m(t) = c(mt)$ for a positive integer m are closed geodesics too. We shall call a closed geodesic *prime* if it is not the iterate of a shorter curve. Analogously a closed orbit of ϕ_t^g of period a is called prime if a is the minimal period. A closed orbit γ (or the corresponding closed geodesic c) is called *nondegenerate* if 1 is not an eigenvalue of the linearized Poincaré map $P_c := d_{\gamma(0)}\mathcal{P}_g(\Sigma, \gamma)$. In that case, γ is an isolated closed orbit and $\pi \circ \gamma$ an isolated closed geodesic. Moreover, one can apply the implicit function theorem to obtain fixed points of the Poincaré map $\mathcal{P}_g$.

Thus, for a metric $\overline{g}$ near g there is a unique closed orbit $\gamma^{\overline{g}}$ for $\phi^{\overline{g}}$ near γ, given by the implicit function theorem, that we call the *continuation* of c.

A Riemannian metric g is called *bumpy* if all the closed orbits of the geodesic flow are nondegenerate. Since $P_{c^m} = P_c^m$ this is equivalent to saying that if $\exp(2\pi i\lambda)$ is an eigenvalue of P_c, then λ is irrational. Let us denote by $\mathcal{G}^r$ the set of metrics of class C^r endowed with the C^r topology for $r \geq 2$. We state the bumpy metric theorem [1, 3]:

Theorem 2.1. *For $2 \leq r \leq \infty$, the set of bumpy metrics of class C^r is a residual subset of $\mathcal{G}^r$.*

The bumpy metric theorem 2.1 clearly implies the following:

Corollary 2.2. *There exists a residual set $\mathcal{O}$ in $\mathcal{G}^r$ such that if $g \in \mathcal{O}$ then for all $T > 0$, the set of periodic orbits of ϕ^g with period $\leq T$ is finite.*

The canonical symplectic form ω induces a symplectic form on Σ and $\mathcal{P}_g(\Sigma, \gamma)$ becomes a symplectic diffeomorphism. Periodic points of $\mathcal{P}_g(\Sigma, \gamma)$ correspond to periodic orbits near γ. Let N denote the orthogonal complement of $v = \dot{c}(0)$ in the tangent space $T_{\pi(v)}M$. Recall that $N \oplus N$ can be identified with the kernel of the canonical contact form and therefore it is a symplectic vector space with respect to ω. One can choose Σ such that the *linearized Poincaré map* $P_g(\gamma) := d_v\mathcal{P}_g(\Sigma, \gamma)$ is a linear symplectic map of $N \oplus N$ and

$$P_g(\gamma)(J(0), \dot{J}(0)) = (J(a), \dot{J}(a)),$$

where J is a normal Jacobi field along the geodesic $\pi \circ \gamma$ and $\dot{J}$ denotes the covariant derivative along the geodesic. After choosing a symplectic basis for $N \oplus N$ we can identify the group of symplectic linear maps of $N \oplus N$ with the symplectic linear group $\mathrm{Sp}(n-1)$ of $\mathbb{R}^{n-1} \oplus \mathbb{R}^{n-1}$.

Let $J_s^r(n-1)$ be the set of r-jets of symplectic automorphisms of $\mathbb{R}^{n-1} \oplus \mathbb{R}^{n-1}$ that fix the origin. Clearly one can identify $J_s^1(n-1)$ with $\mathrm{Sp}(n-1)$. A set $Q \subset J_s^r(n-1)$ is said to be *invariant* if for all $\sigma \in J_s^r(n-1)$, $\sigma Q \sigma^{-1} = Q$. The property that says that the r-jet of a Poincaré map $\mathcal{P}_g(\Sigma, \gamma)$ belongs to Q is independent of the section Σ.

Let $\gamma = \{\phi_t^{g_0}(v)\}$ be a periodic orbit of period a of the geodesic flow $\phi_t^{g_0}$ of the metric $g_0 \in \mathcal{G}^r$. Let W be an open neighborhood of $\pi(v) \in M$. We choose W so that the geodesic $\pi \circ \gamma$ does not have any self intersection in W. Denote by $\mathcal{G}^r(\gamma, g_0, W)$ the set of metrics $g \in \mathcal{G}^r$

for which γ is a periodic orbit of period a and such that the support of $g - g_0$ lies in W.

We now state the local perturbation result of Klingenberg and Takens [29, Theorem 2].

Theorem 2.3. *If Q is an open and dense invariant subset of $J_s^{r-1}(n-1)$, then there is for every neighborhood $\mathcal{V}$ of g_0 in $\mathcal{G}^r$ a metric $g \in \mathcal{V} \cap \mathcal{G}^r(\gamma, g_0, W)$ such that the $(r-1)$-jet of $\mathcal{P}_g(\Sigma, \gamma)$ belongs to Q.*

As pointed out by Anosov [3], once Theorem 2.1 is proved, combining Corollary 2.2 and Theorem 2.3 one gets:

Theorem 2.4. *Let $Q \subset J_s^{r-1}(n-1)$ be open, dense and invariant. Then there exists a residual subset $\mathcal{O} \subset \mathcal{G}^r$ such that for all $g \in \mathcal{O}$, the $(r-1)$-jet of the Poincaré map of every closed geodesic of g belongs to Q.*

A closed orbit is said to be *hyperbolic* if its linearized Poincaré map has no eigenvalues of modulus 1. If γ is a hyperbolic closed orbit and $\theta = \gamma(0)$, define the strong stable and strong unstable manifolds of γ at θ by

$$W^{ss}(\theta) = \left\{ v \in SM \mid \lim_{t \to +\infty} d\big(\phi_t^g(v), \phi_t^g(\theta)\big) = 0 \right\},$$
$$W^{su}(\theta) = \left\{ v \in SM \mid \lim_{t \to -\infty} d\big(\phi_t^g(v), \phi_t^g(\theta)\big) = 0 \right\}.$$

Define the weak stable and weak unstable manifolds by

$$W^s(\gamma) := \bigcup_{t \in \mathbb{R}} \phi_t\big(W^{ss}(\theta)\big), \qquad W^u(\gamma) := \bigcup_{t \in \mathbb{R}} \phi_t\big(W^{su}(\theta)\big).$$

It turns out that they are immersed submanifolds of dimension

$$\dim W^s(\gamma) = \dim W^u(\gamma) = \dim M.$$

A heteroclinic point is a point in the intersection $W^s(\gamma) \cap W^u(\eta)$ for two hyperbolic closed orbits γ and η. We say that $\theta \in SM$ is a transversal heteroclinic point if $\theta \in W^s(\gamma) \cap W^u(\eta)$, and $T_\theta W^s(\gamma) + T_\theta W^u(\eta) = T_\theta SM$.

In [17], Donnay showed for surfaces how to perturb a heteroclinic point of a metric on a surface to make it transversal. In fact a similar method has been used by Petroll [41] for higher dimensional manifolds and this method actually gives C^r perturbations. Reference [41] is difficult to find, but there is a sketch of the proof in [11].

However, without further analysis, these perturbations do not give control on the size of the subsets where the stable and unstable manifolds are made transversal, as is needed for the proof of the Kupka-Smale theorem. Available proofs of the Kupka-Smale theorem [46, 47] for general hamiltonians do not apply to geodesic flows without further arguments.

Using Corollary 2.2 and Theorem 2.4 we show here how to extend the proof of the Kupka-Smale theorem for hamiltonian flows to the case of geodesic flows, provided that the perturbations used are local. The perturbations in [47] are not local. The perturbations in claim a in [46] are local, they are written for volume preserving flows but they can be adapted to the hamiltonian case. We choose to present in Appendix A another kind of perturbation, suitable for use in the proof of Theorem 2.5 and that could be useful for other types of problems.

Theorem 2.5. *Let $Q \subset J^{r-1}_s(n-1)$ be open, dense and invariant. Then there exists a residual subset $\mathcal{O} \subset \mathcal{G}^r$ such that for all $g \in \mathcal{O}$:*

- *The $(r-1)$-jet of the Poincaré map of every closed geodesic of g belongs to Q.*
- *All heteroclinic points of hyperbolic closed geodesics of g are transversal.*

Proof. We are going to modify the proof of the Kupka-Smale theorem for general hamiltonians to fit our geodesic flow setting. Let $\mathcal{H}^r(N)$ be the set of C^r Riemannian g metrics such that the $(r-1)$-jet of the Poincaré map of every closed geodesic of g with period $\leq N$ belongs to Q. If necessary intersect Q with the set $A \subset J^{r-1}(n-1)$ of jets of symplectic maps whose derivative at the origin has no eigenvalue equal to 1. Then Q is still open, dense and invariant. Since the periodic orbits of period $\leq N$ for such g are generic, there is a finite number of them. Since Q is open and the Poincaré map depends continuously on the Riemannian metric, $\mathcal{H}^r(N)$ is an open subset of $\mathcal{G}^r$. By Theorem 2.4, $\mathcal{H}^r(N)$ is a dense subset of $\mathcal{G}^r$.

Let $\mathcal{K}^r(N)$ be the subset of $\mathcal{H}^r(N)$ of those metrics g such that for any pair of hyperbolic periodic orbits γ and η of g with period $\leq N$, the submanifolds $W^s_N(\gamma)$ and $W^u_N(\eta)$ are transversal, where $W^s_N(\gamma)$ is given by those points $\theta \in W^s(\gamma)$ with $\mathrm{dist}_{W^s(\gamma)}(\theta, \gamma) < N$ and similarly for $W^u_N(\eta)$. Since the stable and unstable manifolds of a hyperbolic orbit depend continuously on compact parts in the C^1 topology with respect to the vector field, $\mathcal{K}^r(N)$ is an open subset of $\mathcal{G}^r$.

It remains to prove that $\mathcal{K}^r(N)$ is dense in $\mathcal{G}^r$, for then the set

$$\mathcal{K}^r := \bigcap_{N\in\mathbb{N}} \mathcal{K}^r(N)$$

is the residual subset we are looking for.

We see first that in order to prove the density of $\mathcal{K}^r(N)$ it is enough to make small local perturbations. Let γ, η be two hyperbolic periodic orbits γ, η of period $\leq N$. Observe that if the two invariant manifolds $W^u(\gamma)$, $W^s(\eta)$ intersect, then they intersect along complete orbits. If they intersect transversally, then they are transversal along the whole orbit of the intersection point.

A fundamental domain for $W^u(\gamma)$ is a compact subset $K \subset W^u(\gamma)$ such that every orbit in $W^u(\gamma)$ intersects K. Such a fundamental domain can be constructed for example inside the strong unstable manifold of γ using Hartman's theorem (one considers the linearization of the Poincaré return map in a neighborhood of γ). Moreover there are fundamental domains which are arbitrarily small and arbitrarily near to γ. Hence it is enough to make $W^u_N(\gamma)$ transversal to $W^s_N(\eta)$ in a fundamental domain for $W^u(\gamma)$.

We will use the following perturbation lemma whose proof will be given after completing the proof of Theorem 2.5:

Lemma 2.6. *For every point $\theta \in W^u(\gamma)$ such that the projection $\pi|_{W^u(\gamma)}$ is a diffeomorphism in a neighborhood of θ, and sufficiently small neighborhoods $\theta \in V \subset \overline{V} \subset U$ in SM, there are Riemannian metrics $\overline{g}$ such that:*

1. *$\overline{g}$ is arbitrarily near g in the C^r-topology;*
2. *g and $\overline{g}$ coincide outside $\pi(U)$;*
3. *γ and η are periodic orbits for $\overline{g}$;*
4. *the connected component of $W^u_N(\gamma) \cap V$ containing θ and the submanifold $W^s(\eta)$ are transversal.*

Let θ be in a fundamental domain K for $W^u(\gamma)$. By the inverse function theorem the projection $\pi|_{W^u(\gamma)}$ is a local diffeomorphism at θ if and only of the tangent space of $W^u(\gamma)$ at θ is transversal to the vertical subspace i.e., $T_\theta W^u(\gamma) \cap \ker d_\theta\pi = \{0\}$.

Observe that the manifolds $W^u(\gamma)$ and $W^s(\gamma)$ are lagrangian. A well-known property of the geodesic flow (cf. [40]) asserts that if W

is a lagrangian subspace, then the set of times t for which $d_\theta\phi_t(W) \cap \ker d_{\phi_t\,\theta}\pi \neq \{0\}$ is discrete and hence at most countable.

By flowing a bit the point θ we obtain another point $\phi_t(\theta)$ satisfying the conditions of the lemma. We can also choose t such that $\pi_t(\theta)$ does not intersect any closed geodesic of period $\leq N$. One chooses the neighborhood U in Lemma 2.6 such that the support of the perturbation $\pi(U)$ does not intersect any closed geodesic of period $\leq N$. Choose a neighborhood V such that $\phi_t(\theta) \in V \subset \overline{V} \subset U$. Applying Lemma 2.6 we obtain a new Riemannian metric $\overline{g}$ such that $\overline{g}|_{\pi(U)^c} = g|_{\pi(U)^c}$ and the connected component of $W^u_N(\gamma) \cap \overline{V}$ containing $\phi_t(\theta)$ is transversal to $W^s(\eta)$. If the perturbation is small enough, flowing backwards a bit we obtain a neighborhood $\overline{V_1}$ of θ, where $W^u_N(\gamma)$ and $W^s(\eta)$ are transversal.

Now cover the compact fundamental domain K by a finite number of these neighborhoods $\overline{V_1}$ and call them, let us say, $W_1, \dots W_r$. Observe that in Lemma 2.6 the perturbations are arbitrarily small but the neighborhood V of transversality is fixed. Since transversality of compact parts of stable (unstable) manifolds is an open condition on g, one can make the perturbation on W_{i+1} small enough so that the invariant manifolds are still transversal on $W_1, \dots, W_i$.

In order to make now $W^u_N(\eta)$ transverse to $W^s_N(\gamma)$ one can use the invariance of the geodesic flow under the flip $F(x,v) = (x,-v)$, so that $W^s(\gamma) = W^u(F(\gamma))$ or repeat the same arguments for the geodesic flow with the time reversed.

This completes the proof of the density of $\mathcal{K}^r(N)$. q.e.d.

Proof of Lemma 2.6. Perhaps, the easiest way to prove Lemma 2.6 is to use a perturbation result for general hamiltonian systems. The Legendre transform $\mathcal{L}(x,v) = g_x(v,\cdot)$ conjugates the geodesic flow with the hamiltonian flow of

$$H(x,p) := \frac{1}{2}\sum_{ij} g^{ij}(x)\, p_i\, p_j$$

on the cotangent bundle T^*M with the canonical (and *fixed*) symplectic form $\omega = \sum_i dp_i \wedge dx_i$. Here $g^{ij}(x)$ is the inverse of the matrix of the Riemannian metric.

Observe that the stable and unstable manifolds are lagrangian submanifolds of T^*M.

Now use a local perturbation result for the hamiltonian flow (e.g., [46, Claim a, Th. 3] or A.3 in Appendix A) to obtain a new hamiltonian

flow which has $W^u(\gamma)$ transversal to the old $W^s(\eta)$ in a neighborhood V. The stable manifold $W^s(\eta)$ only depends on the future times and on the future it only accumulates on the periodic orbit η so up to the perturbation it does not change.

If the perturbation is small enough, then the new piece of unstable manifold $\widetilde{W}^u(\gamma)$ in the support of the perturbation U still projects injectively to M. Let $p : \pi(U) \to T^*_{\pi(U)}M$ be such that the connected component of $\widetilde{W}^u(\gamma) \cap U$ containing γ is

$$\mathrm{Graph}(p) = \big\{ \big(x, p(x)\big) \,\big|\, x \in \pi(U) \big\}.$$

Define a new Riemannian metric by

$$\overline{g}_{ij}(x) = \begin{cases} 2\,H(x,p(x))\; g_{ij}(x) & \text{if } x \in \pi(U), \\ g_{ij}(x) & \text{if } x \notin \pi(U). \end{cases}$$

Then $\overline{g}$ is C^r near g, coincides with g on the complement of $\pi(U)$ and its hamiltonian satisfies

$$\begin{aligned} \overline{H}\big(x,p(x)\big) &= \frac{1}{2} \sum_{ij} \overline{g}^{ij}(x)\, p_i(x)\, p_j(x) \\ &= \frac{1}{2} \sum_{ij} \frac{g^{ij}(x)}{2\,H(x,p(x))}\, p_i(x)\, p_j(x) \\ &= \frac{H(x,p(x))}{2\,H(x,p(x))} = \frac{1}{2}, \qquad \text{for } x \in \pi(U). \end{aligned} \tag{2}$$

Then $\widetilde{W}^u(\gamma)$ is a lagrangian submanifold of T^*M which is in the energy level $\overline{H} \equiv \frac{1}{2}$ of the hamiltonian for $\overline{g}$ and which coincides with the unstable manifold of γ in a neighborhood of γ. By Lemma A.1, $\widetilde{W}^u(\gamma)$ is invariant under the geodesic flow of $\overline{g}$ and hence is the unstable manifold of γ for $\overline{g}$. q.e.d.

3. Twist maps and topological entropy

We say that a homeomorphism of the annulus $f : [0,1] \times S^1 \hookleftarrow$ is a *twist map* if for all $\theta \in S^1$ the function $[0,1] \ni r \mapsto \pi_2 \circ f(r,\theta) \in S^1$ is strictly monotonic.

For a proof of the Birkhoff's normal form below see Birkhoff [7], Siegel and Moser [49] or Le Calvez [32, Th. 1.1]. For a higher dimensional version for symplectic maps see Klingenberg [28, p. 101].

Birkhoff's Normal Form 3.1. *Let f be a C^∞ diffeomorphism defined on a neighborhood of 0 in $\mathbb{R}^2$ such that $f(0) = 0$, f preserves the area form $dx \wedge dy$, and the eigenvalues of $d_0 f$ satisfy $|\lambda| = 1$ and $\lambda^n \neq 1$ for all $n \in \{1, \dots, q\}$ for some $q \geq 4$.*

Then there exists a C^∞ diffeomorphism h, defined on a neighborhood of 0 such that $h(0) = 0$, h preserves the form $dx \wedge dy$ and in complex coordinates $z = x + iy \approx (x, y)$ one has:

$$h \circ f \circ h^{-1}(z) = \lambda\, z\, e^{2\pi i P(z\bar{z})} + o(|z|^{q-1}),$$

where $P(X) = a_1 X + \cdots + a_m X^m$ is a real polynomial of degree m with $2m + 1 < q$.

The coefficients a_i, $1 \leq i \leq m \leq \frac{q}{2} - 1$ are uniquely determined by f.

In polar coordinates the function $g = h \circ f \circ h^{-1}$ is written as

$$(r, \theta) \longmapsto \big(r + \mu(r, \theta),\ \ \theta + \alpha + a_1 r^2 + \cdots + a_m r^{2m} + \nu(r, \theta)\big),$$

where $\lambda = e^{2\pi i\, \alpha}$. If $a_1 \neq 0$ and $|r| \leq \varepsilon$ is small enough, then $\frac{\partial}{\partial r}(\pi_2 \circ g)$ has the same nonzero sign as a_1 and hence g is a twist map in $[0, \varepsilon] \times S^1$.

We shall use following result:

Proposition 3.2 (Le Calvez [33, Remarques p. 34])**.** *Let f be a diffeomorphism of the annulus $\mathbb{R} \times S^1$ such that it is a twist map, it is area preserving, the form $f^*(R\, d\theta) - R\, d\theta$ is exact and:*

(i) *If x is a periodic point for f and q is its least period, the eigenvalues of $d_x f^q$ are not roots of unity.*

(ii) *The stable and unstable manifolds of hyperbolic periodic orbits of f intersect transversally (i.e., whenever they meet, they meet transversally).*

Then f has periodic orbits with homoclinic points.

We are now ready to show:

Proposition 3.3. *Let g_0 be a metric of class C^r, $r \geq 4$, on a surface M with a nonhyperbolic closed geodesic. Then there exists a C^∞ metric g arbitrarily close to g_0 in the C^r topology with a nontrivial hyperbolic basic set. In particular, $h_{\mathrm{top}}(g) > 0$.*

Proof. Let $Q_h \subset J^3_s(1)$ be the subset of 3-jets of symplectic automorphisms which are hyperbolic at the origin. Given $k \in \mathbb{Z}$, $k \neq 0$, let $Q_k \subset J^3_s(1)$ be

$$Q_k = \left\{ \sigma f_{\alpha,a_1} \sigma^{-1} \,\middle|\, \sigma \in J^3_s(1),\, a_1 \neq 0,\, \alpha \notin \{\tfrac{n}{k} \,|\, n \in \mathbb{Z}\} \right\} \cup Q_h,$$

where $f_{\alpha,a_1} : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $f_{\alpha,a_1}(r,\theta) = (r, \theta + \alpha + a_1 r^2) + o(r^3)$ in polar coordinates. Now let $Q = \bigcap_k Q_k$. By Birkhoff's normal form (with $q = 4$), the set Q_k is open, dense and invariant.

By the Kupka-Smale Theorem 2.5 there is a residual subset $\mathcal{O}_k \subset \mathcal{G}^r$ ($r \geq 4$) such that any metric in $\mathcal{O}_k$ has the two properties stated in the theorem. Since $\bigcap_k \mathcal{O}_k$ is a residual subset, we can C^r-approximate g_0, $r \geq 4$, by a C^∞ metric g with a nonhyperbolic closed orbit γ such that the 3-jet of its Poincaré map is in Q and g satisfies the conditions (i) and (ii) in Proposition 3.2.

The symplectic form on TM induced by the Riemannian metric, induces a symplectic form on a local transverse section Σ to γ, which is preserved by the Poincaré map $\mathcal{P}_g(\Sigma, \gamma)$. By Darboux's theorem, using a change of coordinates we can assume that Σ is a neighborhood of 0 in $\mathbb{R}^2$ and that the symplectic form on Σ is the area form of $\mathbb{R}^2$.

By the definition of Q, the Poincaré map $f = \mathcal{P}_g(\Sigma, \gamma)$ is conjugate to a twist map $f_0 = h\, f\, h^{-1}$ when written in polar coordinates. In order to apply Proposition 3.2 we show below a change of coordinates which transforms f_0 into an exact twist map of the annulus $\mathbb{R}^+ \times S^1$. Then the existence of a homoclinic orbit implies the existence of a nontrivial hyperbolic basic set.

Consider the following maps

$$\begin{array}{ccccc}
(x,y) & \longrightarrow & (r,\theta) & \longrightarrow & (\frac{1}{2}r^2, \theta) = (R,\theta) \\
\mathbb{D} & \xrightarrow{P} & \mathbb{R}^+ \times S^1 & \longrightarrow & \mathbb{R}^+ \times S^1 \\
{\scriptstyle f_0}\downarrow & & \downarrow & & \downarrow{\scriptstyle T} \\
\mathbb{D} & \longrightarrow & \mathbb{R}^+ \times S^1 & \longrightarrow & \mathbb{R}^+ \times S^1
\end{array}$$

where $\mathbb{D} = \{z \in \mathbb{C} \,|\, |z| < 1\}$, $P^{-1}(r,\theta) = (r\cos\theta, r\sin\theta)$. Write $G(x,y) = (\frac{1}{2}r^2, \theta) = (R,\theta)$, the upper composition. Then $G^*(R\,d\theta) = \frac{1}{2}(x\,dy - y\,dx) =: \lambda$. Observe that $d\lambda = dx \wedge dy$ is the area form in $\mathbb{D}$. Since $\mathbb{D}$ is contractible, ${f_0}^*(\lambda) - \lambda$ is exact. Then $T^*(R\,d\theta) - R\,d\theta$ is exact. Since $R(r) = \frac{1}{2}r^2$ is strictly increasing on $r > 0$, T is a twist map iff f_0 is a twist map. q.e.d.

4. Franks' lemma for geodesic flows of surfaces

Let $\gamma = \{\phi_t^g(v) \,|\, t \in [0,1]\}$ be a piece of an orbit of length 1 of the geodesic flow ϕ_t^g of the metric $g \in \mathcal{G}^r$. Let Σ_0 and Σ_t be sections at v and $\phi_t(v)$ respectively. We have a Poincaré map $\mathcal{P}_g(\Sigma_0, \Sigma_t, \gamma)$ going from Σ_0 to Σ_t. One can choose Σ_t such that the *linearized Poincaré map*

$$P_g(\gamma)(t) \stackrel{\text{def}}{=} d_v \mathcal{P}_g(\Sigma_0, \Sigma_t, \gamma)$$

is a linear symplectic map from $\mathcal{N}_0 := N(v) \oplus N(v)$ to $\mathcal{N}_t := N(\phi_t v) \oplus N(\phi_t v)$ and

$$P_g(\gamma)(t)(J(0), \dot{J}(0)) = (J(t), \dot{J}(t)),$$

where J is a normal Jacobi field along the geodesic $\pi \circ \gamma$ and $\dot{J}$ denotes the covariant derivative along the geodesic. Let us identify the set of all linear symplectic maps from $\mathcal{N}_0$ to $\mathcal{N}_t$ with the symplectic group

$$\mathrm{Sp}(1) := \{X \in \mathbb{R}^{2\times 2} \,|\, X^* \mathbb{J} X = \mathbb{J}\},$$

where $\mathbb{J} = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$.

Suppose that the geodesic arc $\pi \circ \gamma(t)$, $t \in [0,1]$, does not have any self intersection and let W be a tubular neighborhood of it. We denote by $\mathcal{G}^r(\gamma, g, W)$ the set of metrics $\bar{g} \in \mathcal{G}^r$ for which γ is a piece of orbit of length 1 and such that the support of $\bar{g} - g$ lies in W.

When we apply the following theorem to a piece of a closed geodesic we cannot avoid to have self intersections of the whole geodesic. Suppose given a finite set of non-self intersecting geodesic segments $\mathfrak{F} = \{\eta_1, \ldots, \eta_m\}$ with the following properties:

1. The endpoints of η_i are not contained in W;
2. The segment $\pi \circ \gamma|_{[0,1]}$ intersects each η_i transversally.

Denote by $\mathcal{G}^r(\gamma, g, W, \mathfrak{F})$ the set of metrics $\bar{g} \in \mathcal{G}^r(\gamma, g, W)$ such that $\bar{g} = g$ in a small neighborhood of $W \cap \cup_{i=1}^m \eta_i([0,1])$.

Consider the map $S : \mathcal{G}^r(\gamma, g, W) \to \mathrm{Sp}(1)$ given by $S(\bar{g}) = P_{\bar{g}}(\gamma)(1)$. The following result is the analogue for geodesic flows of the infinitesimal part of Franks' lemma [18, lem. 1.1] (whose proof for general diffeomorphisms is quite simple).

Theorem 4.1. *Let $g_0 \in \mathcal{G}^r$, $r \geq 4$. Given $\mathcal{U} \subset \mathcal{G}^2$ a neighborhood of g_0, there exists $\delta = \delta(g_0, \mathcal{U}) > 0$ such that given $g \in \mathcal{U}$, γ, W and $\mathfrak{F}$ as above, the image of $\mathcal{U} \cap \mathcal{G}^r(\gamma, g, W, \mathfrak{F})$ under the map S contains the ball of radius δ centered at $S(g_0)$.*

The time 1 in the preceding statement was chosen to simplify the exposition and the same result holds for any time τ chosen in a closed interval $[a,b] \subset]0,+\infty[$; now with $\delta = \delta(g_0, \mathcal{U}, a, b) > 0$. In order to fix the setting, take $[a,b] = [\frac{1}{2}, 1]$ and assume that the injectivity radius of M is larger than 1. This implies that there are no periodic orbits with period smaller than 2 and that any periodic orbit can be cut into non self-intersecting geodesic segments of length τ with $\tau \in [\frac{1}{2}, 1]$. We shall apply Theorem 4.1 to such segments of a periodic orbit choosing the supporting neighborhoods carefully as we now describe.

Given $g \in \mathcal{G}^r$ and γ a prime periodic orbit of g let $\tau \in [\frac{1}{2}, 1]$ be such that $m\tau = \text{period}(\gamma)$ with $m \in \mathbb{N}$. For $0 \leq k < m$, let $\gamma_k(t) := \gamma(t + k\tau)$ with $t \in [0, \tau]$. Given a tubular neighborhood W of $\pi \circ \gamma$ and $0 \leq k < m$ let $S_k : \mathcal{G}^r(\gamma, g, W) \to \text{Sp}(1)$ be the map $S_k(\overline{g}) = P_{\overline{g}}(\gamma_k)(\tau)$.

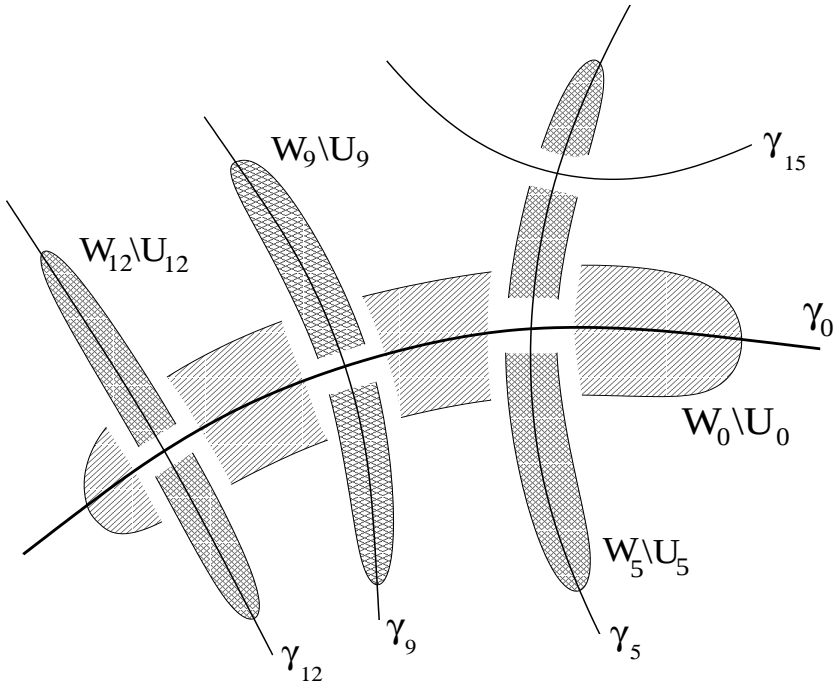


Figure 1: Avoiding self-intersections.

Let W_0 be a small tubular neighborhood of γ_0 contained in W. Let $\mathcal{F}_0 = \{\eta_1^0, \ldots, \eta_{m_0}^0\}$ be the set of geodesic segments η given by

those subsegments of γ of length τ whose endpoints are outside W_0 and which intersect γ_0 transversally at $\eta(\tau/2)$ (see Figure 1). We now apply Theorem 4.1 to γ_0, W_0 and $\mathcal{F}_0$. The proof of this theorem also selects a neighborhood U_0 of $W_0 \cap \cup_{i=1}^{m_0} \eta_i^0([0,\tau])$. We now consider γ_1 and we choose a tubular neighborhood W_1 of γ_1 small enough so that if γ_1 intersects γ_0 transversally, then W_1 intersected with W_0 is contained in U_0 (see Figure 1). By continuing in this fashion we select recursively tubular neighborhoods $W_0, \dots, W_{m-1}$, all contained in W, to which we successively apply Theorem 4.1. This choice of neighborhoods ensures that there is no interference between one perturbation and the next. In the end we obtain the following:

Corollary 4.2. *Let $g_0 \in \mathcal{G}^r$, $r \geq 4$. Given a neighborhood $\mathcal{U}$ of g_0 in $\mathcal{G}^2$, there exists $\delta = \delta(g_0, \mathcal{U}) > 0$ such that if $g \in \mathcal{U}$, γ is a prime closed orbit of ϕ^g and W is a tubular neighborhood of $c = \pi \circ \gamma$, then the image of $\mathcal{U} \cap \mathcal{G}^r(\gamma, g_0, W) \to \Pi_{k=0}^{m-1} \mathrm{Sp}(1)$ under the map $(S_0, \dots, S_{m-1})$ contains the product of balls of radius δ centered at $S_k(g_0)$ for $0 \leq k < m$.*

The arguments below can be used to show that $\overline{g} - g$ can be supported not only outside a finite number of intersecting segments but outside any given compact set[2] of measure zero in γ. This is done by adjusting the choice of the function h in (10).

The nature of these results (i.e., the independence on the size of the neighborhood W) forces us to use the C^1 topology on the perturbation of the geodesic flow, thus the C^2 topology on the metric. The size $\delta(g_0, \mathcal{U}) > 0$ in Theorem 4.1 and Corollary 4.2 depends on the C^4-norm of g_0.

Proof of Theorem 4.1. Let us begin by describing informally the strategy that we shall follow to prove Theorem 4.1. At the beginning we fix most of the constants and bump functions that are needed. Using Fermi coordinates along the geodesic $c = \pi \circ \gamma$, we consider a family of perturbations following Klingenberg and Takens in [29]. We show that the map S is a submersion when restricted to a suitable submanifold of the set of perturbations. To obtain a size δ that depends only on g_0 and $\mathcal{U}$ and that works for all $g \in \mathcal{U}$, γ and W we find a uniform lower bound for the norm of the derivative of S using the constants and the

[2]But to use this argument to support $\overline{g} - g$ outside a given infinite set of geodesic segments of length $\geq \frac{1}{2}$ one needs to bound from below their angle of intersection with c.

bump functions that we fixed before. This uniform estimate can only be obtained in the C^2 topology.

The technicalities of the proof can be summarized as follows. To obtain a C^2 perturbation of the metric preserving the geodesic segment $c = \pi\circ\gamma$ one needs a perturbation of the form (12), with $\alpha(t,x) = \varphi(x)\,\beta_A(t)$, where $\varphi(x)$ is a bump function supported in an ε-neighborhood in the transversal direction to c and $\beta_A(t)$ is given by formula (31). The derivative of $\beta_A(t)$ with respect to A is given by formula (20). The second factor in (20) is used to make the derivative of S surjective,[3] and the first factor $h(t)$ is an approximation of a characteristic function used to support the perturbation outside a neighborhood of the intersecting segments in $\mathfrak{F} = \{\eta_1, \ldots, \eta_m\}$. Then inequality (8) shows that if the neighborhood W of c is taken small enough, the C^2 norm of the perturbation is essentially bounded by only the C^0 norm of $\beta_A(t)$. In order to bound the C^2 norm of β_A from (31) in Equation (8), we use the hypothesis $g_0 \in \mathcal{G}^4$ to have a bound for the second derivative of the curvature $K_0(t,0)$ of g_0 along the geodesic c.

By shrinking $\mathcal{U}$ if necessary, we can assume that

$$\|g\|_{C^2} \leq \|g_0\|_{C^2} + 1 \quad \text{for all } g \in \mathcal{U}. \tag{3}$$

Let $k_1 = k_1(\mathcal{U}) > 1$ be such that if $g \in \mathcal{U}$ and ϕ_t is the geodesic flow of g, then

$$\|d_v\phi_t\| \leq k_1 \quad \text{and} \quad \|d_v\phi_t^{-1}\| \leq k_1 \quad \text{for all } t \in [0,1] \tag{4}$$

and all $v \in S^1_g M$. Let $0 < \lambda \ll \frac{1}{2}$ and let $k_2 = k_2(\mathcal{U}, \lambda) > 0$ be such that

$$\max_{|t-1/2|\leq\lambda} \|d_v\phi_t - d_v\phi_{1/2}\| \leq k_2 \quad \text{and} \quad \max_{|t-1/2|\leq\lambda} \|d_v\phi_t^{-1} - d_v\phi_{1/2}^{-1}\| \leq k_2 \tag{5}$$

for all $g \in \mathcal{U}$ and all $v \in S^1_g M$. If $\lambda = \lambda(g_0, \mathcal{U})$ is small enough, then

$$0 < k_2 < \frac{1}{16\,k_1^3} < 1 < k_1. \tag{6}$$

Let δ_λ and $\Delta_\lambda : [0,1] \to [0, +\infty[$ be C^∞ functions such that δ_λ has support on $[\frac{1}{2} - \lambda, \frac{1}{2}[$, Δ_λ has support on $]\frac{1}{2}, \frac{1}{2} + \lambda]$, $\int \delta_\lambda(t)\, dt = \int \Delta_\lambda(t)\, dt = 1$ and the support of Δ_λ is an interval.

[3]The functions $\delta_\lambda(t)$ and $\Delta_\lambda(t)$ are approximations to a Dirac delta at $t = \frac{1}{2}$.

Let $k_3 = k_3(g_0, \mathcal{U}, \lambda) = k_3(g_0, \mathcal{U}) = k_3(\lambda)$ be

$$k_3 := k_1^2 \left[\|\delta_\lambda\|_{C^0} + \left\|\delta'_\lambda\right\|_{C^0} + \|\Delta_\lambda\|_{C^0} \left(1 + \|g_0\|_{C^2}\right) + \left\|\Delta''_\lambda\right\|_{C^0} \right] \tag{7}$$

where δ'_λ and Δ''_λ are the first and second derivatives of the functions δ_λ and Δ_λ with respect to t.

Given ε with $0 < \varepsilon < 1$, let $\varphi_\varepsilon : \mathbb{R} \to [0,1]$ be a C^∞ function such that $\varphi_\varepsilon(x) = 1$ if $x \in [-\frac{\varepsilon}{4}, \frac{\varepsilon}{4}]$ and $\varphi_\varepsilon(x) = 0$ if $x \notin [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$. In Lemma 4.5 it is proven that $\varphi_\varepsilon(x)$ can be chosen such that

$$\left\|\varphi_\varepsilon(x)\, \beta(t)\, x^2\right\|_{C^2} \leq k_4 \|\beta\|_{C^0} + k_4\, \varepsilon \|\beta\|_{C^1} + \varepsilon^2 \|\beta\|_{C^2}. \tag{8}$$

for some fixed $k_4 > 0$ (independent of ε) and any $\beta : [0,1] \to \mathbb{R}$ of class C^2.

Choose $0 < \varrho \ll 1/(4k_1^2 k_3)$. From (6), we have that

$$\frac{1}{k_1^2} - k_3\, \varrho - 4\, k_1 k_2 > \frac{1}{2k_1^2}. \tag{9}$$

Let $h : [0,1] \to [0,1]$ be a C^∞ function supported outside a neighborhood of the intersecting points and the endpoints of the support of Δ_λ,

$$\operatorname{supp}(h) \subset [0,1] \setminus \left[\gamma^{-1}(\cup_{i=1}^m \eta_i) \cup \partial \operatorname{supp}(\Delta_\lambda) \right]$$

and such that

$$\int_0^1 |h(t) - 1|\, dt \leq \varrho. \tag{10}$$

We now introduce Fermi coordinates along the geodesic arc $c = \pi \circ \gamma$. All the facts that we will use about Fermi coordinates can be found in [21, 28]. Take an orthonormal frame $\{\dot{c}(0), E\}$ in $T_{c(0)}M$. Let $E(t)$ denote the parallel translation of E along c. Consider the differentiable map $\Phi : [0,1] \times \mathbb{R} \to M$ given by

$$\Phi(t,x) = \exp_{c(t)} \big(xE(t)\big).$$

This map has maximal rank at $(t,0)$, $t \in [0,1]$. Since $c(t)$ has no self intersections on $t \in [0,1]$, there exists a neighborhood V of $[0,1] \times \{0\}$ in which $\Phi|_V$ is a diffeomorphism.

Choose

$$\varepsilon_1 = \varepsilon_1(g_0, \mathcal{U}, \gamma, \mathfrak{F}) > 0 \tag{11}$$

such that the segments η_i do not intersect the points with coordinates (t,x) with $|x|<\varepsilon_1$ and $t\in\operatorname{supp}(h)$ and such that $[0,1]\times[-\varepsilon_1,\varepsilon_1]\subset V$ and $\Phi([0,1]\times[-\varepsilon_1,\varepsilon_1])\subset W$.

Let $[g_0(t,x)]_{ij}$ denote the components of the metric g_0 in the chart (Φ,V). Let $\alpha(t,x)$ denote a C^∞ function on $[0,1]\times\mathbb{R}$ with support contained in $V\setminus\Phi^{-1}[\cup_{i=0}^m\eta_i([0,1])]$. We can define a new Riemannian metric g by setting

$$\begin{aligned} g_{00}(t,x) &= [g_0(t,x)]_{00}+\alpha(t,x)\,x^2\,;\\ g_{01}(t,x) &= [g_0(t,x)]_{01};\\ g_{11}(t,x) &= [g_0(t,x)]_{11}; \end{aligned} \tag{12}$$

where we index the coordinates by $x_0=t$ and $x_1=x$.

For any such metric g we have that (cf. [21, 28]):

$$\begin{aligned} g^{ij}(t,0) = \;\; g_{ij}(t,0) \;\; &= \delta_{ij}, \qquad 0\le i,j\le 1;\\ \partial_k\, g^{ij}(t,0) = \partial_k\, g_{ij}(t,0) &= 0, \qquad 0\le i,j,k\le 1; \end{aligned}$$

where $[g^{ij}]$ is the inverse matrix of $[g_{ij}]$.

We need the differential equations for the geodesic flow ϕ_t in hamiltonian form. It is well-known that the geodesic flow is conjugated to the hamiltonian flow of the function

$$H(x,y)=\frac{1}{2}\sum_{ij} g^{ij}(x)\,y_i\,y_j.$$

Hamilton's equations are

$$\begin{aligned} \tfrac{d}{dt}\,x_i = \;\; H_{y_i} \;\; &= \sum_j g^{ij}(x)\,y_j,\\ \tfrac{d}{dt}\,y_k = -H_{x_k} &= -\tfrac{1}{2}\sum_{i,j}\tfrac{\partial}{\partial x_k}g^{ij}(x)\,y_i\,y_j. \end{aligned}$$

Let $\mathcal{F}$ be the set of the Riemannian metrics given by (12) endowed with the C^2 topology. One easily checks that $\mathcal{F}\subset\mathcal{G}^r(\gamma,g_0,W,\mathfrak{F})$. Let

$$\mathcal{V}:=\mathcal{F}\cap\mathcal{U}.$$

Using the identity $\frac{d}{dt}\,(d\phi_t)=(dX\circ\phi_t)\cdot d\phi_t$, with $X=\frac{d}{dt}\phi_t\big|_{t=0}$, we obtain the differential equations for the linearized hamiltonian flow, on

the geodesic $c(t)$ (given by: t, $x=0$, $y_0=1$, $y_1=0$), which we call the Jacobi equation:

$$\frac{d}{dt}\Big|_{(t,x=0)} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} H_{yx} & H_{yy} \\ -H_{xx} & -H_{xy} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 & I \\ 0 & 0 \\ 0 & -K \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}, \tag{13}$$

where

$$K(t,0) = \tfrac{1}{2}\tfrac{\partial^2}{\partial x^2}\, g^{00}(t,0) = -\tfrac{1}{2}\tfrac{\partial^2}{\partial x^2}\, g_{00}(t,0). \tag{14}$$

Let

$$K_0(t,0) := \tfrac{1}{2}\tfrac{\partial^2}{\partial x^2}\, g_0^{00}(t,0).$$

It is easy to check that

$$K(t,0) = K_0(t,0) - \alpha(t,0). \tag{15}$$

By comparison with the usual Jacobi equation[4] we get that $K(t,0)$ is the curvature at the point $c(t)$ for the metric g.

Observe from (12) that the conditions[5]

$$\begin{aligned} a_0(t) = \langle h, \dot{c}\rangle_g = \sum_i g^{0i}(t,0)\, a_i(t) \equiv 0\,, \\ b_0(t) = \dot{a}_0(t) = \langle b, \dot{c}\rangle_g \equiv 0\,, \end{aligned} \tag{16}$$

are invariant among the metrics $g \in \mathcal{F}$ and satisfy (13). In particular the subspaces

$$\mathcal{N}_t = \{(a,b) \in T_{c(t)}TM \,|\, a_0 = b_0 = 0\} \approx \mathbb{R}\times\mathbb{R}$$

are invariant under (13) for all $g \in \mathcal{F}$. From now on reduce the Jacobi Equation (13) to the subspaces $\mathcal{N}_t$.

We need uniform estimates for all $g \in \mathcal{V}$. Fix $g \in \mathcal{V}$ and write

$$\mathbb{A}_t = \mathbb{A}_t^g = \begin{bmatrix} 0 & 1 \\ -K(t,0) & 0 \end{bmatrix}_{2\times 2} \tag{17}$$

[4]The geometric notion of curvature is not really used. The reader might just use Equation (14) as the definition of curvature in this section.

[5]Here the products $\langle h, \dot{c}\rangle_g$ and $\langle b, \dot{c}\rangle_g$ are not needed to follow the argument. In fact, here $\dot{c}(t)$ is the hamiltonian orbit corresponding to the geodesic $c(t)$ in the cotangent bundle and $\langle\ ,\ \rangle_g$ is the Riemannian metric in the cotangent bundle induced by g, whose coefficients are those of the inverse matrix $[g^{ij}]$. These products are included in (16) to suggest the reader that the following subspace $\mathcal{N}_t$ is just the reduction of the space of Jacobi fields to those Jacobi fields which are orthogonal to the geodesic.

where K is from (15). Let $X_t = X_t^g = d\phi_t|_{\mathcal{N}_0} : \mathcal{N}_0 \to \mathcal{N}_t$ be the solution of the Jacobi Equation (13) for g:

$$\dot{X}_t = \mathbb{A}_t\, X_t. \tag{18}$$

The *time* 1 map X_1 is a symplectic linear isomorphism: $X_1^* \mathbb{J} X_1 = \mathbb{J}$, where $\mathbb{J} = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$. Differentiating this equation we get the tangent space of the symplectic isomorphisms at X_1: $\mathcal{T}_{X_1} = \{Y \in \mathbb{R}^{2\times 2} \,|\, X_1^* \mathbb{J} Y$ is symmetric$\}$. Observe that, since X_1 is symplectic:

$$\mathcal{T}_{X_1} = X_1 \cdot \mathcal{T}_I \tag{19}$$

and that $\mathcal{T}_I$ is the space of 2×2 matrices of the form[6] $Z = \left[\begin{smallmatrix} b & c \\ a & -b \end{smallmatrix}\right]$.

Let us consider the map given by

$$\mathcal{F} \ni g \overset{H}{\longmapsto} X_1^g \in \mathrm{Sp}(1).$$

Equivalently, H is the restriction of $S : \mathcal{G}^r(\gamma, g_0, W) \to \mathrm{Sp}(1)$ to $\mathcal{F}$. We shall show that H is a submersion at any $g \in \mathcal{V}$. We start by finding a uniform lower bound for the norm of $d_g H$ restricted to a suitable subspace.

Lemma 4.3. *Consider a small parameter s near zero and write $g_s = g + \alpha^s x_1^2\, dx_0 \otimes dx_0 \in \mathcal{F}$ where*

$$\alpha^s(t,x) := \varphi_\varepsilon(x)\ \beta^s(t),$$

where $\beta^s(t)$ satisfies $\beta^{s=0}(t) \equiv 0$ and

$$\left.\frac{\partial \beta^s(t)}{\partial s}\right|_{s=0} = h(t)\ \left\{\delta(t)\ a + \delta'(t)\, b - \left(\Delta_\lambda(t)\, K(t,0) + \tfrac{1}{2}\,\Delta_\lambda''(t)\right) c\right\}, \tag{20}$$

where $a, b, c \in \mathbb{R}$, $0 \le h(t) \le 1$ satisfies (10), $K(t,0)$ is the curvature of g at $(t,0)$ and $0 < \varepsilon < \varepsilon_1$. In particular α^s has support contained in V.

Then

$$\left\| d_g H\left(\tfrac{d}{ds} g_s\big|_{s=0}\right)\right\| \ge \frac{1}{2k_1^3}\ \left\|\left[\begin{smallmatrix} b & c \\ a & -b \end{smallmatrix}\right]\right\|.$$

[6] If $\dim M > 2$ the elements of $\mathcal{T}_I$ have the form $\left[\begin{smallmatrix} b & c \\ a & d \end{smallmatrix}\right]$, with a and c symmetric and $d = -b^*$. The arguments shown here are not sufficient to cover this case.

We use $h(t)$ to support the perturbation of the Riemannian metric outside the intersecting segments and also to bound the C^2 norm of the term $\frac{\Delta_\lambda''(t)}{\Delta_\lambda(t)}\left(e^{-h\Delta_\lambda c}-1\right)$ in Equation (31).

Proof. From (13) and $K_{g_s}(t,0) = K_g(t,0) - \alpha^s(t,0)$, we see that $X_t^{g_s}$ satisfies

$$\dot{X}_t^{g_s} = (\mathbb{A}_t + D_t^s)\, X_t^{g_s},$$

where $\mathbb{A}_t$ is from (17) and $D_t = \left[\begin{smallmatrix} 0 & 0 \\ \alpha^s(t,0) & 0 \end{smallmatrix}\right]$. Thus the derivative of the map H satisfies $d_g H\big(\frac{d}{ds} g_s\big|_{s=0}\big) = Z_1$, where

$$\dot{Z}_t = \mathbb{A}_t\, Z_t + E_t\, X_t,$$

where $E_t = \frac{d}{ds}\big|_{s=0} D_t^s = h(t) \left[\begin{smallmatrix} 0 & 0 \\ \frac{\partial\beta}{\partial s}\big|_{s=0}(t) & 0 \end{smallmatrix}\right]$. Writing $Z_t = X_t W_t$ and using that $\dot{X}_t = \mathbb{A}_t\, X_t$, we get that $\dot{W}_t = X_t^{-1}\, E_t\, X_t$. Hence

$$Z_1 = X_1 \int_0^1 X_t^{-1}\, E_t\, X_t\, dt\,. \tag{21}$$

Write $A := \left[\begin{smallmatrix} b & c \\ a & -b \end{smallmatrix}\right]$. We have to prove that

$$\|Z_1\| \geq \frac{1}{2k_1^3}\, \|A\| \qquad \text{for all } g \in \mathcal{V}.$$

We compute the integral in (21). Write $B = \left[\begin{smallmatrix} 0 & 0 \\ b & 0 \end{smallmatrix}\right]$ and $C = \left[\begin{smallmatrix} 0 & 0 \\ -c/2 & 0 \end{smallmatrix}\right]$. Then, using (18),

$$\begin{aligned}
\int_0^1 X_t^{-1}\, \delta_\lambda'(t)\, B\, X_t\, dt &= -\int_0^1 \delta_\lambda(t) \left[(X_t^{-1})'\, B\, X_t + X_t^{-1}\, B\, X_t'\right] dt \\
&= \int_0^1 \delta_\lambda(t)\, X_t^{-1} \left[\mathbb{A}_t\, B - B\, \mathbb{A}_t\right] X_t\, dt \\
&= \int_0^1 \delta_\lambda(t)\, X_t^{-1} \left[\begin{smallmatrix} b & 0 \\ 0 & -b \end{smallmatrix}\right] X_t\, dt\,.
\end{aligned}$$

$$\begin{aligned}
&\int_0^1 X_t^{-1}\, \Delta_\lambda''(t)\, C\, X_t\, dt \\
&= \int_0^1 \Delta_\lambda'(t)\, X_t^{-1} \left[\begin{smallmatrix} -\frac{c}{2} & 0 \\ 0 & \frac{c}{2} \end{smallmatrix}\right] X_t\, dt \\
&= \int_0^1 \Delta_\lambda(t)\, X_t^{-1} \left[\mathbb{A}_t \left[\begin{smallmatrix} -c/2 & 0 \\ 0 & c/2 \end{smallmatrix}\right] - \left[\begin{smallmatrix} -c/2 & 0 \\ 0 & c/2 \end{smallmatrix}\right] \mathbb{A}_t\right] X_t\, dt \\
&= \int_0^1 \Delta_\lambda(t)\, X_t^{-1} \left[\begin{smallmatrix} 0 & c \\ \frac{1}{2}(K(t,0)\, c + c\, K(t,0)) & 0 \end{smallmatrix}\right] X_t\, dt\,.
\end{aligned}$$

Hence,

$$\int_0^1 X_t^{-1}\,\frac{E_t}{h(t)}\,X_t\,dt$$
$$=\int_0^1 \delta_\lambda(t)\,X_t^{-1}\,\left[\begin{smallmatrix} b & 0\\ a & -b\end{smallmatrix}\right]\,X_t\,dt+\int_0^1 \Delta_\lambda(t)\,X_t^{-1}\,\left[\begin{smallmatrix} 0 & c\\ 0 & 0\end{smallmatrix}\right]\,X_t\,dt\,.$$

Write $P(t)=X_t^{-1}\,\frac{E_t}{h(t)}\,X_t$, $Q_1(t)=X_t^{-1}\,\left[\begin{smallmatrix} b & 0\\ a & -b\end{smallmatrix}\right]\,X_t$, $Q_2(t)=X_t^{-1}\,\left[\begin{smallmatrix} 0 & c\\ 0 & 0\end{smallmatrix}\right]\,X_t$ and $\mathcal{Q}(t)=X_t^{-1}\,\left[\begin{smallmatrix} b & c\\ a & -b\end{smallmatrix}\right]\,X_t$. Then

$$\int_0^1 P(t)\,dt=\int_0^1 \delta_\lambda(t)\,Q_1(t)\,dt+\int_0^1 \Delta_\lambda(t)\,Q_2(t)\,dt. \tag{22}$$

Using (4) we have that

$$\begin{aligned}\|\delta_\lambda(t)\,Q_1(t)\| &\le \|\delta_\lambda\|_{C^0}\,\left\|X_t^{-1}\right\|\,\sqrt{2}\,\max\{|a|,|b|\}\,\|X_t\|\\ &\le \|\delta_\lambda\|_{C^0}\;\cdot\;k_1\;\cdot\;\|A\|\;\cdot\;k_1.\end{aligned}$$

Similarly

$$\|\Delta_\lambda(t)\,Q_1(t)\|\le\|\Delta_\lambda\|_{C^0}\,k_1^2\,|c|\le\|\Delta_\lambda\|_{C^0}\,k_1^2\,\|A\|\,.$$

Hence, using (7), we have that

$$\begin{aligned}\|P\|_0 &\le k_1^2\,(\|\delta_\lambda\|_{C^0}+\|\Delta_\lambda\|_{C^0})\,\|A\|\\ &\le k_3(\lambda)\,\|A\|\,.\end{aligned} \tag{23}$$

From (21), we have that

$$Z_1=X_1\int_0^1 h(t)\,P(t)\,dt. \tag{24}$$

Observe that

$$\begin{aligned}\left\|\int_0^1 \delta_\lambda(t)\,Q_1(t)\,dt-Q_1\left(\tfrac12\right)\right\| &\le \int_0^1 \delta_\lambda(t)\,\left\|Q_1(t)-Q_1(\tfrac12)\right\|\,dt\\ &\le \mathcal{O}_\lambda(Q_1,\tfrac12),\\ \left\|\int_0^1 \Delta_\lambda(t)\,Q_2(t)\,dt-Q_2\left(\tfrac12\right)\right\| &\le \int_0^1 \Delta_\lambda(t)\,\left\|Q_2(t)-Q_2(\tfrac12)\right\|\,dt\\ &\le \mathcal{O}_\lambda(Q_2,\tfrac12),\end{aligned}$$

where $\mathcal{O}_\lambda(Q_i, \frac{1}{2}) := \max_{|t-1/2|\leq\lambda} \left\|Q_i(t) - Q_i(\frac{1}{2})\right\|$. Thus, using (22),

$$\begin{aligned}
&\left\|\int_0^1 h(t)\,P(t)\,dt - Q\big(\tfrac{1}{2}\big)\right\| \\
&= \left\|\int h\,P\,dt - \int_0^1 P\,dt + \int P\,dt - Q\big(\tfrac{1}{2}\big)\right\| \\
&\leq \left\|\int (h-1)\,P\,\right\| + \left\|\int \delta_\lambda(t)\,Q_1(t) - Q_1\big(\tfrac{1}{2}\big)\right\| \\
&\quad + \left\|\int \Delta_\lambda(t)\,Q_2(t) - Q_2\big(\tfrac{1}{2}\big)\right\|, \\
&\leq \|P\|_0 \int |h-1| + \mathcal{O}_\lambda(Q_1, \tfrac{1}{2}) + \mathcal{O}_\lambda(Q_2, \tfrac{1}{2}) \\
&\leq \|P\|_0 \int |h-1| + 2\,\mathcal{O}_\lambda(\mathcal{Q}, \tfrac{1}{2})\,, \qquad \text{because } \mathcal{Q} = Q_1 + Q_2.
\end{aligned}$$

If $f, g : [0,1] \to \mathbb{R}^{2\times 2}$, by adding and subtracting $f(t)\,g(\frac{1}{2})$, we obtain the formula

$$\mathcal{O}_\lambda(f\,g, \tfrac{1}{2}) \leq \|f\|_0\ \mathcal{O}_\lambda(g, \tfrac{1}{2}) + \mathcal{O}_\lambda(f, \tfrac{1}{2})\ \left\|g(\tfrac{1}{2})\right\|. \tag{25}$$

Also, if $e \in \mathbb{R}^{2\times 2}$ is constant, then

$$\mathcal{O}_\lambda(e\,f, \tfrac{1}{2}) \leq \|e\|\ \mathcal{O}_\lambda(f, \tfrac{1}{2}). \tag{26}$$

Write $A = \left[\begin{smallmatrix} b & c \\ a & -b \end{smallmatrix}\right]$. Using formulas (25), (26), we obtain from (4) and (5) that

$$\begin{aligned}
\mathcal{O}_\lambda(\mathcal{Q}, \tfrac{1}{2}) &= \mathcal{O}_\lambda(X_t^{-1} A\, X_t, \tfrac{1}{2}) \\
&\leq \|X_t^{-1}\|_0\, \mathcal{O}_\lambda(AX_t, \tfrac{1}{2}) + \mathcal{O}_\lambda(X_t^{-1}, \tfrac{1}{2})\ \|A\|\ \|X_{1/2}\|_0 \\
&\leq \|X_t^{-1}\|_0\ \|A\|\ \mathcal{O}_\lambda(X_t, \tfrac{1}{2}) + \mathcal{O}_\lambda(X_t^{-1}, \tfrac{1}{2})\ \|A\|\ \|X_{1/2}\|_0 \\
&\leq 2\,k_1\,k_2\ \|A\|\,.
\end{aligned}$$

Also, from (23) and (10),

$$\|P\|_0 \int |h-1| \leq \|A\|\ k_3(\lambda) \int |h-1| \leq k_3\ \|A\|\,\varrho.$$

Moreover

$$\|A\| = \left\|X_{1/2}\ \mathcal{Q}\big(\tfrac{1}{2}\big)\ X_{1/2}^{-1}\right\| \leq k_1^2\ \left\|\mathcal{Q}\big(\tfrac{1}{2}\big)\right\|.$$

Hence, using (9),

$$\begin{aligned}\left\|\int h\,P\,dt\right\| &\geq \left\|\mathcal{Q}(\tfrac{1}{2})\right\| - \left\|\int h\,P - \mathcal{Q}(\tfrac{1}{2})\right\| \\ &\geq \left(\tfrac{1}{k_1^2} - k_3\,\varrho - 4\,k_1\,k_2\right)\,\|A\| \\ &\geq \frac{1}{2k_1^2}\,\|A\|\,.\end{aligned}$$

This implies that the transformation $\mathcal{T}_I \ni A \mapsto \int_0^1 h(t)\,P(t)\,dt \in \mathcal{T}_I$ is onto. From (19) and (24), the map $\mathcal{T}_I \ni A \mapsto Z_1 \in \mathcal{T}_{X_1}$ is surjective. Moreover, using (4) and (24),

$$k_1\,\|Z_1\| \geq \left\|X_1^{-1}Z_1\right\| = \left\|\int_0^1 h\,P\,dt\right\| \geq \frac{1}{2k_1^2}\,\|A\|\,.$$

Thus

$$\|Z_1\| \geq \frac{1}{2k_1^3}\,\|A\| \qquad \text{for all } g \in \mathcal{V}.$$

q.e.d.

We shall combine Lemma 4.3 with the next lemma to prove the theorem.

Lemma 4.4. *Let $\mathcal{N}$ be a smooth connected Riemannian 3-manifold and let $F : \mathbb{R}^3 \to \mathcal{N}$ be a smooth map such that*

$$|d_xF(v)| \geq a > 0 \quad \textit{for all } (x,v) \in T\mathbb{R}^3 \textit{ with } |v| = 1 \textit{ and } |x| \leq r. \tag{27}$$

Then for all $0 < b < a\,r$,

$$\big\{w \in \mathcal{N} \,|\, d\big(w, F(0)\big) < b\,\big\} \subseteq F\big\{x \in \mathbb{R}^3 \,|\, |x| < \tfrac{b}{a}\,\big\}.$$

Proof. Let $w \in \mathcal{N}$ with $d\big(w, F(0)\big) < b$. Let $\beta : [0,1] \to \mathcal{N}$ be a differentiable curve with $\beta(0) = F(0)$, $\beta(1) = w$ and $|\dot\beta| < b$. Let $\tau = \sup(A)$, where $A \subset [0,1]$ is the set of $t \in [0,1]$ such that there exist a unique C^1 curve $\alpha : [0,t] \to \mathbb{R}^3$ such that $\alpha(0) = 0$, $|\alpha(s)| < r$ and $F(\alpha(s)) = \beta(s)$ for all $s \in [0,t]$. By the inverse function theorem $\tau > 0$, A is open in $[0,1]$ and there exist a unique $\alpha : [0,\tau[\to \mathbb{R}^3$ such that $F \circ \alpha = \beta$. By (27),

$$\big|\dot\beta(s)\big| = \big\|d_{\alpha(s)}F\big\| \cdot |\dot\alpha(s)| \geq a\;|\dot\alpha(s)|\,, \qquad \text{for all } s \in [0,\tau[. \tag{28}$$

Thus, $|\dot\alpha| \leq \frac{1}{a}\max_{0\leq t\leq 1}\left|\dot\beta(t)\right|$. This implies that α is Lipschitz and hence it can be extended continuously to $[0,\tau]$. Observe that $|\alpha(\tau)| < r$, for if $|\alpha(\tau)| \geq r$, then

$$b \geq b\,\tau \geq \int_0^\tau \left|\dot\beta(s)\right|\,ds \geq a\int_0^\tau |\dot\alpha(s)|\;ds \geq a\,r,$$

contradicting the hypothesis $b < ar$. This implies that the set A is also closed in $[0,1]$. Thus $A = [0,1]$ and $\tau = 1$. From (28), writing $x = \alpha(1) \in F^{-1}\{w\}$,

$$|x| \leq \text{length}(\alpha) = \int_0^1 |\dot\alpha(t)|\,dt \leq \frac{1}{a}\int_0^1 |\dot\beta(t)|\,dt < \frac{b}{a}.$$

q.e.d.

We now see that the condition (27) of Lemma 4.4 holds in our setting. Let $k_5 = k_5(g_0,\mathcal{U},\gamma,\mathfrak{F})$ and $k_6 = k_6(g_0,\mathcal{U},\gamma,\mathfrak{F})$ be

$$\begin{aligned} k_5 :&= \|\delta_\lambda\|_0 + \left\|\delta'_\lambda\right\|_0 + \left[\,\|\Delta_\lambda\|_0\,\|g_0\|_{C^2} + \tfrac{1}{2}\left\|\Delta''_\lambda\right\|_0\right] e^{\|\Delta_\lambda\|_0},\\ k_6 :&= \max_{|c|\leq 1}\left\{2\,\|h\|_{C^2}\left[\|\delta_\lambda\|_{C^2} + \left\|\delta'_\lambda\right\|_{C^2}\right] + 2\,\|g_0\|_{C^4}\left\|\left(e^{-h\Delta_\lambda c}-1\right)\right\|_{C^2}\right.\\ &\qquad\left. + \left\|\frac{\Delta''_\lambda}{2\Delta_\lambda}(e^{-h\Delta_\lambda c}-1)\right\|_{C^2}\right\}, \end{aligned}$$

observe that since $\Delta_\lambda > 0$ on $\text{supp}(h)$, the last term in k_6 is finite.

Let $0 < \rho_1 < 1$ be such that the closed ball

$$\overline{B_{\mathcal{G}^2}(g_0,\rho_1)} \subseteq \mathcal{U}. \tag{29}$$

Choose $0 < \varepsilon = \varepsilon(g_0,\mathcal{U},\gamma,\mathfrak{F}) < \varepsilon_1$, small enough so that

$$(\varepsilon\,k_4 + \varepsilon^2)\,k_6 \leq \tfrac{1}{2}\,\rho_1.$$

Choose $0 < \delta < 1$ such that

$$k_4\,k_5\,(2k_1^3\,\delta) + (\varepsilon\,k_4 + \varepsilon^2)\,k_6 \leq \rho_1 < 1 \quad \text{and} \quad 2\,k_1^3\delta \leq 1. \tag{30}$$

For $A = \left[\begin{smallmatrix} b & c \\ a & -b\end{smallmatrix}\right]$, let

$$\begin{aligned} \beta_A(t) := h(t)\left\{\delta_\lambda(t)\,a + \delta'_\lambda(t)\,b\right\}&\\ + \left(K_0(t,0) + \tfrac{\Delta''_\lambda(t)}{2\,\Delta_\lambda(t)}\right)\left(e^{-h(t)\,\Delta_\lambda(t)\,c}-1\right);& \end{aligned} \tag{31}$$

and let $g_A \in \mathcal{G}^{r-2}(\gamma, g_0, W, \mathfrak{F})$ be the Riemannian metric

$$g_A := g_0 + \varphi_\varepsilon(x)\,\beta_A(t)\,x^2\,dt \otimes dt.$$

Observe that $\beta_A = 0$ when $h(t) = 0$, so that $g = g_0$ in a neighborhood of the intersections of the segments η_i with $c = \pi \circ \gamma$. Then the choice of $\varepsilon < \varepsilon_1$ from (11), ensures that $g = g_0$ in a neighborhood of the intersecting segments.

Observe that

$$\frac{\partial \beta_A}{\partial a} = h(t)\,\delta_\lambda(t), \qquad \frac{\partial \beta_A}{\partial b} = h(t)\,\delta'_\lambda(t),$$
$$\frac{\partial \beta_A}{\partial c} = -h(t)\left\{\Delta_\lambda(t)\big(K_0(t,0) + \beta_A(t)\big) + \tfrac{1}{2}\,\Delta''_\lambda(t)\right\},$$

In particular, the directional derivatives of the map $\mathcal{T}_I \ni A \mapsto \beta_A$ are given by formula (20). (Note that $K_0(t,0) + \beta_A(t)$ is the curvature of g_A at $(t,0)$.) Indeed,

$$\begin{aligned}\frac{\partial \beta_A}{\partial c} + h(t)\,\Delta_\lambda(t)\,\beta_A(t) &= h(t)^2\,\Delta_\lambda(t)\left\{\delta_\lambda(t)\,a + \delta'_\lambda(t)\,b\right\} \\ &\quad - h(t)\left\{K_0(t,0)\,\Delta_\lambda(t) + \tfrac{1}{2}\,\Delta''_\lambda(t)\right\} \\ &= -h(t)\left\{K_0(t,0)\,\Delta_\lambda(t) + \tfrac{1}{2}\,\Delta''_\lambda(t)\right\},\end{aligned}$$

because $\Delta_\lambda(t)\,\delta_\lambda(t) \equiv 0$ and $\Delta_\lambda(t)\,\delta'_\lambda(t) \equiv 0$.

Define $F : \mathcal{T}_I \to \mathrm{Sp}(1)$ by

$$F(A) = S(g_A) = d_{\dot c(0)}\phi_1^{g_A}\big|_{\mathcal{N}_1}.$$

Applying Lemma 4.3, we get that if $g_B \in \mathcal{V}$, then the derivative $d_B F$ satisfies

$$\|(d_B F)\cdot A\| \geq \frac{1}{2\,k_1^3}\,\|A\|, \quad \text{if } g_B \in \mathcal{V}. \tag{32}$$

Let $G : \mathcal{T}_I \to \mathcal{G}^{r-2}(\gamma, g_0, W, \mathfrak{F})$ be the map $G(A) = g_A$. By Lemma 4.5, we have that

$$\begin{aligned}\|G(A) - g_0\|_{C^2} &= \big\|\varphi_\varepsilon(x)\,\beta_A(t)\,x^2\big\|_{C^2} \\ &\leq k_4\,\|\beta_A\|_{C^0} + \varepsilon\,k_4\,\|\beta_A\|_{C^1} + \varepsilon^2\,\|\beta_A\|_{C^2}\end{aligned} \tag{33}$$

Observe that for $|c| \leq 1$ we have that

$$\big|\,e^{-h\Delta_\lambda c} - 1\big| \leq |c|\,\max_{|c|\leq 1}\Big|\tfrac{\partial}{\partial c}\big(e^{-h\Delta_\lambda c} - 1\big)\Big| \leq |c|\,\Delta_\lambda\,e^{\|\Delta_\lambda\|_0}, \qquad \text{if } |c| \leq 1.$$

Then, if $|c| \leq 1$,

$$\begin{aligned}\left|K_0(t,0) + \tfrac{\Delta_\lambda''(t)}{2\,\Delta_\lambda(t)}\right| \left| e^{-h\Delta_\lambda c} - 1\right| &\leq \left[\, |K_0|\,\Delta_\lambda + \tfrac{1}{2}\,|\Delta_\lambda''|\,\right]\; e^{\|\Delta_\lambda\|_0}\,|c| \\ &\leq |c|\Big[\,\|g_0\|_{C^2}\,\|\Delta_\lambda\|_0 + \tfrac{1}{2}\big\|\Delta_\lambda''\big\|_0\Big] e^{\|\Delta_\lambda\|_0}\end{aligned}$$

Hence

$$\|\beta_A\|_{C^0} \leq k_5\;\|A\|\,, \quad \text{ if } \;\|A\| \leq 1.$$

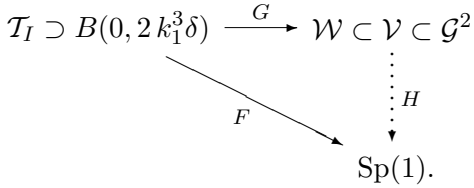
Since $\|f\cdot g\|_{C^2} \leq 2\,\|f\|_{C^2}\,\|g\|_{C^2}$, $\|\beta_A\|_{C^1} \leq \|\beta_A\|_{C^2} \leq k_6$. Then from (33) we get that

$$\|G(A) - g_0\|_{C^2} \leq k_4\,k_5\;\|A\| + (\varepsilon\,k_4 + \varepsilon^2)\,k_6, \quad \text{ if } \;\|A\| \leq 1 \text{ and } G(A) \in \mathcal{U}.$$

By definition of ρ_1 in (29), we can write $\mathcal{W} := B_{\mathcal{G}^2}(g_0,\rho_1) \cap G(\mathcal{T}_I) \subset \mathcal{V} \subset \mathcal{U}$. Then (30) implies that

$$G\big(B_{\mathcal{T}_I}(0, 2\,k_1^3\,\delta)\big) \subseteq \mathcal{W} \subset \mathcal{V}.$$

Thus the hypothesis $g_B \in \mathcal{V}$ of (32) is satisfied and we can consider the following diagram.



Applying Lemma 4.4 to F in (32), with $r = 2\,k_1^3\,\delta$ and $a = \frac{1}{2\,k_1^3}$, we get that

$$\begin{aligned}B_{\mathrm{Sp}(1)}(S(g_0),\delta) &\subseteq F\big(B_{\mathcal{T}_I}(0, 2\,k_1^3\,\delta)\big) \\ &\subseteq F(G^{-1}(\mathcal{W})) \subset S\big(\mathcal{U} \cap \mathcal{G}^r(\gamma, g_0, W, \mathfrak{F})\big).\end{aligned}$$

q.e.d.

Bump functions

Lemma 4.5. *There exist $k_4 > 0$ and a family of C^∞ functions $\varphi_\varepsilon : [-\varepsilon,\varepsilon]^{n-1} \to [0,1]$ such that $\varphi_\varepsilon(x) \equiv 1$ if $x \in [-\frac{\varepsilon}{4},\frac{\varepsilon}{4}]^{n-1}$, $\varphi_\varepsilon(x) \equiv 0$ if $x \notin [-\frac{\varepsilon}{2},\frac{\varepsilon}{2}]^{n-1}$ and for any C^2 map $B : [0,1] \to \mathbb{R}^{(n-1)\times(n-1)}$ the function $\alpha(t,x) := \varphi_\varepsilon(x)\; x^* B(t)\, x$ satisfies,*

$$\|\alpha\|_{C^2} \leq k_4\;\|B\|_{C^0} + \varepsilon\,k_4\,\|B\|_{C^1} + \varepsilon^2\,\|B\|_{C^2},$$

with k_4 independent of $0 < \varepsilon < 1$.

Proof. Let $\psi : [-1,1] \to [0,1]$ be a C^∞ function such that $\psi(x) \equiv 1$ for $|x| \leq \frac{1}{4}$ and $\psi(x) \equiv 0$ for $|x| \geq \frac{1}{2}$. Given $\varepsilon > 0$ let $\varphi = \varphi_\varepsilon : [-\varepsilon, \varepsilon]^{n-1} \to [0,1]$ be defined by $\varphi(x) = \prod_{i=1}^{n-1} \psi\left(\frac{x_i}{\varepsilon}\right)$. Let $B \in \mathbb{R}^{(n-1)\times(n-1)}$ and let $\beta(x) = \varphi(x)\, x^* B x$. Then

$$\|\beta\|_0 \leq \varepsilon^2 \, \|B\| \tag{34}$$

$$d_x\beta = (d_x\varphi)\, x^* B x + \varphi(x)\, x^*(B + B^*)$$

$$\frac{\partial \varphi}{\partial x_i} = \tfrac{1}{\varepsilon}\, \psi'\left(\tfrac{x_i}{\varepsilon}\right) \prod_{k\neq i}^{n-1} \psi(\tfrac{x_k}{\varepsilon})$$

$$\|d_x\varphi\| \leq \tfrac{1}{\varepsilon} \, \|d\psi\|_0 \tag{35}$$

$$\|d_x\beta\| \leq 3\,\varepsilon \, \|B\| \, \|\psi\|_{C^1} \tag{36}$$

$$d_x^2\beta = (d_x^2\varphi)\, x^* B x + 2\,(d_x\varphi)\, x^*\,(B + B^*) + \varphi(x)\,(B + B^*)$$

$$\frac{\partial^2 \psi}{\partial x_i\, \partial x_j} = \tfrac{1}{\varepsilon^2}\, \psi''\big(\tfrac{x_i}{\varepsilon}\big) \prod_{k\neq i} \psi\big(\tfrac{x_k}{\varepsilon}\big)\, \delta_{ij} + \tfrac{1}{\varepsilon^2}\, \psi'\big(\tfrac{x_i}{\varepsilon}\big)\, \psi'\big(\tfrac{x_i}{\varepsilon}\big) \prod_{k\neq i,j} \psi\big(\tfrac{x_k}{\varepsilon}\big)\,(1 - \delta_{ij}).$$

$$\left\|d_x^2\varphi\right\| \leq \tfrac{1}{\varepsilon^2} \, \max\big\{ \, \left\|d^2\psi\right\|_0 \, , \, \|d\psi\|_0^2 \, \big\} \leq \tfrac{1}{\varepsilon^2} \, \|\psi\|_{C^2}^2 \, .$$

$$\left\|d_x^2\beta\right\| \leq \|\psi\|_{C^2}^2 \, \|B\|\,(1 + 4 + 2)$$

$$\leq 7 \, \|\psi\|_{C^2}^2 \, \|B\| \, . \tag{37}$$

Let $k_4 := 4 + 3\,\|\psi\|_{C^1} + 7\,\|\psi\|_{C^2}^2$. Then from (34), (36) and (37), we have that

$$\|\beta\|_{C^2} \leq k_4 \, \|B\| \, . \tag{38}$$

Now let $\alpha(t,x) := \varphi(x)\, x^*\, B(t)\, x$. Observe that

$$\|\alpha\|_{C^2} \leq \sup_t \|\alpha(t,\cdot)\|_{C^2} + \sup_x \|\alpha(\cdot,x)\|_{C^2} + 2 \, \left\| \tfrac{\partial^2 \alpha}{\partial x\, \partial t} \right\|_0 .$$

$$\leq \|\beta\|_{C^2} + \varepsilon^2 \, \|B\|_{C^2} + 2 \, \left\| \tfrac{\partial^2 \alpha}{\partial x\, \partial t} \right\|_0 .$$

But, using (35),

$$\tfrac{\partial^2 \alpha}{\partial x\, \partial t} = d_x\varphi \cdot x^* B'(t)\, x + \varphi(x) \left[\, x^*\, B'(t) + B'(t)\, x \,\right]$$

$$\left\| \tfrac{\partial^2 \alpha}{\partial x\, \partial t} \right\| \leq \varepsilon \, \|\psi\|_{C^1} \, \|B'\|_0 + 2\,\varepsilon\, \|B'\|_0$$

$$\leq \tfrac{1}{2}\, k_4\, \varepsilon\, \|B\|_{C^1} .$$

Hence, using (38),

$$\|\alpha\|_{C^2} \leq k_4 \, \|B\|_{C^0} + k_4\, \varepsilon \, \|B\|_{C^1} + \varepsilon^2 \, \|B\|_{C^2} \, .$$

q.e.d.

5. Dominated splittings for geodesic flows

We say that a linear map $T : \mathbb{R}^N \to \mathbb{R}^N$ is *hyperbolic* if it has no eigenvalue of modulus 1. The *stable* and *unstable* subspaces of T are

$$E^s(T) := \left\{ v \in \mathbb{R}^N \mid \lim_{n\to+\infty} T^n(v) = 0 \right\},$$
$$E^u(T) := \left\{ v \in \mathbb{R}^N \mid \lim_{n\to+\infty} T^{-n}(v) = 0 \right\}.$$

5.1 Periodic sequences of symplectic maps

Let $\mathrm{GL}(\mathbb{R}^N)$ be the group of linear isomorphisms of $\mathbb{R}^N$. We say that a sequence $\xi : \mathbb{Z} \to \mathrm{GL}(\mathbb{R}^N)$ is *periodic* if there exists $n_0 \geq 1$ such that $\xi_{j+n_0} = \xi_j$ for all $j \in \mathbb{Z}$. We say that a periodic sequence ξ is *hyperbolic* if the linear map $\prod_{i=0}^{n_0-1} \xi_i$ is hyperbolic. In this case the stable and unstable subspaces of $\prod_{i=0}^{n_0-1} \xi_{i+j}$ are denoted by $E^s_j(\xi)$ and $E^u_j(\xi)$ respectively.

Given two periodic families of sequences in $\mathrm{GL}(\mathbb{R}^N)$, $\xi = \{\xi^{(\alpha)} \,|\, \alpha \in \mathcal{A}\}$ and $\eta = \{\eta^{(\alpha)} \,|\, \alpha \in \mathcal{A}\}$, define

$$d(\xi,\eta) = \sup\left\{ \|\xi^{(\alpha)}_n - \eta^{(\alpha)}_n\| \;\middle|\; \alpha \in \mathcal{A},\ n \in \mathbb{Z} \right\}.$$

We say that two periodic families are *periodically equivalent* if they have the same indexing set $\mathcal{A}$ and for all $\alpha \in \mathcal{A}$ the minimum periods of $\xi^{(\alpha)}$ and $\eta^{(\alpha)}$ coincide. We say that a family ξ is *hyperbolic* if for all $\alpha \in \mathcal{A}$, the periodic sequence $\xi^{(\alpha)}$ is hyperbolic. Finally, we say that a hyperbolic periodic family ξ is *stably hyperbolic* if there exists $\varepsilon > 0$ such that any periodically equivalent family η satisfying $d(\eta,\xi) < \varepsilon$ is also hyperbolic.

Theorem 5.1 (Mañé, [34, Lemma II.3])**.** *If $\{\xi^{(\alpha)} \,|\, \alpha \in \mathcal{A}\}$ is a stably hyperbolic family of periodic sequences of linear isomorphisms of $\mathbb{R}^N$, then there exist constants $m \in \mathbb{Z}^+$ and $0 < \lambda < 1$ such that for all $\alpha \in \mathcal{A}$, $j \in \mathbb{Z}$:*

$$\left\| \prod_{i=0}^{m-1} \xi^{(\alpha)}_{j+i} \Big| E^s_j(\xi^{(\alpha)}) \right\| \cdot \left\| \Big[\prod_{i=0}^{m-1} \xi^{(\alpha)}_{j+i} \Big]^{-1} \Big| E^u_{j+m}(\xi^{(\alpha)}) \right\| \leq \lambda.$$

Denote by $\mathrm{Sp}(1) = \mathrm{SL}(2,\mathbb{R})$ the group of symplectic linear maps in $\mathbb{R}^2$. Lemma 5.4 below shows that if a periodic sequence ξ of symplectic maps in $\mathbb{R}^2$ is stably hyperbolic among the periodic sequences in

$\mathrm{Sp}(1)$ and $\sup_\alpha \|\xi^{(\alpha)}\| < \infty$, then it is also stably hyperbolic among the sequences in $\mathrm{GL}(\mathbb{R}^2)$. Thus we get:

Corollary 5.2. *If $\{\xi^{(\alpha)} \mid \alpha \in \mathcal{A}\}$ is a family of periodic sequences in $\mathrm{Sp}(1)$ which is stably hyperbolic in $\mathrm{Sp}(1)$, and $\sup_\alpha \|\xi^{(a)}\| < \infty$. Then there exist constants $m \in \mathbb{Z}^+$ and $0 < \lambda < 1$ such that for all $\alpha \in \mathcal{A}$, $j \in \mathbb{Z}$:*

$$\left\| \prod_{i=0}^{m-1} \xi_{j+i}^{(\alpha)} \Big| E_j^s(\xi^{(\alpha)}) \right\| \cdot \left\| \Big[\prod_{i=0}^{m-1} \xi_{j+i}^{(\alpha)} \Big]^{-1} \Big| E_{j+m}^u(\xi^{(\alpha)}) \right\| \le \lambda.$$

Remark 5.3. Write $T_j^N := \prod_{i=0}^{N-1} \xi_{j+i}^{(\alpha)}$. Using that $\|AB\| \le \|A\| \, \|B\|$ for $A, B \in \mathrm{GL}(\mathbb{R}^2)$ we get that for all $N \ge 1$ and all $\alpha \in \mathcal{A}$, $j \in \mathbb{Z}$,

$$\left\| T_j^{mN} \big| E_j^s(\xi^{(\alpha)}) \right\| \left\| \big[T_j^{mN} \big]^{-1} \big| E_{j+Nm}^u(\xi^{(\alpha)}) \right\| < \lambda^N.$$

Lemma 5.4. *Let $F_k \in \mathrm{GL}(\mathbb{R}^2)$, $T_k \in \mathrm{Sp}(1)$ with $\|F_k - T_k\| < \varepsilon$ for $k = 1, \dots, N$, where*

$$2\,\varepsilon \left(1 + 2 \max_{1 \le j \le N} \|T_j\| \right) < \tfrac{1}{2}.$$

Suppose that $\mathbb{F} = F_N \circ F_{N-1} \circ \cdots \circ F_1$ is not hyperbolic. Then there exist $A_k \in \mathrm{Sp}(1)$ such that

$$\|A_k - T_k\| < 16\,\varepsilon \left(2 + \max_{1 \le j \le N} \|T_j\| \right)^2$$

and $\mathbb{A} := A_N \circ A_{N-1} \circ \cdots \circ A_1$ is not hyperbolic.

Proof. Suppose first that $\mathbb{F}$ has complex eigenvalues λ and $\overline{\lambda}$. Since $\mathbb{F}$ is not hyperbolic, $|\lambda| = |\overline{\lambda}| = 1$, and hence $\det \mathbb{F} = +1$.

Let $e_1 = (1, 0)$, $e_2 = (0, 1)$ and

$$\lambda_k := \det F_k = \omega(F_k\, e_1, F_k\, e_2).$$

Since $\omega(a, b) \le |a|\,|b|$, we have that

$$\begin{aligned} |\lambda_k - 1| &= |\omega(F_k e_1, F_k e_2) - \omega(T_k e_1, T_k e_2)| \\ &\le |\omega(F_k e_1 - T_k e_1, F_k e_2) - \omega(T_k e_1, T_k e_2 - F_k e_2)| \\ &\le \varepsilon \, \|F_k\| + \varepsilon \, \|T_k\| \\ &\le 2\,\varepsilon \big[2 \, \|T_k\| + 1 \big] < \tfrac{1}{2}, \end{aligned}$$

in particular, λ_k is positive. Since $\left|1-\frac{1}{\sqrt{x}}\right| \leq 2\,|x-1|$ for $\frac{1}{2} \leq x \leq \frac{3}{2}$, we obtain

$$\left|1-\tfrac{1}{\sqrt{\lambda_k}}\right| \leq 4\,\varepsilon\big[2\,\|T_k\|+1\big].$$

Since $\prod_{k=1}^{N} \lambda_k = \det \mathbb{F} = 1$,

$$\prod_{k=1}^{N} \tfrac{1}{\sqrt{\lambda_k}} = 1.$$

Observe that $\mathrm{Sp}(1) = \{A \in \mathrm{GL}(\mathbb{R}^2) \mid \det A = +1\}$. Write

$$A_k := \tfrac{1}{\sqrt{\lambda_k}}\, F_k.$$

Then $A_k \in \mathrm{Sp}(1)$. Also

$$\mathbb{A} = A_N \circ \cdots \circ A_1 = \left(\textstyle\prod_{k=1}^{N} \frac{1}{\sqrt{\lambda_k}}\right) \mathbb{F} = \mathbb{F}$$

is not hyperbolic. Finally,

$$\begin{aligned}
\|A_k - T_k\| &\leq \|A_k - F_k\| + \|F_k - T_k\| \\
&\leq \left|1-\tfrac{1}{\sqrt{\lambda_k}}\right|\, \|F_k\| + \varepsilon \\
&\leq 4\,\varepsilon\, \left[2\,\|T_k\|+1\right]^2 + \varepsilon \\
&\leq 4\,\varepsilon\, \left[2\,\|T_k\|+2\right]^2.
\end{aligned}$$

Now suppose that $\mathbb{F}$ has an eigenvalue 1. The case of an eigenvalue -1 follows from this case using $-T_1$ and $-F_1$ instead of T_1 and F_1.

Take $a_1 \neq 0$ such that $\mathbb{F}(a_1) = a_1$. Define inductively

$$a_{k+1} := F_k(a_k), \qquad u_k := \frac{a_k}{|a_k|}.$$

We shall construct a symplectic map $A_k \in \mathrm{Sp}(1)$ such that $\|A_k - T_k\| < \big(3+\|T_k\|\big)\,\varepsilon$ and $A_k(u_k) = F_k(u_k)$. This will imply that $A_k(a_k) = a_{k+1}$, $\mathbb{A}(a_1) = a_1$ and thus that $\mathbb{A}$ is not hyperbolic.

Let $J(x,y) := (-y,x)$ and

$$\lambda_k := \frac{\omega(T_k J u_k, T_k u_k)}{\omega(F_k J u_k, F_k u_k)} = \frac{1}{\omega(F_k J u_k, F_k u_k)}.$$

Define $A_k \in \mathrm{GL}(\mathbb{R}^2)$ by $A_k(u_k) = F_k(u_k)$ and $A_k(Ju_k) = \lambda_k\, F_k(Ju_k)$. Then

$$\omega(A_k J u_k, A_k u_k) = \lambda_k\, \omega(F_k J u_k, F_k u_k) = 1 = \omega(Ju_k, u_k),$$

so that $A_k \in \mathrm{Sp}(1)$.

Since $\omega(a,b) \leq |a|\,|b|$, we have that

$$\begin{aligned}
|\tfrac{1}{\lambda_k} - 1| &= \big|\omega(F_k Ju_k, F_k u_k) - \omega(T_k Ju_k, T_k u_k)\big| \\
&= \big|\omega(F_k Ju_k, F_k u_k - T_k u_k) + \omega(F_k Ju_k - T_k Ju_k, T_k u_k)\big| \\
&\leq \varepsilon\,(\|F_k\| + \|T_k\|) \\
&\leq \varepsilon\,(2\,\|T_k\| + 1).
\end{aligned}$$

Since $|x-1| \leq 4\,\left|1 - \frac{1}{x}\right|$ for $\frac{1}{2} < x < \frac{3}{2}$,

$$\begin{aligned}
|A_k(Ju_k) - T_k(Ju_k)| &= |\lambda_k\,F_k(Ju_k) - T_k(Ju_k)| \\
&\leq |\lambda_k - 1|\;\;|F_k(Ju_k)| + |F_k(Ju_k) - T_k(Ju_k)| \\
&\leq |\lambda_k - 1|\;\|F_k\| + \|F_k - T_k\| \\
&\leq 4\varepsilon\left[\,2\,\|T_k\| + 1\,\right](\|T_k\| + 1) + \varepsilon \\
&\leq 4\varepsilon\left(2 + 2\;\|T_k\|\,\right)^2.
\end{aligned}$$

Also,

$$|A_k(u_k) - T_k(u_k)| = |F_k(u_k) - T_k(u_k)| \leq \varepsilon.$$

Since the basis $\{u_k, Ju_k\}$ is orthonormal, we have that

$$\|A_k - T_k\| \leq 16\,\varepsilon\left(1 + \|T_k\|\,\right)^2 + \varepsilon.$$

q.e.d.

5.2 The hyperbolic splitting

Let M be a closed 2-dimensional smooth manifold and let $\mathcal{R}^1(M)$ be the set of C^r Riemannian metrics on M, $r \geq 4$, all of whose closed geodesics are hyperbolic, endowed with the C^2 topology and let $\mathcal{F}^1(M) = \mathrm{int}\big(\mathcal{R}^1(M)\big)$ be the interior of $\mathcal{R}^1(M)$ in the C^2 topology.

Given $g \in \mathcal{G}^r(M)$ let $\mathrm{Per}(g)$ be the union of the hyperbolic (prime) periodic orbits of g. We say that a closed ϕ^g -invariant subset $\Lambda \subset SM$ is hyperbolic if there exists a (continuous) splitting $T_\Lambda(SM) = E^s \oplus E^c \oplus E^u$ such that:

- $E^c = \langle X^g \rangle$ is generated by the vector field of ϕ^g.

- There exist constants $K > 0$ and $0 < \lambda < 1$ such that:

$$|d_\theta \phi_t^g(\xi)| \leq K\,\lambda^t\,|\xi|\,, \qquad \text{for all } t \geq 0,\ \ \theta \in \Lambda,\ \ \xi \in E^s(\theta);$$
$$\left|d_\theta \phi_{-t}^g(\xi)\right| \leq K\,\lambda^t\,|\xi|\,, \qquad \text{for all } t \geq 0,\ \ \theta \in \Lambda,\ \ \xi \in E^u(\theta).$$

We shall show now:

Theorem D. *If $g \in \mathcal{F}^1(M)$, then the closure $\overline{\mathrm{Per}(g)}$ is a hyperbolic set.*

We state a local version which implies Theorem D. Let $U \subseteq SM$ be an open subset and let $\mathcal{R}^1(U)$ be the set of Riemannian metrics $g \in \mathcal{G}^r(M)$ such that all the periodic orbits of ϕ^g contained in U are hyperbolic. Let $\mathrm{Per}(g, U)$ be the union of the periodic orbits of ϕ^g entirely contained in U. Let $\mathcal{F}^1(U) = \mathrm{int}_{C^2}\big(\mathcal{R}^1(U)\big)$.

Proposition 5.5. *If $g \in \mathcal{F}^1(U)$, then the closure $\overline{\mathrm{Per}(g,U)}$ is hyperbolic.*

Proof. Observe that on a C^2 neighborhood $\mathcal{U}$ of g each periodic orbit in $\mathrm{Per}(g, U)$ can be continued and its continuation (see Section 2) is hyperbolic, because otherwise one could produce a non-hyperbolic orbit.

Cut the closed orbits in $\mathrm{Per}(g, U)$ into segments of length in $[\frac{1}{4}\ell, \frac{1}{2}\ell]$ where ℓ is the injectivity radius of g. Given a closed orbit γ in $\mathrm{Per}(g, U)$ construct normal local transversal sections Σ_i to ϕ^g passing through the cutting points $\gamma(t_i)$ of γ. Given a nearby metric $\overline{g}$, cut the continuation $\gamma^{\overline{g}}$ of γ along the Σ_i's: $\gamma^{\overline{g}}(t_i^{\overline{g}}) \in \Sigma_i$. Then $\gamma^{\overline{g}}$ is cut in the same number of segments as γ is, so that the families

$$\mathcal{F}(\overline{g}) = \left\{ d_{\gamma^{\overline{g}}(t_i^{\overline{g}})} \phi^{\overline{g}}_{t_{i+1}^{\overline{g}} - t_i^{\overline{g}}} |_{\mathcal{N}^{\overline{g}}(\gamma^{\overline{g}}(t_i^{\overline{g}}))} \;\middle|\; \gamma \in \mathrm{Per}(g, U),\ 0 \leq i \leq n(\gamma) \right\}$$

in $\mathrm{Sp}(\mathcal{N}^{\overline{g}}(\theta))$ are periodically equivalent, where

$$\mathcal{N}^{\overline{g}}(\theta) = \{\xi \in T_\theta S(M, \overline{g}) \mid \langle d\pi\xi, \theta\rangle_{\overline{g}} = 0\}$$

and $n(\gamma)$ is the number of segments in which we cut γ.

Lemma 5.6. *If $g \in \mathcal{F}(U)$ then the family $\mathcal{F}(g)$ is stably hyperbolic.*

Proof. Since $g \in \mathcal{F}^1(U)$, there exists a C^2-neighborhood $\mathcal{U}$ of g in $\mathcal{G}^r(M)$ such that for all $\overline{g} \in \mathcal{U}$, the family $\mathcal{F}(\overline{g})$ is hyperbolic. Let $\delta = \delta(g, \mathcal{U}) > 0$ be given by Corollary 4.2. For $\gamma \in \mathrm{Per}(g, U)$, write

$$\xi_i^{(\gamma)} := d_{\gamma(t_i)} \phi^g_{t_{i+1} - t_i} |_{\mathcal{N}_i}, \qquad t_i := t_i^g, \qquad \mathcal{N}_i := \mathcal{N}^g(\gamma^g(t_i)).$$

Suppose that the family

$$\mathcal{F}(g) = \left\{\xi_i^{(\gamma)} \,\middle|\, \gamma \in \mathrm{Per}(g,U),\, 1 \le i \le n_i(\gamma)\right\}$$

is not stably hyperbolic. Then there exist a periodic orbit $\gamma \in \mathrm{Per}(g,U)$ for g and a sequence of symplectic linear maps $\eta_i : \mathcal{N}_i \to \mathcal{N}_{i+1}$ such that $\left\| \eta_i - \xi_i^{(\gamma)} \right\| < \delta$ and $\prod_{i=1}^{n(\gamma)} \eta_i$ is not hyperbolic. Observe that the perturbations of Franks' Lemma 4.1 do not change the subspaces $\mathcal{N}(\theta)$ along the selected segment of $c(t)$. By Corollary 4.2 there is another Riemannian metric $\overline{g} \in \mathcal{U}$ such that γ is also a closed orbit for $\overline{g}$, $t_i^{\overline{g}} = t_i$, $\mathcal{N}^{\overline{g}}(t_i^{\overline{g}}) = \mathcal{N}_i$ and $d_{\gamma(t_i)}\phi^{\overline{g}}_{t_{i+1}-t_i}|_{\mathcal{N}_i} = \eta_i$. Let $\tau(\gamma) := \sum_{i=1}^{n(\gamma)} t_i$ be the period of γ. Since the linearized Poincaré map for $(\gamma, \overline{g})$ is

$$d_{\gamma(0)}\phi^{\overline{g}}_{\tau(\gamma)}\big|_{\mathcal{N}_1} = \textstyle\prod_{i=1}^{n(\gamma)} \eta_i,$$

the closed orbit γ is not hyperbolic for the metric $\overline{g} \in \mathcal{U}$. This contradicts the choice of $\mathcal{U}$. q.e.d.

Applying Corollary 5.2 — and Remark 5.3 if necessary (the time spacing between cut points may vary) — we get that there exist $\lambda < 1$ and $T > 0$ such that

(39)
$$\left\|d_\theta\phi_T\big|E^s(\theta)\right\| \cdot \left\|d_{\phi_T\theta}\,\phi_{-T}\big|E^u(\phi_T\,\theta)\right\| < \lambda \qquad \text{for all } \theta \in \mathrm{Per}(g,U);$$

where $\phi = \phi^g$.

Write $\Lambda(g) = \overline{\mathrm{Per}(g,U)}$. For $\theta \in \Lambda(g)$ let

$$S(\theta) := \mathrm{span}\left\{\xi \in \mathcal{N}^g(\theta) \;\middle|\; \begin{smallmatrix} \exists \langle\theta_n\rangle \subseteq \mathrm{Per}(g,U),\ \lim_n \theta_n = \theta; \\ \exists \xi_n \in E^s(\theta_n), \quad \lim_n \xi_n = \xi. \end{smallmatrix}\right\}$$
$$U(\theta) := \mathrm{span}\left\{\xi \in \mathcal{N}^g(\theta) \;\middle|\; \begin{smallmatrix} \exists \langle\theta_n\rangle \subseteq \mathrm{Per}(g,U),\ \lim_n \theta_n = \theta; \\ \exists \xi_n \in E^u(\theta_n), \quad \lim_n \xi_n = \xi. \end{smallmatrix}\right\}$$

Then the domination condition (39) implies that

$$\left\|d_\theta\phi_T\big|_{S(\theta)}\right\| \cdot \left\|d_{\phi_T\theta}\,\phi_{-T}\big|_{U(\phi_T\theta)}\right\| \le \lambda, \qquad \text{for all } \theta \in \Lambda(g). \tag{40}$$

We show now that the domination condition (40) implies that $S \oplus U$ is a continuous splitting of $\mathcal{N}|_{\Lambda(g)} = S \oplus U$. First observe that $S(\theta) \cap U(\theta) = \{0\}$ for all $\theta \in \Lambda(g)$; because if $\xi_0 \in S(\theta) \cap U(\theta)$, writing $\xi_T := d_\theta\phi_T(\xi_0)$, we would have that

$$|\xi_T| \le \left\|d_\theta\phi_T\big|_{S(\theta)}\right\| \cdot |\xi_0| \le \left\|d_\theta\phi_T\big|_{S(\theta)}\right\| \cdot \left\|d_{\phi_T\theta}\,\phi_{-T}\big|_{U(\phi_T\theta)}\right\| \cdot |\xi_T| \le \lambda\,|\xi_T|.$$

But the definitions of S and U imply that $\dim S(\theta) \geq \dim E^s(\theta_n)$ and that $\dim U(\theta) \geq \dim E^u(\theta_n)$ if $\lim_n \theta_n = \theta$ and $\theta_n \in \mathrm{Per}(g, U)$. Therefore $\mathcal{N}(\theta) = S(\theta) \oplus U(\theta)$ and $\lim_n S(\theta_n) = S(\theta)$, $\lim U(\theta_n) = U(\theta)$ in the appropriate Grassmann manifold.

The continuity of the bundles S and U and their definition imply that $S(\theta) = E^s(\theta)$ and $U(\theta) = E^u(\theta)$ when $\theta \in \mathrm{Per}(g, U)$. Observe that if $\theta \in \mathrm{Per}(g, U)$ then $E^s(\theta)$ and $E^u(\theta)$ are[7] lagrangian subspaces of $\mathcal{N}(\theta)$ because

$$\omega_g(u, v) = \lim_{t \to +\infty} \omega_g\big(d_\theta \phi_t(u), d_\theta \phi_t(v)\big) = 0,$$

where ω_g is the symplectic form induced by g. The continuity of the bundles S and U and the continuity of ω_g imply that the subspaces $S(\theta)$ and $U(\theta)$ are lagrangian for all $\theta \in \Lambda(g)$. Then the next proposition due to Ruggiero [48, Proposition 2.1] (cf. also [14]) shows that $\overline{\mathrm{Per}(g, U)}$ is hyperbolic.

Proposition 5.7. *Let $S(\theta) \oplus U(\theta)$ be a continuous, invariant lagrangian splitting defined on a compact invariant set $X \subset SM$. The splitting is dominated if and only if it is hyperbolic.*

A hyperbolic set Λ is said to be *locally maximal* if there exists an open neighborhood U of Λ, such that Λ is the maximal invariant subset of U, i.e.,

$$\Lambda = \bigcap_{t \in \mathbb{R}} d\phi_t^g(U).$$

A *basic set* is a locally maximal hyperbolic set with a dense orbit. It is *nontrivial* if it is not a single closed orbit.

Given a continuous flow ϕ_t on a topological space X a point $x \in X$ is said *wandering* if there is an open neighborhood U of x and $T > 0$ such that $\phi_t(U) \cap U = \emptyset$ for all $t > T$. Denote by $\Omega(\phi_t|_X)$ the set of non-wandering points for (X, ϕ_t). Recall:

Smale's Spectral Decomposition Theorem for Flows 5.8 ([50, 27])**.** *If Λ is a locally maximal hyperbolic set for a flow ϕ_t, then there exists a finite collection of basic sets $\Lambda_1, \ldots \Lambda_N$ such that the non-wandering set of the restriction $\phi_t\big|_\Lambda$ satisfies*

$$\Omega\big(\phi_t\big|_\Lambda\big) = \bigcup_{i=1}^{N} \Lambda_i.$$

[7] This is trivial in our case of $\dim \mathcal{N}(\theta) = 2$ and $\dim E^s(\theta) = \dim E^u(\theta) = 1$.

Corollary 5.9. *If the number of geometrically distinct periodic geodesics is infinite and $g \in \mathcal{F}^1(M)$, then $\overline{\mathrm{Per}(g)}$ contains a nontrivial hyperbolic basic set.*

Proof. Let $\Lambda = \overline{\mathrm{Per}(g)}$. Since $g \in \mathcal{F}^1(M)$, Theorem D implies that Λ is a hyperbolic set. By Proposition 6.4.6 in [27], there exists an open neighborhood U of Λ such that the set

$$\Lambda_U := \bigcap\nolimits_{t\in\mathbb{R}} \phi_t^g(\overline{U})$$

is hyperbolic. Since $\Lambda = \overline{\mathrm{Per}(g)}$, its non-wandering set is $\Omega(\phi_t|_\Lambda) = \Lambda$. By definition of Λ_U, $\Lambda \subseteq \Lambda_U$ and hence $\Lambda = \Omega(\phi_t|_\Lambda) \subseteq \Omega(\phi_t|_{\Lambda_U})$. By Corollary 6.4.20 in [27], the periodic orbits are dense in the non-wandering set $\Omega(\phi_t|_{\Lambda_U})$ of the locally maximal hyperbolic set Λ_U. Thus $\Lambda \subseteq \Omega(\phi_t|_{\Lambda_U}) \subseteq \overline{\mathrm{Per}(g)} = \Lambda$. By Theorem 5.8, the set $\Lambda = \Omega(\phi_t|_{\Lambda_U})$ decomposes into a finite collection of basic sets. Since the number of periodic orbits in Λ is infinite, at least one of the basic sets Λ_i is not a single periodic orbit, i.e., it is nontrivial. q.e.d.

N. Hingston proves in [24] that if M is a simply-connected manifold rational homotopy equivalent to a compact rank-one symmetric space with a metric all of whose closed geodesics are hyperbolic then

$$\liminf_{\ell\to\infty} n(\ell)\,\frac{\log(\ell)}{\ell} > 0,$$

where $n(\ell)$ is the number of geometrically distinct closed geodesics of length $\leq \ell$.

Rademacher proves:

Theorem 5.10 (Rademacher [44, Cor. 2])**.** *For a C^4-generic metric on a compact Riemannian manifold with finite fundamental group there are infinitely many geometrically distinct closed geodesics.*

Thus we have (As we mentioned in the introduction, we could have also used the stronger results of Franks and Bangert [5, 19] which assert that any metric on S^2 has infinitely many geometrically distinct closed geodesics.):

Theorem 1.1. *If a C^4 metric on a closed surface cannot be C^2-approximated by one having an elliptic closed geodesic, then it has a nontrivial hyperbolic basic set.*

Theorem 1.1 together with Proposition 3.3 completes the proof of Theorem A.

Appendix A. Perturbation of lagrangian manifolds

In this appendix we prove a perturbation lemma for invariant lagrangian submanifolds of an autonomous hamiltonian suitable for use in the proof of the Kupka-Smale theorem for geodesic flows.

Let V be a $2n$-dimensional vector space. A symplectic form ω on V is an antisymmetric bilinear map which is nondegenerate, i.e., for all $v \in V \setminus \{0\}$ there exists $w \in V$ such that $\omega(v, w) \neq 0$. We say that a subspace $E \subset V$ is *isotropic* if $\omega|_E \equiv 0$ and that it is *lagrangian* if E is isotropic and $\dim E = n = \frac{1}{2} \dim V$. This is the maximal dimension that an isotropic subspace can have.

A symplectic manifold $(\mathcal{M}, \omega)$ is a $2n$-dimensional smooth manifold together with a symplectic form ω, i.e., a 2-form which is nondegenerate at each tangent space. A *lagrangian* submanifold $\mathcal{N} \subset M$ is a submanifold such that each tangent space $T_x\mathcal{N}$ is a lagrangian subspace of $T_x\mathcal{M}$. In particular, $\dim \mathcal{N} = n$.

Lemma A.1. *Let $(\mathcal{M}, \omega)$ be a symplectic manifold and $H : \mathcal{M} \to \mathbb{R}$ be a smooth function. If $\mathcal{N}$ is a lagrangian submanifold of $(\mathcal{M}, \omega)$ such that $\mathcal{N} \subset H^{-1}\{k\}$ for some $k \in \mathbb{R}$ then the hamiltonian vector field X of H is tangent to $\mathcal{N}$. In particular, $\mathcal{N}$ is a union of orbit segments of the hamiltonian flow.*

Proof. The hamiltonian vector field X is defined by $i_X\omega = -dH$. In particular, on the level set $\Sigma = H^{-1}\{k\}$ we have that $i_X\omega|_\Sigma = dH|_\Sigma \equiv 0$. Then $i_X\omega|_\mathcal{N} \equiv 0$. Then for all $x \in \mathcal{N}$ the subspace $E_x := T_x\mathcal{N} \oplus \langle X(x) \rangle$ is isotropic. If $X(x) \notin T_x\mathcal{N}$ then $\dim E_x = n + 1$ which is impossible. Thus $X(x) \in T_xN$. q.e.d.

We shall use a special coordinate system associated to a lagrangian submanifold that we shall call Darboux coordinates for the lagrangian manifold.

Lemma A.2. *Let $\mathcal{N}$ be a lagrangian submanifold contained in an energy level $H^{-1}\{k\}$ of a hamiltonian $H : \mathcal{M} \to \mathbb{R}$ on a symplectic manifold $(\mathcal{M}, \omega)$. Let $\theta \in \mathcal{N}$ and suppose that θ is not a singularity of the hamiltonian vector field of H. Then there exist a neighborhood U in $\mathcal{M}$ and a coordinate system $(x, p) : U \to \mathbb{R}^n \times \mathbb{R}^n$ such that:*

(a) $\omega = \sum_i dp_i \wedge dx_i$.

(b) $\mathcal{N} \cap U = [p \equiv 0]$.

(c) *The hamiltonian vector field of H on $\mathcal{N}$ is given by $X_H|_\mathcal{N} = \frac{\partial}{\partial x_0}$.*

Proof. By Weinstein's theorem [53], [2, Th. 5.3.18], [36, Th. 3.32] there is a neighborhood W_1 of $\mathcal{N}$ which is symplectomorphic to a neighborhood of the zero section of $T^*\mathcal{N}$ with its canonical symplectic form, sending $\mathcal{N}$ to the zero section $\mathcal{N} \times 0$.

By Lemma A.1, $\mathcal{N}$ is invariant under the hamiltonian flow ϕ_t of H. Let V be a flow box for the restriction $\phi_t|_{\mathcal{N}}$ containing $\theta \in V$ and choose a local chart $x : V \to \mathbb{R}^n$ for $\mathcal{N}$ such that $x(\theta) = 0 \in \mathbb{R}^n$ and $X|_V = \frac{\partial}{\partial x_0}$, where $X|_V$ is the restriction of the hamiltonian vector field to $\mathcal{N} \cap V$.

The canonical symplectic coordinates associated to the chart (V, x) are given by $(x, p) : V \times \mathbb{R}^n \to T^*_V\mathcal{N}$, $p_i = dx_i$. The pull-back of the canonical symplectic form for $T^*\mathcal{N}$ in these coordinates is $\sum_i dp_i \wedge dx_i$. The zero section $V \times 0 \subset \mathcal{N} \times 0 \subset T^*\mathcal{N}$ is given by $[p \equiv 0]$. Now compose this chart (x, p) with the symplectomorphism to obtain the required chart. q.e.d.

This is our perturbation lemma for invariant lagrangian submanifolds.

Lemma A.3. *Let $\mathcal{N}$ and $\mathcal{K}$ be two lagrangian submanifolds inside an energy level $H^{-1}\{k\}$ of a hamiltonian $H : \mathcal{M} \to \mathbb{R}$ of a symplectic manifold $(\mathcal{M}, \omega)$. Let $\theta \in \mathcal{N}$ be a nonsingular point for the hamiltonian vector field. Let $(t, x; p)$, $t \equiv x_0$, be Darboux coordinates coordinates for $\mathcal{N}$, $0 \le t \le 1$, $|x| < \varepsilon$ as in Lemma* A.2. *Choose $0 < \varepsilon_2 < \varepsilon_1 < \varepsilon$. Then there exist a sequence $\mathcal{N}_n$ of lagrangian submanifolds of $(\mathcal{M}, \omega)$ such that:*

(a) *$\mathcal{N}_n \to \mathcal{N}$ in the C^∞ topology.*

(b) *$\mathcal{N}_n \cap A = \mathcal{N} \cap A$, where $A := \big\{(t, x; p) \,\big|\, \max_i |x_i| \ge \varepsilon_1 \quad \text{or} \quad 0 \le t \le \frac{1}{4}\big\}$.*

(c) *$H(\mathcal{N}_n \cap B) = \{k\}$, where $B = A \cup \big\{(t, x; p) \,\big|\, \frac{1}{2} \le t \le 1\big\}$.*

(d) *$\mathcal{N}_n \cap D$ is transversal to $\mathcal{K}$, where $D = \{(t, x; p) \,|\, t = 1$, and $\max_i |x_i| < \varepsilon_2\}$.*

Proof. Let $\varphi : [-\varepsilon, \varepsilon]^{n-1} \to [0, 1]$ be a C^∞ function such that $\varphi(x) = 0$ if $\max_i |x_i| > \varepsilon_1$ and $\varphi(x) = 1$ if $x \in [-\varepsilon_2, \varepsilon_2]^{n-1}$. Given $s = (s_1, \dots, s_{n-1}) \in \mathbb{R}^{n-1}$ with $|s|$ small, let $h_s : [-\varepsilon, \varepsilon]^{n-1} \to \mathbb{R}$ be the function $h_s(x) := 1 + \varphi(x) \sum_{i=1}^{n-1} s_i\, x_i$. Then

$$\begin{aligned} &d_x h_s = (s_1, \dots, s_{n-1}), && \text{if} \quad x \in [-\varepsilon_2, \varepsilon_2]^{n-1}; \\ &d_x h_s = 0, && \text{if} \quad \max_i |x_i| > \varepsilon_1. \end{aligned}$$

Let $p^s : \{1\} \times [-\varepsilon,\varepsilon]^{n-1} \to \mathbb{R}^n$ be defined by $p^s(1,x) := (p_0^s(x), d_x h_s)$, where $p_0^s(x) : [-\varepsilon,\varepsilon]^{n-1} \to \mathbb{R}$ is defined by the equation

$$H\big((1,x); p_0^s(x)\big) = k. \tag{41}$$

Since the curves $t \mapsto (t,x,p=0)$ are solutions of the hamiltonian equations,

$$H_{p_0}\big((1,x),0\big) \equiv 1 \neq 0.$$

By the implicit function theorem, for s small we can solve Equation (41) for $(s,x) \mapsto p_0^s(x)$ and this is a C^∞ function on s and x.

The graph of p^s:

$$\mathrm{Graph}(p^s) := \big\{\big((1,x); p^s(x)\big) \,\big|\, x \in [-\varepsilon,\varepsilon]^{n-1}\big\} \subset H^{-1}\{k\}$$

is an isotropic submanifold of $H^{-1}\{\frac{1}{2}\}$. Indeed, its tangent vectors are generated by $\xi_i = \big((0,e_i); \frac{\partial p^s}{\partial x_i}\big)$ and

$$\begin{aligned} dp \wedge dx\,(\xi_i,\xi_j) &= \sum_{k=0}^{n-1} \frac{\partial p_k^s}{\partial x_i}\, dx^k(e_j) - \frac{\partial p_k^s}{\partial x_j}\, dx^k(e_i) \\ &= \frac{\partial p_j^s}{\partial x_i} - \frac{\partial p_i^s}{\partial x_j} = \frac{\partial^2 h}{\partial x_i \partial x_j} - \frac{\partial^2 h}{\partial x_j \partial x_i} = 0. \end{aligned}$$

When s is near zero, the submanifold $\mathrm{Graph}(p^s)$ is C^∞ near

$$\mathrm{Graph}(p^0) := \big(\{1\} \times [-\varepsilon,\varepsilon]^{n-1}\big) \times \{e_0\} \subset \mathcal{N}.$$

The tangent subspace to $\mathrm{Graph}(p^0)$ is generated by the vectors $\xi_i^0 = \big((0,e_i);0\big)$. Condition (c) in Lemma A.2 implies that, the hamiltonian vector field on $\mathcal{N}$ is $X = \big((1,0);0\big)$. Then X is transversal to $\mathrm{Graph}(p^0)$. Then for s small, the hamiltonian vector field X is also transversal to $\mathrm{Graph}(p^s)$.

Let

$$N_s = \big[\tfrac{1}{2} \le t \le 1\big] \bigcap \big[|x| < \varepsilon\big] \bigcap \phi_{[-2\,\varepsilon,0]}\big(\mathrm{Graph}(p^s)\big),$$

We are adding the flow direction to the isotropic submanifold $\mathrm{Graph}(p^s)$ of the energy level $[H \equiv k]$. Then N_s is also isotropic. Since $\dim N_s = n$, N_s is a lagrangian submanifold. Since the projection $\pi|_{\mathcal{N}}$ is a diffeomorphism and when $s \to 0$, N_s converges to $\mathcal{N}$ in the C^∞ topology, it follows that $\pi|_{N_s}$ is also a diffeomorphism for s small. Then N_s is the graph of a 1-form $\eta(t,x) \in T^*B$ defined on $[\frac{1}{2},1] \times [-\varepsilon,\varepsilon]^{n-1}$. Since N_s is

a lagrangian submanifold, the 1-form η_s is closed. Since its domain is contractible, η_s is exact: $\eta_s = d_{(t,x)}u_s$. Adding a constant if necessary we can assume that $u_s = h_s$ on $\{1\} \times [-\varepsilon, \varepsilon]^{n-1}$. Extend u_s to a C^∞ function on B such that

$$\begin{aligned} u_s(t,x) &= t, \qquad \text{if} \quad \max_i |x_i| > \varepsilon_1 \quad \text{or} \quad t < \tfrac{1}{4}. \\ \lim_{s\to 0} u_s(t,x) &= t, \qquad \text{in the } C^\infty \text{ topology.} \end{aligned}$$

This can be done using the Whitney extension theorem [55].

By construction $H(du_s) \equiv k \equiv H(\mathcal{K})$ on $t \in [\frac{1}{2}, 1]$. Since $\mathrm{Graph}(du_s)$ and $\mathcal{K}$ are lagrangian submanifolds, they are invariant under the hamiltonian vector field. Hence $\mathrm{Graph}(du_s)$ and $\mathcal{K}$ are transversal over $(t,x) \in [\frac{1}{2}, 1] \times [-\varepsilon_2, \varepsilon_2]^{n-1}$ if and only if their intersections with $[t = 1]$, $\big(x, \partial_x u_s(1,x)\big)$ and $\mathcal{K} \cap [t = 1]$ are transversal over $x \in [-\varepsilon_2, \varepsilon_2]^{n-1}$. By construction of u_s we have that

$$\partial_x u_s(1,x) = s \in \mathbb{R}^{n-1} \qquad \text{for } x \in [-\varepsilon_2, \varepsilon_2]^{n-1}.$$

Observe that the submanifolds $\mathrm{Graph}(du_s)$ on $(t,x) \in [\frac{1}{2}, 1] \times [-\varepsilon_2, \varepsilon_2]$, parametrized by s are a foliation of $(t,x;p) \in [\frac{1}{2}, 1] \times [-\varepsilon_2, \varepsilon_2]^{n-1} \times [-\delta, \delta]^{n-1}$. The projection of $\mathcal{K} \cap [t = 1]$ into the transverse direction to the foliation is given by $[-\varepsilon_2, \varepsilon_2]^{n-1} \ni x \mapsto d_x v$ where the function $v : [-\varepsilon, \varepsilon]^{n-1} \to \mathbb{R}$ is defined by $\mathcal{K} \cap [t = 1] = \mathrm{Graph}(dv)$. Therefore $\mathrm{Graph}(du_s)$ is transversal to $\mathcal{K}$ if and only if s is a regular value for $x \mapsto d_x v$. By Sard's theorem the set of regular values of dv has total measure, in particular there is a sequence $s_n \to 0$ of regular values. The sets $\mathcal{N}_n := \mathrm{Graph}(du_{s_n})$ are the required lagrangian manifolds. q.e.d.

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