J. DIFFERENTIAL GEOMETRY
60 (2002) 389-453

GAUGE-FIXING CONSTANT SCALAR CURVATURE EQUATIONS ON RULED MANIFOLDS AND THE FUTAKI INVARIANTS

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Abstract

In this article we introduce and prove the solvability of the gauge-fixing constant scalar curvature equations on ruled Kaehler manifolds. We prove that when some lifting conditions for holomorphic vector fields on the base manifold are satisfied the solutions for the gauge-fixing constant scalar curvature equations are actually solutions for the constant scalar curvature equations provided the corresponding Futaki invariants vanish.

In this article we will prove that the vanishing of certain natural Futaki invariants would imply the existence results for Kaehler metrics on ruled manifolds with constant scalar curvature. This work extends that of [10, 11] to the case where the base m-dimensional compact Kaehler manifold $(M : \omega_M)$ with constant scalar curvature may admit nontrivial holomorphic vector fields while the holomorphic vector bundle E over M with Einstein-Hermitian connection may not be simple.

In order to state our results properly we recall some facts about the structure of the groups of holomorphic automorphisms of compact Kaehler manifolds with constant scalar curvature. More background material can be found in [15].

Theorem 0. *Assume that $(M : \omega_M)$ is an m-dimensional compact Kaehler manifold with constant scalar curvature. Here ω_M is the Kaehler form of M. Let $\mathfrak{h}(M)$ denote the complex Lie algebra of holomorphic vector fields on M. Then we have the following direct sum decomposition (in the Lie algebra sense) of the Lie algebra $\mathfrak{h}(M)$:*

$$\mathfrak{h}(M) = \mathfrak{h}_o(M) \oplus \mathfrak{c}(M)$$

Received September 10, 2001; revised September 2, 2002.

in which

$$\mathfrak{h}_o(M) \equiv \big\{ Z \in \mathfrak{h}(M) : i_Z \omega_M = \bar{\partial} f \\ \textit{for some smooth } \mathbb{C}\textit{-valued function } f \in \Gamma(M : \mathbb{C}) \textit{ on } M \big\}$$

and

$$\mathfrak{c}(M) \equiv \left\{ Z \in \mathfrak{h}(M) : i_Z \omega_M \in H^{(0:1)}(M : \mathbb{C}) \right\}.$$

Note that the complex Lie algebra $\mathfrak{c}(M)$ is commutative and is a Lie subalgebra of the Lie algebra of the isometry group of $(M : \omega_M)$. Also, $\mathfrak{h}_o(M)$ is the complexification of the intersection $\mathfrak{k}_{(M:\omega_M)}$ of $\mathfrak{h}_o(M)$ with the Lie algebra of the isometry group of $(M : \omega_M)$.

Remark. Elements of $\mathfrak{h}_o(M)$ can be characterized intrinsically as the holomorphic vector fields on M with nonempty zero loci. This characterization of $\mathfrak{h}_o$ is valid for any compact Kaehler manifold (*not* necessarily with constant scalar curvature) and is due to Andre Lichnerowicz.

Remark. Note that the Lie algebra of smooth vector fields on M preserving the complex structure of M is isomorphic to $\mathfrak{h}(M)$ (in the Lie algebra sense). This interpretation of $\mathfrak{h}(M)$ is already used implicitly in the statement of Theorem 0 and will be used in the rest of this article.

Let $\mathrm{Aut}\,(M)$ denote the group of holomorphic automorphisms of M and G the connected component, containing the identity map of M, of the Lie subgroup of $\mathrm{Aut}\,(M)$ generated by $\mathfrak{h}_o(M)$. Let $K_{(M:\omega_M)}$ denote the compact connected component, containing the identity map of M, of the Lie subgroup of $\mathrm{Aut}\,(M)$ generated by the intersection $\mathfrak{k}_{(M:\omega_M)}$ of $\mathfrak{h}_o(M)$ with the Lie algebra of the isometry group of $(M : \omega_M)$ so that G is the complexification of $K_{(M:\omega_M)}$.

Assume that $\pi : E \longrightarrow M$ is a holomorphic vector bundle of rank n over M with Einstein-Hermitian metric H_E. Let A denote the Einstein-Hermitian connection on E induced by H_E. Let $\mathbb{P}(E)$ denote the projectivization of E over M. Then $\mathbb{P}(E)$ is a compact complex manifold with $(-1+m+n)$ dimensions. Let L be the universal line bundle over $\mathbb{P}(E)$. Then the Einstein-Hermitian metric H_E induces a Hermitian metric H_{L^*} on the dual L^* of L over $\mathbb{P}(E)$. Let A_{L^*} denote the Hermitian connection on L^* induced by H_{L^*}. Thus there is a representative

$$\frac{i \cdot F_{A_{L^*}}}{2\pi} = \frac{i}{2\pi} \cdot \bar{\partial}\partial \log H_{L^*} = -\frac{i}{2\pi} \cdot \bar{\partial}\partial \log H_L$$

of the Euler class $e(L^*)$ of L^* on $\mathbb{P}(E)$ induced by the Hermitian connection A_{L^*}. Here H_L is the Hermitian metric on L over $\mathbb{P}(E)$ induced by the Einstein-Hermitian metric H_E on E over M. Note that the representative $\frac{i\cdot F_{A_{L^*}}}{2\pi}$ of $e(L^*)$ on $\mathbb{P}(E)$ induces the Fubini-Study metric on each fiber $\mathbb{P}(\mathbb{C}^n)$ of $\check{\pi} : \mathbb{P}(E) \longrightarrow M$. Thus, for each $k \in \mathbb{N}$ large enough,

$$\frac{i \cdot F_{A_{L^*}}}{2\pi} + k \cdot \check{\pi}^* \omega_M$$

is a Kaehler form on $\mathbb{P}(E)$.

[‡]: Suppose that, for each $k \in \mathbb{N}$ large enough, there exists a corresponding Kaehler form on $\mathbb{P}(E)$, lying in the Kaehler class

$$\left[\frac{i \cdot F_{A_{L^*}}}{2\pi} + k \cdot \check{\pi}^* \omega_M\right],$$

carrying constant scalar curvature. Then, for each $k \in \mathbb{N}$ large enough, the corresponding Futaki character must be zero.

Let $\mathrm{Aut}\,(\mathbb{P}(E))$ denote the group of holomorphic automorphisms of $\mathbb{P}(E)$. Let $\mathrm{Aut}\,(E)$ denote the group of holomorphic automorphisms of E over M. Let G_E denote the natural image of $\mathrm{Aut}\,(E)$ in $\mathrm{Aut}\,(\mathbb{P}(E))$ preserving the holomorphic projection map $\check{\pi} : \mathbb{P}(E) \longrightarrow M$. Then we have

$$G_E = \frac{\mathrm{Aut}\,(E)}{\mathbb{C}^*}.$$

Let $\mathfrak{g}_E$ denote the Lie algebra of G_E. Our Theorem A shows that the converse of [‡] is true when the elements of $\mathfrak{h}_o(M)$ can be lifted to holomorphic vector fields on $\mathbb{P}(E)$ preserving the holomorphic projection map $\check{\pi} : \mathbb{P}(E) \longrightarrow M$.

Theorem A. *Assume that the elements of $\mathfrak{h}_o(M)$ can be lifted to holomorphic vector fields on $\mathbb{P}(E)$ preserving the holomorphic projection map*

$$\check{\pi} : \mathbb{P}(E) \longrightarrow M.$$

Suppose that, for each $k \in \mathbb{N}$ large enough, the corresponding Futaki character associated with

$$\mathfrak{g}_E + \text{ (the lifted action of) } \mathfrak{h}_o(M)$$

and the Kaehler class $\left[\frac{i\cdot F_{A_{L^*}}}{2\pi} + k \cdot \check{\pi}^* \omega_M\right]$ on $\mathbb{P}(E)$ is zero. Then, for each $k \in \mathbb{N}$ large enough, there exists a corresponding Kaehler form on

$\mathbb{P}(E)$, *lying in the Kaehler class*

$$\left[\frac{i\cdot F_{A_{L^*}}}{2\pi}+k\cdot\check{\pi}^*\omega_M\right],$$

carrying constant scalar curvature.

Corollary A. *Assume that the holomorphic vector bundle E with Einstein-Hermitian connection A over M is simple while the elements of $\mathfrak{h}_o(M)$ can be lifted to holomorphic vector fields on $\mathbb{P}(E)$ preserving the holomorphic projection map*

$$\check{\pi}:\mathbb{P}(E)\longrightarrow M.$$

Suppose that, for each $k\in\mathbb{N}$ large enough, the corresponding Futaki character associated with (the lifted action of) $\mathfrak{h}_o(M)$ and the Kaehler class $\left[\frac{i\cdot F_{A_{L^*}}}{2\pi}+k\cdot\check{\pi}^*\omega_M\right]$ on $\mathbb{P}(E)$ is zero. Then, for each $k\in\mathbb{N}$ large enough, there exists a corresponding Kaehler form on $\mathbb{P}(E)$, lying in the Kaehler class

$$\left[\frac{i\cdot F_{A_{L^*}}}{2\pi}+k\cdot\check{\pi}^*\omega_M\right],$$

carrying constant scalar curvature.

Corollary B. *Assume that the compact Kaehler manifold $(M:\omega_M)$ with constant scalar curvature does* not *admit nontrivial infinitesimal deformation of Kaehler forms in the Kaehler class $[\omega_M]$ on M with constant scalar curvature. Suppose that, for each $k\in\mathbb{N}$ large enough, the corresponding Futaki character associated with $\mathfrak{g}_E$ and the Kaehler class $\left[\frac{i\cdot F_{A_{L^*}}}{2\pi}+k\cdot\check{\pi}^*\omega_M\right]$ on $\mathbb{P}(E)$ is zero. Then, for each $k\in\mathbb{N}$ large enough, there exists a corresponding Kaehler form on $\mathbb{P}(E)$, lying in the Kaehler class*

$$\left[\frac{i\cdot F_{A_{L^*}}}{2\pi}+k\cdot\check{\pi}^*\omega_M\right],$$

carrying constant scalar curvature.

In view of Theorem 0 it might seem necessary to add the following invariance assumption of $\frac{i\cdot F_{A_{L^*}}}{2\pi}$ to Theorem A: $\frac{i\cdot F_{A_{L^*}}}{2\pi}$ is invariant under the lifted action of $\mathfrak{k}_{(M:\omega_M)}$ on $\mathbb{P}(E)$. Our Theorem B shows that it is unnecessary to make such extra assumption in Theorem A because the invariance of $\frac{i\cdot F_{A_{L^*}}}{2\pi}$ can be inferred directly from the vanishing of Futaki invariants. In particular, in Corollary A, $\frac{i\cdot F_{A_{L^*}}}{2\pi}$ is automatically

invariant under the lifted action of $\mathfrak{k}_{(M:\omega_M)}$ on $\mathbb{P}(E)$ because there is only one possible lifting of $\mathfrak{h}_o(M)$.

Theorem B. *Assume that the elements of $\mathfrak{h}_o(M)$ can be lifted to holomorphic vector fields on $\mathbb{P}(E)$ preserving the holomorphic projection map*

$$\check{\pi} : \mathbb{P}(E) \longrightarrow M.$$

Suppose that, for each $k \in \mathbb{N}$ large enough, the corresponding Futaki invariants associated with

$$\mathfrak{g}_E + \text{ (the lifted action of) } \mathfrak{k}_{(M:\omega_M)}$$

and the Kaehler class $\left[\frac{i\cdot F_{A_{L^*}}}{2\pi} + k\cdot\check{\pi}^*\omega_M\right]$ on $\mathbb{P}(E)$ are zero. Then the lifting of the intersection $\mathfrak{k}_{(M:\omega_M)}$ of $\mathfrak{h}_o(M)$ with the Lie algebra of the isometry group of $(M:\omega_M)$ can be properly rearranged such that

$$\frac{i\cdot F_{A_{L^*}}}{2\pi}$$

is invariant under the rearranged lifted action of $\mathfrak{k}_{(M:\omega_M)}$ on $\mathbb{P}(E)$.

Let $\mathfrak{k}_E$ denote the maximal compact Lie subalgebra of $\mathfrak{g}_E$. Actually, in Theorem B, the rearranged lifting of $\mathfrak{k}_{(M:\omega_M)}$ is, modulo $\mathrm{Hom}\left(\mathfrak{k}_{(M:\omega_M)} : \mathfrak{k}_E\right)$, uniquely determined. Precise version (Theorem II.B) of this result can be found in Section II.

We will prove Theorem A through solving the "*gauge-fixing* constant scalar curvature equation" depending on k large enough. (The gauge-fixing constant scalar curvature equation and its solvability, when the parameter k is sufficiently large, will be introduced in Section V.) With the solvability of the gauge-fixing constant scalar curvature equation we will then show that the solvability of constant scalar curvature equation can be inferred from the vanishing of Futaki invariants. Actually, by incorporating the vanishing of the corresponding Futaki invariants, we will show that the solutions to the gauge-fixing constant scalar curvature equation are actually solutions to the constant scalar curvature equation, when the parameter k is sufficiently large, in Section VI.

I. Futaki invariants

Here we summarize some basic facts about the Futaki Invariants. The reader can find more background material in [7, 15].

Theorem I.A. *Assume that $(B:\omega_B)$ is a b-dimensional compact Kaehler manifold (not necessarily with constant scalar curvature). Let $\mathfrak{h}(B)$ denote the complex Lie algebra of holomorphic vector fields on B and $[\omega_B]$ the Kaehler class associated with the Kaehler form ω_B on B. Let $c_{[\omega_B]} \in \mathbb{R}$ denote the constant associated with $[\omega_B]$ satisfying the following equality:*

$$c_{[\omega_B]} \cdot \int_B \frac{\omega^m}{m!} = \int_B \frac{i\cdot\rho_\omega}{2\pi} \wedge \frac{\omega^{(-1+m)}}{(-1+m)!} \qquad \forall \omega \in [\omega_B]\,.$$

Here ρ_ω is the curvature form of the holomorphic line bundle $\wedge^b T^{(1:0)}(B)$ on B (the highest degree wedge product of the holomorphic tangent bundle $T^{(1:0)}(B)$ of B) defined by $\omega \in [\omega_B]$. Let

$$\mathfrak{F} : \mathfrak{h}(B) \times [\omega_B] \longrightarrow \mathbb{C}$$

be defined as follows:

$$\mathfrak{F}(Z:\omega) \equiv \int_B \psi_{(Z:\omega)} \cdot \left(-c_{[\omega_B]} \cdot \frac{\omega^m}{m!} + \frac{i\cdot\rho_\omega}{2\pi} \wedge \frac{\omega^{(-1+m)}}{(-1+m)!} \right) \qquad \forall (Z:\omega) \in \mathfrak{h}(B) \times [\omega_B]$$

in which the smooth function $\psi_{(Z:\omega)} \in \Gamma\,(B:\mathbb{C})$ on B satisfies

$$\mathcal{L}_Z \omega = i\bar{\partial}\partial\,\psi_{(Z:\omega)}.$$

Then $\mathfrak{F}$ only depends on $\mathfrak{h}(B)$: $\mathfrak{F}(Z:\bullet)$ is constant on $[\omega_B]$ for each $Z \in \mathfrak{h}(B)$. Besides we have

$$\mathfrak{F}([Z:W]:\omega) = 0 \qquad \forall (Z:W:\omega) \in \mathfrak{h}(B) \times \mathfrak{h}(B) \times [\omega_B]\,.$$

$\mathfrak{F}$ is called the Futaki character associated with $(\mathfrak{h}(B):[\omega_B])$.

It is obvious that when B carries constant scalar curvature the Futaki character associated with $\mathfrak{h}(B)$ and $(B:\omega_B)$ must be zero. We will apply Theorem I.A to the compact complex manifold $\mathbb{P}(E)$.

II. Lifting of the elements of $\mathfrak{h}_o(M)$

In this section we discuss the lifting of the elements of $\mathfrak{h}_o(M)$ to holomorphic vector fields on $\mathbb{P}(E)$ preserving the holomorphic projection map

$$\check{\pi} : \mathbb{P}(E) \longrightarrow M.$$

We will consider the Real aspect of $\mathfrak{h}_o(M)$: Given a holomorphic vector field Z on M with nonempty zero locus we will consider the lifting of the corresponding smooth vector field X_Z on M preserving the complex structure of M. Here the smooth vector field X_Z on M is defined as follows:

$$X_Z \equiv Z + \overline{Z}.$$

Our immediate purpose is to lift X_Z to a smooth vector field on E preserving the vector bundle structure and the holomorphic structure of E over M.

Given the connection A on E there is a convenient lifting of X_Z (induced by the distribution of horizontal spaces on E specified by A) to a smooth vector field on E preserving the vector bundle structure of E over M. But this lifting does not necessarily preserve the holomorphic structure of E over M. Thus we add a smooth section s of $\mathrm{Hom}\,(E:E)$ over M to this lifting. Let F_A denote the curvature form of E induced by the connection A. Then it is easy to see that this modified lifting of X_Z preserves the holomorphic structure of E over M if and only if

$$-\bar{\partial}_A s + F_A\,(Z:\ ) = 0.$$

In particular we infer that X_Z can be lifted to a smooth vector field on E preserving the vector bundle structure and the holomorphic structure of E over M if and only if

$$0 = [F_A\,(Z:\ )] \in H^1_{\bar{\partial}_A}(M : \mathrm{Hom}\,(E:E)).$$

Thus we have the following:

Theorem II.A. *Assume that E is a holomorphic vector bundle with Hermitian connection A over a compact Kaehler manifold $(M:\omega_M)$. Let F_A denote the curvature form of E over M induced by the connection A. Then for any holomorphic vector field Z on M we have*

$$0 = [F_A\,(Z:\ )] \in H^1_{\bar{\partial}_A}(M : \mathrm{Hom}\,(E:E))$$

if and only if the corresponding smooth vector field $X_Z = Z + \overline{Z}$ on M, preserving the complex structure of M, can be lifted to a smooth vector field on $\mathbb{P}(E)$ preserving both the complex structure of $\mathbb{P}(E)$ and the holomorphic projection map

$$\check{\pi} : \mathbb{P}(E) \longrightarrow M.$$

Remark. Theorem II.A is valid for any compact Kaehler manifold (not necessarily with constant scalar curvature). Note that for any holomorphic vector field Z on M the equation

$$0 = [F_A(Z : \ )] \in H^1_{\bar{\partial}_A}(M : \mathrm{Hom}(E : E))$$

only depends on the holomorphic structure of E over M. It does not depend on the Hermitian connection A on E over M.

Now suppose further that the smooth vector field X_Z on M actually preserves the Kaehler form ω_M on M. Since $0 = [F_A(Z : \ )] \in H^1_{\bar{\partial}_A}(M : \mathrm{Hom}(E : E))$ there certainly exists a lifting of X_Z to a smooth vector field $\check{X}_Z$ on $\mathbb{P}(E)$ preserving both the complex structure of $\mathbb{P}(E)$ and the holomorphic projection map $\check{\pi} : \mathbb{P}(E) \longrightarrow M$.

Theorem II.B. *Suppose that the Futaki invariant associated with the lifting $\check{X}_Z$ (of X_Z) and the Kaehler class $\left[\frac{i \cdot F_{A_{L^*}}}{2\pi} + k \cdot \check{\pi}^* \omega_M\right]$ on $\mathbb{P}(E)$ vanishes for any sufficiently large $k \in \mathbb{N}$. Then there exists a lifting of X_Z, preserving the holomorphic projection map $\check{\pi} : \mathbb{P}(E) \longrightarrow M$, to a smooth vector field $\check{\mathbf{X}}_Z$ on $\mathbb{P}(E)$ preserving both $\frac{i \cdot F_{A_{L^*}}}{2\pi}$ and the complex structure of $\mathbb{P}(E)$. Note that the smooth vector field $\check{\mathbf{X}}_Z$ on $\mathbb{P}(E)$ is, modulo the compact Lie subalgebra $\mathfrak{k}_E$ of $\mathfrak{g}_E$, uniquely determined.*

It is obvious that the converse of Theorem II.B is true. Proving Theorem II.B requires more knowledge and will not be given in this section. The reader can find it in Appendix II.

III. Splitting of holomorphic vector bundles with Einstein-Hermitian connections over compact Kaehler manifolds

In this section we will consider the splitting of the holomorphic vector bundle E with Einstein-Hermitian connection A over a compact Kaehler manifold M. Since the assumption that $(M : \omega_M)$ carries constant scalar curvature will not be used the results of this section are valid for any compact Kaehler manifold.

We begin with some basic facts about the structure of holomorphic vector bundles with Einstein-Hermitian connections over compact Kaehler manifolds. The reader can find more background material in [12].

Note that E can be expressed as the direct sum of certain simple holomorphic vector bundles E_θ over M

$$E = \oplus_\theta E_\theta.$$

Besides the Einstein-Hermitian connection A on E can be expressed as

$$A = \oplus_\theta A_\theta$$

with each connection A_θ on E_θ being Einstein-Hermitian. Since any nontrivial holomorphic map between slope-stable holomorphic vector bundles over a compact Kaehler manifold must be an isomorphism it is easy to understand the structure of $\mathrm{Aut}(E)$ completely through the following examples:

Example I. When all the bundles E_θ are lying in the same isomorphism class of a slope-stable holomorphic vector bundle E_o over M we have

$$\mathrm{Aut}\,(E) = \mathrm{GL}\left(\frac{n}{\mathrm{rank}\,E_o} : \mathbb{C}\right).$$

Example II. Let d denote the number of the isomorphism classes defined by these slope-stable holomorphic vector bundles E_θ over M. When the isomorphism classes defined by these bundles E_θ over M are all distinct we have

$$\mathrm{Aut}\,(E) = \mathbb{C}^* \times \cdots \times \mathbb{C}^*$$

in which there are d copies of the multiplicative group

$$\mathbb{C}^* = \{z \in \mathbb{C} : z \neq 0\}.$$

Actually let d denote the number of the isomorphism classes defined by these slope-stable holomorphic vector bundles E_θ over M. Then $\mathrm{Aut}\,(E)$ is the product of d complex general linear groups. Each complex linear group is acting on the direct sum of those slope-stable holomorphic vector bundles E_θ over M lying in the same isomorphism class.

IV. Some basic facts and kernel identification

Since the restriction of $\frac{i \cdot F_{A_{L^*}}}{2\pi}$ on each fiber $\mathbb{P}\,(\mathbb{C}^n)$ of $\check{\pi} : \mathbb{P}(E) \longrightarrow M$ is simply the Fubini-Study Kaehler form there is a well-defined

smooth vector bundle W over M whose fiber (vector space over $\mathbb{R}$) W_z over $z \in M$ is the eigenspace of the lowest nonzero eigenvalue of the (Fubini-Study) Laplacian on the fiber $\mathbb{P}(\mathbb{C}^n)$ of $\mathbb{P}(E)$ over M. Note that the eigenspace of the lowest nonzero eigenvalue $4\pi n$ of the (Fubini-Study) Laplacian on $\mathbb{P}(\mathbb{C}^n)$ simply consists of the quotients of traceless Hermitian quadratic functions on $\mathbb{C}^n$ by the usual Hermitian metric $\sum \delta_{\alpha\beta} \cdot w_\alpha \cdot \overline{w}_\beta$ on $\mathbb{C}^n$. It is well-known in Kaehler geometry that this eigenspace represents the tangent space at the Hermitian metric $\sum \delta_{\alpha\beta} \cdot w_\alpha \cdot \overline{w}_\beta$ of the moduli space of Einstein-Kaehler metrics on $\mathbb{P}(\mathbb{C}^n)$:

$$\frac{\mathfrak{sl}(n:\mathbb{C})}{\mathfrak{su}(n)}.$$

On the other hand integration along the fibers of $\check{\pi} : \mathbb{P}(E) \longrightarrow M$ maps a smooth function on $\mathbb{P}(E)$ onto a smooth function on M. Let $\Gamma(M:W)$ denote the space of smooth sections of W over M. Then for each smooth $\mathbb{R}$-valued function $f \in \Gamma(\mathbb{P}(E):\mathbb{R})$ on $\mathbb{P}(E)$ we have the following corresponding decomposition:

$$f = \hat{\sigma}(f) \oplus \sigma(f) \oplus \widetilde{\sigma}(f)$$

in which $(\hat{\sigma}(f) : \sigma(f)) \in \Gamma(M:\mathbb{R}) \oplus \Gamma(M:W)$ while the restriction of $\widetilde{\sigma}(f)$ on each fiber $\mathbb{P}(\mathbb{C}^n)$ of $\check{\pi} : \mathbb{P}(E) \longrightarrow M$ over $z \in M$ is orthogonal to both the space W_z and the space of constant functions on that fiber (over $z \in M$).

Let $\Gamma_o(\mathbb{P}(\mathbb{C}^n):\mathbb{R})$ denote the space of smooth $\mathbb{R}$-valued functions f on $\mathbb{P}(\mathbb{C}^n)$ satisfying

$$\int_{\mathbb{P}(\mathbb{C}^n)} f \cdot \frac{\omega_{\text{F-S}}^{(-1+n)}}{(-1+n)!} = 0.$$

Now we introduce a basic result about a special kind of quadratic combinations of elements of the eigenspace of the lowest nonzero eigenvalue $(4\pi n)$ of the Fubini-Study Laplacian $\Delta_{\text{F-S}}$ on $\mathbb{P}(\mathbb{C}^n)$.

Proposition IV.A. *Assume that $\mathbb{C}^n$ and $\mathbb{P}(\mathbb{C}^n)$ are respectively endowed with the standard Hermitian metric $H_{\mathbb{C}^n} = \sum \delta_{\alpha\beta} \cdot w_\alpha \cdot \overline{w}_\beta$ on $\mathbb{C}^n$ and the Fubini-Study Kaehler form $\omega_{\text{F-S}} = -\frac{i}{2\pi}\bar{\partial}\partial \log H_{\mathbb{C}^n}$ on $\mathbb{P}(\mathbb{C}^n)$. We define a symmetric quadratic operation Q on the eigenspace of the lowest nonzero eigenvalue $(4\pi n)$ of the Fubini-Study Laplacian $\Delta_{\text{F-S}}$ on $\mathbb{P}(\mathbb{C}^n)$ as follows:*

Given $X \in \mathfrak{sl}(n:\mathbb{C})$ *let* $f_X \in \Gamma_o(\mathbb{P}(\mathbb{C}^n):\mathbb{R})$ *denote the smooth* $\mathbb{R}$*-valued function on* $\mathbb{P}(\mathbb{C}^n)$ *satisfying*

$$\mathcal{L}_X \omega_{\text{F-S}} = i\bar{\partial}\partial f_X.$$

Then we have $f_X = 0$ *when* $X \in \mathfrak{su}(n)$*. Besides we always have*

$$\Delta_{\text{F-S}} f_X = 4\pi n \cdot f_X.$$

We define $Q(f_X : f_X)$ *through the following equality:*

$$Q(f_X : f_X) \cdot \frac{\omega_{\text{F-S}}^{(-1+n)}}{(-1+n)!} = i\bar{\partial}\partial f_X \wedge i\bar{\partial}\partial f_X \wedge \frac{\omega_{\text{F-S}}^{(-3+n)}}{(-3+n)!}.$$

Then for each element f_X *of the eigenspace of the lowest nonzero eigenvalue* $(4\pi n)$ *of the Fubini-Study Laplacian* $\Delta_{\text{F-S}}$ *on* $\mathbb{P}(\mathbb{C}^n)$ *the following smooth function:*

$$-(2\pi \cdot n \cdot f_X) \cdot (2\pi \cdot n \cdot f_X) + Q(f_X : f_X)$$

on $\mathbb{P}(\mathbb{C}^n)$ *is orthogonal to the eigenspace of the lowest nonzero eigenvalue* $(4\pi n)$ *of the Fubini-Study Laplacian* $\Delta_{\text{F-S}}$ *on* $\mathbb{P}(\mathbb{C}^n)$*.*

This result has been proved in [11] through direct computation. Actually it can be inferred from the vanishing of the Futaki character associated with $\mathfrak{sl}(n:\mathbb{C})$ and the Fubini-Study Kaehler class $[\omega_{\text{F-S}}]$ on $\mathbb{P}(\mathbb{C}^n)$.

Proof of Proposition IV.A. Let $\rho_{\omega_{\text{F-S}}}$ denote the curvature form of the dual of the canonical line bundle of $\mathbb{P}(\mathbb{C}^n)$ defined by $\omega_{\text{F-S}}$. Then we have

$$\frac{i \cdot \rho_{\omega_{\text{F-S}}}}{2\pi} = \frac{i \cdot \bar{\partial}\partial \log\det H_{\text{F-S}}}{2\pi} = n \cdot \omega_{\text{F-S}}.$$

Here $H_{\text{F-S}}$ is the Einstein-Kaehler metric on $\mathbb{P}(\mathbb{C}^n)$ induced by $\omega_{\text{F-S}}$. Let

$$\omega_{\text{F-S}:t} \equiv \omega_{\text{F-S}} + t \cdot i\bar{\partial}\partial f_X \qquad \forall t \in \mathbb{R}.$$

Then for each $t \in \mathbb{R}$ with $|t| \geq 0$ being small $\omega_{\text{F-S}:t}$ is a Kaehler form on $\mathbb{P}(\mathbb{C}^n)$ lying in the Fubini-Study Kaehler class $[\omega_{\text{F-S}}]$. We define a symmetric quadratic operation $\mathcal{Q}$ on the eigenspace of the lowest nonzero

eigenvalue $(4\pi n)$ of the Fubini-Study Laplacian $\Delta_{\text{F-S}}$ on $\mathbb{P}(\mathbb{C}^n)$ through the following equality:

$$\begin{aligned}&\mathcal{Q}(f_X : f_X) \cdot \frac{\omega_{\text{F-S}}^{(-1+n)}}{(-1+n)!}\\ &= \frac{d}{dt} \circ \frac{d}{dt} \left(-n \cdot (-1+n) \cdot \frac{\omega_{\text{F-S}:t}^{(-1+n)}}{(-1+n)!} + \frac{i \cdot \rho_{\omega_{\text{F-S}:t}}}{2\pi} \wedge \frac{\omega_{\text{F-S}:t}^{(-2+n)}}{(-2+n)!} \right) \Bigg|_{t=0}.\end{aligned}$$

It can be checked readily that

$$\begin{aligned}&\mathcal{Q}(f_X : f_X) \cdot \frac{\omega_{\text{F-S}}^{(-1+n)}}{(-1+n)!}\\ &= \frac{i}{2\pi} \bar{\partial}\partial \left(-\frac{\Delta_{\text{F-S}} f_X}{2} \cdot \frac{\Delta_{\text{F-S}} f_X}{2} + Q(f_X : f_X) \right) \wedge \frac{\omega_{\text{F-S}}^{(-2+n)}}{(-2+n)!}\end{aligned}$$

and thence

$$\begin{aligned}\mathcal{Q}(f_X : f_X) =& \frac{\Delta_{\text{F-S}}}{4\pi} \left(-\frac{\Delta_{\text{F-S}} f_X}{2} \cdot \frac{\Delta_{\text{F-S}} f_X}{2} + Q(f_X : f_X) \right)\\ =& \frac{\Delta_{\text{F-S}}}{4\pi} \left[-(2\pi n \cdot f_X) \cdot (2\pi n \cdot f_X) + Q(f_X : f_X) \right].\end{aligned}$$

Now for any $Y \in \mathfrak{sl}(n : \mathbb{C})$ the Futaki invariant associated with Y and the Fubini-Study Kaehler class $[\omega_{\text{F-S}}]$ on $\mathbb{P}(\mathbb{C}^n)$ vanishes. Thus we have

$$\begin{aligned}0 =& \mathfrak{F}(Y : \omega_{\text{F-S}:t})\\ =& \int_{\mathbb{P}(\mathbb{C}^n)} \left(-\frac{\mathcal{L}_Y \log H_{\mathbb{C}^n}}{2\pi} + t \cdot \mathcal{L}_Y f_X \right)\\ &\cdot \left(-n \cdot (-1+n) \cdot \frac{\omega_{\text{F-S}:t}^{(-1+n)}}{(-1+n)!} + \frac{i \cdot \rho_{\omega_{\text{F-S}:t}}}{2\pi} \wedge \frac{\omega_{\text{F-S}:t}^{(-2+n)}}{(-2+n)!} \right)\end{aligned}$$

for $t \in \mathbb{R}$ with $|t| \geq 0$ being small. Let us now consider the equality

$$0 = \frac{d}{dt} \circ \frac{d}{dt} \mathfrak{F}(Y : \omega_{\text{F-S}:t}) \Big|_{t=0}.$$

Since $X \in \mathfrak{sl}(n : \mathbb{C})$ preserves the Einstein-Kaehler condition (equivalently the constant scalar curvature condition) on $\mathbb{P}(\mathbb{C}^n)$:

$$\mathcal{L}_X \left(-n \cdot (-1+n) \cdot \frac{\omega_{\text{F-S}}^{(-1+n)}}{(-1+n)!} + \frac{i \cdot \rho_{\omega_{\text{F-S}}}}{2\pi} \wedge \frac{\omega_{\text{F-S}}^{(-2+n)}}{(-2+n)!} \right) = 0$$

and

$$\mathcal{L}_X \omega_{\text{F-S}} = i\bar{\partial}\partial f_X$$

we have

$$\frac{d}{dt}\left(-n\cdot(-1+n)\cdot\frac{\omega_{\text{F-S:}t}^{(-1+n)}}{(-1+n)!}+\frac{i\cdot\rho_{\omega_{\text{F-S:}t}}}{2\pi}\wedge\frac{\omega_{\text{F-S:}t}^{(-2+n)}}{(-2+n)!}\right)\Bigg|_{t=0}=0.$$

Thus

$$\begin{aligned}
0 &= \frac{d}{dt}\circ\frac{d}{dt}\,\mathfrak{F}\left(Y:\omega_{\text{F-S:}t}\right)\Big|_{t=0}\\
&= \int_{\mathbb{P}(\mathbb{C}^n)} -\frac{\mathcal{L}_Y \log H_{\mathbb{C}^n}}{2\pi}\cdot\frac{d}{dt}\circ\frac{d}{dt}\\
&\qquad\cdot\left(-n\cdot(-1+n)\cdot\frac{\omega_{\text{F-S:}t}^{(-1+n)}}{(-1+n)!}+\frac{i\cdot\rho_{\omega_{\text{F-S:}t}}}{2\pi}\wedge\frac{\omega_{\text{F-S:}t}^{(-2+n)}}{(-2+n)!}\right)\Bigg|_{t=0}\\
&= \int_{\mathbb{P}(\mathbb{C}^n)} -\frac{\mathcal{L}_Y \log H_{\mathbb{C}^n}}{2\pi}\cdot\mathcal{Q}\left(f_X:f_X\right)\cdot\frac{\omega_{\text{F-S}}^{(-1+n)}}{(-1+n)!}\\
&= \int_{\mathbb{P}(\mathbb{C}^n)} f_Y\cdot\frac{\Delta_{\text{F-S}}}{4\pi}\left[-(2\pi n\cdot f_X)\cdot(2\pi n\cdot f_X)+Q\left(f_X:f_X\right)\right]\cdot\frac{\omega_{\text{F-S}}^{(-1+n)}}{(-1+n)!}
\end{aligned}$$

for any $Y\in\mathfrak{sl}(n:\mathbb{C})$. Since $\Delta_{\text{F-S}}$ is symmetric with respect to the Fubini-Study Kaehler form $\omega_{\text{F-S}}$ on $\mathbb{P}\left(\mathbb{C}^n\right)$ we conclude from the last equality that for any $Y\in\mathfrak{sl}(n:\mathbb{C})$

$$\begin{aligned}
0 &= \int_{\mathbb{P}(\mathbb{C}^n)} \frac{\Delta_{\text{F-S}} f_Y}{4\pi}\cdot\left[-(2\pi n\cdot f_X)\cdot(2\pi n\cdot f_X)+Q\left(f_X:f_X\right)\right]\cdot\frac{\omega_{\text{F-S}}^{(-1+n)}}{(-1+n)!}\\
&= \int_{\mathbb{P}(\mathbb{C}^n)} n\cdot f_Y\cdot\left[-(2\pi n\cdot f_X)\cdot(2\pi n\cdot f_X)+Q\left(f_X:f_X\right)\right]\cdot\frac{\omega_{\text{F-S}}^{(-1+n)}}{(-1+n)!}
\end{aligned}$$

and thence the assertion of Proposition IV.A is true. q.e.d.

Corollary IV.A. *Assume that $\mathbb{C}^n$ and $\mathbb{P}\left(\mathbb{C}^n\right)$ are respectively endowed with the standard Hermitian metric $H_{\mathbb{C}^n}=\sum\delta_{\alpha\beta}\cdot w_\alpha\cdot\overline{w}_\beta$ on $\mathbb{C}^n$ and the Fubini-Study Kaehler form $\omega_{\text{F-S}}=-\frac{i}{2\pi}\bar{\partial}\partial\log H_{\mathbb{C}^n}$ on $\mathbb{P}\left(\mathbb{C}^n\right)$. We define a symmetric quadratic operation Q on the eigenspace of the lowest nonzero eigenvalue $(4\pi n)$ of the Fubini-Study Laplacian $\Delta_{\text{F-S}}$ on $\mathbb{P}\left(\mathbb{C}^n\right)$ as follows:*

Given $X \in \mathfrak{sl}(n : \mathbb{C})$ *and* $Y \in \mathfrak{sl}(n : \mathbb{C})$ *let* $f_X \in \Gamma_o(\mathbb{P}(\mathbb{C}^n) : \mathbb{R})$ *and* $f_Y \in \Gamma_o(\mathbb{P}(\mathbb{C}^n) : \mathbb{R})$ *denote the smooth* $\mathbb{R}$*-valued functions on* $\mathbb{P}(\mathbb{C}^n)$ *satisfying*

$$\mathcal{L}_X \omega_{\text{F-S}} = i\bar{\partial}\partial f_X \qquad \text{and} \qquad \mathcal{L}_Y \omega_{\text{F-S}} = i\bar{\partial}\partial f_Y.$$

We define $Q(f_X : f_Y)$ *through the following equality:*

$$Q(f_X : f_Y) \cdot \frac{\omega_{\text{F-S}}^{(-1+n)}}{(-1+n)!} = i\bar{\partial}\partial f_X \wedge i\bar{\partial}\partial f_Y \wedge \frac{\omega_{\text{F-S}}^{(-3+n)}}{(-3+n)!}.$$

Then for each pair $(f_X : f_Y)$ *of elements of the eigenspace of the lowest nonzero eigenvalue* $(4\pi n)$ *of the Fubini-Study Laplacian* $\Delta_{\text{F-S}}$ *on* $\mathbb{P}(\mathbb{C}^n)$ *the following smooth function:*

$$-(2\pi \cdot n \cdot f_X) \cdot (2\pi \cdot n \cdot f_Y) + Q(f_X : f_Y)$$

on $\mathbb{P}(\mathbb{C}^n)$ *is orthogonal to the eigenspace of the lowest nonzero eigenvalue* $(4\pi n)$ *of the Fubini-Study Laplacian* $\Delta_{\text{F-S}}$ *on* $\mathbb{P}(\mathbb{C}^n)$.

Note that the Einstein-Hermitian connection A on E over M defines a smooth distribution $\mathcal{H}$ of horizontal spaces on $\mathbb{P}(E)$:

$$T(\mathbb{P}(E)) = V \oplus \mathcal{H}.$$

Here V is the subbundle of $T(\mathbb{P}(E))$ over $\mathbb{P}(E)$ consisting of tangent vectors which are tangential to the fibers of $\check{\pi} : \mathbb{P}(E) \longrightarrow M$. Let $V^{[*]}$ denote the maximal subbundle of $T^*(\mathbb{P}(E))$ over $\mathbb{P}(E)$ whose action on $\mathcal{H}$ is identically zero. Then the decomposition $T(\mathbb{P}(E)) = V \oplus \mathcal{H}$ of $T(\mathbb{P}(E))$ over $\mathbb{P}(E)$ induces the following corresponding decomposition:

$$T^*(\mathbb{P}(E)) = V^{[*]} \oplus \check{\pi}^*(T^*(M))$$

of $T^*(\mathbb{P}(E))$ over $\mathbb{P}(E)$. Thus we have the following decomposition:

$$\wedge^* T^*(\mathbb{P}(E)) = \mathcal{C}_V \oplus \mathcal{C}_m \oplus \mathcal{C}_M$$

of $\wedge^* T^*(\mathbb{P}(E))$ over $\mathbb{P}(E)$. Here $\mathcal{C}_V = \wedge^* V^{[*]}$ and $\mathcal{C}_M = \wedge^* \check{\pi}^* T^*(M)$ while $\mathcal{C}_m$ is the subbundle of $\wedge^* T^*(\mathbb{P}(E))$ over $\mathbb{P}(E)$ consisting of the mixed components of $\wedge^* T^*(\mathbb{P}(E))$. Thus we have the following diagram:

$$\begin{array}{ccccc} \mathcal{C}_V & \xleftarrow{\Pi_{\mathcal{C}_V}} & \wedge^* T^*(\mathbb{P}(E)) & \xrightarrow{\Pi_{\mathcal{C}_M}} & \mathcal{C}_M \\ & & \downarrow \Pi_{\mathcal{C}_m} & & \\ & & \mathcal{C}_m & & \end{array}$$

of projection maps over $\mathbb{P}(E)$ such that $\mathrm{id} = \Pi_{\mathcal{C}_V} \oplus \Pi_{\mathcal{C}_m} \oplus \Pi_{\mathcal{C}_M}$ on $\wedge^* T^*(\mathbb{P}(E))$. Since the decomposition

$$T^*(\mathbb{P}(E)) = V^{[*]} \oplus \check{\pi}^*(T^*(M))$$

of $T^*(\mathbb{P}(E))$ is defined by the Einstein-Hermitian connection A on E over M we note that the representative

$$\frac{i \cdot F_{A_{L^*}}}{2\pi}$$

of the Euler class $e(L^*)$ of L^* on $\mathbb{P}(E)$ has no nontrivial mixed components of $\wedge^* T^*(\mathbb{P}(E))$:

$$\frac{i \cdot F_{A_{L^*}}}{2\pi} = \Pi_{\mathcal{C}_V}\left(\frac{i \cdot F_{A_{L^*}}}{2\pi}\right) \oplus \Pi_{\mathcal{C}_M}\left(\frac{i \cdot F_{A_{L^*}}}{2\pi}\right).$$

Now we introduce a Hermitian form (metric) $\check{\omega}$ on $\mathbb{P}(E)$ by setting

$$\check{\omega} = \Pi_{\mathcal{C}_V}\left(\frac{i \cdot F_{A_{L^*}}}{2\pi}\right) + \check{\pi}^* \omega_M.$$

Remark. It should be noted that

$$\check{\omega}^{(-1+m+n)} = \lim_{k \to +\infty} k^{-m} \cdot \left(\frac{i \cdot F_{A_{L^*}}}{2\pi} + k \cdot \omega_M\right)^{(-1+m+n)}.$$

Actually $\check{\omega}$ can be realized as a modified limit of $\frac{i \cdot F_{A_{L^*}}}{2\pi} + k \cdot \omega_M$, as $k \to +\infty$, with the base directions of $\mathbb{P}(E)$ being properly rescaled.

Note that the derivation operator

$$d : \Gamma(\mathbb{P}(E) : \mathbb{R}) \longrightarrow \Gamma(\mathbb{P}(E) : T^*(\mathbb{P}(E)) \otimes \mathbb{R})$$

can be expressed as

$$d = d_V + d_M$$

in which $d_V : \Gamma(\mathbb{P}(E) : \mathbb{R}) \longrightarrow \Gamma\left(\mathbb{P}(E) : \mathbb{R} \otimes V^{[*]}\right)$ and $d_M : \Gamma(\mathbb{P}(E) : \mathbb{R}) \longrightarrow \Gamma(\mathbb{P}(E) : \mathbb{R} \otimes \check{\pi}^*(T^*(M)))$. Let d_V^* and d_M^* be respectively the adjoint operators of d_V and d_M with respect to the Hermitian form (metric) $\check{\omega}$ on $\mathbb{P}(E)$. Then we have

$$\Delta = d^* \circ d = \Delta_V + \Delta_M$$

in which $\Delta_V \equiv d_V^* \circ d_V$ and $\Delta_M \equiv d_M^* \circ d_M$. Similarly we have $\bar{\partial} = \bar{\partial}_V + \bar{\partial}_M$ and $\partial = \partial_V + \partial_M$. Let Λ_V and Λ_M be respectively the adjoint operators of

$$\bullet \longmapsto \Pi_{\mathcal{C}_V} \frac{i \cdot F_{A_{L^*}}}{2\pi} \wedge \bullet$$

and

$$\bullet \longmapsto \breve{\pi}^* \omega_M \wedge \bullet$$

on $\mathbb{P}(E)$ with respect to the Hermitian form (metric) $\breve{\omega}$. We use the symbols $\left(\bar{\partial}\partial\right)_V$ and $\left(\bar{\partial}\partial\right)_M$ to denote respectively $\Pi_{\mathcal{C}_V} \circ (\bar{\partial}\partial)$ and $\Pi_{\mathcal{C}_M} \circ (\bar{\partial}\partial)$:

$$\left(\bar{\partial}\partial\right)_V = \Pi_{\mathcal{C}_V} \circ (\bar{\partial}\partial) \text{ and } \left(\bar{\partial}\partial\right)_M = \Pi_{\mathcal{C}_M} \circ (\bar{\partial}\partial).$$

Similarly we use the symbol $\left(\bar{\partial}\partial\right)_m$ to denote $\Pi_{\mathcal{C}_m} \circ (\bar{\partial}\partial)$: $\left(\bar{\partial}\partial\right)_m = \Pi_{\mathcal{C}_m} \circ (\bar{\partial}\partial)$. Then we have the following results (proved in the Appendix of [11]):

Proposition IV.B. *Given $f \in \Gamma(\mathbb{P}(E) : \mathbb{R})$ we have the equalities*

$$i \cdot \Lambda_V \circ \left(\bar{\partial}\partial\right)_V f = \frac{\Delta_V f}{2}$$

and

$$i \cdot \Lambda_M \circ \left(\bar{\partial}\partial\right)_M f = \frac{\Delta_M f}{2}.$$

Proposition IV.C. $\Delta_M \circ \Delta_V = \Delta_V \circ \Delta_M$.

In particular we have

$$\Delta_M \circ (-4\pi n \cdot \mathrm{id} + \Delta_V) = (-4\pi n \cdot \mathrm{id} + \Delta_V) \circ \Delta_M$$

and thence Δ_M preserves $\Gamma(M : W)$. In [11] it is shown that the invertibility of the linear partial differential operator Δ_M acting on $\Gamma(M : W)$ is equivalent to the simplicity of the holomorphic vector bundle E over M. Actually each smooth section s of W over M can be realized as a smooth Hermitian section of Hom $(E : E)$ over M. Using the Einstein-Hermitian condition of A on E over M it can be checked readily that the smooth $\mathbb{R}$-valued function s on $\mathbb{P}(E)$ satisfies $\Delta_M s = 0$ if and only if its corresponding smooth Hermitian section of Hom $(E : E)$ over M is harmonic (and thence holomorphic by the Einstein-Hermitian condition of A on E over M). Thus the kernel of Δ_M acting on $\Gamma(M : W)$ is isomorphic to

$$\frac{\mathfrak{g}_E}{\mathfrak{k}_E}.$$

It should be noted that the linear partial differential operator Δ_M acting on $\Gamma(M:W)$ is both nonnegative and symmetric (with respect to the Hermitian form (metric) $\breve{\omega}$ on $\mathbb{P}(E)$).

We can realize this picture more concretely as follows. Let G_E denote the natural image of $\mathrm{Aut}(E)$ in $\mathrm{Aut}(\mathbb{P}(E))$ preserving the holomorphic projection map $\breve{\pi}:\mathbb{P}(E)\longrightarrow M$. Then we have

$$G_E=\frac{\mathrm{Aut}(E)}{\mathbb{C}^*}.$$

Let $\mathfrak{g}_E$ denote the Lie algebra of G_E over $\mathbb{C}$. Let $\mathfrak{k}_E$ denote the compact Lie algebra generated by the elements of $\mathfrak{g}_E$ preserving the representative

$$\frac{i\cdot F_{A_{L^*}}}{2\pi}$$

of the Euler class of L^* on $\mathbb{P}(E)$ so that $\mathfrak{g}_E$ is the complexification of $\mathfrak{k}_E$. Let K_E denote the compact subgroup of G_E generated by $\mathfrak{k}_E$. Given a smooth vector field $Y\in\mathfrak{g}_E$ on $\mathbb{P}(E)$ we denote by $f_Y\in\Gamma(M:W)$ the corresponding smooth $\mathbb{R}$-valued function on $\mathbb{P}(E)$ satisfying

$$\mathcal{L}_Y\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right)=i\bar{\partial}\partial f_Y.$$

(Note that when $Y\in\mathfrak{k}_E$ we have $f_Y=0$.) Let $\mathbf{N}_W$ denote the kernel of Δ_M acting on $\Gamma(M:W)$. Then we have

$$\mathbf{N}_W=\left\{f_Y\in\Gamma(M:W):\mathcal{L}_Y\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right)=i\bar{\partial}\partial f_Y \text{ for some } Y\in\frac{\mathfrak{g}_E}{\mathfrak{k}_E}\right\}.$$

We can now decompose the function space $\Gamma(M:W)$ into the direct sum of $\mathbf{N}_W$ and the orthogonal complement of $\mathbf{N}_W$ in $\Gamma(M:W)$. Thus for $f\in\Gamma(M:W)$ we have

$$f=\tau^+_{\mathbf{N}_W}(f)\oplus\tau_{\mathbf{N}_W}(f)$$

in which $\tau^+_{\mathbf{N}_W}(f)$ is orthogonal to $\mathbf{N}_W$ while $\tau_{\mathbf{N}_W}(f)$ is the $\mathbf{N}_W$-component of f.

Let $\mathcal{V}_M$ denote the infinitesimal deformation operator for the con-

stant scalar curvature equation on $(M:\omega_M)$:

$$\begin{aligned}\mathcal{V}_M\bullet &= i\bar{\partial}\partial\left(\frac{\Delta_M\bullet}{4\pi}\right)\wedge\frac{\omega_M^{(-1+m)}}{(-1+m)!}\\ &\quad+\left[-\Lambda_M\mathrm{trace}\left(\frac{i\cdot F_{\omega_M}}{2\pi}\right)\right]\cdot i\bar{\partial}\partial\bullet\wedge\frac{\omega_M^{(-1+m)}}{(-1+m)!}\\ &\quad+i\cdot\bar{\partial}\partial\bullet\wedge\mathrm{trace}\left(\frac{i\cdot F_{\omega_M}}{2\pi}\right)\wedge\frac{\omega_M^{(-2+m)}}{(-2+m)!}\\ &=\frac{\Delta_M\circ\Delta_M\bullet}{8\pi}\cdot\frac{\omega_M^m}{m!}+\left[-\Lambda_M\mathrm{trace}\left(\frac{i\cdot F_{\omega_M}}{2\pi}\right)\right]\cdot\frac{\Delta_M\bullet}{2}\cdot\frac{\omega_M^m}{m!}\\ &\quad+i\cdot\bar{\partial}\partial\bullet\wedge\mathrm{trace}\left(\frac{i\cdot F_{\omega_M}}{2\pi}\right)\wedge\frac{\omega_M^{(-2+m)}}{(-2+m)!}.\end{aligned}$$

Here F_{ω_M} is the curvature form of the holomorphic tangent bundle of M induced by the Kaehler form ω_M on M while

$$\Lambda_M\mathrm{trace}\left(\frac{i\cdot F_{\omega_M}}{2\pi}\right)$$

is the scalar curvature of $(M:\omega_M)$:

$$\left[\Lambda_M\mathrm{trace}\left(\frac{i\cdot F_{\omega_M}}{2\pi}\right)\right]\cdot\frac{\omega_M^m}{m!}=\mathrm{trace}\left(\frac{i\cdot F_{\omega_M}}{2\pi}\right)\wedge\frac{\omega_M^{(-1+m)}}{(-1+m)!}.$$

Let $\Gamma_o(M:\mathbb{R})$ denote the space of smooth $\mathbb{R}$-valued functions f on M satisfying

$$\int_M f\cdot\Omega_M=0.$$

Here we set $\Omega_M\equiv\frac{\omega_M^m}{m!}$. Then the linear partial differential operator

$$\frac{\mathcal{V}_M}{\Omega_M}=\frac{\mathcal{V}_M}{\frac{\omega_M^m}{m!}}$$

acting on $\Gamma_o(M:\mathbb{R})$ is both nonnegative and symmetric (with respect to the Kaehler form (metric) ω_M on M). Note that the kernel of the linear partial differential operator $\mathcal{V}_M$ acting on $\Gamma_o(M:\mathbb{R})$ is isomorphic to the vector space

$$\frac{\mathfrak{h}_o(M)}{\mathfrak{k}_{(M:\omega_M)}}$$

over $\mathbb{R}$. Let $\mathbf{N}_{\mathcal{V}_M}$ denote the vector space over $\mathbb{R}$ of the kernel of $\mathcal{V}_M$ acting on $\Gamma_o(M:\mathbb{R})$. We can now decompose the function space $\Gamma_o(M:\mathbb{R})$ into the direct sum of $\mathbf{N}_{\mathcal{V}_M}$ and the orthogonal complement of $\mathbf{N}_{\mathcal{V}_M}$ in $\Gamma_o(M:\mathbb{R})$. Thus for $f \in \Gamma_o(M:\mathbb{R})$ we have

$$f = \tau^{+}_{\mathbf{N}_{\mathcal{V}_M}}(f) \oplus \tau_{\mathbf{N}_{\mathcal{V}_M}}(f)$$

in which $\tau^{+}_{\mathbf{N}_{\mathcal{V}_M}}(f)$ is orthogonal to $\mathbf{N}_{\mathcal{V}_M}$ while $\tau_{\mathbf{N}_{\mathcal{V}_M}}(f)$ is the $\mathbf{N}_{\mathcal{V}_M}$-component of f:

$$\mathcal{V}_M\left(\tau_{\mathbf{N}_{\mathcal{V}_M}}(f)\right) = 0 \Longleftrightarrow \frac{\mathcal{V}_M\left(\tau_{\mathbf{N}_{\mathcal{V}_M}}(f)\right)}{\frac{\omega_M^m}{m!}} = 0.$$

V. Gauge-fixing constant scalar curvature equation

In this section we will introduce the gauge-fixing constant scalar curvature equation, depending on the parameter $k \in \mathbb{N}$, and prove its solvability when k is large enough.

Let ${}_oH_{\#k}$ denote the Kaehler metric on $\mathbb{P}(E)$ induced by the Kaehler form

$${}_o\omega_{\#k} \equiv \frac{i \cdot F_{A_{L^*}}}{2\pi} + k \cdot \check{\pi}^* \omega_M.$$

Suppose that, for each $k \in \mathbb{N}$ large enough, $\omega_{\#k}$ is a Kaehler form on $\mathbb{P}(E)$ lying in the Kaehler class $\left[{}_o\omega_{\#k}\right]$ so that

$$\omega_{\#k} = {}_o\omega_{\#k} + i \cdot \bar{\partial}\partial\psi_k$$

with $\psi_k \in \Gamma(\mathbb{P}(E):\mathbb{R})$ satisfying

$$\int_{\mathbb{P}(E)} \psi_k \cdot \Omega_{\mathbb{P}(E)} = 0 \Longleftrightarrow \int_M \hat{\sigma}(\psi_k) \cdot \Omega_M = 0$$

in which $\Omega_M = \frac{\omega_M^m}{m!}$ and $\Omega_{\mathbb{P}(E)} \equiv \frac{\check{\omega}^{(-1+m+n)}}{(-1+m+n)!} = \frac{\left(\frac{i \cdot F_{A_{L^*}}}{2\pi}\right)^{(-1+n)}}{(-1+n)!} \wedge \frac{\omega_M^m}{m!}$. Let $\check{c}_k \in \mathbb{R}$, depending on the parameter $k \in \mathbb{N}$, be the topological invariant satisfying the following equality:

$$\check{c}_k \cdot \int_{\mathbb{P}(E)} \frac{{}_o\omega_{\#k}^{(-1+m+n)}}{(-1+m+n)!} = \int_{\mathbb{P}(E)} \frac{i \cdot \bar{\partial}\partial \log\det {}_oH_{\#k}}{2\pi} \wedge \frac{{}_o\omega_{\#k}^{(-2+m+n)}}{(-2+m+n)!}.$$

Let $H_{\#k}$ be the Kaehler metric on $\mathbb{P}(E)$ induced by the Kaehler form $\omega_{\#k}$. Then the Constant Scalar Curvature Equation for $\omega_{\#k}$ is

$$\mathcal{S}\left(\omega_{\#k}\right) = 0$$

in which

$$\mathcal{S}\left(\omega_{\#k}\right) \equiv -\check{c}_k \cdot \frac{\omega_{\#k}^{(-1+m+n)}}{(-1+m+n)!} + \frac{i \cdot \bar{\partial}\partial \log\det H_{\#k}}{2\pi} \wedge \frac{\omega_{\#k}^{(-2+m+n)}}{(-2+m+n)!}.$$

Based on the work [10, 11] we might want to solve the constant scalar curvature equation, depending on $k \in \mathbb{N}$ large enough, directly. However it is impractical to do so as there exist nontrivial kernels of linear partial differential operators associated with the constant scalar curvature equation. These nontrivial kernels exist simply because the constant scalar curvature equation is invariant under the action of the group Aut $(\mathbb{P}(E))$ of holomorphic automorphisms of $\mathbb{P}(E)$. In order to tackle this difficulty we add the following "*gauge-fixing*" term:

$$\left(\frac{n \cdot \tau_{\mathbf{N}_W} \circ \sigma\left(\psi_k\right)}{k} + \frac{\tau_{\mathbf{N}_{\mathcal{V}_M}} \circ \hat{\sigma}\left(\psi_k\right)}{k \cdot k}\right) \cdot k^m \cdot \Omega_{\mathbb{P}(E)}$$

to the constant scalar curvature equation and define the "gauge-fixing constant scalar curvature equation" as

$$\mathcal{S}_{\text{G-F}}\left(\omega_{\#k}\right) = 0$$

in which

$$\begin{aligned}
&\mathcal{S}_{\text{G-F}}\left(\omega_{\#k}\right) \\
&\equiv \mathcal{S}\left(\omega_{\#k}\right) + \left(\frac{n \cdot \tau_{\mathbf{N}_W} \circ \sigma\left(\psi_k\right)}{k} + \frac{\tau_{\mathbf{N}_{\mathcal{V}_M}} \circ \hat{\sigma}\left(\psi_k\right)}{k \cdot k}\right) \cdot k^m \cdot \Omega_{\mathbb{P}(E)} \\
&= -\check{c}_k \cdot \frac{\omega_{\#k}^{(-1+m+n)}}{(-1+m+n)!} + \frac{i \cdot \bar{\partial}\partial \log\det H_{\#k}}{2\pi} \wedge \frac{\omega_{\#k}^{(-2+m+n)}}{(-2+m+n)!} \\
&\quad + \left(\frac{n \cdot \tau_{\mathbf{N}_W} \circ \sigma\left(\psi_k\right)}{k} + \frac{\tau_{\mathbf{N}_{\mathcal{V}_M}} \circ \hat{\sigma}\left(\psi_k\right)}{k \cdot k}\right) \cdot k^m \cdot \Omega_{\mathbb{P}(E)}.
\end{aligned}$$

Let $\Gamma_o\left(\mathbb{P}(E) : \mathbb{R}\right)$ denote the space of smooth $\mathbb{R}$-valued functions f on $\mathbb{P}(E)$ satisfying

$$\int_{\mathbb{P}(E)} f \cdot \Omega_{\mathbb{P}(E)} = 0 \Longleftrightarrow \int_M \hat{\sigma}(f) \cdot \Omega_M = 0.$$

We will solve the gauge-fixing constant scalar curvature equation for $k \in \mathbb{N}$ large enough by considering $\psi_k \in \Gamma_o(\mathbb{P}(E) : \mathbb{R})$ admitting asymptotic expansion of the following form:

$$\psi_k \sim \phi_0 + \sum_{\theta \in \mathbb{N}} \frac{\phi_\theta}{k^\theta}$$

as $k \to +\infty$. Here each $\phi_\bullet \in \Gamma_o(\mathbb{P}(E) : \mathbb{R})$ is a smooth $\mathbb{R}$-valued function, independent of the parameter k, on $\mathbb{P}(E)$. Besides the following induction condition:

$$\sigma(\phi_0) = \widetilde{\sigma}(\phi_0) = 0 = \widetilde{\sigma}(\phi_1) \Longleftrightarrow \phi_0 \in \Gamma_o(M : \mathbb{R}) \quad \text{and} \quad \phi_1 \in \Gamma_o(M : \mathbb{R}) \oplus \Gamma(M : W)$$

is imposed on the leading terms ϕ_0 and ϕ_1.

Before solving the gauge-fixing constant scalar curvature equation, for k large enough, we collect some relevant basic facts which can be checked readily. Note that it is virtually better to rewrite the term

$$\frac{i \cdot \bar{\partial}\partial \log \det H_{\#k}}{2\pi}$$

of the gauge-fixing constant scalar curvature equation $\mathcal{S}_{\text{G-F}}\left(\omega_{\#k}\right) = 0$ as

$$\frac{i}{2\pi} \cdot \bar{\partial}\partial \log\left(k^m \cdot \det \check{H}\right) + \frac{i}{2\pi} \cdot \bar{\partial}\partial \log\left(\frac{\omega_{\#k}^{(-1+m+n)}}{k^m \cdot \check{\omega}^{(-1+m+n)}}\right).$$

Here $\check{H}$ is the Hermitian metric on $\mathbb{P}(E)$ induced by $\check{\omega}$ and thence

$$k^m \cdot \det \check{H}$$

is a Hermitian metric on the dual of the canonical line bundle of $\mathbb{P}(E)$. It can be shown that

$$\begin{aligned} &\frac{i}{2\pi} \cdot \bar{\partial}\partial \log\left(k^m \cdot \det \check{H}\right) \\ &= n \cdot \frac{i \cdot F_{A_{L^*}}}{2\pi} + \check{\pi}^* \text{trace}\left(\frac{i \cdot F_A}{2\pi}\right) + \check{\pi}^* \text{trace}\left(\frac{i \cdot F_{\omega_M}}{2\pi}\right). \end{aligned}$$

Here F_A is the curvature form of E induced by the Einstein-Hermitian connection A on E over M while F_{ω_M} is the curvature form of the

holomorphic tangent bundle of M induced by the Kaehler form ω_M on M. Since

$$\frac{i\cdot\bar{\partial}\partial\log\det\ {}_{o}H_{\#k}}{2\pi}$$
$$=\frac{i}{2\pi}\cdot\bar{\partial}\partial\log\left(k^m\cdot\det\check{H}\right)+\frac{i}{2\pi}\cdot\bar{\partial}\partial\log\left(\frac{{}_{o}\omega_{\#k}^{(-1+m+n)}}{k^m\cdot\check{\omega}^{(-1+m+n)}}\right)$$

we have

$$\check{c}_k\cdot\int_{\mathbb{P}(E)}\frac{{}_{o}\omega_{\#k}^{(-1+m+n)}}{(-1+m+n)!}$$
$$=\int_{\mathbb{P}(E)}\frac{i\cdot\bar{\partial}\partial\log\det\ {}_{o}H_{\#k}}{2\pi}\wedge\frac{{}_{o}\omega_{\#k}^{(-2+m+n)}}{(-2+m+n)!}$$
$$=\int_{\mathbb{P}(E)}\frac{i}{2\pi}\cdot\bar{\partial}\partial\log\left(k^m\cdot\det\check{H}\right)\wedge\frac{{}_{o}\omega_{\#k}^{(-2+m+n)}}{(-2+m+n)!}$$
$$=\int_{\mathbb{P}(E)}\left[n\cdot\frac{i\cdot F_{A_{L^*}}}{2\pi}+\mathrm{trace}\left(\frac{i\cdot F_A}{2\pi}\right)+\mathrm{trace}\left(\frac{i\cdot F_{\omega_M}}{2\pi}\right)\right]\wedge$$
$$\frac{{}_{o}\omega_{\#k}^{(-2+m+n)}}{(-2+m+n)!}.$$

Thus the topological invariant $\check{c}_k$ is a rational function of the parameter k. By using the Einstein-Hermitian condition of the connection A on E over M

$$n\cdot\Pi_{\mathcal{C}_M}\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right)\wedge\frac{\omega_M^{(-1+m)}}{(-1+m)!}+\mathrm{trace}\left(\frac{i\cdot F_A}{2\pi}\right)\wedge\frac{\omega_M^{(-1+m)}}{(-1+m)!}=0$$

it can be shown readily that the power series expansion of $\check{c}_k$ in $\frac{1}{k}$ is

$$\check{c}_k=(-1+n)\cdot n+\frac{\Lambda_M\mathrm{trace}\left(\frac{i\cdot F_{\omega_M}}{2\pi}\right)}{k}+\frac{\check{c}_{k:2}}{k\cdot k}+\text{ higher order terms}$$

in which $\check{c}_{k:2}$ is a constant, independent of the parameter k, while $\Lambda_M\mathrm{trace}\left(\frac{i\cdot F_{\omega_M}}{2\pi}\right)$ is the scalar curvature of $(M:\omega_M)$:

$$\left[\Lambda_M\mathrm{trace}\left(\frac{i\cdot F_{\omega_M}}{2\pi}\right)\right]\cdot\frac{\omega_M^m}{m!}=\mathrm{trace}\left(\frac{i\cdot F_{\omega_M}}{2\pi}\right)\wedge\frac{\omega_M^{(-1+m)}}{(-1+m)!}.$$

Now let us consider the asymptotic expansion of ψ_k as $k \to +\infty$. By substituting

$$\omega_{\#k} = {}_o\omega_{\#k} + i\bar{\partial}\partial\psi_k \sim {}_o\omega_{\#k} + i\bar{\partial}\partial\phi_0 + \sum_{\theta\in\mathbb{N}} \frac{i\bar{\partial}\partial\phi_\theta}{k^\theta} \qquad \text{as } k \to +\infty$$

into $\mathcal{S}_{\text{G-F}}\left(\omega_{\#k}\right)$ we have

$$\mathcal{S}_{\text{G-F}}\left(\omega_{\#k}\right) \sim k^m \cdot \left(\mathbf{B}_0 + \sum_{\theta\in\mathbb{N}} \frac{\mathbf{B}_\theta}{k^\theta}\right) \qquad \text{as } k \to +\infty$$

in which each $\mathbf{B}_\bullet$ is independent of the parameter k. In order to show that the asymptotic expansion of ψ_k (as $k \to +\infty$) exists we simply need to show the solvability of the following system of equations:

$$\mathbf{B}_\theta = 0 \Longleftrightarrow \frac{\mathbf{B}_\theta}{\Omega_{\mathbb{P}(E)}} = 0$$

for any integer $\theta \geq 0$. By using the induction condition

$$\phi_0 \in \Gamma_o\left(M : \mathbb{R}\right) \text{ and } \phi_1 \in \Gamma_o\left(M : \mathbb{R}\right) \oplus \Gamma\left(M : W\right)$$

it can be checked readily that

$$\mathbf{B}_0 = 0 = \mathbf{B}_1.$$

We can then find

$$\hat{\sigma}\left(\phi_\theta\right) \oplus \sigma\left(\phi_{\theta+1}\right) \oplus \widetilde{\sigma}\left(\phi_{\theta+2}\right)$$

through solving the equation

$$\frac{\mathbf{B}_{\theta+2}}{\Omega_{\mathbb{P}(E)}} = \hat{\sigma}\left(\frac{\mathbf{B}_{\theta+2}}{\Omega_{\mathbb{P}(E)}}\right) \oplus \sigma\left(\frac{\mathbf{B}_{\theta+2}}{\Omega_{\mathbb{P}(E)}}\right) \oplus \widetilde{\sigma}\left(\frac{\mathbf{B}_{\theta+2}}{\Omega_{\mathbb{P}(E)}}\right) = 0$$

by induction on integers $\theta \geq 0$. Actually we have the following result (proved in Appendix I):

Proposition V.A. *By choosing the induction condition*

$$\phi_0 \in \Gamma_o\left(M : \mathbb{R}\right) \text{ and } \phi_1 \in \Gamma_o\left(M : \mathbb{R}\right) \oplus \Gamma\left(M : W\right)$$

there exists a unique family of smooth $\mathbb{R}$-valued functions $\phi_\theta \in \Gamma_o\left(\mathbb{P}(E) : \mathbb{R}\right)$ on $\mathbb{P}(E)$, depending on integers $\theta \geq 0$, such that $\mathbf{B}_\theta = 0$ for any integer $\theta \geq 0$.

Now for each large $N \in \mathbb{N}$ we define a Kaehler form ${}_N\omega_{\#k}$ on $\mathbb{P}(E)$, depending on $k \in \mathbb{N}$ large enough, as follows:

$$\begin{aligned} {}_N\omega_{\#k} &\equiv {}_o\omega_{\#k} + i\bar{\partial}\partial\phi_0 + \sum_{\theta\in\mathbb{N} \text{ with } \theta\leq N} \frac{i\bar{\partial}\partial\phi_\theta}{k^\theta} \\ &= \frac{i\cdot F_{A_{L^*}}}{2\pi} + k\cdot\omega_M + i\bar{\partial}\partial\phi_0 + \sum_{\theta\in\mathbb{N} \text{ with } \theta\leq N} \frac{i\bar{\partial}\partial\phi_\theta}{k^\theta}. \end{aligned}$$

Here each $\phi_\bullet$ is taken from the unique family of smooth $\mathbb{R}$-valued functions on $\mathbb{P}(E)$ of Proposition V.A. Then we have the following result:

Corollary V.A. *Given $\gamma \geq 0$ we denote by $\| \bullet \|_{\mathcal{C}^\gamma(\mathbb{P}(E):\check{\omega})}$ the $\mathcal{C}^\gamma$-norm of $\bullet$ with respect to the Hermitian form (metric) $\check{\omega}$ on $\mathbb{P}(E)$. Given $p \in \mathbb{N}$ there exists a corresponding constant $C_{(\gamma:p)} > 0$ such that for each $N \geq p$ we have*

$$\left\| \frac{\mathcal{S}_{\text{G-F}}\left({}_N\omega_{\#k}\right)}{k^m\cdot\Omega_{\mathbb{P}(E)}} \right\|_{\mathcal{C}^\gamma(\mathbb{P}(E):\check{\omega})} \leq \frac{C_{(\gamma:p)}}{k^p}$$

whenever $k \geq k_{(\gamma:p:N)}$. Here the choice of $k_{(\gamma:p:N)} \in \mathbb{N}$ depends on N.

Actually when $N \geq p$ we have, by Proposition V.A,

$$\mathbf{B}_0 = \cdots = \mathbf{B}_p = 0$$

and therefore the smooth $\mathbb{R}$-valued function $\frac{\mathcal{S}_{\text{G-F}}\left({}_N\omega_{\#k}\right)}{k^m\cdot\Omega_{\mathbb{P}(E)}}$ on $\mathbb{P}(E)$ must carry the factor $\frac{1}{k\cdot k^p}$ intrinsically when $k > 0$ is large enough (equivalently when $\frac{1}{k} > 0$ is small enough). Corollary V.A then follows immediately from standard results of calculus.

We define for each large $k \in \mathbb{N}$ a functional $\mathbf{R}$ on the Kaehler class $\left[{}_o\omega_{\#k}\right] = \left[\frac{i\cdot F_{A_{L^*}}}{2\pi} + k\cdot\omega_M\right]$ as follows:

$$\mathbf{R}\left({}_o\omega_{\#k} + i\bar{\partial}\partial\bullet\right) \equiv \frac{\left({}_o\omega_{\#k} + i\bar{\partial}\partial\bullet\right)^{(-1+m+n)}}{k^m\cdot\check{\omega}^{(-1+m+n)}} = \frac{\frac{\left({}_o\omega_{\#k}+i\bar{\partial}\partial\bullet\right)^{(-1+m+n)}}{(-1+m+n)!}}{k^m\cdot\Omega_{\mathbb{P}(E)}}$$

for any Kaehler form $\left({}_o\omega_{\#k} + i\bar{\partial}\partial\bullet\right)$ on $\mathbb{P}(E)$ lying in the Kaehler class $\left[{}_o\omega_{\#k}\right]$. Then the gauge-fixing constant scalar curvature equation for $\omega_{\#k} = {}_o\omega_{\#k} + i\bar{\partial}\partial\psi_k$ can be expressed equivalently as follows:

$$\frac{\mathcal{S}_{\text{G-F}}\left(\omega_{\#k}\right)}{k^m\cdot\Omega_{\mathbb{P}(E)}} = 0$$

in which

$$\begin{aligned}
&\frac{\mathcal{S}_{\text{G-F}}\left(\omega_{\#k}\right)}{k^m\cdot\Omega_{\mathbb{P}(E)}}\\
&\equiv-\check{c}_k\cdot\mathbf{R}\left(\omega_{\#k}\right)+\frac{\frac{i\cdot\bar{\partial}\partial\log\mathbf{R}\left(\omega_{\#k}\right)}{2\pi}\wedge\frac{\omega_{\#k}^{(-2+m+n)}}{(-2+m+n)!}}{k^m\cdot\Omega_{\mathbb{P}(E)}}\\
&\quad+\frac{\left[n\cdot\frac{i\cdot F_{A_{L^*}}}{2\pi}+\check{\pi}^*\text{trace}\left(\frac{i\cdot F_A}{2\pi}\right)+\check{\pi}^*\text{trace}\left(\frac{i\cdot F_{\omega_M}}{2\pi}\right)\right]\wedge\frac{\omega_{\#k}^{(-2+m+n)}}{(-2+m+n)!}}{k^m\cdot\Omega_{\mathbb{P}(E)}}\\
&\quad+\left(\frac{n\cdot\tau_{\mathbf{N}_W}\circ\sigma\left(\psi_k\right)}{k}+\frac{\tau_{\mathbf{N}_{\mathcal{V}_M}}\circ\hat{\sigma}\left(\psi_k\right)}{k\cdot k}\right).
\end{aligned}$$

Now for each large $N\in\mathbb{N}$ we define a corresponding 4^{th} order (elliptic) linear partial differential operator $\mathbf{L}_N$, depending on the parameter k, acting on $\psi\in\Gamma_o\left(\mathbb{P}(E):\mathbb{R}\right)$ as follows:

$$\begin{aligned}
\mathbf{L}_N\left(\psi\right)=-\check{c}_k\cdot\,{}_A\mathbf{L}_N\left(\psi\right)&+\frac{\frac{i\bar{\partial}\partial}{2\pi}\left(\frac{{}_A\mathbf{L}_N(\psi)}{\mathbf{R}\left({}_N\omega_{\#k}\right)}\right)\wedge\frac{{}_N\omega_{\#k}^{(-2+m+n)}}{(-2+m+n)!}}{k^m\cdot\Omega_{\mathbb{P}(E)}}\\
&+\frac{\frac{i\cdot\bar{\partial}\partial\log\mathbf{R}\left({}_N\omega_{\#k}\right)}{2\pi}\wedge i\bar{\partial}\partial\psi\wedge\frac{{}_N\omega_{\#k}^{(-3+m+n)}}{(-3+m+n)!}}{k^m\cdot\Omega_{\mathbb{P}(E)}}+\,{}_B\mathbf{L}_N\left(\psi\right)\\
&+\left(\frac{n\cdot\tau_{\mathbf{N}_W}\circ\sigma\left(\psi\right)}{k}+\frac{\tau_{\mathbf{N}_{\mathcal{V}_M}}\circ\hat{\sigma}\left(\psi\right)}{k\cdot k}\right).
\end{aligned}$$

Here ${}_A\mathbf{L}_N$ is the corresponding 2^{nd} order linear partial differential operator (without the 0^{th} order part) acting on $\psi\in\Gamma_o\left(\mathbb{P}(E):\mathbb{R}\right)$ defined as follows:

$$\,{}_A\mathbf{L}_N\left(\psi\right)\equiv\frac{i\bar{\partial}\partial\psi\wedge\frac{{}_N\omega_{\#k}^{(-2+m+n)}}{(-2+m+n)!}}{k^m\cdot\Omega_{\mathbb{P}(E)}}$$

while ${}_B\mathbf{L}_N$ is the corresponding 2^{nd} order linear partial differential operator (without the 0^{th} order part) acting on $\psi\in\Gamma_o\left(\mathbb{P}(E):\mathbb{R}\right)$ defined as follows:

$$\begin{aligned}
&\,{}_B\mathbf{L}_N\left(\psi\right)\\
&\equiv\frac{i\bar{\partial}\partial\psi\wedge\left[n\cdot\frac{i\cdot F_{A_{L^*}}}{2\pi}+\check{\pi}^*\text{trace}\left(\frac{i\cdot F_A}{2\pi}\right)+\check{\pi}^*\text{trace}\left(\frac{i\cdot F_{\omega_M}}{2\pi}\right)\right]\wedge\frac{{}_N\omega_{\#k}^{(-3+m+n)}}{(-3+m+n)!}}{k^m\cdot\Omega_{\mathbb{P}(E)}}.
\end{aligned}$$

Proposition V.B. *Let* $\| \bullet \|_{L^2(\mathbb{P}(E):\check{\omega})}$ *denote the* L^2*-norm of* $\bullet$ *with respect to the Hermitian form (metric)* $\check{\omega}$ *on* $\mathbb{P}(E)$*. Then for each large* $N \in \mathbb{N}$ *there exists a corresponding* $k_N \in \mathbb{N}$ *such that for any* $\psi \in \Gamma_o(\mathbb{P}(E) : \mathbb{R})$ *we have*

$$\begin{aligned} C \cdot \|\mathbf{L}_N(\psi)\|_{L^2(\mathbb{P}(E):\check{\omega})} \geq & \left\| \left(\Delta_V + \frac{\Delta_M}{k}\right) \circ \left(\Delta_V + \frac{\Delta_M}{k}\right) \widetilde{\sigma}\psi \right\|_{L^2(\mathbb{P}(E):\check{\omega})} \\ & + \left\| \frac{(\Delta_M + \mathrm{id})}{k} \circ \left(\Delta_V + \frac{\Delta_M}{k}\right) \sigma\psi \right\|_{L^2(\mathbb{P}(E):\check{\omega})} \\ & + \left\| \frac{(\Delta_M + \mathrm{id}) \circ (\Delta_M + \mathrm{id})\, \hat{\sigma}\psi}{k \cdot k} \right\|_{L^2(\mathbb{P}(E):\check{\omega})} \end{aligned}$$

whenever $k \geq k_N$*. Here the constant* $C > 0$ *depends on* $(M : \omega_M)$ *and the Einstein-Hermitian structure of the holomorphic vector bundle* E *over* M *but not on* $N \in \mathbb{N}$*.*

Let $\mathbf{P}_{\#k}$ denote the 4^{th} order (elliptic) linear partial differential operator, depending on the parameter k, acting on $\psi \in \Gamma_o(\mathbb{P}(E) : \mathbb{R})$ defined as follows:

$$\begin{aligned} \mathbf{P}_{\#k}(\psi) = & \frac{\Delta_V}{8\pi} \circ (-4\pi n \cdot \mathrm{id} + \Delta_V)\, \psi \\ & + \frac{\frac{\Delta_M \circ \Delta_V \psi}{8\pi} + \frac{\Delta_V \circ \Delta_M \psi}{8\pi}}{k} + \frac{n \cdot \tau_{\mathbf{N}_W} \circ \sigma\psi}{k} \\ & + \frac{\frac{\Delta_M \circ \Delta_M \psi}{8\pi} + \left[-\Lambda_M \mathrm{trace}\left(\frac{i \cdot F_{\omega_M}}{2\pi}\right)\right] \cdot \frac{\Delta_M \psi}{2}}{k \cdot k} \\ & + \frac{\frac{i(\bar{\partial}\partial)_M \psi \wedge \left[\mathrm{trace}\left(\frac{i \cdot F_{\omega_M}}{2\pi}\right)\right] \wedge \frac{\omega_M^{(-2+m)}}{(-2+m)!}}{\Omega_M}}{k \cdot k} \\ & + \frac{\tau_{\mathbf{N}_{\mathcal{V}_M}} \circ \hat{\sigma}\psi}{k \cdot k}. \end{aligned}$$

Actually, for each large $N \in \mathbb{N}$, it can be shown that $\mathbf{L}_N$ is dominated by $\mathbf{P}_{\#k}$ when the parameter k is sufficiently large.

Corollary V.B. *Given integer* $\gamma \geq 0$ *we define the Sobolev norm* $\| \bullet \|_{H^{[2\gamma]}(\mathbb{P}(E):\check{\omega})}$ *of* $\bullet$ *as follows:*

$$\begin{aligned} \| \bullet \|_{H^{[2\gamma]}(\mathbb{P}(E):\check{\omega})} \equiv & \| \bullet \|_{L^2(\mathbb{P}(E):\check{\omega})} + \|(\Delta_V + \Delta_M) \bullet\|_{L^2(\mathbb{P}(E):\check{\omega})} \\ & + \cdots + \|(\Delta_V + \Delta_M)^{\gamma} \bullet\|_{L^2(\mathbb{P}(E):\check{\omega})}. \end{aligned}$$

Then for each large $N \in \mathbb{N}$ there exists a corresponding $k_{(\gamma:N)} \in \mathbb{N}$ such that for any $\psi \in \Gamma_o(\mathbb{P}(E) : \mathbb{R})$ we have

$$\begin{aligned} & C_\gamma \cdot \|\mathbf{L}_N(\psi)\|_{H^{[2\gamma]}(\mathbb{P}(E):\breve{\omega})} \\ & \geq \left\| \left(\Delta_V + \frac{\Delta_M}{k} \right) \circ \left(\Delta_V + \frac{\Delta_M}{k} \right) \widetilde{\sigma}\psi \right\|_{H^{[2\gamma]}(\mathbb{P}(E):\breve{\omega})} \\ & \quad + \left\| \frac{(\Delta_M + \mathrm{id})}{k} \circ \left(\Delta_V + \frac{\Delta_M}{k} \right) \sigma\psi \right\|_{H^{[2\gamma]}(\mathbb{P}(E):\breve{\omega})} \\ & \quad + \left\| \frac{(\Delta_M + \mathrm{id}) \circ (\Delta_M + \mathrm{id})\, \hat{\sigma}\psi}{k \cdot k} \right\|_{H^{[2\gamma]}(\mathbb{P}(E):\breve{\omega})} \end{aligned}$$

whenever $k \geq k_{(\gamma:N)}$. Here the constant $C_\gamma > 0$ depends on γ but not on N. In particular we have for any $\psi \in \Gamma_o(\mathbb{P}(E) : \mathbb{R})$ the following estimate:

$$C_\gamma \cdot \|\mathbf{L}_N(\psi)\|_{H^{[2\gamma]}(\mathbb{P}(E):\breve{\omega})} \geq \frac{\|(\Delta_V + \Delta_M) \circ (\Delta_V + \Delta_M)\, \psi\|_{H^{[2\gamma]}(\mathbb{P}(E):\breve{\omega})}}{k \cdot k}$$

whenever $k \geq k_{(\gamma:N)}$.

Remark. Note that the linear 2^{nd} order elliptic linear partial differential operator Δ_M is coercive when acting on $\Gamma_o(M : \mathbb{R})$. Besides the linear 2^{nd} order elliptic linear partial differential operator $\Delta_M + \Delta_V$ is coercive when acting on $\Gamma_o(\mathbb{P}(E) : \mathbb{R})$.

We will prove these results in Appendix III.

Given $\gamma \geq 0$ we denote by $H_o^{[2\gamma]}(\mathbb{P}(E) : \breve{\omega})$ the Sobolev space consisting of $\mathbb{R}$-valued functions $f \in H^{[2\gamma]}(\mathbb{P}(E) : \breve{\omega})$ on $\mathbb{P}(E)$ satisfying

$$\int_{\mathbb{P}(E)} f \cdot \Omega_{\mathbb{P}(E)} = 0.$$

Note that Corollary V.B implies the invertibility of the 4^{th} order (elliptic) linear partial differential operator

$$\mathbf{L}_N : H_o^{[2\gamma+4]}(\mathbb{P}(E) : \breve{\omega}) \longrightarrow H_o^{[2\gamma]}(\mathbb{P}(E) : \breve{\omega})$$

whenever k is sufficiently large. Let $\mathbf{I}_N$ denote the inverse of $\mathbf{L}_N$. Then we have, for any $f \in H_o^{[2\gamma]}(\mathbb{P}(E) : \breve{\omega})$, the following estimate:

$$\|\mathbf{I}_N f\|_{H_o^{[2\gamma+4]}(\mathbb{P}(E):\breve{\omega})} \leq C_\gamma \cdot k \cdot k \cdot \|f\|_{H_o^{[2\gamma]}(\mathbb{P}(E):\breve{\omega})}$$

whenever $k \geq k_{(\gamma:N)}$. Note that in this estimate for $\mathbf{I}_N$ the constant $C_\gamma > 0$ does not depend on $N \in \mathbb{N}$ though $k_{(\gamma:N)}$ must be chosen larger for large N.

Given $\psi \in \Gamma_o(\mathbb{P}(E) : \mathbb{R})$ we note that, for $t \in \mathbb{R}$,

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{\mathcal{S}_{\text{G-F}}\left({}_N\omega_{\#k} + t\cdot i\bar{\partial}\partial\psi\right)}{k^m\cdot\Omega_{\mathbb{P}(E)}}\right]\\
&= -\check{c}_k\cdot\ {}_A\mathbf{L}^t_N(\psi) + {}_B\mathbf{L}^t_N(\psi)\\
&\quad + \frac{\frac{i\bar{\partial}\partial}{2\pi}\left(\frac{{}_A\mathbf{L}^t_N(\psi)}{\mathbf{R}\left({}_N\omega_{\#k}+t\cdot i\bar{\partial}\partial\psi\right)}\right)\wedge\frac{\left({}_N\omega_{\#k}+t\cdot i\bar{\partial}\partial\psi\right)^{(-2+m+n)}}{(-2+m+n)!}}{k^m\cdot\Omega_{\mathbb{P}(E)}}\\
&\quad + \frac{\frac{i\cdot\bar{\partial}\partial\log\mathbf{R}\left({}_N\omega_{\#k}+t\cdot i\bar{\partial}\partial\psi\right)}{2\pi}\wedge i\bar{\partial}\partial\psi\wedge\frac{\left({}_N\omega_{\#k}+t\cdot i\bar{\partial}\partial\psi\right)^{(-3+m+n)}}{(-3+m+n)!}}{k^m\cdot\Omega_{\mathbb{P}(E)}}\\
&\quad + \left(\frac{n\cdot\tau_{\mathbf{N}_W}\circ\sigma(\psi)}{k} + \frac{\tau_{\mathbf{N}_{\mathcal{V}_M}}\circ\hat{\sigma}(\psi)}{k\cdot k}\right)
\end{aligned}$$

in which ${}_A\mathbf{L}^t_N$ and ${}_B\mathbf{L}^t_N$ are the corresponding 2^{nd} order nonlinear partial differential operators acting on $\psi \in \Gamma_o(\mathbb{P}(E):\mathbb{R})$ defined respectively as follows:

$${}_A\mathbf{L}^t_N(\psi) \equiv \frac{i\bar{\partial}\partial\psi\wedge\frac{\left({}_N\omega_{\#k}+t\cdot i\bar{\partial}\partial\psi\right)^{(-2+m+n)}}{(-2+m+n)!}}{k^m\cdot\Omega_{\mathbb{P}(E)}}$$

and

$$\begin{aligned}
&{}_B\mathbf{L}^t_N(\psi)\\
&\equiv\frac{i\bar{\partial}\partial\psi\wedge\left[n\cdot\frac{i\cdot F_{A_{L^*}}}{2\pi}+\check{\pi}^*\text{trace}\left(\frac{i\cdot F_A}{2\pi}\right)+\check{\pi}^*\text{trace}\left(\frac{i\cdot F_{\omega_M}}{2\pi}\right)\right]\wedge\frac{\left({}_N\omega_{\#k}+t\cdot i\bar{\partial}\partial\psi\right)^{(-3+m+n)}}{(-3+m+n)!}}{k^m\cdot\Omega_{\mathbb{P}(E)}}.
\end{aligned}$$

From the shape of $\frac{d}{dt}\left[\frac{\mathcal{S}_{\text{G-F}}\left({}_N\omega_{\#k}+t\cdot i\bar{\partial}\partial\psi\right)}{k^m\cdot\Omega_{\mathbb{P}(E)}}\right]$ it is easy to see that, for bounded $t \in \mathbb{R}$, the nonlinear partial differential operator

$$\frac{d}{dt}\circ\frac{d}{dt}\left[\frac{\mathcal{S}_{\text{G-F}}\left({}_N\omega_{\#k}+t\cdot i\bar{\partial}\partial\psi\right)}{k^m\cdot\Omega_{\mathbb{P}(E)}}\right] = \frac{d}{dt}\circ\frac{d}{dt}\left[\frac{\mathcal{S}\left({}_N\omega_{\#k}+t\cdot i\bar{\partial}\partial\psi\right)}{k^m\cdot\Omega_{\mathbb{P}(E)}}\right]$$

is genuinely nonlinear. Actually, by the Sobolev Embedding Theorem, for each sufficiently large $\gamma \in \mathbb{N}$ there exists a corresponding $k_{(\gamma:N)} \in \mathbb{N}$ such that for any pair $(f : g)$ of elements of $H_o^{[2\gamma]}(\mathbb{P}(E) : \breve{\omega})$ satisfying $\|f\|_{H^{[2\gamma+4]}(\mathbb{P}(E):\breve{\omega})} \leq \frac{1}{k}$ and $\|g\|_{H^{[2\gamma+4]}(\mathbb{P}(E):\breve{\omega})} \leq \frac{1}{k}$ we have the following estimate:

$$\begin{aligned}
&\left\| -\frac{d}{dt} \circ \frac{d}{dt} \left[\frac{\mathcal{S}_{\text{G-F}}\left({}_N\omega_{\#k} + t \cdot i\bar{\partial}\partial f \right)}{k^m \cdot \Omega_{\mathbb{P}(E)}} \right] \right. \\
&\qquad \left. + \frac{d}{dt} \circ \frac{d}{dt} \left[\frac{\mathcal{S}_{\text{G-F}}\left({}_N\omega_{\#k} + t \cdot i\bar{\partial}\partial g \right)}{k^m \cdot \Omega_{\mathbb{P}(E)}} \right] \right\|_{H^{[2\gamma]}(\mathbb{P}(E):\breve{\omega})} \\
&\leq C_\gamma \cdot \left(\|f\|_{H^{[2\gamma+4]}(\mathbb{P}(E):\breve{\omega})} + \|g\|_{H^{[2\gamma+4]}(\mathbb{P}(E):\breve{\omega})} \right) \cdot \|-f + g\|_{H^{[2\gamma+4]}(\mathbb{P}(E):\breve{\omega})}
\end{aligned}$$

whenever $k \geq k_{(\gamma:N)}$. Here the constant $C_\gamma > 0$ can be chosen to be independent of $N \in \mathbb{N}$.

Now we note that for $\psi \in \Gamma_o(\mathbb{P}(E) : \mathbb{R})$ the gauge-fixing constant scalar curvature equation for $\left({}_N\omega_{\#k} + i\bar{\partial}\partial\psi \right)$ lying in the Kaehler class $\left[{}_o\omega_{\#k} \right]$ can be expressed as

$$0 = \frac{\mathcal{S}_{\text{G-F}}\left({}_N\omega_{\#k} + i\bar{\partial}\partial\psi \right)}{k^m \cdot \Omega_{\mathbb{P}(E)}} = \frac{\mathcal{S}_{\text{G-F}}\left({}_N\omega_{\#k} \right)}{k^m \cdot \Omega_{\mathbb{P}(E)}} + \mathbf{L}_N(\psi) + \mathbf{GN}_N(\psi)$$

in which $\mathbf{GN}_N$ is the genuinely nonlinear partial differential operator acting on $\psi \in \Gamma_o(\mathbb{P}(E) : \mathbb{R})$ defined as follows:

$$\begin{aligned}
\mathbf{GN}_N(\psi) &\equiv \int_0^1 (-t+1) \cdot \frac{d}{dt} \circ \frac{d}{dt} \left[\frac{\mathcal{S}_{\text{G-F}}\left({}_N\omega_{\#k} + t \cdot i\bar{\partial}\partial\psi \right)}{\Omega_{\mathbb{P}(E)}} \right] \cdot dt \\
&= \int_0^1 (-t+1) \cdot \frac{d}{dt} \circ \frac{d}{dt} \left[\frac{\mathcal{S}\left({}_N\omega_{\#k} + t \cdot i\bar{\partial}\partial\psi \right)}{\Omega_{\mathbb{P}(E)}} \right] \cdot dt.
\end{aligned}$$

Since for any Kaehler form $\left({}_N\omega_{\#k} + i\bar{\partial}\partial\bullet \right)$ lying in the Kaehler class $\left[{}_o\omega_{\#k} \right]$ the integral

$$\int_{\mathbb{P}(E)} \frac{\mathcal{S}_{\text{G-F}}\left({}_N\omega_{\#k} + i\bar{\partial}\partial\bullet \right)}{\Omega_{\mathbb{P}(E)}} \cdot \Omega_{\mathbb{P}(E)} = \int_{\mathbb{P}(E)} \mathcal{S}_{\text{G-F}}\left({}_N\omega_{\#k} + i\bar{\partial}\partial\bullet \right)$$

always vanishes we can apply the inverse $\mathbf{I}_N$ of $\mathbf{L}_N$ to the last expression of the gauge-fixing constant scalar curvature equation and obtain the following equivalent

$$0 = \mathbf{I}_N \left(\frac{\mathcal{S}_{\text{G-F}}\left({}_N\omega_{\#k} \right)}{k^m \cdot \Omega_{\mathbb{P}(E)}} \right) + \psi + \mathbf{I}_N \circ \mathbf{GN}_N(\psi).$$

We will now solve this equation through the Contraction Mapping Theorem. Given sufficiently large $\gamma \in \mathbb{N}$ we may choose large $q \in \mathbb{N}$ such that the genuinely nonlinear operator $\mathbf{I}_N \circ \mathbf{GN}_N$, when acting on the complete metric space

$$\left\{ \psi \in H_o^{[2\gamma+4]} \left(\mathbb{P}(E) : \check{\omega} \right) : \|\psi\|_{H^{[2\gamma+4]}(\mathbb{P}(E):\check{\omega})} \leq \frac{1}{k^q} \right\},$$

is contractive with contraction constant $\leq \frac{1}{2}$ whenever the parameter k is large enough. On the other hand, by Corollary V.A and Corollary V.B, we may choose sufficiently large $N \in \mathbb{N}$ such that

$$\left\| \mathbf{I}_N \left(\frac{\mathcal{S}_{\text{G-F}} \left({}_N\omega_{\#k} \right)}{k^m \cdot \Omega_{\mathbb{P}(E)}} \right) \right\|_{H^{[2\gamma+4]}(\mathbb{P}(E):\check{\omega})} \leq \frac{1}{k \cdot k^q}$$

whenever the parameter k is large enough. Thus by the Contraction Mapping Theorem we conclude that for each suitable choice of $(\gamma : q : N) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ the gauge-fixing constant scalar curvature equation can be solved uniquely, whenever the parameter k is large enough, by a Kaehler form $\left({}_N\omega_{\#k} + i\bar{\partial}\partial\psi \right)$ on $\mathbb{P}(E)$ with $\psi \in H_o^{[2\gamma+4]} \left(\mathbb{P}(E) : \check{\omega} \right)$ satisfying

$$\|\psi\|_{H^{[2\gamma+4]}(\mathbb{P}(E):\check{\omega})} \leq \frac{1}{k^q}.$$

With standard results of partial differential equations it can be shown readily that the solutions $\psi \in H_o^{[2\gamma+4]} \left(\mathbb{P}(E) : \check{\omega} \right)$ to the gauge-fixing constant scalar curvature equation, depending on sufficiently large k, found in this way are actually smooth because we already have high regularity and good approximation results. Hence we have:

Theorem V.A. *When the parameter $k \in \mathbb{N}$ is sufficiently large the corresponding gauge-fixing constant scalar curvature equation*

$$\frac{\mathcal{S}_{\text{G-F}} \left({}_o\omega_{\#k} + i\bar{\partial}\partial\psi_k \right)}{k^m \cdot \Omega_{\mathbb{P}(E)}} = 0$$

can be solved by some smooth $\mathbb{R}$-valued function $\psi_k \in \Gamma_o \left(\mathbb{P}(E) : \mathbb{R} \right)$ on $\mathbb{P}(E)$. Besides this family of smooth $\mathbb{R}$-valued functions $\psi_k \in \Gamma_o(\mathbb{P}(E) : \mathbb{R})$ on $\mathbb{P}(E)$ admits asymptotic expansion of the following form

$$\psi_k \sim \phi_0 + \sum_{\theta \in \mathbb{N}} \frac{\phi_\theta}{k^\theta}$$

as $k \to +\infty$. Here each $\phi_\bullet$ is taken from the unique family of smooth $\mathbb{R}$-valued functions on $\mathbb{P}(E)$ of Proposition V.A. *Actually, for each pair $(\gamma : q) \in \mathbb{N} \times \mathbb{N}$ of large enough integers, we may even require, when $N \in \mathbb{N}$ is chosen sufficiently large, that*

$$ {}_o\omega_{\#k} + i\bar{\partial}\partial\psi_k = {}_N\omega_{\#k} + i\bar{\partial}\partial\psi_{(k:N)} $$

with $\psi_{(k:N)} \in \Gamma_o\left(\mathbb{P}(E) : \mathbb{R}\right)$ satisfying

$$ \left\|\psi_{(k:N)}\right\|_{H^{[2\gamma+4]}(\mathbb{P}(E):\check{\omega})} \leq \frac{1}{k^q} $$

whenever k is large enough. In this case the choice of the solution

$$ {}_N\omega_{\#k} + i\bar{\partial}\partial\psi_{(k:N)} $$

to the gauge-fixing constant scalar curvature equation

$$ \frac{\mathcal{S}_{\text{G-F}}\left({}_N\omega_{\#k} + i\bar{\partial}\partial\psi_{(k:N)}\right)}{k^m \cdot \Omega_{\mathbb{P}(E)}} = 0 $$

with $\psi_{(k:N)} \in \Gamma_o\left(\mathbb{P}(E) : \mathbb{R}\right)$ satisfying $\left\|\psi_{(k:N)}\right\|_{H^{[2\gamma+4]}(\mathbb{P}(E):\check{\omega})} \leq \frac{1}{k^q}$ is, for each sufficiently large k, unique.

Now we conclude this section with a remark. Suppose that, for each $k \in \mathbb{N}$ large enough, the corresponding Futaki invariants associated with

$$ \mathfrak{g}_E + \text{ (the lifted action of) } \mathfrak{k}_{(M:\omega_M)} $$

and the Kaehler class $\left[\frac{i \cdot F_{A_{L^*}}}{2\pi} + k \cdot \check{\pi}^* \omega_M\right]$ on $\mathbb{P}(E)$ are zero. Then, by Theorem II.B, we may assume that

$$ \frac{i \cdot F_{A_{L^*}}}{2\pi} $$

is invariant under the lifted action of $\mathfrak{k}_{(M:\omega_M)}$ on $\mathbb{P}(E)$. Thus both $\frac{i \cdot F_{A_{L^*}}}{2\pi}$ and ω_M are invariant under the action of $\left(\mathfrak{k}_E + \text{ the lifting of } \mathfrak{k}_{(M:\omega_M)}\right)$ on $\mathbb{P}(E)$. In particular the gauge-fixing constant scalar curvature equation is invariant under the action of

$$ \left(\mathfrak{k}_E + \text{ the lifting of } \mathfrak{k}_{(M:\omega_M)}\right) $$

on $\mathbb{P}(E)$. (Note that $\left[\mathfrak{g}_E : \text{ the lifting of } \mathfrak{k}_{(M:\omega_M)}\right] \subset \mathfrak{g}_E$.) Hence, by Proposition V.A and the uniqueness result of Theorem V.A, the solutions

$$ {}_o\omega_{\#k} + i\bar{\partial}\partial\psi_k = {}_N\omega_{\#k} + i\bar{\partial}\partial\psi $$

to the gauge-fixing constant scalar curvature equation, depending on sufficiently large k, are invariant under the action of $\big(\mathfrak{k}_E+$ the lifting of $\mathfrak{k}_{(M:\omega_M)}\big)$ on $\mathbb{P}(E)$.

VI. Solving the constant scalar curvature equation

In this section our main purpose is to prove Theorem A based on the solvability results for the gauge-fixing constant scalar curvature equation — Theorem V.A.

By Theorem II.B we may assume that the lifting of $\mathfrak{k}_{(M:\omega_M)}$ on $\mathbb{P}(E)$ preserves $\frac{i\cdot F_{A_{L^*}}}{2\pi}$. Moreover such lifting of $\mathfrak{k}_{(M:\omega_M)}$ on $\mathbb{P}(E)$ is, modulo the compact Lie subalgebra $\mathfrak{k}_E$ of $\mathfrak{g}_E$, uniquely determined. Thus, by complexification, there is a preferred lifting of $\mathfrak{h}_o(M)$ on $\mathbb{P}(E)$ which is essentially uniquely determined. In this section will fix one such preferred lifting of $\mathfrak{h}_o(M)$. Besides, for each smooth vector field X on M preserving the complex structure of M, we will use the same symbol X to denote the lifting of X on $\mathbb{P}(E)$ when there is no confusion.

Now we fix respectively a $K_{(M:\omega_M)}$-invariant metric on $\mathfrak{h}_o(M)$ and a K_E-invariant metric on $\mathfrak{g}_E$. Here K_E is the maximal compact subgroup of G_E. By doing so the $\mathbb{R}$-linear subspace of $\mathfrak{h}_o(M)$, orthogonal to $\mathfrak{k}_{(M:\omega_M)}$, is isomorphic to the $\mathbb{R}$-linear space $\mathbf{N}_{\mathcal{V}_M}$ while the $\mathbb{R}$-linear subspace of $\mathfrak{g}_E$, orthogonal to $\mathfrak{k}_E$, is isomorphic to the $\mathbb{R}$-linear space $\mathbf{N}_W$. In particular, for each $f\in\mathbf{N}_{\mathcal{V}_M}$, $\|f\|_{\mathcal{C}^0(\mathbb{P}(E):\breve{\omega})}$ is comparable with the $\mathcal{C}^0$-norm of its correspondent in the orthogonal complement of $\mathfrak{k}_{(M:\omega_M)}$ in $\mathfrak{h}_o(M)$. Moreover this comparability is uniform on $\mathbf{N}_{\mathcal{V}_M}$. Actually fixing a $K_{(M:\omega_M)}$-invariant metric on $\mathfrak{h}_o(M)$ simply means that we have fixed the uniform comparability between $\mathbf{N}_{\mathcal{V}_M}$ and

$$\frac{\mathfrak{h}_o(M)}{\mathfrak{k}_{(M:\omega_M)}}.$$

Similar results are valid for $\mathbf{N}_{\mathcal{V}_M}$.

Let $\omega_{\#k}={}_o\omega_{\#k}+i\bar{\partial}\partial\psi_k$ denote the solution of Theorem V.A to the gauge-fixing constant scalar curvature equation. We will denote by $\langle:\rangle$ the inner product on $L^2\left(\mathbb{P}(E):\breve{\omega}\right)$ defined by the Hermitian (metric) form $\breve{\omega}$ on $\mathbb{P}(E)$:

$$\langle f:g\rangle\equiv\int_{\mathbb{P}(E)}f\cdot g\cdot\Omega_{\mathbb{P}(E)}$$

$\forall(f:g)\in L^2\left(\mathbb{P}(E):\breve{\omega}\right)\times L^2\left(\mathbb{P}(E):\breve{\omega}\right)$. Besides we will use the symbol $\mathbf{c}$ to denote a sufficiently large constant >0 independent of the

parameter k.

Suppose that $X_k \in \Gamma(M : T(M))$, orthogonal to $\mathfrak{k}_{(M:\omega_M)}$, is the smooth vector field on M, preserving the complex structure of M, such that

$$\mathcal{L}_{X_k}\omega_M = i\bar{\partial}\partial\left(\tau_{\mathbf{N}_{\mathcal{V}_M}} \circ \hat{\sigma}\psi_k\right).$$

Let $f_{X_k} \in \Gamma_o(\mathbb{P}(E) : \mathbb{R})$ be the corresponding smooth $\mathbb{R}$-valued function on $\mathbb{P}(E)$ satisfying

$$\mathcal{L}_{X_k}\left(\frac{i \cdot F_{A_{L^*}}}{2\pi}\right) = i\bar{\partial}\partial f_{X_k}.$$

Then we have

$$\mathcal{L}_{X_k}\omega_{\#k} = i\bar{\partial}\partial\left(k \cdot \tau_{\mathbf{N}_{\mathcal{V}_M}} \circ \hat{\sigma}\psi_k + f_{X_k} + \mathcal{L}_{X_k}\psi_k\right).$$

Since the Futaki invariant, corresponding to X_k and the Kaehler class $\left[\frac{i \cdot F_{A_{L^*}}}{2\pi} + k \cdot \check{\pi}^*\omega_M\right]$ on $\mathbb{P}(E)$, vanishes we have, by incorporating the gauge-fixing constant scalar curvature equation, the following equality:

$$\int_{\mathbb{P}(E)} \left(k \cdot \tau_{\mathbf{N}_{\mathcal{V}_M}} \circ \hat{\sigma}\psi_k + f_{X_k} + \mathcal{L}_{X_k}\psi_k\right) \cdot \left[\frac{n \cdot \tau_{\mathbf{N}_W} \circ \sigma\psi_k}{k} + \frac{\tau_{\mathbf{N}_{\mathcal{V}_M}} \circ \hat{\sigma}\psi_k}{k \cdot k}\right] \cdot \Omega_{\mathbb{P}(E)} = 0.$$

Since $\Gamma(M : \mathbb{R})$ is orthogonal to $\Gamma(M : W)$ with respect to the inner product $\langle : \rangle$ on $L^2(\mathbb{P}(E) : \check{\omega})$ we note that the term

$$\int_{\mathbb{P}(E)} k \cdot \tau_{\mathbf{N}_{\mathcal{V}_M}} \circ \hat{\sigma}(\psi_k) \cdot \frac{n \cdot \tau_{\mathbf{N}_W} \circ \sigma\psi_k}{k} \cdot \Omega_{\mathbb{P}(E)}$$

of the above equality vanishes. Thus, when the parameter k is sufficiently large, we infer from Theorem V.A that there exists a constant $\mathbf{c} > 0$, independent of k, such that

$$\|f_{X_k}\|_{\mathcal{C}^0(\mathbb{P}(E):\check{\omega})} + \|\mathcal{L}_{X_k}\psi_k\|_{\mathcal{C}^0(\mathbb{P}(E):\check{\omega})} \leq \mathbf{c} \cdot \|X_k\|_{\mathcal{C}^0(\mathbb{P}(E):\check{\omega})}.$$

On the other hand

$$\|X_k\|_{\mathcal{C}^0(\mathbb{P}(E):\check{\omega})}$$

is comparable with $\|\tau_{\mathbf{N}_{\mathcal{V}_M}}\circ\hat{\sigma}\left(\psi_k\right)\|_{\mathcal{C}^0(\mathbb{P}(E):\check{\omega})}$. Thus, by using the Schwarz inequality, we infer from the above equality

$$\int_{\mathbb{P}(E)}\left(k\cdot\tau_{\mathbf{N}_{\mathcal{V}_M}}\circ\hat{\sigma}\psi_k+f_{X_k}+\mathcal{L}_{X_k}\psi_k\right)\cdot\left[\frac{n\cdot\tau_{\mathbf{N}_W}\circ\sigma\psi_k}{k}+\frac{\tau_{\mathbf{N}_{\mathcal{V}_M}}\circ\hat{\sigma}\psi_k}{k\cdot k}\right]\cdot\Omega_{\mathbb{P}(E)}=0$$

that, when k is sufficiently large, there exists a constant $\hat{\mathbf{c}}>0$, independent of k, such that

$$\|\tau_{\mathbf{N}_{\mathcal{V}_M}}\circ\hat{\sigma}\left(\psi_k\right)\|^2_{\mathcal{C}^0(\mathbb{P}(E):\check{\omega})}\leq\hat{\mathbf{c}}\cdot\|\tau_{\mathbf{N}_W}\circ\sigma\left(\psi_k\right)\|^2_{\mathcal{C}^0(\mathbb{P}(E):\check{\omega})}.$$

Suppose that $Y_k\in\mathfrak{g}_E$, orthogonal to $\mathfrak{k}_E$, is the smooth vector field on $\mathbb{P}(E)$ such that

$$\mathcal{L}_{Y_k}\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right)=i\bar{\partial}\partial\left(\tau_{\mathbf{N}_W}\circ\sigma\psi_k\right).$$

Then we have

$$\mathcal{L}_{Y_k}\omega_{\#k}=i\bar{\partial}\partial\left(\tau_{\mathbf{N}_W}\circ\sigma\psi_k+\mathcal{L}_{Y_k}\psi_k\right).$$

Since the Futaki invariant, corresponding to Y_k and the Kaehler class $\left[\frac{i\cdot F_{A_{L^*}}}{2\pi}+k\cdot\check{\pi}^*\omega_M\right]$ on $\mathbb{P}(E)$, vanishes we have, by incorporating the gauge-fixing constant scalar curvature equation, the following equality:

$$\int_{\mathbb{P}(E)}\left(\tau_{\mathbf{N}_W}\circ\sigma\psi_k+\mathcal{L}_{Y_k}\psi_k\right)\cdot\left[\frac{n\cdot\tau_{\mathbf{N}_W}\circ\sigma\psi_k}{k}+\frac{\tau_{\mathbf{N}_{\mathcal{V}_M}}\circ\hat{\sigma}\psi_k}{k\cdot k}\right]\cdot\Omega_{\mathbb{P}(E)}=0.$$

Since $\Gamma\left(M:\mathbb{R}\right)$ is orthogonal to $\Gamma\left(M:W\right)$ with respect to the inner product $\langle:\rangle$ on $L^2\left(\mathbb{P}(E):\check{\omega}\right)$ we note that the term

$$\int_{\mathbb{P}(E)}\tau_{\mathbf{N}_W}\circ\sigma\left(\psi_k\right)\cdot\frac{\tau_{\mathbf{N}_{\mathcal{V}_M}}\circ\hat{\sigma}\psi_k}{k\cdot k}\cdot\Omega_{\mathbb{P}(E)}$$

of the above equality vanishes. Since the zeroth order term ϕ_0 in the asymptotic expansion of ψ_k, as $k\to+\infty$,

$$\psi_k\sim\phi_0+\sum_{\theta=1}^{+\infty}\frac{\phi_\theta}{k^\theta}$$

satisfies

$$\sigma(\phi_0) = \widetilde{\sigma}(\phi_0) = 0 \Longleftrightarrow \phi_0 \in \Gamma_o(M : \mathbb{R})$$

we infer from Theorem V.A that, when the parameter k is sufficiently large, there exists a constant $\mathbf{c} > 0$, independent of k, such that

$$\|\mathcal{L}_{Y_k}\psi_k\|_{\mathcal{C}^0(\mathbb{P}(E):\breve{\omega})} \leq \mathbf{c} \cdot \frac{\|Y_k\|_{\mathcal{C}^0(\mathbb{P}(E):\breve{\omega})}}{k}$$

because $\mathcal{L}_{Y_k}\phi_0 = 0$. On the other hand

$$\|Y_k\|_{\mathcal{C}^0(\mathbb{P}(E):\breve{\omega})}$$

is comparable with $\|\tau_{\mathbf{N}_W} \circ \sigma(\psi_k)\|_{\mathcal{C}^0(\mathbb{P}(E):\breve{\omega})}$. Thus, by using the Schwarz inequality, we infer from the above equality

$$\int_{\mathbb{P}(E)} (\tau_{\mathbf{N}_W} \circ \sigma\psi_k + \mathcal{L}_{Y_k}\psi_k) \cdot \left[\frac{n \cdot \tau_{\mathbf{N}_W} \circ \sigma\psi_k}{k} + \frac{\tau_{\mathbf{N}_{\mathcal{V}_M}} \circ \hat{\sigma}\psi_k}{k \cdot k}\right] \cdot \Omega_{\mathbb{P}(E)} = 0$$

that, when k is sufficiently large, there exists a constant $\widetilde{\mathbf{c}} > 0$, independent of k, such that

$$\|\tau_{\mathbf{N}_W} \circ \sigma(\psi_k)\|^2_{\mathcal{C}^0(\mathbb{P}(E):\breve{\omega})} \leq \widetilde{\mathbf{c}} \cdot \frac{\|\tau_{\mathbf{N}_{\mathcal{V}_M}} \circ \hat{\sigma}(\psi_k)\|^2_{\mathcal{C}^0(\mathbb{P}(E):\breve{\omega})}}{k \cdot k}.$$

By comparing this inequality

$$\|\tau_{\mathbf{N}_W} \circ \sigma(\psi_k)\|^2_{\mathcal{C}^0(\mathbb{P}(E):\breve{\omega})} \leq \widetilde{\mathbf{c}} \cdot \frac{\|\tau_{\mathbf{N}_{\mathcal{V}_M}} \circ \hat{\sigma}(\psi_k)\|^2_{\mathcal{C}^0(\mathbb{P}(E):\breve{\omega})}}{k \cdot k}$$

with the previous inequality

$$\|\tau_{\mathbf{N}_{\mathcal{V}_M}} \circ \hat{\sigma}(\psi_k)\|^2_{\mathcal{C}^0(\mathbb{P}(E):\breve{\omega})} \leq \hat{\mathbf{c}} \cdot \|\tau_{\mathbf{N}_W} \circ \sigma(\psi_k)\|^2_{\mathcal{C}^0(\mathbb{P}(E):\breve{\omega})}$$

we conclude immediately that

$$\tau_{\mathbf{N}_{\mathcal{V}_M}} \circ \hat{\sigma}(\psi_k) = 0 = \tau_{\mathbf{N}_W} \circ \sigma(\psi_k)$$

when the parameter k is sufficiently large.

Hence the (uniquely determined, for each k) smooth $\mathbb{R}$-valued solutions ψ_k of Theorem V.A to the gauge-fixing constant scalar curvature equation are actually solutions to the constant scalar curvature equation, without gauge-fixing, when the parameter k is sufficiently large.

q.e.d.

Appendix I. Induction scheme

We will prove Proposition V.A by induction on integers $\theta \geq 0$. Before proceeding we note that the integral of $\mathcal{S}_{\text{G-F}}(\bullet)$ on $\mathbb{P}(E)$ is always zero:

$$\int_{\mathbb{P}(E)} \mathcal{S}_{\text{G-F}}(\bullet) = \int_{\mathbb{P}(E)} \frac{\mathcal{S}_{\text{G-F}}(\bullet)}{\Omega_{\mathbb{P}(E)}} \cdot \Omega_{\mathbb{P}(E)} = \int_{\mathbb{P}(E)} \hat{\sigma}\left(\frac{\mathcal{S}_{\text{G-F}}(\bullet)}{\Omega_{\mathbb{P}(E)}}\right) \cdot \Omega_{\mathbb{P}(E)} = 0.$$

Let us start with solving the equation

$$\mathbf{B}_2 = 0 \Longleftrightarrow \frac{\mathbf{B}_2}{\Omega_{\mathbb{P}(E)}} = 0.$$

Assume that

$$\frac{\omega_{\#k}^{(-1+m+n)}}{k^m \cdot \breve{\omega}^{(-1+m+n)}} \sim \frac{{}_o\omega_{\#k}^{(-1+m+n)}}{k^m \cdot \breve{\omega}^{(-1+m+n)}} + \frac{R_1}{k} + \frac{R_2}{k \cdot k} + \text{ higher order terms}$$

in which R_1 and R_2 are independent of the parameter k. Since the connection A on E over M is Einstein-Hermitian we have

$$n \cdot \Pi_{\mathcal{C}_M}\left(\frac{i \cdot F_{A_{L^*}}}{2\pi}\right) \wedge \frac{\omega_M^{(-1+m)}}{(-1+m)!} + \text{trace}\left(\frac{i \cdot F_A}{2\pi}\right) \wedge \frac{\omega_M^{(-1+m)}}{(-1+m)!} = 0$$

and thence

$$\frac{{}_o\omega_{\#k}^{(-1+m+n)}}{k^m \cdot \breve{\omega}^{(-1+m+n)}} = \frac{\Omega_{\mathbb{P}(E)}}{\Omega_{\mathbb{P}(E)}} + \frac{-\frac{\Lambda_M \text{trace}\left(\frac{i \cdot F_A}{2\pi}\right)}{n}}{k} + \frac{\frac{\frac{\left(\frac{i \cdot F_{A_{L^*}}}{2\pi}\right)^{(n+1)}}{(n+1)!} \wedge \frac{\omega_M^{(-2+m)}}{(-2+m)!}}{\Omega_{\mathbb{P}(E)}}}{k \cdot k} + \text{ higher order terms}$$

in which $\Lambda_M \text{trace}\left(\frac{i \cdot F_A}{2\pi}\right)$ is the constant associated with the Einstein-Hermitian connection A on E over M:

$$\left[\Lambda_M \text{trace}\left(\frac{i \cdot F_A}{2\pi}\right)\right] \cdot \frac{\omega_M^m}{m!} = \text{trace}\left(\frac{i \cdot F_A}{2\pi}\right) \wedge \frac{\omega_M^{(-1+m)}}{(-1+m)!}.$$

Besides it can be checked readily that

$$R_1 = \frac{\Delta_M \phi_0}{2} + \frac{\Delta_V \phi_1}{2} = \frac{\Delta_M \phi_0}{2} + \frac{\Delta_V \sigma(\phi_1)}{2} = \frac{\Delta_M \phi_0}{2} + 2\pi n \cdot \sigma(\phi_1)$$

and

$$R_2 = \frac{\Delta_V \phi_2}{2} + \frac{Q\left(\sigma\left(\phi_1\right) : \sigma\left(\phi_1\right)\right)}{2!} + \frac{\Delta_M \phi_0}{2} \cdot \frac{\Delta_V \phi_1}{2}$$
$$+ \frac{\frac{i\bar{\partial}\partial\phi_0 \wedge i\bar{\partial}\partial\phi_0}{2!} \wedge \frac{\omega_M^{(-2+m)}}{(-2+m)!}}{\Omega_M} + \frac{i\bar{\partial}\partial\phi_0 \wedge \frac{\left(\frac{i \cdot F_{A_{L^*}}}{2\pi}\right)^n}{n!} \wedge \frac{\omega_M^{(-2+m)}}{(-2+m)!}}{\Omega_{\mathbb{P}(E)}}$$
$$+ \frac{i\bar{\partial}\partial\phi_1 \wedge \frac{\left(\frac{i \cdot F_{A_{L^*}}}{2\pi}\right)^{(-1+n)}}{(-1+n)!} \wedge \frac{\omega_M^{(-1+m)}}{(-1+m)!}}{\Omega_{\mathbb{P}(E)}}.$$

Here the symmetric quadratic operator $Q\left(\bullet : \bullet\right)$ is defined along each fiber $\mathbb{P}\left(\mathbb{C}^n\right)$ of the holomorphic projection map $\check{\pi} : \mathbb{P}(E) \longrightarrow M$ as in Proposition IV.A and Corollary IV.A. Thus

$$\frac{\mathbf{B}_2}{\Omega_{\mathbb{P}(E)}} = -\left((-1+n) \cdot n \cdot R_2 + \left[\Lambda_M \mathrm{trace}\left(\frac{i \cdot F_{\omega_M}}{2\pi}\right)\right] \cdot R_1\right)$$
$$+ n \cdot (-2+n) \cdot \frac{\Delta_V \phi_2}{2}$$
$$+ n \cdot (-1+n) \cdot \frac{i\bar{\partial}\partial\phi_1 \wedge \frac{\left(\frac{i \cdot F_{A_{L^*}}}{2\pi}\right)^{(-1+n)}}{(-1+n)!} \wedge \frac{\omega_M^{(-1+m)}}{(-1+m)!}}{\Omega_{\mathbb{P}(E)}}$$
$$+ n \cdot n \cdot \frac{i\bar{\partial}\partial\phi_0 \wedge \frac{\left(\frac{i \cdot F_{A_{L^*}}}{2\pi}\right)^n}{n!} \wedge \frac{\omega_M^{(-2+m)}}{(-2+m)!}}{\Omega_{\mathbb{P}(E)}}$$
$$+ n \cdot (-3+n) \cdot \frac{Q\left(\sigma\left(\phi_1\right) : \sigma\left(\phi_1\right)\right)}{2!}$$
$$+ n \cdot (-2+n) \cdot \frac{\Delta_M \phi_0}{2} \cdot \frac{\Delta_V \phi_1}{2}$$
$$+ n \cdot (-1+n) \cdot \frac{\frac{i\bar{\partial}\partial\phi_0 \wedge i\bar{\partial}\partial\phi_0}{2!} \wedge \frac{\omega_M^{(-2+m)}}{(-2+m)!}}{\Omega_M}$$
$$+ \left[\Lambda_M \mathrm{trace}\left(\frac{i \cdot F_A}{2\pi}\right) + \Lambda_M \mathrm{trace}\left(\frac{i \cdot F_{\omega_M}}{2\pi}\right)\right] \cdot \frac{\Delta_V \phi_1}{2}$$
$$+ \frac{\left[\mathrm{trace}\left(\frac{i \cdot F_A}{2\pi}\right) + \mathrm{trace}\left(\frac{i \cdot F_{\omega_M}}{2\pi}\right)\right] \wedge i\bar{\partial}\partial\phi_0 \wedge \frac{\omega_M^{(-2+m)}}{(-2+m)!}}{\Omega_M}$$

$$
\begin{aligned}
&+\frac{\frac{i\bar{\partial}\partial R_1}{2\pi}\wedge\frac{\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right)^{(-1+n)}}{(-1+n)!}\wedge\frac{\omega_M^{(-1+m)}}{(-1+m)!}}{\Omega_{\mathbb{P}(E)}}\\
&+Q\left(\frac{\sigma\left(R_1\right)}{2\pi}:\sigma\left(\phi_1\right)\right)+\frac{\Delta_V R_1}{4\pi}\cdot\frac{\Delta_M\phi_0}{2}\\
&+\frac{\Delta_V R_2}{4\pi}+\frac{-\Delta_V}{8\pi}\left(-\frac{\Lambda_M\mathrm{trace}\left(\frac{i\cdot F_A}{2\pi}\right)}{n}+R_1\right)^2\\
&+n\cdot\tau_{\mathbf{N}_W}\circ\sigma\left(\phi_1\right)+\tau_{\mathbf{N}_{\mathcal{V}_M}}\circ\hat{\sigma}\left(\phi_0\right)+\text{ known terms.}
\end{aligned}
$$

By substituting the formulae for R_1 and R_2 into the above expression of $\frac{\mathbf{B}_2}{\Omega_{\mathbb{P}(E)}}$ we see that

$$
\begin{aligned}
\frac{\mathbf{B}_2}{\Omega_{\mathbb{P}(E)}}=&-\left[\Lambda_M\mathrm{trace}\left(\frac{i\cdot F_{\omega_M}}{2\pi}\right)\right]\cdot\frac{\Delta_M\phi_0}{2}+(-n)\cdot\frac{\Delta_V\phi_2}{2}\\
&+n\cdot\frac{i\bar{\partial}\partial\phi_0\wedge\frac{\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right)^n}{n!}\wedge\frac{\omega_M^{(-2+m)}}{(-2+m)!}}{\Omega_{\mathbb{P}(E)}}\\
&+\left[\Lambda_M\mathrm{trace}\left(\frac{i\cdot F_A}{2\pi}\right)\right]\cdot\frac{\Delta_V\phi_1}{2}\\
&+\frac{\left[\mathrm{trace}\left(\frac{i\cdot F_A}{2\pi}\right)+\mathrm{trace}\left(\frac{i\cdot F_{\omega_M}}{2\pi}\right)\right]\wedge i\bar{\partial}\partial\phi_0\wedge\frac{\omega_M^{(-2+m)}}{(-2+m)!}}{\Omega_M}\\
&+\frac{\Delta_M\circ\Delta_M\phi_0}{8\pi}+\frac{n\cdot i\bar{\partial}\partial\sigma\left(\phi_1\right)\wedge\frac{\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right)^{(-1+n)}}{(-1+n)!}\wedge\frac{\omega_M^{(-1+m)}}{(-1+m)!}}{\Omega_{\mathbb{P}(E)}}\\
&+\frac{\Delta_V R_2}{4\pi}+\left[\Lambda_M\mathrm{trace}\left(\frac{i\cdot F_A}{2\pi}\right)\right]\cdot\frac{\Delta_V\sigma\left(\phi_1\right)}{2}\\
&+(-n)\cdot\frac{\Delta_M\phi_0}{2}\cdot\frac{\Delta_V\sigma\left(\phi_1\right)}{2}\\
&+(-n\cdot n\cdot\pi)\cdot\frac{\Delta_V}{2}\left[\sigma\left(\phi_1\right)\cdot\sigma\left(\phi_1\right)\right]\\
&+n\cdot\tau_{\mathbf{N}_W}\circ\sigma\left(\phi_1\right)+\tau_{\mathbf{N}_{\mathcal{V}_M}}\circ\hat{\sigma}\left(\phi_0\right)+\text{known terms.}
\end{aligned}
$$

Since $\frac{i\cdot F_{A_{L^*}}}{2\pi}=\Pi_{\mathcal{C}_V}\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right)\oplus\Pi_{\mathcal{C}_M}\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right)$ we have (by using the

Einstein-Hermitian condition of A on E over M)

$$\begin{aligned}
&\frac{i\bar{\partial}\partial\sigma\left(\phi_1\right)\wedge\frac{\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right)^{(-1+n)}}{(-1+n)!}\wedge\frac{\omega_M^{(-1+m)}}{(-1+m)!}}{\Omega_{\mathbb{P}(E)}}\\
&=\frac{\Delta_M\sigma\left(\phi_1\right)}{2}+\frac{\Delta_V\sigma\left(\phi_1\right)}{2}\cdot\frac{\Pi_{\mathcal{C}_M}\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right)\wedge\frac{\omega_M^{(-1+m)}}{(-1+m)!}}{\Omega_M}\\
&=\frac{\Delta_M\sigma\left(\phi_1\right)}{2}+\frac{\left[-\Lambda_M\mathrm{trace}\left(\frac{i\cdot F_A}{2\pi}\right)\right]}{n}\cdot\frac{\Delta_V\sigma\left(\phi_1\right)}{2}
\end{aligned}$$

and thence

$$\begin{aligned}
\frac{\mathbf{B}_2}{\Omega_{\mathbb{P}(E)}}=&-\left[\Lambda_M\mathrm{trace}\left(\frac{i\cdot F_{\omega_M}}{2\pi}\right)\right]\cdot\frac{\Delta_M\phi_0}{2}+(-n)\cdot\frac{\Delta_V\phi_2}{2}\\
&+n\cdot\frac{i\bar{\partial}\partial\phi_0\wedge\frac{\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right)^n}{n!}\wedge\frac{\omega_M^{(-2+m)}}{(-2+m)!}}{\Omega_{\mathbb{P}(E)}}\\
&+\frac{\left[\mathrm{trace}\left(\frac{i\cdot F_A}{2\pi}\right)+\mathrm{trace}\left(\frac{i\cdot F_{\omega_M}}{2\pi}\right)\right]\wedge i\bar{\partial}\partial\phi_0\wedge\frac{\omega_M^{(-2+m)}}{(-2+m)!}}{\Omega_M}\\
&+\frac{\Delta_M\circ\Delta_M\phi_0}{8\pi}+n\cdot\frac{\Delta_M\sigma\left(\phi_1\right)}{2}\\
&+\frac{\Delta_V R_2}{4\pi}+\left[\Lambda_M\mathrm{trace}\left(\frac{i\cdot F_A}{2\pi}\right)\right]\cdot\frac{\Delta_V\sigma\left(\phi_1\right)}{2}\\
&+(-n)\cdot\frac{\Delta_M\phi_0}{2}\cdot\frac{\Delta_V\sigma\left(\phi_1\right)}{2}\\
&+(-n\cdot n\cdot\pi)\cdot\frac{\Delta_V}{2}\left[\sigma\left(\phi_1\right)\cdot\sigma\left(\phi_1\right)\right]\\
&+n\cdot\tau_{\mathbf{N}_W}\circ\sigma\left(\phi_1\right)+\tau_{\mathbf{N}_{\mathcal{V}_M}}\circ\hat{\sigma}\left(\phi_0\right)+\text{known terms.}
\end{aligned}$$

Now we note that

$$\begin{aligned}
&\int_{\mathbb{P}(\mathbb{C}^n)}n\cdot\frac{i\bar{\partial}\partial\phi_0\wedge\frac{\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right)^n}{n!}\wedge\frac{\omega_M^{(-2+m)}}{(-2+m)!}}{\Omega_{\mathbb{P}(E)}}\\
&=\int_{\mathbb{P}(\mathbb{C}^n)}n\cdot\frac{i\bar{\partial}\partial\phi_0\wedge\Pi_{\mathcal{C}_M}\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right)\wedge\frac{\omega_M^{(-2+m)}}{(-2+m)!}}{\Omega_M}
\end{aligned}$$

$$= \int_{\mathbb{P}(\mathbb{C}^n)} - \frac{i\bar{\partial}\partial\phi_0 \wedge \mathrm{trace}\left(\frac{i\cdot F_A}{2\pi}\right) \wedge \frac{\omega_M^{(-2+m)}}{(-2+m)!}}{\Omega_M}$$

along each fiber $\mathbb{P}(\mathbb{C}^n)$ of $\check{\pi} : \mathbb{P}(E) \longrightarrow M$. Thus

$$\begin{aligned}
\hat{\sigma}\left(\frac{\mathbf{B}_2}{\Omega_{\mathbb{P}(E)}}\right) &= -\frac{\left[\Lambda_M \mathrm{trace}\left(\frac{i\cdot F_{\omega_M}}{2\pi}\right)\right]\cdot \Delta_M\phi_0}{2} \\
&\quad + \frac{\mathrm{trace}\left(\frac{i\cdot F_{\omega_M}}{2\pi}\right) \wedge i\bar{\partial}\partial\phi_0 \wedge \frac{\omega_M^{(-2+m)}}{(-2+m)!}}{\Omega_M} + \frac{\Delta_M \circ \Delta_M\phi_0}{8\pi} \\
&\quad + \tau_{\mathbf{N}_{\mathcal{V}_M}}\phi_0 + \hat{\sigma}\,(\text{known terms}) \\
&= \left(\frac{\mathcal{V}_M}{\Omega_M} + \tau_{\mathbf{N}_{\mathcal{V}_M}}\right)\phi_0 + \hat{\sigma}\,(\text{known terms})\,.
\end{aligned}$$

Remark. Let ξ_A denote the smooth $(-1+m+n)$-form on $\mathbb{P}(E)$ defined by A as follows:

$$\begin{aligned}
\xi_A \equiv (n+n)\cdot \frac{\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right)^{(n+1)}}{(n+1)!} \wedge \frac{\omega_M^{(-2+m)}}{(-2+m)!} \\
+ \check{\pi}^*\mathrm{trace}\left(\frac{i\cdot F_A}{2\pi}\right) \wedge \frac{\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right)^{n}}{n!} \wedge \frac{\omega_M^{(-2+m)}}{(-2+m)!} \\
+ \check{\pi}^*\mathrm{trace}\left(\frac{i\cdot F_{\omega_M}}{2\pi}\right) \wedge \frac{\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right)^{n}}{n!} \wedge \frac{\omega_M^{(-2+m)}}{(-2+m)!}.
\end{aligned}$$

Let $\frac{\xi_A}{\Omega_{\mathbb{P}(E)}}$ denote the smooth $\mathbb{R}$-valued function on $\mathbb{P}(E)$ satisfying

$$\xi_A = \frac{\xi_A}{\Omega_{\mathbb{P}(E)}} \cdot \Omega_{\mathbb{P}(E)}.$$

Then, using the constancy of the scalar curvature of $(M : \omega_M)$ and the Einstein-Hermitian condition satisfied by A, it can be shown that

$$\hat{\sigma}\left(\frac{\mathbf{B}_2}{\Omega_{\mathbb{P}(E)}}\right) = \left(\frac{\mathcal{V}_M}{\Omega_M} + \tau_{\mathbf{N}_{\mathcal{V}_M}}\right)\phi_0 + \hat{\sigma}\left(-c_{\xi_A} + \frac{\xi_A}{\Omega_{\mathbb{P}(E)}}\right)$$

in which $c_{\xi_A} \in \mathbb{R}$ is the constant satisfying

$$c_{\xi_A} \cdot \int_{\mathbb{P}(E)} \Omega_{\mathbb{P}(E)} = \int_{\mathbb{P}(E)} \xi_A.$$

Thus in the equality

$$\begin{aligned}\hat{\sigma}\left(\frac{\mathbf{B}_2}{\Omega_{\mathbb{P}(E)}}\right) &= -\frac{\left[\Lambda_M \mathrm{trace}\left(\frac{i\cdot F_{\omega_M}}{2\pi}\right)\right]\cdot \Delta_M\phi_0}{2}\\ &\quad + \frac{\mathrm{trace}\left(\frac{i\cdot F_{\omega_M}}{2\pi}\right)\wedge i\bar{\partial}\partial\phi_0\wedge\frac{\omega_M^{(-2+m)}}{(-2+m)!}}{\Omega_M} + \frac{\Delta_M\circ\Delta_M\phi_0}{8\pi}\\ &\quad + \tau_{\mathbf{N}_{\mathcal{V}_M}}\phi_0 + \hat{\sigma}\left(\text{known terms}\right)\\ &= \left(\frac{\mathcal{V}_M}{\Omega_M} + \tau_{\mathbf{N}_{\mathcal{V}_M}}\right)\phi_0 + \hat{\sigma}\left(\text{known terms}\right).\end{aligned}$$

the phrase "known terms" simply means

$$-c_{\xi_A} + \frac{\xi_A}{\Omega_{\mathbb{P}(E)}}$$

which is, given A, obviously a known smooth $\mathbb{R}$-valued function on M.

Since the elliptic linear partial differential operator $\left(\frac{\mathcal{V}_M}{\Omega_M} + \tau_{\mathbf{N}_{\mathcal{V}_M}}\right)$ acting on $\Gamma_o\left(M:\mathbb{R}\right)$ is both symmetric and positive we infer that

$$\phi_0 \in \Gamma_o\left(M:\mathbb{R}\right)$$

can be uniquely solved from the equation $\hat{\sigma}\left(\frac{\mathbf{B}_2}{\Omega_{\mathbb{P}(E)}}\right) = 0$.

With $\phi_0 \in \Gamma_o\left(M:\mathbb{R}\right)$ being known we have (by the Einstein-Hermitian condition of A on E over M)

$$\begin{aligned}R_2 &= \frac{\Delta_V\phi_2}{2} + \frac{Q\left(\sigma\left(\phi_1\right):\sigma\left(\phi_1\right)\right)}{2!} + \frac{\Delta_M\phi_0}{2}\cdot\frac{\Delta_V\sigma\left(\phi_1\right)}{2}\\ &\quad + \frac{\Delta_M\sigma\left(\phi_1\right)}{2} + \frac{\left[-\Lambda_M\mathrm{trace}\left(\frac{i\cdot F_A}{2\pi}\right)\right]}{n}\cdot\frac{\Delta_V\sigma\left(\phi_1\right)}{2} + \frac{\Delta_M\hat{\sigma}\left(\phi_1\right)}{2}\\ &\quad + \text{known terms}\end{aligned}$$

and thence

$$\begin{aligned}\frac{\mathbf{B}_2}{\Omega_{\mathbb{P}(E)}} &= \frac{\Delta_V}{8\pi}\circ\left(-4\pi n\cdot\mathrm{id} + \Delta_V\right)\phi_2\\ &\quad + \frac{\Delta_V}{8\pi}\left[-\left(2\pi n\cdot\sigma\phi_1\right)\cdot\left(2\pi n\cdot\sigma\phi_1\right) + Q\left(\sigma\phi_1:\sigma\phi_1\right)\right]\\ &\quad + n\cdot\left(\Delta_M + 8\pi\cdot\tau_{\mathbf{N}_W}\right)\sigma\left(\phi_1\right) + \text{ known terms.}\end{aligned}$$

Now by Proposition IV.A we have

$$\sigma\left(\frac{\mathbf{B}_2}{\Omega_{\mathbb{P}(E)}}\right)=n\cdot\left(\Delta_M+8\pi\cdot\tau_{\mathbf{N}_W}\right)\sigma\left(\phi_1\right)+\ \sigma\left(\text{known terms}\right).$$

Since the elliptic linear partial differential operator $\left(\Delta_M+8\pi\cdot\tau_{\mathbf{N}_W}\right)$ acting on $\Gamma\left(M:W\right)$ is both symmetric and positive we infer that $\sigma\left(\phi_1\right)$ $\in\Gamma\left(M:W\right)$ can be uniquely solved from the equation $\sigma\left(\frac{\mathbf{B}_2}{\Omega_{\mathbb{P}(E)}}\right)=0$. With both $\phi_0\in\Gamma_o\left(M:\mathbb{R}\right)$ and $\sigma\left(\phi_1\right)\in\Gamma\left(M:W\right)$ being known we have

$$\widetilde{\sigma}\left(\frac{\mathbf{B}_2}{\Omega_{\mathbb{P}(E)}}\right)=\frac{\Delta_V}{8\pi}\circ\left(-4\pi n\cdot\mathrm{id}+\Delta_V\right)\widetilde{\sigma}\left(\phi_2\right)+\ \widetilde{\sigma}\left(\text{known terms}\right)$$

and thence $\widetilde{\sigma}\left(\phi_2\right)$ can be uniquely solved from the equation $\widetilde{\sigma}\left(\frac{\mathbf{B}_2}{\Omega_{\mathbb{P}(E)}}\right)=0$ fiberwisely.

Now given $\theta\in\mathbb{N}$ we will solve the equation

$$\mathbf{B}_{\theta+2}=0\Longleftrightarrow\frac{\mathbf{B}_{\theta+2}}{\Omega_{\mathbb{P}(E)}}=0$$

under the hypothesis that

$$\hat{\sigma}\left(\phi_\mu\right)\oplus\sigma\left(\phi_{\mu+1}\right)\oplus\widetilde{\sigma}\left(\phi_{\mu+2}\right)$$

is already known for any integer $0\leq\mu<\theta$. In particular $\hat{\sigma}\left(\phi_0\right)\oplus$ $\sigma\left(\phi_1\right)\oplus\widetilde{\sigma}\left(\phi_2\right)$ and $R_1=\frac{\Delta_M\phi_0}{2}+\frac{\Delta_V\sigma(\phi_1)}{2}=\frac{\Delta_M\phi_0}{2}+2\pi n\cdot\sigma\left(\phi_1\right)$ are already known.

Suppose that

$$\begin{aligned}\frac{\omega_{\#k}^{(-1+m+n)}}{k^m\cdot\breve{\omega}^{(-1+m+n)}}\sim\frac{{}_o\omega_{\#k}^{(-1+m+n)}}{k^m\cdot\breve{\omega}^{(-1+m+n)}}+\frac{R_1}{k}\\+\cdots+\frac{R_{\theta+1}}{k^\theta\cdot k}+\frac{R_{\theta+2}}{k^\theta\cdot k\cdot k}+\ \text{higher order terms}\end{aligned}$$

in which each $R_\bullet$ is independent of the parameter k. Then by our induction hypothesis we have R_μ being known for any $\mu\in\mathbb{N}$ satisfying $\mu\leq\theta$. It can be checked readily that

$$R_{\theta+1}=\frac{\Delta_M\hat{\sigma}\left(\phi_\theta\right)}{2}+\frac{\Delta_V\sigma\left(\phi_{\theta+1}\right)}{2}+\ \text{known terms}$$

and

$$\begin{aligned}R_{\theta+2} = \frac{\Delta_V \phi_{\theta+2}}{2} &+ \frac{i\bar{\partial}\partial\phi_{\theta+1} \wedge \frac{\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right)^{(-1+n)}}{(-1+n)!} \wedge \frac{\omega_M^{(-1+m)}}{(-1+m)!}}{\Omega_{\mathbb{P}(E)}} \\ &+ \frac{i\bar{\partial}\partial\hat{\sigma}\left(\phi_\theta\right) \wedge \Pi_{\mathcal{C}_M}\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right) \wedge \frac{\omega_M^{(-2+m)}}{(-2+m)!}}{\Omega_M} + Q\left(\sigma\phi_1 : \sigma\phi_{\theta+1}\right) \\ &+ \frac{\Delta_V \sigma\left(\phi_1\right)}{2} \cdot \frac{\Delta_M \hat{\sigma}\left(\phi_\theta\right)}{2} + \frac{\Delta_M \phi_0}{2} \cdot \frac{\Delta_V \sigma\left(\phi_{\theta+1}\right)}{2} \\ &+ \frac{i\bar{\partial}\partial\phi_0 \wedge i\bar{\partial}\partial\hat{\sigma}\left(\phi_\theta\right) \wedge \frac{\omega_M^{(-2+m)}}{(-2+m)!}}{\Omega_M} + \text{ known terms.}\end{aligned}$$

Note that $\widetilde{\sigma}\left(R_{\theta+1}\right)$ is already known. Thus

$$\begin{aligned}\frac{\mathbf{B}_{\theta+2}}{\Omega_{\mathbb{P}(E)}} = &-\left((-1+n)\cdot n\cdot R_{\theta+2} + \left[\Lambda_M \text{trace}\left(\frac{i\cdot F_{\omega_M}}{2\pi}\right)\right]\cdot R_{\theta+1}\right) \\ &+ n\cdot(-2+n)\cdot\frac{\Delta_V\phi_{\theta+2}}{2} \\ &+ n\cdot(-1+n)\cdot\frac{i\bar{\partial}\partial\phi_{\theta+1}\wedge\frac{\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right)^{(-1+n)}}{(-1+n)!}\wedge\frac{\omega_M^{(-1+m)}}{(-1+m)!}}{\Omega_{\mathbb{P}(E)}} \\ &+ n\cdot n\cdot\frac{i\bar{\partial}\partial\hat{\sigma}\left(\phi_\theta\right)\wedge\Pi_{\mathcal{C}_M}\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right)\wedge\frac{\omega^{(-2+m)}}{(-2+m)!}}{\Omega_M} \\ &+ n\cdot(-3+n)\cdot Q\left(\sigma\phi_1:\sigma\phi_{\theta+1}\right) \\ &+ n\cdot(-2+n)\cdot\frac{\Delta_M\phi_0}{2}\cdot\frac{\Delta_V\sigma\left(\phi_{\theta+1}\right)}{2} \\ &+ n\cdot(-2+n)\cdot\frac{\Delta_V\sigma\left(\phi_1\right)}{2}\cdot\frac{\Delta_M\hat{\sigma}\left(\phi_\theta\right)}{2} \\ &+ n\cdot(-1+n)\cdot\frac{i\bar{\partial}\partial\phi_0\wedge i\bar{\partial}\partial\hat{\sigma}\left(\phi_\theta\right)\wedge\frac{\omega_M^{(-2+m)}}{(-2+m)!}}{\Omega_M} \\ &+ \left[\Lambda_M\text{trace}\left(\frac{i\cdot F_A}{2\pi}\right) + \Lambda_M\text{trace}\left(\frac{i\cdot F_{\omega_M}}{2\pi}\right)\right]\cdot\frac{\Delta_V\sigma\left(\phi_{\theta+1}\right)}{2} \\ &+ \frac{\left[\text{trace}\left(\frac{i\cdot F_A}{2\pi}\right) + \text{trace}\left(\frac{i\cdot F_{\omega_M}}{2\pi}\right)\right]\wedge i\bar{\partial}\partial\hat{\sigma}\left(\phi_\theta\right)\wedge\frac{\omega_M^{(-2+m)}}{(-2+m)!}}{\Omega_M} \\ &+ Q\left(\frac{\sigma\left(R_1\right)}{2\pi}:\sigma\left(\phi_{\theta+1}\right)\right) + \frac{\Delta_V R_1}{4\pi}\cdot\frac{\Delta_M\hat{\sigma}\left(\phi_\theta\right)}{2}\end{aligned}$$

$$+\frac{\frac{i\bar{\partial}\partial R_{\theta+1}}{2\pi}\wedge\frac{\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right)^{(-1+n)}}{(-1+n)!}\wedge\frac{\omega_M^{(-1+m)}}{(-1+m)!}}{\Omega_{\mathbb{P}(E)}}+Q\left(\sigma\left(\phi_1\right):\frac{\sigma\left(R_{\theta+1}\right)}{2\pi}\right)$$
$$+\frac{\Delta_M\phi_0}{2}\cdot\frac{\Delta_V R_{\theta+1}}{4\pi}+\frac{\Delta_V R_{\theta+2}}{4\pi}$$
$$+\frac{-\Delta_V}{4\pi}\left(\left[-\frac{\Lambda_M\mathrm{trace}\left(\frac{i\cdot F_A}{2\pi}\right)}{n}+R_1\right]\cdot R_{\theta+1}\right)$$
$$+n\cdot\tau_{\mathbf{N}_W}\circ\sigma\left(\phi_{\theta+1}\right)+\tau_{\mathbf{N}_{\mathcal{V}_M}}\circ\hat{\sigma}\left(\phi_\theta\right)+\text{ known terms.}$$

Substituting the formulae for $R_{\theta+1}$ and $R_{\theta+2}$ into this expression of $\frac{\mathbf{B}_{\theta+2}}{\Omega_{\mathbb{P}(E)}}$ we have

$$\begin{aligned}\frac{\mathbf{B}_{\theta+2}}{\Omega_{\mathbb{P}(E)}}=&-\left[\Lambda_M\mathrm{trace}\left(\frac{i\cdot F_{\omega_M}}{2\pi}\right)\right]\cdot\frac{\Delta_M\hat{\sigma}\left(\phi_\theta\right)}{2}+(-n)\cdot\frac{\Delta_V\phi_{\theta+2}}{2}\\&+n\cdot\frac{i\bar{\partial}\partial\hat{\sigma}\left(\phi_\theta\right)\wedge\Pi_{\mathcal{C}_M}\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right)\wedge\frac{\omega^{(-2+m)}}{(-2+m)!}}{\Omega_M}\\&+\left[\Lambda_M\mathrm{trace}\left(\frac{i\cdot F_A}{2\pi}\right)\right]\cdot\frac{\Delta_V\sigma\left(\phi_{\theta+1}\right)}{2}\\&+\frac{\left[\mathrm{trace}\left(\frac{i\cdot F_A}{2\pi}\right)+\mathrm{trace}\left(\frac{i\cdot F_{\omega_M}}{2\pi}\right)\right]\wedge i\bar{\partial}\partial\hat{\sigma}\left(\phi_\theta\right)\wedge\frac{\omega_M^{(-2+m)}}{(-2+m)!}}{\Omega_M}\\&+\frac{\Delta_M\circ\Delta_M\hat{\sigma}\left(\phi_\theta\right)}{8\pi}\\&+\frac{n\cdot i\bar{\partial}\partial\sigma\left(\phi_{\theta+1}\right)\wedge\frac{\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right)^{(-1+n)}}{(-1+n)!}\wedge\frac{\omega_M^{(-1+m)}}{(-1+m)!}}{\Omega_{\mathbb{P}(E)}}\\&+\frac{\Delta_V R_{\theta+2}}{4\pi}+\frac{-\Delta_V}{4\pi}\left(\left[-\frac{\Lambda_M\mathrm{trace}\left(\frac{i\cdot F_A}{2\pi}\right)}{n}+R_1\right]\cdot R_{\theta+1}\right)\\&+n\cdot\tau_{\mathbf{N}_W}\circ\sigma\left(\phi_{\theta+1}\right)+\tau_{\mathbf{N}_{\mathcal{V}_M}}\circ\hat{\sigma}\left(\phi_\theta\right)+\text{ known terms.}\end{aligned}$$

Since $\frac{i\cdot F_{A_{L^*}}}{2\pi}=\Pi_{\mathcal{C}_V}\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right)\oplus\Pi_{\mathcal{C}_M}\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right)$ we have (by using the

Einstein-Hermitian condition of A on E over M)

$$\frac{i\bar{\partial}\partial\sigma\left(\phi_{\theta+1}\right)\wedge\frac{\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right)^{(-1+n)}}{(-1+n)!}\wedge\frac{\omega_M^{(-1+m)}}{(-1+m)!}}{\Omega_{\mathbb{P}(E)}}$$
$$=\frac{\Delta_M\sigma\left(\phi_{\theta+1}\right)}{2}+\frac{\left[-\Lambda_M\mathrm{trace}\left(\frac{i\cdot F_A}{2\pi}\right)\right]}{n}\cdot\frac{\Delta_V\sigma\left(\phi_{\theta+1}\right)}{2}$$

and thence

$$\begin{aligned}\frac{\mathbf{B}_{\theta+2}}{\Omega_{\mathbb{P}(E)}}=&-\left[\Lambda_M\mathrm{trace}\left(\frac{i\cdot F_{\omega_M}}{2\pi}\right)\right]\cdot\frac{\Delta_M\hat{\sigma}\left(\phi_\theta\right)}{2}+(-n)\cdot\frac{\Delta_V\phi_{\theta+2}}{2}\\&+n\cdot\frac{i\bar{\partial}\partial\hat{\sigma}\left(\phi_\theta\right)\wedge\Pi_{\mathcal{C}_M}\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right)\wedge\frac{\omega^{(-2+m)}}{(-2+m)!}}{\Omega_M}\\&+\frac{\left[\mathrm{trace}\left(\frac{i\cdot F_A}{2\pi}\right)+\mathrm{trace}\left(\frac{i\cdot F_{\omega_M}}{2\pi}\right)\right]\wedge i\bar{\partial}\partial\hat{\sigma}\left(\phi_\theta\right)\wedge\frac{\omega_M^{(-2+m)}}{(-2+m)!}}{\Omega_M}\\&+\frac{\Delta_M\circ\Delta_M\hat{\sigma}\left(\phi_\theta\right)}{8\pi}+n\cdot\frac{\Delta_M\sigma\left(\phi_{\theta+1}\right)}{2}\\&+\frac{\Delta_V R_{\theta+2}}{4\pi}+\frac{-\Delta_V}{4\pi}\left(\left[-\frac{\Lambda_M\mathrm{trace}\left(\frac{i\cdot F_A}{2\pi}\right)}{n}+R_1\right]\cdot R_{\theta+1}\right)\\&+n\cdot\tau_{\mathbf{N}_W}\circ\sigma\left(\phi_{\theta+1}\right)+\tau_{\mathbf{N}_{\mathcal{V}_M}}\circ\hat{\sigma}\left(\phi_\theta\right)+\text{ known terms.}\end{aligned}$$

Since

$$\int_{\mathbb{P}(\mathbb{C}^n)}\left(\frac{n\cdot i\bar{\partial}\partial\hat{\sigma}\left(\phi_\theta\right)\wedge\Pi_{\mathcal{C}_M}\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right)\wedge\frac{\omega_M^{(-2+m)}}{(-2+m)!}}{\Omega_M}\right.$$
$$\left.+\frac{i\bar{\partial}\partial\hat{\sigma}\left(\phi_\theta\right)\wedge\mathrm{trace}\left(\frac{i\cdot F_A}{2\pi}\right)\wedge\frac{\omega_M^{(-2+m)}}{(-2+m)!}}{\Omega_M}\right)=0$$

along each fiber $\mathbb{P}\left(\mathbb{C}^n\right)$ of $\check{\pi}:\mathbb{P}(E)\longrightarrow M$ we have:

$$\begin{aligned}\hat{\sigma}\left(\frac{\mathbf{B}_{\theta+2}}{\Omega_{\mathbb{P}(E)}}\right) &= -\frac{\left[\Lambda_M \mathrm{trace}\left(\frac{i\cdot F_{\omega_M}}{2\pi}\right)\right]\cdot \Delta_M\hat{\sigma}\left(\phi_\theta\right)}{2}\\ &\quad+\frac{\mathrm{trace}\left(\frac{i\cdot F_{\omega_M}}{2\pi}\right)\wedge i\bar{\partial}\partial\hat{\sigma}\left(\phi_\theta\right)\wedge\frac{\omega_M^{(-2+m)}}{(-2+m)!}}{\Omega_M}\\ &\quad+\frac{\Delta_M\circ\Delta_M\hat{\sigma}\left(\phi_\theta\right)}{8\pi}+\tau_{\mathbf{N}_{\mathcal{V}_M}}\circ\hat{\sigma}\left(\phi_\theta\right)+\ \hat{\sigma}\left(\text{known terms}\right)\\ &= \left(\frac{\mathcal{V}_M}{\Omega_M}+\tau_{\mathbf{N}_{\mathcal{V}_M}}\right)\hat{\sigma}\left(\phi_\theta\right)+\ \hat{\sigma}\left(\text{known terms}\right).\end{aligned}$$

Thus $\hat{\sigma}\left(\phi_\theta\right)\in\Gamma_o\left(M:\mathbb{R}\right)$ can be uniquely solved from the equation $\hat{\sigma}\left(\frac{\mathbf{B}_{\theta+2}}{\Omega_{\mathbb{P}(E)}}\right)=0$.

With $\hat{\sigma}\left(\phi_\theta\right)\in\Gamma_o\left(M:\mathbb{R}\right)$ being known we have (by the Einstein-Hermitian condition of A on E over M)

$$\begin{aligned}R_{\theta+2} &= \frac{\Delta_V\phi_{\theta+2}}{2}+\frac{\Delta_M\sigma\left(\phi_{\theta+1}\right)}{2}+\frac{\Delta_M\hat{\sigma}\left(\phi_{\theta+1}\right)}{2}\\ &\quad+\frac{\left[-\Lambda_M\mathrm{trace}\left(\frac{i\cdot F_A}{2\pi}\right)\right]}{n}\cdot\frac{\Delta_V\sigma\left(\phi_{\theta+1}\right)}{2}\\ &\quad+Q\left(\sigma\phi_1:\sigma\phi_{\theta+1}\right)+\frac{\Delta_M\phi_0}{2}\cdot\frac{\Delta_V\sigma\left(\phi_{\theta+1}\right)}{2}+\ \text{known terms}\end{aligned}$$

and thence

$$\begin{aligned}\frac{\mathbf{B}_{\theta+2}}{\Omega_{\mathbb{P}(E)}} &= \frac{\Delta_V}{8\pi}\circ\left(-4\pi n\cdot\mathrm{id}+\Delta_V\right)\phi_{\theta+2}+n\cdot\Delta_M\sigma\left(\phi_{\theta+1}\right)\\ &\quad+\frac{\Delta_V}{4\pi}\left[-\left(2\pi n\cdot\sigma\phi_1\right)\cdot\left(2\pi n\cdot\sigma\phi_{\theta+1}\right)+Q\left(\sigma\phi_1:\sigma\phi_{\theta+1}\right)\right]\\ &\quad+n\cdot\tau_{\mathbf{N}_W}\circ\sigma\left(\phi_{\theta+1}\right)+\ \text{known terms}.\end{aligned}$$

Now by Corollary IV.A we have

$$\sigma\left(\frac{\mathbf{B}_{\theta+2}}{\Omega_{\mathbb{P}(E)}}\right)=n\cdot\left(\Delta_M+8\pi\cdot\tau_{\mathbf{N}_W}\right)\sigma\left(\phi_{\theta+1}\right)+\ \sigma\left(\text{known terms}\right).$$

Thus $\sigma\left(\phi_{\theta+1}\right)\in\Gamma\left(M:W\right)$ can be uniquely solved from the equation $\sigma\left(\frac{\mathbf{B}_{\theta+2}}{\Omega_{\mathbb{P}(E)}}\right)=0$. With both $\hat{\sigma}\left(\phi_\theta\right)\in\Gamma_o\left(M:\mathbb{R}\right)$ and $\sigma\left(\phi_{\theta+1}\right)\in\Gamma\left(M:W\right)$ being known we have

$$\widetilde{\sigma}\left(\frac{\mathbf{B}_{\theta+2}}{\Omega_{\mathbb{P}(E)}}\right)=\frac{\Delta_V}{8\pi}\circ\left(-4\pi n\cdot\mathrm{id}+\Delta_V\right)\phi_{\theta+2}+\ \widetilde{\sigma}\left(\text{known terms}\right)$$

and thence $\widetilde{\sigma}(\phi_{\theta+2})$ can be uniquely solved from the equation $\widetilde{\sigma}\left(\frac{\mathbf{B}_{\theta+2}}{\Omega_{\mathbb{P}(E)}}\right) = 0$ fiberwise. q.e.d.

Appendix II. Proof of Theorem II.B

Since

$$0 = [F_A(Z:\ )] \in H^1_{\bar{\partial}_A}(M : \mathrm{Hom}\ (E:E))$$

there exists, by Theorem II.A, a smooth vector field $\check{X}_Z$ on $\mathbb{P}(E)$ preserving both the complex structure of $\mathbb{P}(E)$ and the holomorphic projection map $\check{\pi} : \mathbb{P}(E) \longrightarrow M$ such that

$$\check{\pi}_* \check{X}_Z = X_Z.$$

Since X_Z preserves the Kaehler form ω_M on M we have

$$\mathcal{L}_{\check{X}_Z} \omega_M = 0.$$

Let $f_{\check{X}_Z} \in \Gamma_o(\mathbb{P}(E) : \mathbb{R})$ denote the smooth $\mathbb{R}$-valued function on $\mathbb{P}(E)$ satisfying

$$\mathcal{L}_{\check{X}_Z}\left(\frac{i \cdot F_{A_{L^*}}}{2\pi}\right) = i\bar{\partial}\partial f_{\check{X}_Z}.$$

Since the Futaki invariant associated with the lifting $\check{X}_Z$ (of X_Z) and the Kaehler class $\left[\frac{i \cdot F_{A_{L^*}}}{2\pi} + k \cdot \check{\pi}^* \omega_M\right]$ on $\mathbb{P}(E)$ vanishes for any sufficiently large $k \in \mathbb{N}$ we have, for any smooth $\mathbb{R}$-valued function g on $\mathbb{P}(E)$,

$$\int_{\mathbb{P}(E)} \left(f_{\check{X}_Z} + \hat{\sigma}g + \frac{\sigma g}{k} + \frac{\widetilde{\sigma} g}{k \cdot k}\right) \cdot \mathcal{S}\left({}_o\omega_{\#k} + i\bar{\partial}\partial \circ \hat{\sigma} g + \frac{i\bar{\partial}\partial \circ \sigma g}{k} + \frac{i\bar{\partial}\partial \circ \widetilde{\sigma} g}{k \cdot k}\right) = 0$$

whenever the parameter $k \in \mathbb{N}$ is large enough. Thus, by the expansion results of Appendix I, we have, by choosing $\hat{\sigma} g = 0 = \sigma g$, the following power series expansion result for $\frac{\mathcal{S}\left({}_o\omega_{\#k} + \frac{i\bar{\partial}\partial \circ \widetilde{\sigma} g}{k \cdot k}\right)}{\Omega_{\mathbb{P}(E)}}$ in $\frac{1}{k}$:

$$\frac{\mathcal{S}\left({}_o\omega_{\#k} + \frac{i\bar{\partial}\partial \circ \widetilde{\sigma} g}{k \cdot k}\right)}{\Omega_{\mathbb{P}(E)}} = \frac{\frac{\Delta_V}{8\pi} \circ (-4\pi n \cdot \mathrm{id} + \Delta_V)\, \widetilde{\sigma} g}{k \cdot k} + \frac{\text{known terms}}{k \cdot k} + \text{higher order terms}$$

for any smooth $\mathbb{R}$-valued function g on $\mathbb{P}(E)$ with $\hat{\sigma}g = 0 = \sigma g$. Hence we infer from the vanishing of Futaki invariants:

$$\int_{\mathbb{P}(E)} \left(f_{\check{X}_Z} + \frac{\widetilde{\sigma} g}{k \cdot k} \right) \cdot \mathcal{S} \left({}_o\omega_{\#k} + \frac{i\bar{\partial}\partial \circ \widetilde{\sigma} g}{k \cdot k} \right) = 0$$

for any sufficiently large $k \in \mathbb{N}$

that

$$\frac{\Delta_V}{8\pi} \circ (-4\pi n \cdot \mathrm{id} + \Delta_V) f_{\check{X}_Z} = 0 \Longleftrightarrow \widetilde{\sigma} \left(f_{\check{X}_Z} \right) = 0.$$

Similarly, by using the expansion results of Appendix I, we have, by choosing $\hat{\sigma} g = 0 = \widetilde{\sigma} g$, the following power series expansion result for $\frac{\mathcal{S}\left({}_o\omega_{\#k} + \frac{i\bar{\partial}\partial \circ \sigma g}{k} \right)}{\Omega_{\mathbb{P}(E)}}$ in $\frac{1}{k}$:

$$\begin{aligned} \frac{\mathcal{S} \left({}_o\omega_{\#k} + \frac{i\bar{\partial}\partial \circ \sigma g}{k} \right)}{\Omega_{\mathbb{P}(E)}} &= \frac{n \cdot \Delta_M \sigma g}{k \cdot k} \\ &\quad + \frac{\frac{\Delta_V}{8\pi} \left[-(2\pi n \cdot \sigma g) \cdot (2\pi n \cdot \sigma g) + Q\left(\sigma g : \sigma g\right) \right]}{k \cdot k} \\ &\quad + \frac{\text{known terms}}{k \cdot k} + \text{ higher order terms} \end{aligned}$$

for any smooth section $g \in \Gamma\left(M : W\right)$. Thus we infer from Proposition IV.A and the vanishing of Futaki invariants:

$$\int_{\mathbb{P}(E)} \left(f_{\check{X}_Z} + \frac{\sigma g}{k} \right) \cdot \mathcal{S} \left({}_o\omega_{\#k} + \frac{i\bar{\partial}\partial \circ \sigma g}{k} \right) = 0$$

for any sufficiently large $k \in \mathbb{N}$

that

$$\Delta_M \, \sigma \left(f_{\check{X}_Z} \right) = 0 \Longleftrightarrow \sigma \left(f_{\check{X}_Z} \right) \in \Gamma\left(M : W\right) \text{ is a harmonic section.}$$

Now we consider the isometry condition of X_Z:

$$\mathcal{L}_{X_Z} \omega_M = 0 \Longleftrightarrow \mathcal{L}_{\check{X}_Z} \omega_M = 0.$$

Since $\Delta_M \, \sigma \left(f_{\check{X}_Z} \right) = 0$ there exists a corresponding element $Y_{\check{X}_Z} \in \mathfrak{g}_E$ such that

$$\mathcal{L}_{Y_{\check{X}_Z}} \left(\frac{i \cdot F_{A_{L^*}}}{2\pi} \right) = i\bar{\partial}\partial \circ \sigma \left(f_{\check{X}_Z} \right).$$

(Note that $Y_{\check{X}_Z} \in \mathfrak{g}_E$ is uniquely determined modulo the compact Lie subalgebra $\mathfrak{k}_E$ of $\mathfrak{g}_E$.) Let $\check{\mathbf{X}}_Z \equiv \left(-Y_{\check{X}_Z} + \check{X}_Z\right)$. Then we have

$$\begin{aligned}\mathcal{L}_{\check{\mathbf{X}}_Z}\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right) &= -\mathcal{L}_{Y_{\check{X}_Z}}\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right) + \mathcal{L}_{\check{X}_Z}\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right)\\ &= i\bar{\partial}\partial\circ\hat{\sigma}\left(f_{\check{X}_Z}\right).\end{aligned}$$

We claim that $\Delta_M\hat{\sigma}\left(f_{\check{X}_Z}\right) = 0$. To see this we note that

$$\mathcal{L}_{\check{\mathbf{X}}_Z}\omega_M = -\mathcal{L}_{Y_{\check{X}_Z}}\omega_M + \mathcal{L}_{\check{X}_Z}\omega_M = 0 + 0 = 0$$

and thence

$$\begin{aligned}&\frac{\Delta_M\hat{\sigma}\left(f_{\check{X}_Z}\right)}{2}\cdot\Omega_{\mathbb{P}(E)}\\ &= \frac{\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right)^{(-1+n)}}{(-1+n)!}\wedge i\bar{\partial}\partial\circ\hat{\sigma}\left(f_{\check{X}_Z}\right)\wedge\frac{\omega_M^{(-1+m)}}{(-1+m)!}\\ &= \frac{\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right)^{(-1+n)}}{(-1+n)!}\wedge\mathcal{L}_{\check{\mathbf{X}}_Z}\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right)\wedge\frac{\omega_M^{(-1+m)}}{(-1+m)!}\\ &= \mathcal{L}_{\check{\mathbf{X}}_Z}\left[\frac{\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right)^{n}}{n!}\wedge\frac{\omega_M^{(-1+m)}}{(-1+m)!}\right].\end{aligned}$$

However, by the Einstein-Hermitian condition of A on E over M, we have

$$\begin{aligned}&\frac{\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right)^{n}}{n!}\wedge\frac{\omega_M^{(-1+m)}}{(-1+m)!}\\ &= \frac{\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right)^{(-1+n)}}{(-1+n)!}\wedge\Pi_{\mathcal{C}_M}\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right)\wedge\frac{\omega_M^{(-1+m)}}{(-1+m)!}\\ &= -\frac{\left[\Lambda_M\mathrm{trace}\left(\frac{i\cdot F_A}{2\pi}\right)\right]}{n}\cdot\frac{\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right)^{(-1+n)}}{(-1+n)!}\wedge\frac{\omega_M^{m}}{m!}.\end{aligned}$$

Thus

$$\begin{aligned}&\frac{\Delta_M\hat{\sigma}\left(f_{\check{X}_Z}\right)}{2}\cdot\Omega_{\mathbb{P}(E)}\\&=-\frac{\left[\Lambda_M\mathrm{trace}\left(\frac{i\cdot F_A}{2\pi}\right)\right]}{n}\cdot\mathcal{L}_{\check{\mathbf{X}}_Z}\left[\frac{\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right)^{(-1+n)}}{(-1+n)!}\wedge\frac{\omega_M^m}{m!}\right]=0\end{aligned}$$

because

$$\begin{aligned}&\mathcal{L}_{\check{\mathbf{X}}_Z}\left[\frac{\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right)^{(-1+n)}}{(-1+n)!}\wedge\frac{\omega_M^m}{m!}\right]\\&=\frac{\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right)^{(-2+n)}}{(-2+n)!}\wedge\mathcal{L}_{\check{\mathbf{X}}_Z}\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right)\wedge\frac{\omega_M^m}{m!}\\&=\frac{\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right)^{(-2+n)}}{(-2+n)!}\wedge i\bar{\partial}\partial\circ\hat{\sigma}\left(f_{\check{X}_Z}\right)\wedge\frac{\omega_M^m}{m!}\end{aligned}$$

naturally vanishes. Hence $\Delta_M\hat{\sigma}\left(f_{\check{X}_Z}\right)=0$, as claimed, and so $\hat{\sigma}\left(f_{\check{X}_Z}\right)=0$. That is : $\frac{i\cdot F_{A_{L^*}}}{2\pi}$ is invariant under the action of $\check{\mathbf{X}}_Z=-Y_{\check{X}_Z}+\check{X}_Z$.

q.e.d.

Appendix III. Large k behavior of $\mathbf{L}_N$

In this section we will investigate, for each large $N\in\mathbb{N}$, the behavior of the 4^{th} order elliptic linear partial differential operator $\mathbf{L}_N$ as the parameter k goes to infinity. Here the 4^{th} order (elliptic) linear partial differential operator $\mathbf{L}_N$, depending on the parameter k, acting on $\psi\in\Gamma_o\left(\mathbb{P}(E):\mathbb{R}\right)$ is defined as follows:

$$\begin{aligned}\mathbf{L}_N(\psi) = -\check{c}_k \cdot {}_A\mathbf{L}_N(\psi) + \frac{\frac{i\bar{\partial}\partial}{2\pi}\left(\frac{{}_A\mathbf{L}_N(\psi)}{\mathbf{R}\left({}_N\omega_{\#k}\right)}\right) \wedge \frac{{}_N\omega_{\#k}^{(-2+m+n)}}{(-2+m+n)!}}{k^m \cdot \Omega_{\mathbb{P}(E)}} \\ + \frac{\frac{i\cdot\bar{\partial}\partial \log \mathbf{R}\left({}_N\omega_{\#k}\right)}{2\pi} \wedge i\bar{\partial}\partial\psi \wedge \frac{{}_N\omega_{\#k}^{(-3+m+n)}}{(-3+m+n)!}}{k^m \cdot \Omega_{\mathbb{P}(E)}} + {}_B\mathbf{L}_N(\psi) \\ + \left(\frac{n \cdot \tau_{\mathbf{N}_W} \circ \sigma(\psi)}{k} + \frac{\tau_{\mathbf{N}_{\mathcal{V}_M}} \circ \hat{\sigma}(\psi)}{k \cdot k}\right).\end{aligned}$$

in which the 2^{nd} order linear partial differential operators ${}_A\mathbf{L}_N$ and ${}_B\mathbf{L}_N$ (both without the 0^{th} order parts), acting on $\psi \in \Gamma_o(\mathbb{P}(E) : \mathbb{R})$, are respectively defined as follows:

$${}_A\mathbf{L}_N(\psi) \equiv \frac{i\bar{\partial}\partial\psi \wedge \frac{{}_N\omega_{\#k}^{(-2+m+n)}}{(-2+m+n)!}}{k^m \cdot \Omega_{\mathbb{P}(E)}}$$

and

$$\begin{aligned}&{}_B\mathbf{L}_N(\psi) \\ &\equiv \frac{i\bar{\partial}\partial\psi \wedge \left[n \cdot \frac{i\cdot F_{A_{L^*}}}{2\pi} + \check{\pi}^*\text{trace}\left(\frac{i\cdot F_A}{2\pi}\right) + \check{\pi}^*\text{trace}\left(\frac{i\cdot F_{\omega_M}}{2\pi}\right)\right] \wedge \frac{{}_N\omega_{\#k}^{(-3+m+n)}}{(-3+m+n)!}}{k^m \cdot \Omega_{\mathbb{P}(E)}}.\end{aligned}$$

Note that

$$\begin{aligned}\mathbf{R}\left({}_N\omega_{\#k}\right) &= \frac{{}_o\omega_{\#k}^{(-1+m+n)}}{k^m \cdot \check{\omega}^{(-1+m+n)}} + \frac{R_1}{k} + \frac{R_2}{k \cdot k} + \text{ higher order terms} \\ &= \frac{\Omega_{\mathbb{P}(E)}}{\Omega_{\mathbb{P}(E)}} + \frac{\mathbf{R}_1}{k} + \frac{\mathbf{R}_2}{k \cdot k} + \text{ higher order terms}\end{aligned}$$

in which

$$\mathbf{R}_1 = \frac{\left[-\Lambda_M \text{trace}\left(\frac{i\cdot F_A}{2\pi}\right)\right]}{n} + \frac{\Delta_M \phi_0}{2} + \frac{\Delta_V \sigma(\phi_1)}{2}$$

and

$$\mathbf{R}_2 = \frac{\frac{\Pi_{\mathcal{C}_M}\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right) \wedge \Pi_{\mathcal{C}_M}\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right)}{2!} \wedge \frac{\omega_M^{(-2+m)}}{(-2+m)!}}{\Omega_M} + R_2$$

are smooth $\mathbb{R}$-valued functions, independent of the parameter k, on $\mathbb{P}(E)$. (More expansion results for $\mathbf{R}\left({}_N\omega_{\#k}\right)$ can be found in Appendix I.) We define a symmetric operator $\mathbf{Q}_V(\bullet:\bullet)$ as follows:

$$\mathbf{Q}_V(f:g) \equiv \frac{i\bar{\partial}\partial f \wedge i\bar{\partial}\partial g \wedge \frac{\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right)^{(-3+n)}}{(-3+n)!} \wedge \frac{\omega_M^m}{m!}}{\Omega_{\mathbb{P}(E)}}$$

for any pair $(f:g)$ of smooth $\mathbb{R}$-valued functions on $\mathbb{P}(E)$. In particular we have

$$\mathbf{Q}_V(f:g) = Q(f:g)$$

when both f and g are smooth sections of W over M. It should be noted that $\mathbf{Q}_V(f:g)$ only depends on the fiber-directional differentiation of f and g:

$$\mathbf{Q}_V(f:g) = \mathbf{Q}_V(\sigma f + \widetilde{\sigma} f : \sigma g + \widetilde{\sigma} g).$$

Let ${}_E\mathbf{L}_N$ denote the 2^{nd} order linear partial differential operator (without the 0^{th} order part) acting on $\psi \in \Gamma_o(\mathbb{P}(E):\mathbb{R})$ defined as follows:

$${}_E\mathbf{L}_N(\psi) \equiv \frac{\frac{i\cdot\bar{\partial}\partial\log\mathbf{R}\left({}_N\omega_{\#k}\right)}{2\pi} \wedge i\bar{\partial}\partial\psi \wedge \frac{{}_N\omega_{\#k}^{(-3+m+n)}}{(-3+m+n)!}}{k^m\cdot\Omega_{\mathbb{P}(E)}}.$$

Then ${}_E\mathbf{L}_N$ depends on $\frac{1}{k}$ in the polynomial manner and we have

$$\begin{aligned}
{}_E\mathbf{L}_N(\psi) &= \frac{\mathbf{Q}_V\left(\frac{\Delta_V\sigma\phi_1}{4\pi}:\psi\right)}{k} + \frac{{}_E\mathbf{L}_{N:2}(\psi)}{k\cdot k} \\
&\quad + \text{those terms of } {}_E\mathbf{L}_N(\psi) \text{ carrying higher order powers of } \frac{1}{k} \\
&= \frac{n\cdot\mathbf{Q}_V(\sigma\phi_1:\psi)}{k} + \frac{{}_E\mathbf{L}_{N:2}(\psi)}{k\cdot k} \\
&\quad + \text{those terms of } {}_E\mathbf{L}_N(\psi) \text{ carrying higher order powers of } \frac{1}{k}
\end{aligned}$$

in which

$$
\begin{aligned}
&{}_{E}\mathbf{L}_{N:2}\left(\psi\right)\\
&=\frac{\frac{i\bar{\partial}\partial\mathbf{R}_{1}}{2\pi}\wedge i\bar{\partial}\partial\psi\wedge\frac{\left(\frac{i\cdot F_{A_{L^{*}}}}{2\pi}\right)^{(-2+n)}}{(-2+n)!}\wedge\frac{\omega_{M}^{(-1+m)}}{(-1+m)!}}{\Omega_{\mathbb{P}(E)}}\\
&\quad+\frac{\frac{i\bar{\partial}\partial\mathbf{R}_{1}}{2\pi}\wedge i\bar{\partial}\partial\psi\wedge i\bar{\partial}\partial\phi_{1}\wedge\frac{\left(\frac{i\cdot F_{A_{L^{*}}}}{2\pi}\right)^{(-4+n)}}{(-4+n)!}\wedge\frac{\omega_{M}^{m}}{m!}}{\Omega_{\mathbb{P}(E)}}\\
&\quad+\frac{\Delta_{M}\phi_{0}}{2}\cdot\mathbf{Q}_{V}\left(\frac{\Delta_{V}\sigma\phi_{1}}{4\pi}:\psi\right)+\frac{\mathbf{Q}_{V}}{2\pi}\left(-\frac{\mathbf{R}_{1}\cdot\mathbf{R}_{1}}{2}+\mathbf{R}_{2}:\psi\right).
\end{aligned}
$$

Thus

$$
\begin{aligned}
\mathbf{L}_{N}\left(\psi\right)&=-\check{c}_{k}\cdot\ {}_{A}\mathbf{L}_{N}\left(\psi\right)+\frac{\frac{i\bar{\partial}\partial\ {}_{A}\mathbf{L}_{N}(\psi)}{2\pi}\wedge\frac{{}_{N}\omega_{\#k}^{(-2+m+n)}}{(-2+m+n)!}}{k^{m}\cdot\Omega_{\mathbb{P}(E)}}+\ {}_{B}\mathbf{L}_{N}\left(\psi\right)\\
&\quad+\frac{n\cdot\mathbf{Q}_{V}\left(\sigma\phi_{1}:\psi\right)}{k}+\frac{\frac{i\bar{\partial}\partial}{2\pi}\left[-\mathbf{R}_{1}\cdot\ {}_{A}\mathbf{L}_{N}\left(\psi\right)\right]\wedge\frac{{}_{N}\omega_{\#k}^{(-2+m+n)}}{(-2+m+n)!}}{k\cdot k^{m}\cdot\Omega_{\mathbb{P}(E)}}\\
&\quad+\frac{n\cdot\tau_{\mathbf{N}_{W}}\circ\sigma\left(\psi\right)}{k}+\frac{\tau_{\mathbf{N}\nu_{M}}\circ\hat{\sigma}\left(\psi\right)}{k\cdot k}\\
&\quad+\frac{\frac{i\bar{\partial}\partial}{2\pi}\left[\left(-\mathbf{R}_{2}+\mathbf{R}_{1}\cdot\mathbf{R}_{1}\right)\cdot\ {}_{A}\mathbf{L}_{N}\left(\psi\right)\right]\wedge\frac{{}_{N}\omega_{\#k}^{(-2+m+n)}}{(-2+m+n)!}}{k\cdot k\cdot k^{m}\cdot\Omega_{\mathbb{P}(E)}}\\
&\quad+\frac{{}_{E}\mathbf{L}_{N:2}\left(\psi\right)}{k\cdot k}\\
&\quad+\text{those terms of }\mathbf{L}_{N}\left(\psi\right)\text{ carrying}\\
&\qquad\qquad\text{higher order powers of }\frac{1}{k}\text{ intrinsically}\\
&=-\check{c}_{k}\cdot\ {}_{A}\mathbf{L}_{N}\left(\psi\right)+\frac{{}_{A}\mathbf{L}_{N}}{2\pi}\circ\ {}_{A}\mathbf{L}_{N}\left(\psi\right)+\ {}_{B}\mathbf{L}_{N}\left(\psi\right)\\
&\quad+\frac{n\cdot\mathbf{Q}_{V}\left(\sigma\phi_{1}:\psi\right)}{k}+\frac{\frac{{}_{A}\mathbf{L}_{N}}{2\pi}\left[-\mathbf{R}_{1}\cdot\ {}_{A}\mathbf{L}_{N}\left(\psi\right)\right]}{k}\\
&\quad+\frac{n\cdot\tau_{\mathbf{N}_{W}}\circ\sigma\left(\psi\right)}{k}+\frac{\tau_{\mathbf{N}\nu_{M}}\circ\hat{\sigma}\left(\psi\right)}{k\cdot k}\\
&\quad+\frac{\frac{{}_{A}\mathbf{L}_{N}}{2\pi}\left[\left(-\mathbf{R}_{2}+\mathbf{R}_{1}\cdot\mathbf{R}_{1}\right)\cdot\ {}_{A}\mathbf{L}_{N}\left(\psi\right)\right]}{k\cdot k}+\frac{{}_{E}\mathbf{L}_{N:2}\left(\psi\right)}{k\cdot k}\\
&\quad+\text{those terms of }\mathbf{L}_{N}\left(\psi\right)\text{ carrying}\\
&\qquad\qquad\text{higher order powers of }\frac{1}{k}\text{ intrinsically.}
\end{aligned}
$$

Now we note that the 2^{nd} order linear partial differential operators ${}_{A}\mathbf{L}_{N}$ and ${}_{B}\mathbf{L}_{N}$ both depend on $\frac{1}{k}$ in the polynomial manner. Actually we have

$$\begin{aligned}{}_{A}\mathbf{L}_{N}\bullet = & \frac{\Delta_{V}\bullet}{2}+\frac{\frac{\Delta_{M}\bullet}{2}+\frac{\left[-\Lambda_{M}\text{trace}\left(\frac{i\cdot F_{A}}{2\pi}\right)\right]}{n}\cdot\frac{\Delta_{V}\bullet}{2}}{k}\\ & +\frac{\mathbf{Q}_{V}\left(\sigma\phi_{1}:\bullet\right)}{k}+\frac{\frac{\Delta_{M}\phi_{0}}{2}\cdot\frac{\Delta_{V}\bullet}{2}}{k}+\frac{{}_{A}\mathbf{L}_{N:2}\bullet}{k\cdot k}\\ & +\text{those terms of }{}_{A}\mathbf{L}_{N}\bullet\text{ carrying higher order powers of }\frac{1}{k}\end{aligned}$$

and

$$\begin{aligned}{}_{B}\mathbf{L}_{N}\left(\psi\right) = & n\cdot\left(-2+n\right)\cdot\frac{\Delta_{V}\psi}{2}\\ & +n\cdot\left(-1+n\right)\cdot\frac{\frac{\Delta_{M}\psi}{2}+\frac{\left[-\Lambda_{M}\text{trace}\left(\frac{i\cdot F_{A}}{2\pi}\right)\right]}{n}\cdot\frac{\Delta_{V}\psi}{2}}{k}\\ & +\frac{n\cdot\left(-3+n\right)\cdot\mathbf{Q}_{V}\left(\sigma\phi_{1}:\psi\right)}{k}+\frac{n\cdot\left(-2+n\right)\cdot\frac{\Delta_{M}\phi_{0}}{2}\cdot\frac{\Delta_{V}\psi}{2}}{k}\\ & +\frac{\left[\Lambda_{M}\text{trace}\left(\frac{i\cdot F_{A}}{2\pi}\right)+\Lambda_{M}\text{trace}\left(\frac{i\cdot F_{\omega_{M}}}{2\pi}\right)\right]\cdot\frac{\Delta_{V}\psi}{2}}{k}\\ & +\frac{{}_{B}\mathbf{L}_{N:2}\left(\psi\right)}{k\cdot k}+\text{those terms of }{}_{B}\mathbf{L}_{N}\left(\psi\right)\text{ carrying}\\ & \qquad\qquad\text{higher order powers of }\frac{1}{k}\end{aligned}$$

in which ${}_{A}\mathbf{L}_{N:2}$ and ${}_{B}\mathbf{L}_{N:2}$ are 2^{nd} order linear partial differential operators (both without the 0^{th} order parts), independent of the parameter k, acting on $\Gamma_{o}\left(\mathbb{P}(E):\mathbb{R}\right)$. Let ${}_{C\text{-}A}\mathbf{L}$ denote the 2^{nd} order linear partial differential operator acting on $\Gamma_{o}\left(\mathbb{P}(E):\mathbb{R}\right)$ defined as follows:

$${}_{C\text{-}A}\mathbf{L}\bullet\equiv\frac{\Delta_{M}\bullet}{2}+\frac{\left[-\Lambda_{M}\text{trace}\left(\frac{i\cdot F_{A}}{2\pi}\right)\right]}{n}\cdot\frac{\Delta_{V}\bullet}{2}+\mathbf{Q}_{V}\left(\sigma\phi_{1}:\bullet\right)+\frac{\Delta_{M}\phi_{0}}{2}\cdot\frac{\Delta_{V}\bullet}{2}$$

so that

$$\begin{aligned}{}_{A}\mathbf{L}_{N}\bullet = & \frac{\Delta_{V}\bullet}{2}+\frac{{}_{C\text{-}A}\mathbf{L}\bullet}{k}+\frac{{}_{A}\mathbf{L}_{N:2}\bullet}{k\cdot k}\\ & +\text{those terms of }{}_{A}\mathbf{L}_{N}\bullet\text{ carrying higher order powers of }\frac{1}{k}.\end{aligned}$$

Then (by substituting

$$\bullet = [-\mathbf{R}_1 \cdot {}_A\mathbf{L}_N(\psi)] \quad \text{and} \quad \bullet = [(-\mathbf{R}_2 + \mathbf{R}_1 \cdot \mathbf{R}_1) \cdot {}_A\mathbf{L}_N(\psi)]$$

respectively into the last formula for ${}_A\mathbf{L}_N\bullet$) we have

$$\begin{aligned}
\mathbf{L}_N(\psi) = {} & -\check{c}_k \cdot {}_A\mathbf{L}_N(\psi) + \frac{{}_A\mathbf{L}_N}{2\pi} \circ {}_A\mathbf{L}_N(\psi) + {}_B\mathbf{L}_N(\psi) \\
& + \frac{n \cdot \tau_{\mathbf{N}_W} \circ \sigma(\psi)}{k} + \frac{n \cdot \mathbf{Q}_V(\sigma\phi_1 : \psi)}{k} \\
& + \frac{-\frac{\Delta_V}{4\pi}[\mathbf{R}_1 \cdot {}_A\mathbf{L}_N(\psi)]}{k} + \frac{\tau_{\mathbf{N}_{\mathcal{V}_M}} \circ \hat{\sigma}(\psi)}{k \cdot k} \\
& + \frac{-\frac{{}_{C\text{-}A}\mathbf{L}}{2\pi}[\mathbf{R}_1 \cdot {}_A\mathbf{L}_N(\psi)]}{k \cdot k} + \frac{\frac{\Delta_V[(-\mathbf{R}_2 + \mathbf{R}_1 \cdot \mathbf{R}_1) \cdot {}_A\mathbf{L}_N(\psi)]}{4\pi}}{k \cdot k} \\
& + \frac{{}_E\mathbf{L}_{N:2}(\psi)}{k \cdot k} + \text{those terms of } \mathbf{L}_N(\psi) \text{ carrying} \\
& \qquad \text{higher order powers of } \frac{1}{k} \text{ intrinsically}
\end{aligned}$$

and thence (by substituting $\bullet = {}_A\mathbf{L}_N(\psi)$ into the last formula for ${}_A\mathbf{L}_N\bullet$)

$$\begin{aligned}
\mathbf{L}_N(\psi) = {} & -\check{c}_k \cdot {}_A\mathbf{L}_N(\psi) + \frac{\Delta_V[{}_A\mathbf{L}_N(\psi)]}{4\pi} + {}_B\mathbf{L}_N(\psi) + \frac{n \cdot \tau_{\mathbf{N}_W} \circ \sigma(\psi)}{k} \\
& + \frac{n \cdot \mathbf{Q}_V(\sigma\phi_1 : \psi)}{k} + \frac{\frac{{}_{C\text{-}A}\mathbf{L}}{2\pi} \circ {}_A\mathbf{L}_N(\psi)}{k} \\
& + \frac{-\frac{\Delta_V}{4\pi}\left[\mathbf{R}_1 \cdot \frac{\Delta_V\psi}{2}\right]}{k} + \frac{\tau_{\mathbf{N}_{\mathcal{V}_M}} \circ \hat{\sigma}(\psi)}{k \cdot k} \\
& + \frac{\frac{{}_A\mathbf{L}_{N:2}}{2\pi} \circ {}_A\mathbf{L}_N(\psi)}{k \cdot k} + \frac{-\frac{\Delta_V}{4\pi}[\mathbf{R}_1 \cdot {}_{C\text{-}A}\mathbf{L}_N(\psi)]}{k \cdot k} \\
& + \frac{-\frac{{}_{C\text{-}A}\mathbf{L}}{2\pi}\left[\mathbf{R}_1 \cdot \frac{\Delta_V\psi}{2}\right]}{k \cdot k} + \frac{\frac{\Delta_V[(-\mathbf{R}_2 + \mathbf{R}_1 \cdot \mathbf{R}_1) \cdot \Delta_V\psi]}{8\pi}}{k \cdot k} + \frac{{}_E\mathbf{L}_{N:2}(\psi)}{k \cdot k} \\
& + \text{those terms of } \mathbf{L}_N(\psi) \text{ carrying} \\
& \qquad \text{higher order powers of } \frac{1}{k} \text{ intrinsically} \\
= {} & -\check{c}_k \cdot {}_A\mathbf{L}_N(\psi) + \frac{\Delta_V \circ \Delta_V\psi}{8\pi} + {}_B\mathbf{L}_N(\psi) + \frac{n \cdot \tau_{\mathbf{N}_W} \circ \sigma\psi}{k} \\
& + \frac{n \cdot \mathbf{Q}_V(\sigma\phi_1 : \psi)}{k} + \frac{\frac{\Delta_V}{4\pi} \circ {}_{C\text{-}A}\mathbf{L}(\psi)}{k} + \frac{{}_{C\text{-}A}\mathbf{L} \circ \frac{\Delta_V}{4\pi}(\psi)}{k}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{-\frac{\Delta_V}{4\pi}\left[\mathbf{R}_1\cdot\frac{\Delta_V\psi}{2}\right]}{k}+\frac{\tau_{\mathbf{N}_{\mathcal{V}_M}}\circ\hat{\sigma}\psi}{k\cdot k}+\frac{\frac{\Delta_V}{4\pi}\circ\ {}_A\mathbf{L}_{N:2}(\psi)}{k\cdot k}\\
&+\frac{\frac{{}_{C\text{-}A}\mathbf{L}}{2\pi}\circ\ {}_{C\text{-}A}\mathbf{L}(\psi)}{k\cdot k}+\frac{\frac{{}_A\mathbf{L}_{N:2}}{4\pi}\circ\Delta_V(\psi)}{k\cdot k}+\frac{\frac{-\Delta_V}{4\pi}\left[\mathbf{R}_1\cdot\ {}_{C\text{-}A}\mathbf{L}(\psi)\right]}{k\cdot k}\\
&+\frac{-\frac{{}_{C\text{-}A}\mathbf{L}}{2\pi}\left[\mathbf{R}_1\cdot\frac{\Delta_V\psi}{2}\right]}{k\cdot k}+\frac{\frac{\Delta_V[(-\mathbf{R}_2+\mathbf{R}_1\cdot\mathbf{R}_1)\cdot\Delta_V\psi]}{8\pi}}{k\cdot k}+\frac{{}_E\mathbf{L}_{N:2}(\psi)}{k\cdot k}\\
&+\text{those terms of }\ \mathbf{L}_N(\psi)\ \ \text{carrying}\\
&\qquad\qquad\text{higher order powers of }\ \frac{1}{k}\ \ \text{intrinsically.}
\end{aligned}
$$

Hence by using the detailed formulae for ${}_A\mathbf{L}_N(\psi)$ and ${}_B\mathbf{L}_N(\psi)$ we have

$$
\begin{aligned}
\mathbf{L}_N(\psi)=&\frac{\Delta_V}{8\pi}\circ(-4\pi n\cdot\mathrm{id}+\Delta_V)\psi+\frac{\frac{\Delta_M\circ\Delta_V\psi}{8\pi}+\frac{\Delta_V\circ\Delta_M\psi}{8\pi}}{k}\\
&+\frac{n\cdot\tau_{\mathbf{N}_W}\circ\sigma\psi}{k}+\frac{{}_U\mathbf{L}_{N:1}(\psi)}{k}+\frac{\mathbf{L}_{N:2}(\psi)}{k\cdot k}\\
&+\text{those terms of }\ \mathbf{L}_N(\psi)\ \ \text{carrying}\\
&\qquad\qquad\text{higher order powers of }\ \frac{1}{k}\ \ \text{intrinsically}
\end{aligned}
$$

in which ${}_U\mathbf{L}_{N:1}$ and $\mathbf{L}_{N:2}$ are the 4^{th} order linear partial differential operators (both without the 0^{th} order parts), independent of the parameter k, acting on $\Gamma_o(\mathbb{P}(E):\mathbb{R})$ defined respectively as follows:

$$
\begin{aligned}
\frac{{}_U\mathbf{L}_{N:1}(\psi)}{k}=&\frac{\frac{\Delta_V}{4\pi}\left(-\frac{\Delta_V\sigma\phi_1}{2}\cdot\frac{\Delta_V\psi}{2}+\mathbf{Q}_V(\sigma\phi_1:\psi)\right)}{k}\\
&+\frac{\mathbf{Q}_V\left(\sigma\phi_1:\frac{(-4\pi n\cdot\mathrm{id}+\Delta_V)\psi}{4\pi}\right)}{k}\\
&+\frac{\left[-\Lambda_M\mathrm{trace}\left(\frac{i\cdot F_A}{2\pi}\right)\right]}{n}\cdot\frac{\Delta_V}{8\pi}\circ(-4\pi n\cdot\mathrm{id}+\Delta_V)\psi}{k}\\
&+\frac{\frac{\Delta_M\phi_0}{2}\cdot\frac{(-4\pi n\cdot\mathrm{id}+\Delta_V)}{4\pi}\circ\frac{\Delta_V\psi}{2}}{k}
\end{aligned}
$$

and

$$\begin{aligned}\frac{\mathbf{L}_{N:2}(\psi)}{k\cdot k}=&\frac{\tau_{\mathbf{N}\nu_M}\circ\hat{\sigma}\psi}{k\cdot k}+\frac{\frac{\Delta_V}{4\pi}\circ\ {}_A\mathbf{L}_{N:2}(\psi)}{k\cdot k}+\frac{\frac{{}_{C\text{-}A}\mathbf{L}}{2\pi}\circ\ {}_{C\text{-}A}\mathbf{L}(\psi)}{k\cdot k}\\&+\frac{\frac{{}_A\mathbf{L}_{N:2}}{4\pi}\circ\Delta_V\psi}{k\cdot k}+\frac{\frac{-\Delta_V}{4\pi}\left[\mathbf{R}_1\cdot\ {}_{C\text{-}A}\mathbf{L}(\psi)\right]}{k\cdot k}\\&+\frac{-\frac{{}_{C\text{-}A}\mathbf{L}}{2\pi}\left[\mathbf{R}_1\cdot\frac{\Delta_V\psi}{2}\right]}{k\cdot k}+\frac{\frac{\Delta_V[(-\mathbf{R}_2+\mathbf{R}_1\cdot\mathbf{R}_1)\cdot\Delta_V\psi]}{8\pi}}{k\cdot k}\\&+\frac{{}_E\mathbf{L}_{N:2}(\psi)}{k\cdot k}+\frac{-(-1+n)\cdot n\cdot\ {}_A\mathbf{L}_{N:2}(\psi)}{k\cdot k}+\frac{{}_B\mathbf{L}_{N:2}(\psi)}{k\cdot k}\\&+\frac{-\left[\Lambda_M\mathrm{trace}\left(\frac{i\cdot F_{\omega_M}}{2\pi}\right)\right]\cdot\ {}_{C\text{-}A}\mathbf{L}(\psi)}{k\cdot k}+\frac{-\check{c}_{k:2}\cdot\frac{\Delta_V\psi}{2}}{k\cdot k}.\end{aligned}$$

We are particularly interested in those parts of $\mathbf{L}_{N:2}$ acting nontrivially on $\Gamma_o(M:\mathbb{R})$. Note that

$${}_{C\text{-}A}\mathbf{L}\bullet=\frac{\Delta_M\bullet}{2}+\ {}_{C\text{-}A}\mathbf{L}^\star\bullet$$

in which the 2nd order linear partial differential operator

$${}_{C\text{-}A}\mathbf{L}^\star\bullet=\frac{\left[-\Lambda_M\mathrm{trace}\left(\frac{i\cdot F_A}{2\pi}\right)\right]}{n}\cdot\frac{\Delta_V\bullet}{2}+\mathbf{Q}_V(\sigma\phi_1:\bullet)+\frac{\Delta_M\phi_0}{2}\cdot\frac{\Delta_V\bullet}{2}$$

has trivial action on $\Gamma(M:\mathbb{R})$:

$${}_{C\text{-}A}\mathbf{L}^\star(f)=0$$

for any $f\in\Gamma(M:\mathbb{R})$. Similarly for the 2nd order linear partial differential operators ${}_A\mathbf{L}_{N:2}$ and ${}_B\mathbf{L}_{N:2}$ we have

$$\begin{aligned}{}_A\mathbf{L}_{N:2}\bullet=&\frac{i\cdot\left(\bar{\partial}\partial\right)_M\bullet\ \wedge\Pi_{\mathcal{C}_M}\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right)\wedge\frac{\omega_M^{(-2+m)}}{(-2+m)!}}{\Omega_M}+\frac{\Delta_V\sigma\phi_1}{2}\cdot\frac{\Delta_M\bullet}{2}\\&+\frac{i\cdot\left(\bar{\partial}\partial\right)_M\bullet\ \wedge i\bar{\partial}\partial\phi_0\wedge\frac{\omega_M^{(-2+m)}}{(-2+m)!}}{\Omega_M}+\ {}_A\mathbf{L}^\star_{N:2}\bullet\end{aligned}$$

and

$$
\begin{aligned}
{}_{B}\mathbf{L}_{N:2}\bullet = n\cdot n\cdot \frac{i\cdot\left(\bar{\partial}\partial\right)_{M}\bullet\ \wedge\Pi_{\mathcal{C}_{M}}\left(\frac{i\cdot F_{A_{L^{*}}}}{2\pi}\right)\wedge\frac{\omega_{M}^{(-2+m)}}{(-2+m)!}}{\Omega_{M}}\\
+ n\cdot(-2+n)\cdot\frac{\Delta_{V}\sigma\phi_{1}}{2}\cdot\frac{\Delta_{M}\bullet}{2}\\
+ n\cdot(-1+n)\cdot\frac{i\cdot\left(\bar{\partial}\partial\right)_{M}\bullet\ \wedge i\bar{\partial}\partial\phi_{0}\wedge\frac{\omega_{M}^{(-2+m)}}{(-2+m)!}}{\Omega_{M}}\\
+\frac{i\left(\bar{\partial}\partial\right)_{M}\bullet\ \wedge\left[\mathrm{trace}\left(\frac{i\cdot F_{A}}{2\pi}\right)+\mathrm{trace}\left(\frac{i\cdot F_{\omega_{M}}}{2\pi}\right)\right]\wedge\frac{\omega_{M}^{(-2+m)}}{(-2+m)!}}{\Omega_{M}}\\
+\ {}_{B}\mathbf{L}_{N:2}^{\star}\bullet
\end{aligned}
$$

in which both ${}_{A}\mathbf{L}_{N:2}^{\star}$ and ${}_{B}\mathbf{L}_{N:2}^{\star}$ act trivially on $\Gamma\left(M:\mathbb{R}\right)$:

$$
{}_{A}\mathbf{L}_{N:2}^{\star}\left(f\right)=0={}_{B}\mathbf{L}_{N:2}^{\star}\left(f\right)
$$

for any $f\in\Gamma\left(M:\mathbb{R}\right)$. Besides we note that

$$
\begin{aligned}
{}_{E}\mathbf{L}_{N:2}\bullet &= \frac{\Delta_{V}\mathbf{R}_{1}}{4\pi}\cdot\frac{\Delta_{M}\bullet}{2}+\ {}_{E}\mathbf{L}_{N:2}^{\star}\bullet\\
&= n\cdot\frac{\Delta_{V}\sigma\phi_{1}}{2}\cdot\frac{\Delta_{M}\bullet}{2}+\ {}_{E}\mathbf{L}_{N:2}^{\star}\bullet
\end{aligned}
$$

in which ${}_{E}\mathbf{L}_{N:2}^{\star}$ acts trivially on $\Gamma\left(M:\mathbb{R}\right)$. Thus

$$
\begin{aligned}
\mathbf{L}_{N:2}\left(\psi\right)=\frac{\Delta_{M}\circ\Delta_{M}\psi}{8\pi}+\frac{i\left(\bar{\partial}\partial\right)_{M}\psi\ \wedge\mathrm{trace}\left(\frac{i\cdot F_{\omega_{M}}}{2\pi}\right)\wedge\frac{\omega_{M}^{(-2+m)}}{(-2+m)!}}{\Omega_{M}}\\
+\left[-\Lambda_{M}\mathrm{trace}\left(\frac{i\cdot F_{\omega_{M}}}{2\pi}\right)\right]\cdot\frac{\Delta_{M}\psi}{2}+\tau_{\mathbf{N}\mathcal{V}_{M}}\circ\hat{\sigma}\psi+\ {}_{U}\mathbf{L}_{N:2}\left(\psi\right)
\end{aligned}
$$

in which

$$
\begin{aligned}
&{}_{U}\mathbf{L}_{N:2}(\psi)\\
&=\frac{\Delta_V}{4\pi}\circ{}_{A}\mathbf{L}_{N:2}(\psi)+\frac{\Delta_M\circ{}_{C\text{-}A}\mathbf{L}^{\star}(\psi)}{4\pi}+\frac{{}_{C\text{-}A}\mathbf{L}^{\star}\circ\Delta_M\psi}{4\pi}\\
&\quad+\frac{{}_{C\text{-}A}\mathbf{L}^{\star}\circ{}_{C\text{-}A}\mathbf{L}^{\star}(\psi)}{2\pi}+\frac{{}_{A}\mathbf{L}_{N:2}}{4\pi}\circ\Delta_V\psi+\frac{-\Delta_V}{4\pi}\left[\mathbf{R}_1\cdot{}_{C\text{-}A}\mathbf{L}(\psi)\right]\\
&\quad+\frac{-{}_{C\text{-}A}\mathbf{L}}{2\pi}\left[\mathbf{R}_1\cdot\frac{\Delta_V\psi}{2}\right]+\frac{\Delta_V\left[(-\mathbf{R}_2+\mathbf{R}_1\cdot\mathbf{R}_1)\cdot\Delta_V\psi\right]}{8\pi}\\
&\quad+\frac{i\left(\bar{\partial}\partial\right)_M\psi\ \wedge\left[n\cdot\Pi_{\mathcal{C}_M}\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right)+\operatorname{trace}\left(\frac{i\cdot F_A}{2\pi}\right)\right]\wedge\frac{\omega_M^{(-2+m)}}{(-2+m)!}}{\Omega_M}\\
&\quad+\left[-(-1+n)\cdot n\cdot{}_{A}\mathbf{L}^{\star}_{N:2}(\psi)+{}_{E}\mathbf{L}^{\star}_{N:2}(\psi)+{}_{B}\mathbf{L}^{\star}_{N:2}(\psi)\right]\\
&\quad+\left[-\Lambda_M\operatorname{trace}\left(\frac{i\cdot F_{\omega_M}}{2\pi}\right)\right]\cdot{}_{C\text{-}A}\mathbf{L}^{\star}(\psi)+\left(-\check{c}_{k:2}\cdot\frac{\Delta_V\psi}{2}\right).
\end{aligned}
$$

Let $\mathbf{P}_{\#k}$ denote the 4^{th} order elliptic linear partial differential operator, depending on the parameter k, acting on $\psi\in\Gamma_o(\mathbb{P}(E):\mathbb{R})$ defined as follows:

$$
\begin{aligned}
&\mathbf{P}_{\#k}(\psi)\\
&=\frac{\Delta_V}{8\pi}\circ(-4\pi n\cdot\mathrm{id}+\Delta_V)\psi+\frac{\frac{\Delta_M\circ\Delta_V\psi}{8\pi}+\frac{\Delta_V\circ\Delta_M\psi}{8\pi}}{k}+\frac{n\cdot\tau_{\mathbf{N}_W}\circ\sigma\psi}{k}\\
&\quad+\frac{\frac{\Delta_M\circ\Delta_M\psi}{8\pi}+\left[-\Lambda_M\operatorname{trace}\left(\frac{i\cdot F_{\omega_M}}{2\pi}\right)\right]\cdot\frac{\Delta_M\psi}{2}}{k\cdot k}\\
&\quad+\frac{\frac{i\left(\bar{\partial}\partial\right)_M\psi\wedge\left[\operatorname{trace}\left(\frac{i\cdot F_{\omega_M}}{2\pi}\right)\right]\wedge\frac{\omega_M^{(-2+m)}}{(-2+m)!}}{\Omega_M}}{k\cdot k}+\frac{\tau_{\mathbf{N}_{\mathcal{V}_M}}\circ\hat{\sigma}\psi}{k\cdot k}.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\mathbf{L}_N(\psi)=\mathbf{P}_{\#k}(\psi)+\frac{{}_{U}\mathbf{L}_{N:1}(\psi)}{k}+\frac{{}_{U}\mathbf{L}_{N:2}(\psi)}{k\cdot k}\\
+\text{those terms of }\ \mathbf{L}_N(\psi)\ \ \text{carrying}\\
\text{higher order powers of }\ \frac{1}{k}\ \ \text{intrinsically.}
\end{aligned}
$$

We will show that $\mathbf{L}_N$ is dominated by $\mathbf{P}_{\#k}$ as the parameter k is sufficiently large. Before doing so let us look at $\mathbf{P}_{\#k}$ more closely. Note that, for any $\psi\in\Gamma_o(\mathbb{P}(E):\mathbb{R})$, we have

$$
\mathbf{P}_{\#k}(\psi)=\mathbf{P}_{\#k}(\hat{\sigma}\psi)+\mathbf{P}_{\#k}(\sigma\psi)+\mathbf{P}_{\#k}(\widetilde{\sigma}\psi)
$$

and thence

$$\begin{aligned}
&\mathbf{P}_{\#k}\left(\psi\right)\\
&=\frac{\Delta_V}{8\pi}\circ\left(-4\pi n\cdot\mathrm{id}+\Delta_V\right)\widetilde{\sigma}\psi+\frac{\frac{\Delta_M\circ\Delta_V\widetilde{\sigma}\psi}{8\pi}+\frac{\Delta_V\circ\Delta_M\widetilde{\sigma}\psi}{8\pi}}{k}+\frac{\frac{\Delta_M\circ\Delta_M\widetilde{\sigma}\psi}{8\pi}}{k\cdot k}\\
&\quad+\frac{n\cdot\Delta_M\sigma\psi}{k}+\frac{n\cdot\tau_{\mathbf{N}_W}\circ\sigma\psi}{k}+\frac{\frac{\Delta_M\circ\Delta_M\sigma\psi}{8\pi}}{k\cdot k}+\frac{\left(\frac{\mathcal{V}_M}{\Omega_M}+\tau_{\mathbf{N}_{\mathcal{V}_M}}\right)\hat{\sigma}\psi}{k\cdot k}\\
&\quad+\frac{\left[-\Lambda_M\mathrm{trace}\left(\frac{i\cdot F_{\omega_M}}{2\pi}\right)\right]\cdot\frac{\Delta_M\widetilde{\sigma}\psi}{2}+\frac{i\left(\bar{\partial}\partial\right)_M\widetilde{\sigma}\psi\wedge\left[\mathrm{trace}\left(\frac{i\cdot F_{\omega_M}}{2\pi}\right)\right]\wedge\frac{\omega_M^{(-2+m)}}{(-2+m)!}}{\Omega_M}}{k\cdot k}\\
&\quad+\frac{\left[-\Lambda_M\mathrm{trace}\left(\frac{i\cdot F_{\omega_M}}{2\pi}\right)\right]\cdot\frac{\Delta_M\sigma\psi}{2}+\frac{i\left(\bar{\partial}\partial\right)_M\sigma\psi\wedge\left[\mathrm{trace}\left(\frac{i\cdot F_{\omega_M}}{2\pi}\right)\right]\wedge\frac{\omega_M^{(-2+m)}}{(-2+m)!}}{\Omega_M}}{k\cdot k}
\end{aligned}$$

in which $\mathcal{V}_M$ is the infinitesimal deformation operator for the constant scalar curvature equation on $(M:\omega_M)$. Now we set

$$\begin{aligned}
{}_o\mathbf{P}_{\#k}\left(\psi\right)\equiv&\frac{\Delta_V}{8\pi}\circ\left(-4\pi n\cdot\mathrm{id}+\Delta_V\right)\widetilde{\sigma}\psi+\frac{\frac{\Delta_M\circ\Delta_V\widetilde{\sigma}\psi}{8\pi}+\frac{\Delta_V\circ\Delta_M\widetilde{\sigma}\psi}{8\pi}}{k}\\
&+\frac{\frac{\Delta_M\circ\Delta_M\widetilde{\sigma}\psi}{8\pi}}{k\cdot k}+\frac{n\cdot\Delta_M\sigma\psi}{k}+\frac{n\cdot\tau_{\mathbf{N}_W}\circ\sigma\psi}{k}\\
&+\frac{\frac{\Delta_M\circ\Delta_M\sigma\psi}{8\pi}}{k\cdot k}+\frac{\left(\frac{\mathcal{V}_M}{\Omega_M}+\tau_{\mathbf{N}_{\mathcal{V}_M}}\right)\hat{\sigma}\psi}{k\cdot k}
\end{aligned}$$

so that

$$\begin{aligned}
-{}_o\mathbf{P}_{\#k}\left(\psi\right)+\mathbf{P}_{\#k}\left(\psi\right)=&\frac{\frac{\left(i\left(\bar{\partial}\partial\right)_M\widetilde{\sigma}\psi+i\left(\bar{\partial}\partial\right)_M\sigma\psi\right)\wedge\left[\mathrm{trace}\left(\frac{i\cdot F_{\omega_M}}{2\pi}\right)\right]\wedge\frac{\omega_M^{(-2+m)}}{(-2+m)!}}{\Omega_M}}{k\cdot k}\\
&+\frac{\left[-\Lambda_M\mathrm{trace}\left(\frac{i\cdot F_{\omega_M}}{2\pi}\right)\right]\cdot\frac{\Delta_M\widetilde{\sigma}\psi+\Delta_M\sigma\psi}{2}}{k\cdot k}.
\end{aligned}$$

Let $\check{C}>0$ denote a sufficiently large constant independent of k (and $N\in\mathbb{N}$). Then it is obvious that

$$\begin{aligned}
\left\|-{}_o\mathbf{P}_{\#k}\left(\psi\right)+\mathbf{P}_{\#k}\left(\psi\right)\right\|_{L^2(\mathbb{P}(E):\check{\omega})}\leq&\check{C}\cdot\frac{\left\|\left(\Delta_M+\Delta_V\right)\widetilde{\sigma}\psi\right\|_{L^2(\mathbb{P}(E):\check{\omega})}}{k\cdot k}\\
&+\check{C}\cdot\frac{\left\|\left(\Delta_V+\Delta_M\right)\sigma\psi\right\|_{L^2(\mathbb{P}(E):\check{\omega})}}{k\cdot k}.
\end{aligned}$$

Let $\langle : \rangle$ denote the inner product on $L^2(\mathbb{P}(E) : \check{\omega})$ defined by the Hermitian form (metric) $\check{\omega}$ on $\mathbb{P}(E)$:

$$\langle f : g \rangle = \int_{\mathbb{P}(E)} f \cdot g \cdot \Omega_{\mathbb{P}(E)}$$

$\forall (f : g) \in L^2(\mathbb{P}(E) : \check{\omega}) \times L^2(\mathbb{P}(E) : \check{\omega})$. Note that the decomposition

$$\Gamma_o(\mathbb{P}(E) : \mathbb{R}) = \hat{\sigma}\Gamma_o(\mathbb{P}(E) : \mathbb{R}) \oplus \sigma\Gamma_o(\mathbb{P}(E) : \mathbb{R}) \oplus \widetilde{\sigma}\Gamma_o(\mathbb{P}(E) : \mathbb{R})$$

of $\Gamma_o(\mathbb{P}(E) : \mathbb{R})$ is orthogonal with respect to this inner product $\langle : \rangle$ on $L^2(\mathbb{P}(E) : \check{\omega})$. Moreover, by Proposition IV.C, this orthogonal decomposition of $\Gamma_o(\mathbb{P}(E) : \mathbb{R})$ is preserved by Δ_M (and Δ_V). Now, by using the Stokes Theorem, it can be shown readily that

$$\begin{aligned}
&\check{C} \cdot \|\widetilde{\sigma} \circ {}_o\mathbf{P}_{\#k}(\psi)\|_{L^2(\mathbb{P}(E):\check{\omega})} \\
&\geq \left\| \left(\Delta_V + \frac{\Delta_M}{k} \right) \circ \left(\Delta_V + \frac{\Delta_M}{k} \right) \widetilde{\sigma}\psi \right\|_{L^2(\mathbb{P}(E):\check{\omega})}
\end{aligned}$$

and

$$\begin{aligned}
&\check{C} \cdot \|\sigma \circ {}_o\mathbf{P}_{\#k}(\psi)\|_{L^2(\mathbb{P}(E):\check{\omega})} \\
&\geq \left\| \frac{(\Delta_M + \mathrm{id})}{k} \circ \left(\Delta_V + \frac{\Delta_M}{k} \right) \sigma\psi \right\|_{L^2(\mathbb{P}(E):\check{\omega})}.
\end{aligned}$$

Remark. Note that, on each fiber $\mathbb{P}(\mathbb{C}^n)$ of $\check{\pi} : \mathbb{P}(E) \longrightarrow M$, the L^2 norm of $\Delta_V \widetilde{\sigma}\psi$ is always bounded by some universal multiple of the L^2 norm of $(-4\pi n \cdot \mathrm{id} + \Delta_V)\widetilde{\sigma}\psi$. Besides we always have $\Delta_V \sigma\psi = 4\pi n \cdot \sigma\psi$.

Similarly, by standard results of Partial Differential Equations or the Stokes Theorem, we have

$$\check{C} \cdot \|\hat{\sigma} \circ {}_o\mathbf{P}_{\#k}(\psi)\|_{L^2(\mathbb{P}(E):\check{\omega})} \geq \left\| \frac{(\Delta_M + \mathrm{id}) \circ (\Delta_M + \mathrm{id})\hat{\sigma}\psi}{k \cdot k} \right\|_{L^2(\mathbb{P}(E):\check{\omega})}.$$

Thus, by the Schwarz inequality, there exists a constant $C > 0$, independent of the parameter k, such that, when k is sufficiently large,

$$\begin{aligned}
C \cdot \|\mathbf{P}_{\#k}(\psi)\|_{L^2(\mathbb{P}(E):\check{\omega})} \geq &\left\| \left(\Delta_V + \frac{\Delta_M}{k} \right) \circ \left(\Delta_V + \frac{\Delta_M}{k} \right) \widetilde{\sigma}\psi \right\|_{L^2(\mathbb{P}(E):\check{\omega})} \\
&+ \left\| \frac{(\Delta_M + \mathrm{id})}{k} \circ \left(\Delta_V + \frac{\Delta_M}{k} \right) \sigma\psi \right\|_{L^2(\mathbb{P}(E):\check{\omega})} \\
&+ \left\| \frac{(\Delta_M + \mathrm{id}) \circ (\Delta_M + \mathrm{id})\hat{\sigma}\psi}{k \cdot k} \right\|_{L^2(\mathbb{P}(E):\check{\omega})}
\end{aligned}$$

is true for any $\psi \in \Gamma_o\left(\mathbb{P}(E):\mathbb{R}\right)$.

We can now use similar ideas to derive estimates for $\left\|\mathbf{L}_N\left(\psi\right)\right\|_{L^2(\mathbb{P}(E):\check{\omega})}$. Actually, by Corollary IV.A, we have

$$\begin{aligned}\frac{{}_U\mathbf{L}_{N:1}\left(\psi\right)}{k} &= \frac{{}_U\mathbf{L}_{N:1}\left(\sigma\psi\right)}{k} + \frac{{}_U\mathbf{L}_{N:1}\left(\widetilde{\sigma}\psi\right)}{k}\\ &= \frac{\frac{\Delta_V}{4\pi}\left(-\frac{\Delta_V\sigma\phi_1}{2}\cdot\frac{\Delta_V\sigma\psi}{2}+\mathbf{Q}_V\left(\sigma\phi_1:\sigma\psi\right)\right)}{k} + \frac{{}_U\mathbf{L}_{N:1}\left(\widetilde{\sigma}\psi\right)}{k}\\ &= \frac{\widetilde{\sigma}\circ{}_U\mathbf{L}_{N:1}\left(\sigma\psi\right)}{k} + \frac{{}_U\mathbf{L}_{N:1}\left(\widetilde{\sigma}\psi\right)}{k}\end{aligned}$$

in which

$$\begin{aligned}\left\|\frac{\widetilde{\sigma}\circ{}_U\mathbf{L}_{N:1}\left(\sigma\psi\right)}{k}\right\|_{L^2(\mathbb{P}(E):\check{\omega})} &\leq \check{C}\cdot\frac{\left\|\Delta_V\circ\left(\Delta_V+\mathrm{id}\right)\sigma\psi\right\|_{L^2(\mathbb{P}(E):\check{\omega})}}{k}\\ &= C_*\cdot\frac{\left\|\sigma\psi\right\|_{L^2(\mathbb{P}(E):\check{\omega})}}{k}\end{aligned}$$

while

$$\left\|\frac{{}_U\mathbf{L}_{N:1}\left(\widetilde{\sigma}\psi\right)}{k}\right\|_{L^2(\mathbb{P}(E):\check{\omega})} \leq \check{C}\cdot\frac{\left\|\Delta_V\circ\left(\Delta_V+\mathrm{id}\right)\widetilde{\sigma}\psi\right\|_{L^2(\mathbb{P}(E):\check{\omega})}}{k}.$$

Here the constant C_* is defined as $C_*\equiv\check{C}\cdot\left(4\pi n\cdot 4\pi n+4\pi n\right)$. On the other hand we have

$$\begin{aligned}\frac{{}_U\mathbf{L}_{N:2}\left(\psi\right)}{k\cdot k} &= \frac{{}_U\mathbf{L}_{N:2}\left(\hat{\sigma}\psi\right)}{k\cdot k} + \frac{{}_U\mathbf{L}_{N:2}\left(\sigma\psi+\widetilde{\sigma}\psi\right)}{k\cdot k}\\ &= \frac{\sigma\circ{}_U\mathbf{L}_{N:2}\left(\hat{\sigma}\psi\right)}{k\cdot k} + \frac{\widetilde{\sigma}\circ{}_U\mathbf{L}_{N:2}\left(\hat{\sigma}\psi\right)}{k\cdot k} + \frac{{}_U\mathbf{L}_{N:2}\left(\sigma\psi+\widetilde{\sigma}\psi\right)}{k\cdot k}\end{aligned}$$

because $\hat{\sigma}\circ{}_U\mathbf{L}_{N:2}\left(\hat{\sigma}\psi\right)=0$. (Actually we have

$$\begin{aligned}&{}_U\mathbf{L}_{N:2}\left(\hat{\sigma}\psi\right)\\ &= \frac{\Delta_V}{4\pi}\circ{}_A\mathbf{L}_{N:2}\left(\hat{\sigma}\psi\right) + \frac{-\Delta_V}{4\pi}\left[\mathbf{R}_1\cdot\frac{\Delta_M\hat{\sigma}\psi}{2}\right]\\ &\quad+\frac{i\left(\bar{\partial}\partial\right)_M\hat{\sigma}\psi\ \wedge\left[n\cdot\Pi_{\mathcal{C}_M}\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right)+\mathrm{trace}\left(\frac{i\cdot F_A}{2\pi}\right)\right]\wedge\frac{\omega_M^{(-2+m)}}{(-2+m)!}}{\Omega_M}.\end{aligned}$$

However the integral of the term

$$\frac{i\left(\bar{\partial}\partial\right)_M\psi\ \wedge\left[n\cdot\Pi_{\mathcal{C}_M}\left(\frac{i\cdot F_{A_{L^*}}}{2\pi}\right)+\mathrm{trace}\left(\frac{i\cdot F_A}{2\pi}\right)\right]\wedge\frac{\omega_M^{(-2+m)}}{(-2+m)!}}{\Omega_M}$$

along each fiber $\mathbb{P}(\mathbb{C}^n)$ of $\check{\pi} : \mathbb{P}(E) \longrightarrow M$ simply vanishes.) Note that

$$\left\| \frac{\sigma \circ {}_U\mathbf{L}_{N:2}(\hat{\sigma}\psi)}{k \cdot k} \right\|_{L^2(\mathbb{P}(E):\check{\omega})} + \left\| \frac{\widetilde{\sigma} \circ {}_U\mathbf{L}_{N:2}(\hat{\sigma}\psi)}{k \cdot k} \right\|_{L^2(\mathbb{P}(E):\check{\omega})}$$
$$\leq \check{C} \cdot \frac{\|\Delta_M \hat{\sigma}\psi\|_{L^2(\mathbb{P}(E):\check{\omega})}}{k \cdot k}$$

while

$$\left\| \frac{{}_U\mathbf{L}_{N:2}(\sigma\psi + \widetilde{\sigma}\psi)}{k \cdot k} \right\|_{L^2(\mathbb{P}(E):\check{\omega})}$$
$$\leq \check{C} \cdot \frac{\|(\Delta_V + \mathrm{id}) \circ (\Delta_M + \Delta_V)(\sigma\psi + \widetilde{\sigma}\psi)\|_{L^2(\mathbb{P}(E):\check{\omega})}}{k \cdot k}.$$

Let $\hat{C} \equiv 2\check{C} \cdot \left(2\check{C} \cdot C_*\right)$ and $\widetilde{C} = 2\check{C} \cdot \left[\check{C} \cdot (2\check{C} + \hat{C})\right]$. We claim that, when the parameter k is sufficiently large, the following estimate:

$$2\check{C} \cdot \|\widetilde{\sigma} \circ \mathbf{L}_N(\psi)\|_{L^2(\mathbb{P}(E):\check{\omega})} + \hat{C} \cdot \|\sigma \circ \mathbf{L}_N(\psi)\|_{L^2(\mathbb{P}(E):\check{\omega})}$$
$$+ \widetilde{C} \cdot \|\hat{\sigma} \circ \mathbf{L}_N(\psi)\|_{L^2(\mathbb{P}(E):\check{\omega})}$$
$$\geq \left\| \left(\Delta_V + \frac{\Delta_M}{k}\right) \circ \left(\Delta_V + \frac{\Delta_M}{k}\right) \widetilde{\sigma}\psi \right\|_{L^2(\mathbb{P}(E):\check{\omega})}$$
$$+ \left\| \frac{(\Delta_M + \mathrm{id})}{k} \circ \left(\Delta_V + \frac{\Delta_M}{k}\right) \sigma\psi \right\|_{L^2(\mathbb{P}(E):\check{\omega})}$$
$$+ \left\| \frac{(\Delta_M + \mathrm{id}) \circ (\Delta_M + \mathrm{id})\,\hat{\sigma}\psi}{k \cdot k} \right\|_{L^2(\mathbb{P}(E):\check{\omega})}$$

is valid for any $\psi \in \Gamma_o(\mathbb{P}(E) : \mathbb{R})$. To see this we simply note that

$$2\check{C} \cdot \left\| \frac{\widetilde{\sigma} \circ {}_U\mathbf{L}_{N:1}(\sigma\psi)}{k} \right\|_{L^2(\mathbb{P}(E):\check{\omega})} \leq \frac{\hat{C}}{2} \cdot \|\sigma \circ {}_o\mathbf{P}_{\#k}(\psi)\|_{L^2(\mathbb{P}(E):\check{\omega})}$$

while

$$2\check{C} \cdot \left\| \frac{\widetilde{\sigma} \circ {}_U\mathbf{L}_{N:2}(\hat{\sigma}\psi)}{k \cdot k} \right\|_{L^2(\mathbb{P}(E):\check{\omega})} + \hat{C} \cdot \left\| \frac{\sigma \circ {}_U\mathbf{L}_{N:2}(\hat{\sigma}\psi)}{k \cdot k} \right\|_{L^2(\mathbb{P}(E):\check{\omega})}$$
$$\leq \frac{\widetilde{C}}{2} \cdot \|\hat{\sigma} \circ {}_o\mathbf{P}_{\#k}(\psi)\|_{L^2(\mathbb{P}(E):\check{\omega})}.$$

With these estimates it can be shown readily, by using the Schwarz inequality, that our claim is true. Proposition V.B then follows immediately.

To prove Corollary V.B we simply apply $(\Delta_V + \Delta_M)$ iteratively to $\mathbf{L}_N(\psi)$. Since the orthogonal decomposition

$$\Gamma_o(\mathbb{P}(E):\mathbb{R}) = \hat{\sigma}\Gamma_o(\mathbb{P}(E):\mathbb{R}) \oplus \sigma\Gamma_o(\mathbb{P}(E):\mathbb{R}) \oplus \widetilde{\sigma}\Gamma_o(\mathbb{P}(E):\mathbb{R})$$

of $\Gamma_o(\mathbb{P}(E):\mathbb{R})$ is preserved by Δ_M (and Δ_V) we can establish Corollary V.B easily through the same method as demonstrated above.

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