J. DIFFERENTIAL GEOMETRY
59 (2001) 301-351

EQUIVALENCES OF REAL SUBMANIFOLDS IN COMPLEX SPACE

M.S. BAOUENDI, LINDA PREISS ROTHSCHILD
& DMITRI ZAITSEV

Abstract

We show that for any real-analytic submanifold M in $\mathbb{C}^N$ there is a proper real-analytic subvariety $V \subset M$ such that for any $p \in M \setminus V$, any real-analytic submanifold M' in $\mathbb{C}^N$, and any $p' \in M'$, the germs (M,p) and (M',p') of the submanifolds M and M' at p and p' respectively are formally equivalent if and only if they are biholomorphically equivalent. As an application, for $p \in M \setminus V$, the problem of biholomorphic equivalence of the germs (M,p) and (M',p') is reduced to that of solving a system of polynomial equations. More general results for k-equivalences are also stated and proved.

1. Introduction

This paper studies equivalences between real-analytic submanifolds in complex vector spaces. If M and M' are two such submanifolds with $p \in M$ and $p' \in M'$, the germs (M,p) and (M',p') of M and M' at p and p' respectively are said to be *biholomorphically equivalent* if there exists a germ of a biholomorphism at p sending (M,p) onto (M',p'). The problem of determining when two such germs are biholomorphically equivalent has been extensively studied for many years. It was already observed by Poincaré [21] that there exist infinitely many germs of real hypersurfaces in $\mathbb{C}^2$ that are pairwise biholomorphically inequivalent. For Levi-nondegenerate hypersurfaces in $\mathbb{C}^2$, E. Cartan [11] constructed a complete system of analytic invariants. Tanaka [22] and Chern and

The first and second authors are partially supported by National Science Foundation grant DMS 98-01258.

Received July 13, 2001.

Moser [13] obtained a deep generalization of Cartan's work in higher dimensional complex spaces. The present paper gives an appproach to the equivalence problem that is valid for any pair of real-analytic submanifolds without any nondegeneracy assumptions, whenever the reference point (in one of them) is outside an explicitly described exceptional real-analytic subvariety. This subvariety is defined in terms of integer biholomorphic invariants obtained by taking Lie brackets of $(0,1)$ vector fields (see §2). Our results reduce the problem of the existence of a biholomorphic equivalence between germs (M,p) and (M',p') to that of the existence of a biholomorphic map sending (M,p) into (M',p') up to a finite order k. We call such a map a *k-equivalence* (see below for precise definitions). The existence of a k-equivalence H is reduced to the solvability of a system of polynomial equations in finitely many coefficients of the Taylor series of H. Reducing biholomorphic equivalence to k-equivalence is not possible for an arbitrary reference point $p \in M$ since there exist pairs of germs of real-analytic submanifolds (M,p) and (M',p') which are k-equivalent for each k (and even formally equivalent) but biholomorphically inequivalent (see the reference to the work of Moser and Webster [18] mentioned below). We consider such points $p \in M$ as exceptional and prove that, for nonexceptional points, the existence of k-equivalences for each k is necessary and sufficient for the existence of a biholomorphic equivalence.

We now give precise definitions needed for the statement of our main results. A *formal map* $H : (\mathbb{C}^N, p) \to (\mathbb{C}^N, p')$, with p and p' in $\mathbb{C}^N$, is a $\mathbb{C}^N$-valued formal power series

$$H(Z) = p' + \sum_{|\alpha| \geq 1} a_\alpha (Z-p)^\alpha, \qquad a_\alpha \in \mathbb{C}^N, \quad Z = (Z_1, \dots, Z_N).$$

The map H is *invertible* if there exists a formal map $H^{-1} : (\mathbb{C}^N, p') \to (\mathbb{C}^N, p)$ such that $H(H^{-1}(Z)) \equiv H^{-1}(H(Z)) \equiv Z$ (which is equivalent to the nonvanishing of the Jacobian of H at p). Suppose M and M' are real-analytic submanifolds in $\mathbb{C}^N$ of the same dimension given by real-analytic (vector valued) local defining functions $\rho(Z, \overline{Z})$ and $\rho'(Z, \overline{Z})$ near $p \in M$ and $p' \in M'$ respectively. A formal invertible map H as above is called a *formal equivalence* between the germs (M,p) and (M',p') if

$$\rho'\big(H(Z(x)), \overline{H(Z(x))}\big) \equiv 0$$

in the sense of formal power series in x for some (and hence for any) real-analytic parametrization $x \mapsto Z(x)$ of M near $p = Z(0)$. If, in addition, H is convergent, we say that H is a *biholomorphic equivalence* between (M,p) and (M',p'). More generally, for any integer $k>1$, we call a formal invertible mapping $H:(\mathbb{C}^N,p)\to(\mathbb{C}^N,p')$ a *k-equivalence* between (M,p) and (M',p') if

$$\rho'\big(H(Z(x)),\overline{H(Z(x))}\big)=O(|x|^k);$$

see Lemma 4.2 for equivalent definitions. Hence a formal invertible map H is a formal equivalence between (M,p) and (M',p') if and only if it is a k-equivalence for every $k>1$.

If M and M' are as above, we shall say that (M,p) and (M',p') are *formally equivalent* (resp. *biholomorphically equivalent* or *k-equivalent*) if there exists a formal equivalence (resp. biholomorphic equivalence or k-equivalence) between (M,p) and (M',p'). As mentioned above, our main result shows, in particular, that for "most" points $p\in M$, the notions of formal and biholomorphic equivalences coincide. More precisely, we prove the following.

Theorem 1.1. *Let $M\subset\mathbb{C}^N$ be a connected real-analytic submanifold. Then there exists a closed proper real-analytic subvariety $V\subset M$ such that for every $p\in M\backslash V$, every real-analytic submanifold $M'\subset\mathbb{C}^N$, every $p'\in M'$, and every integer $\kappa>1$, there exists an integer $k>1$ such that if H is a k-equivalence between (M,p) and (M',p') then there exists a biholomorphic equivalence $\widehat{H}$ between (M,p) and (M',p') with $\widehat{H}(Z)=H(Z)+O(|Z-p|^\kappa)$.*

In fact, a real-analytic subvariety $V\subset M$, for which Theorem 1.1 holds will be explicitly described in §2 below. An immediate consequence of Theorem 1.1 is the following corollary.

Corollary 1.2. *Let $M\subset\mathbb{C}^N$ be a connected real-analytic submanifold, $V\subset M$ the real-analytic subvariety given by Theorem 1.1, and $p\in M\setminus V$. Then for every real-analytic submanifold $M'\subset\mathbb{C}^N$, and every $p'\in M'$, the following are equivalent:*

(i) *(M,p) and (M',p') are k-equivalent for all $k>1$.*

(ii) *(M,p) and (M',p') are formally equivalent.*

(iii) *(M,p) and (M',p') are biholomorphically equivalent.*

It should be noted that in general, the integer k in Theorem 1.1 must be chosen bigger than κ. For example, if $M=M'=\{Z=(z,w)\colon \operatorname{Im} w=|z|^2\}\subset\mathbb{C}^2$, one can easily check that the map $H(z,w):=$

$(z, w+w^3)$ is a 4-equivalence between $(M, 0)$ and $(M', 0)$. However, there is no biholomorphic equivalence $\widehat{H}$ between $(M, 0)$ and $(M', 0)$ such that $\widehat{H}(Z) - H(Z) = O(|Z|^4)$. Indeed, it is known that any biholomorphic equivalence $\widehat{H}$ between $(M, 0)$ and $(M', 0)$ that differs from the identity (and hence from H) by $O(|Z|^3)$ must be the identity (see [13]), and hence necessarily $\widehat{H}(z, w) - H(z, w) = -w^3$. This proves that for this example if $\kappa = 4$, one cannot take $k = 4$.

It follows from M. Artin's celebrated approximation theorems, [1, 2], that systems of analytic equations which have solutions of arbitrarily high (but finite) order necessarily have convergent solutions. However, Artin's general theory cannot be directly applied to the case of mappings between real submanifolds, because the equations are real-analytic, whereas the solutions are complex-analytic and hence defined over a different ground field. The main part of the proof of Theorem 1.1 is devoted to the derivation of an equivalent system of real-analytic equations in appropriate jet spaces whose real-analytic solutions may be used to construct biholomorphic equivalences (see Theorem 11.1). The Artin and Wavrik [24] theorems can then be applied to the latter system of equations to obtain a real-analytic solution and hence the conclusion of Theorem 1.1.

The problem of formal versus biholomorphic equivalence has been studied by a number of mathematicians. It has been known since the fundamental work of Chern and Moser [13] that if M and M' are real-analytic hypersurfaces in $\mathbb{C}^N$ which are Levi nondegenerate at p and p' respectively, then the germs (M, p) and (M', p') are formally equivalent if and only if they are biholomorphically equivalent. It should be mentioned here that Theorem 1.1 and its corollary are new even in the case of a hypersurface. In fact, we believe that the equivalence of (i) with (ii) and (iii) in Corollary 1.2 is new even for Levi nondegenerate hypersurfaces. (See also Remark 5.2 below.) Although it had been known (e.g., in dynamical systems, celestial mechanics, and partial differential equations) that there exist pairs of structures which are formally equivalent (in an appropriate sense) but not biholomorphically equivalent, to our knowledge the first examples of pairs (M, p) and (M', p') of germs of real-analytic submanifolds in $\mathbb{C}^N$ which are formally equivalent but not biholomorphically equivalent are due to Moser and Webster [18]. The examples in that paper consist of real-analytic surfaces M and M' in $\mathbb{C}^2$ with isolated "complex tangent" at p and p' respectively. (It is fairly easy to prove Theorem 1.1 above in the case of real-analytic surfaces in $\mathbb{C}^2$, since outside a real-analytic set such a surface is either totally real or complex.) The work [18] also contains positive results for surfaces in $\mathbb{C}^2$, i.e., cases in which formal and biholomorphic equivalence coincide at some complex tangent points. We should also mention further related work by X. Gong [16] as well as recent work by Beloshapka [9] and Coffman [14].

In other recent work of the first two authors jointly with Ebenfelt [7], [5] and [6], it has been shown that there are many classes of pairs (M, p) and (M', p'), where M and M' are real-analytic generic submanifolds of $\mathbb{C}^N$, for which any formal equivalence is necessarily convergent (see also Corollary 10.3). In particular it follows that the notions of formal equivalence and biholomorphic equivalence for such pairs coincide. The present paper treats the more general case where nonconvergent formal equivalences may exist between (M, p) and (M', p'). Given such a formal equivalence H, Theorem 1.1 implies the existence of a possibly different biholomorphic equivalence that coincides with H up to an arbitrarily high preassigned order. For instance, any formal power series in one variable of the form $\sum_{j=1}^{\infty} a_j z^j$, $a_1 \neq 0$, $a_j \in \mathbb{R}$, may be regarded as a formal equivalence between $(\mathbb{R}, 0)$ (considered as a germ of a real submanifold in $\mathbb{C}$) and itself . By truncating this power series to any order, one obtains a biholomorphic equivalence which agrees with the formal equivalence to that order.

The organization of the paper is as follows. In §2 through §5 the variety V is constructed, and a local description of M near a point $p \in M \setminus V$ is given. The proof of Theorem 1.1 is then reduced to the case where M and M' are generic submanifolds which are finitely nondegenerate at p and p' respectively. In §6 through §13 we prove Theorem 1.1 in that case. For the proof, we first obtain a universal parametrization of k-equivalences between (M, p) and (M', p') in terms of their jets. The construction of this parametrization is in the spirit of that given in [7] for formal equivalences between hypersurfaces, and in [25] and [5] for formal equivalences between generic submanifolds of higher codimension. However, the approach used here is somewhat different and deals with more general situations. The main difference is due to the fact that the parametrization is obtained in terms of finite order jets *along a certain submanifold* rather than in terms of *single* jets at a point p. From this parametrization we obtain a system of real-analytic equations in the product of the above submanifold and the space of jets, whose exact solutions correspond to biholomorphic equivalences and whose approximate solutions of finite order correspond to k-equivalences. As mentioned above, for this system we apply approximation theorems due to Artin [1], [2] and a variant due to Wavrik [24]. The proof is then completed in §13. We conclude the paper in §14 by giving a version of Corollary 1.2 for CR maps between CR submanifolds.

The authors wish to thank Leonard Lipshitz for pointing out to us the article [24], as well as his related joint work with Denef [15].

2. Construction of the real subvariety V

For the remainder of this paper M and M' will always denote connected real-analytic submanifolds of $\mathbb{C}^N$ of the same dimension. For any $p \in M$, we shall define three nonnegative integers: $r_1(p)$, the excess codimension of M at p, $r_2(p)$, the degeneracy of M at p, and $r_3(p)$, the orbit codimension of M at p. We shall show that these integers reach their minima outside proper real-analytic subvarieties $V_1, V_2, V_3 \subset M$ respectively and shall prove Theorem 1.1 for $V := V_1 \cup V_2 \cup V_3$.

Let M be as above, d be the codimension of M in $\mathbb{C}^N$, and $p_0 \in M$ be fixed. Recall that a (vector valued) local defining function $\rho = (\rho^1, \dots, \rho^d)$ near p_0 is a collection of real valued real-analytic functions defined in a neighborhood of p_0 in $\mathbb{C}^N$ such that $M = \{Z : \rho(Z, \overline{Z}) = 0\}$ near p_0 and $d\rho^1 \wedge \ldots \wedge d\rho^d \neq 0$. We associate to M a complex submanifold $\mathcal{M} \subset \mathbb{C}^{2N}$ (called the *complexification of* M) in a neighborhood of $(p_0, \overline{p}_0)$ in $\mathbb{C}^N \times \mathbb{C}^N$ defined by $\mathcal{M} := \{(Z, \zeta) : \rho(Z, \zeta) = 0\}$. Observe that a point $p \in \mathbb{C}^N$ is in M if and only if $(p, \overline{p}) \in \mathcal{M}$. We also note that, if $\widetilde{\rho} = (\widetilde{\rho}^1, \dots, \widetilde{\rho}^d)$ is another local defining function of M near p_0, then $\widetilde{\rho}(Z, \overline{Z}) = a(Z, \overline{Z})\rho(Z, \overline{Z})$ in a neighborhood of p_0 in $\mathbb{C}^N$, where $a(Z, \overline{Z})$ is a $d \times d$ invertible matrix, whose entries are real valued, real-analytic functions in a neighborhood of p_0.

2.1 CR points of M

For $p \in M$ near p_0, the *excess codimension* $r_1(p)$ of M at p is defined by

$$r_1(p) := d - \dim\ \mathrm{span}_{\mathbb{C}} \left\{ \rho^j_Z(p, \overline{p}) : 1 \le j \le d \right\}. \tag{2.1}$$

Here $\rho^j_Z = (\partial\rho^j/\partial Z_1, \dots, \partial\rho^j/\partial Z_N) \in \mathbb{C}^N$ denotes the complex gradient of ρ^j with respect to $Z = (Z_1, \dots, Z_N)$. It is easy to see that $r_1(p)$ is independent of the choice of the defining function ρ and of the holomorphic coordinates Z. A point $p_0 \in M$ is called a CR *point* (or M is called CR at p_0) if the mapping $p \mapsto r_1(p)$ is constant for p in a neighborhood of p_0 in M. The submanifold M is called CR if it is CR at all its points and hence, by connectedness, $r_1 := r_1(p)$ is constant on M. If in addition $r_1 = 0$, then M is said to be *generic* in $\mathbb{C}^N$. We set

$$V_1 := \{p \in M : M \text{ is not CR at } p\}. \tag{2.2}$$

It is easy to see that the function $r_1(p)$ is upper-semicontinuous on M and, since M is connected, the complement $M \setminus V_1$ agrees with the set of all points in M, where $r_1(p)$ reaches its minimum. The following lemma is a consequence of the fact that $r_1(p)$ is upper-semicontinuous for the Zariski topology on M and its proof is left to the reader.

Lemma 2.1. *The subset $V_1 \subset M$ defined by (2.2) is proper and real-analytic.*

2.2 The $(0,1)$ vector fields on M

In order to define the functions $r_2(p)$ and $r_3(p)$, we shall need the notion of $(0,1)$ vector fields on a real submanifold $M \subset \mathbb{C}^N$. For M not necessarily CR and $U \subset M$ an open subset, we call a real-analytic vector field of the form $L = \sum_{j=1}^N a_j(Z,\overline{Z})\frac{\partial}{\partial \overline{Z}_j}$, with $a_j(Z,\overline{Z})$ real-analytic functions on U, a $(0,1)$ *vector field* on U if

$$(L\rho)(Z,\overline{Z}) \equiv 0, \tag{2.3}$$

for any local defining function $\rho(Z,\overline{Z})$ of M. For $p \in M$, we denote by $\mathcal{T}^{0,1}_{M,p}$ the vector space of all germs at p of $(0,1)$ vector fields on M and by $\mathcal{T}^{0,1}_M$ the corresponding sheaf on M whose stalk at any p is $\mathcal{T}^{0,1}_{M,p}$. It is easy to see that $\mathcal{T}^{0,1}_M$ is independent of the choice of $\rho(Z,\overline{Z})$. Observe that $\mathcal{T}^{0,1}_{M,p}$ is closed under commutation and hence is a Lie algebra. If L is a $(0,1)$ vector field on an open set U of M, i.e., $L \in \mathcal{T}^{0,1}_M(U)$, denote by $L_p \in \mathcal{T}^{0,1}_{M,p}$ the germ of L at p for $p \in U$. If M is a CR submanifold in $\mathbb{C}^N$, the above definition of $(0,1)$ vector fields on M coincides with the standard one and in this case the sheaf $\mathcal{T}^{0,1}_M$ is the sheaf of sections of a complex vector bundle on M, called the CR *bundle* of M.

The following consequence of the coherence theorem of Oka-Cartan (see [20] and [12], Proposition 4) will be essential for the proof that the subvariety $V \subset M$ is real-analytic.

Lemma 2.2. *Given $p_0 \in M$, there exists a neighborhood $U \subset M$ of p_0, an integer $m > 0$ and $(0,1)$ vector fields $L_1, \dots, L_m \in \mathcal{T}^{0,1}_M(U)$ such* *that for any $p \in U$, any germ $\mathcal{L} \in \mathcal{T}^{0,1}_{M,p}$ can be written in the form*

$$\mathcal{L} = g_1 L_{1,p} + \cdots + g_m L_{m,p}$$

with $g_1, \ldots, g_m$ germs at p of real-analytic functions on M.

Proof. For $p \in M$ denote by $\mathcal{A}_{M,p}$ the ring of germs at p of real-analytic functions on M. For p near p_0, we can think of an element $L = \sum_{j=1}^N a_j \frac{\partial}{\partial \overline{Z}_j}$ in $\mathcal{T}^{0,1}_{M,p}$ as an N-tuple $(a_1, \ldots, a_N) \in \mathcal{A}^N_{M,p}$ satisfying the condition in (2.3). Hence the subsheaf $\mathcal{T}^{0,1}_M \subset \mathcal{A}^N_M$ coincides with the sheaf of relations

$$\sum_{j=1}^N a_j \left(\frac{\partial \rho^r}{\partial \overline{Z}_j} \right)_p = 0, \quad r = 1, \ldots, d.$$

Since the sheaf $\mathcal{A}_M$ is coherent by the theorem of Oka-Cartan, it follows that $\mathcal{T}^{0,1}_M$ is locally finitely generated over $\mathcal{A}_M$ which proves the lemma. q.e.d.

2.3 Degeneracy and orbit codimension

As above let $p_0 \in M$ be fixed and $\rho(Z, \overline{Z})$ be a local defining function of M near p_0. For $p \in M$ near p_0, we define a vector subspace $E(p) \subset \mathbb{C}^N$ by

$$E(p) := \operatorname{span}_{\mathbb{C}} \Big\{ (\mathcal{L}_1 \ldots \mathcal{L}_s \rho^j_Z)(p, \overline{p}) : 1 \le j \le d; 0 \le s < \infty; \mathcal{L}_1, \ldots, \mathcal{L}_s \in \mathcal{T}^{0,1}_{M,p} \Big\}. \tag{2.4}$$

As before $\rho^j_Z(Z, \overline{Z}) \in \mathbb{C}^N$ denotes the complex gradient of ρ^j with respect to Z. We leave it to the reader to check that $E(p)$ is independent of the choice of the defining function ρ and its dimension is independent of the choice of holomorphic coordinates Z near p. We call the number

$$r_2(p) := N - \dim_{\mathbb{C}} E(p), \tag{2.5}$$

the *degeneracy* of M at p. We say that M is of *minimum degeneracy* at p_0 if p_0 is a local minimum of the function $p \mapsto r_2(p)$. If $r_2(p_0) = 0$, we say that M is *finitely nondegenerate* at p_0. We say that M is l-*nondegenerate* at p_0 if M is finitely nondegenerate at p_0 and l is the smallest integer such that the vectors $(\mathcal{L}_1 \ldots \mathcal{L}_s \rho^j_Z)(p_0, \overline{p}_0)$ span $\mathbb{C}^N$ for

$0 \le s \le l$ and $1 \le j \le d$. When M is generic, the latter definition coincides with the one given in [4]. (See also, e.g., [8].)

We denote by $\mathbb{C}T_pM := \mathbb{C} \otimes_{\mathbb{R}} T_pM$ the complexified tangent space of M at p and by $\overline{\mathcal{T}^{0,1}_{M,p}}$ the complex conjugates of elements in $\mathcal{T}^{0,1}_{M,p}$. Let $\mathfrak{g}_M(p)$ be the complex vector subspace of $\mathbb{C}T_pM$ generated by the values at p of the germs of vector fields in $\mathcal{T}^{0,1}_{M,p}$, $\overline{\mathcal{T}^{0,1}_{M,p}}$ and all their commutators. We call

$$r_3(p) := \dim_{\mathbb{R}} M - \dim_{\mathbb{C}} \mathfrak{g}_M(p) \tag{2.6}$$

the *orbit codimension* of M at p and say that M is of *minimum orbit codimension* at p_0 if p_0 is a local minimum of the function $p \mapsto r_3(p)$. The use of this terminology will be justified in §2.4. We say that M is of *finite type* at p_0, if $r_3(p_0) = 0$. When M is generic, this definition coincides with the finite type condition of Kohn [17] and Bloom-Graham [10].

The following result can be obtained by applying Lemma 2.2, using the fact that $\mathcal{T}^{0,1}_{M,p}$ is a Lie algebra and by induction on $s \ge 0$ in (2.4). We leave the details to the reader.

Lemma 2.3. *For $M \subset \mathbb{C}^N$, $p_0 \in M$ and $\rho(Z, \overline{Z})$ as above, there exist an open neighborhood U of p_0 in M, an integer $m > 0$, and $L_1, \ldots, L_m \in \mathcal{T}^{0,1}_M(U)$ such that for every $p \in U$, one has*

$$E(p) = \operatorname{span}_{\mathbb{C}} \left\{ (L^\alpha \rho^j_Z)(p, \overline{p}) : \alpha \in \mathbb{Z}^m_+; 1 \le j \le d \right\}, \tag{2.7}$$

where $L^\alpha := L_1^{\alpha_1} \ldots L_m^{\alpha_m}$, $\alpha = (\alpha_1, \ldots, \alpha_m)$, and

$$\mathfrak{g}_M(p) = \operatorname{span}_{\mathbb{C}} \Big\{ [X_{i_1}, \ldots, [X_{i_{r-1}}, X_{i_r}] \ldots](p) : r \ge 1; \\ X_{i_j} \in \{L_1, \ldots, L_m, \overline{L}_1, \ldots, \overline{L}_m\} \Big\}. \tag{2.8}$$

Proposition 2.4. *Let $M \subset \mathbb{C}^N$ be a connected real-analytic submanifold. Then the subsets $V_2, V_3 \subset M$ given by*

$$V_2 := \{p \in M : M \text{ is not of minimum degeneracy at } p\} \tag{2.9}$$

and

$$V_3 := \{p \in M : M \text{ is not of minimum orbit codimension at } p\} \tag{2.10}$$

are proper and real-analytic.

Proof. Define $r_i := \min_{p\in M} r_i(p)$, $i = 2, 3$, where $r_2(p)$ and $r_3(p)$ are the integer valued functions defined by (2.5) and (2.6) respectively. Given $p_0 \in M$, choose U and $L_1, \ldots, L_m \in \mathcal{T}^{0,1}_M(U)$ as in Lemma 2.3. Now consider the set of vector valued real-analytic functions $L^\alpha \rho^j_Z$ (as in (2.7)) defined in U. For each subset of $N - r_2$ functions in this set, we take all possible $(N - r_2) \times (N - r_2)$ minors extracted from their components. Then by Lemma 2.3, the set $V_2 \cap U$ is given by the vanishing of all such minors. Since $p_0 \in M$ is arbitrary, $V_2 \subset M$ is a real-analytic subvariety. To show that $V_3 \subset M$ is also real-analytic, we repeat the above argument for the set of vector valued real-analytic functions $p \mapsto [X_{i_1}, \ldots, [X_{i_{r-1}}, X_{i_r}] \ldots](p)$ (as in (2.8)). Both subsets $V_2, V_3 \subset M$ are proper by the choices of r_2 and r_3. q.e.d.

Remark 2.5. For $M \subset \mathbb{C}^N$ a connected real-analytic submanifold, it follows from the definition of $r_1(p)$ and from the proof of Proposition 2.4 that the sets $\{p \in M : r_i(p) \le s\}$, $i = 1, 2, 3$, are also real-analytic subvarieties of M for any integer $s \ge 0$. In particular, for each $i = 1, 2, 3$, the function $r_i(p)$ is constant in $M \setminus V_i$.

2.4 CR orbits in real-analytic submanifolds

Let $M \subset \mathbb{C}^N$ be a real-analytic submanifold (not necessarily CR) and $p_0 \in M$. By a CR *orbit of* p_0 *in* M we mean a germ at p_0 of a real-analytic submanifold $\Sigma \subset M$ through p_0 such that $\mathbb{C}T_p\Sigma = \mathfrak{g}_M(p)$ for all $p \in \Sigma$. The existence (and uniqueness) of the CR orbit of any point in M follows by applying a theorem of Nagano ([19], see also [4], §3.1) to the Lie algebra spanned by the real and imaginary parts of the vector fields $L_1, \ldots, L_m$ given by Lemma 2.3. The terminology introduced above for the orbit codimension is justified by the fact that the (real) codimension of the CR orbit of p in M coincides with the (complex) codimension of $\mathfrak{g}_M(p)$ in $\mathbb{C}T_pM$.

3. Local structure of M at a point of $M \setminus V$

We keep the notation introduced in §2. As before we let $r_i = \min_{p\in M} r_i(p)$, $i = 1, 2, 3$. The following proposition gives the local structure of a manifold near a point which is CR and also of minimum degeneracy.

Proposition 3.1. *Let $M \subset \mathbb{C}^N$ be a connected real-analytic submanifold and $p_0 \in M \setminus (V_1 \cup V_2)$. Then there exist local holomorphic* *coordinates* $Z = (Z^1, Z^2, Z^3) \in \mathbb{C}^{N_1} \times \mathbb{C}^{N_2} \times \mathbb{C}^{N_3}$ *vanishing at* p_0 *with*

$$N_1 := N - r_1 - r_2, \quad N_2 := r_2, \quad N_3 := r_1,$$

a generic real-analytic submanifold $M_1 \subset \mathbb{C}^{N_1}$ *through* 0*, finitely nondegenerate at* 0*, and an open neighborhood* $\mathcal{O} \subset \mathbb{C}^N$ *of* p_0 *such that*

$$M \cap \mathcal{O} = \big\{(Z^1, Z^2, Z^3) \in \mathcal{O} : Z^1 \in M_1, Z^3 = 0\big\}.$$

Equivalently, in the coordinates Z*, the germ of* M *at* 0 *and that of* $M_1 \times \mathbb{C}^{N_2} \times \{0\}$ *coincide.*

Remark 3.2. Suppose that $M \subset \mathbb{C}^N$ is a connected real-analytic submanifold and $p_0 \in M \setminus V_2$, i.e., M is of minimum degeneracy at p_0 but not necessarily CR. One can still ask whether there exist local holomorphic coordinates $Z = (Z_1, Z_2) \in \mathbb{C}^{N_1} \times \mathbb{C}^{N_2}$, vanishing at p_0, with $N_2 = r_2$ and a submanfold $M_1 \subset \mathbb{C}^{N_1}$ through 0 which is finitely nondegenerate at 0 such that, in the coordinates Z, the germ of M at 0 and that of $M_1 \times \mathbb{C}^{N_2}$ coincide. Observe that Proposition 3.1 implies that this is the case if, in addition, M is CR at p_0. The following example shows that it is not the case in general. Let $M \subset \mathbb{C}^3$ be given by $M := \{(z_1, z_2, w) \in \mathbb{C}^3 : w = z_1 \bar{z}_2\}$. M is CR precisely at those points where $z_1 \neq 0$. The $(0,1)$ vector fields on M are multiples of $L = \partial_{\bar{z}_1} + z_2 \partial_{\overline{w}}$. The degeneracy is everywhere 1 and the orbit dimension is everywhere 2. However, as the reader can easily check, the answer to the question above is negative in this example with $p_0 = 0$.

Proof of Proposition 3.1. We may assume $p_0 = 0$. Since M is CR at 0, there is a neighborhood of 0 in $\mathbb{C}^N$ such that the piece of M in that neighborhood is contained as a generic submanifold in a complex submanifold of $\mathbb{C}^N$ (called the *intrinsic complexification* of M) of complex dimension $N - r_1$ (see, e.g., [4]). By a suitable choice of holomorphic coordinates $(Z^1, Z^2, Z^3) \in \mathbb{C}^{N_1} \times \mathbb{C}^{N_2} \times \mathbb{C}^{N_3}$ with N_1, N_2, N_3 as in the proposition, we may assume that the intrinsic complexification is given by $Z^3 = 0$ near 0. Then M is a generic submanifold of the subspace $\{Z^3 = 0\}$. Hence in the rest of the proof it suffices to assume that $M \subset \mathbb{C}^N$ is generic and $0 \in M$. We may therefore find holomorphic coordinates $Z = (z, w) \in \mathbb{C}^n \times \mathbb{C}^d$, with d being the codimension of M in $\mathbb{C}^N$ and $n := N - d$, such that if $\rho = (\rho_1, \ldots, \rho_d)$ is a local defining function of M near 0, then $\rho_w(0)$ is an invertible $d \times d$ matrix. By the implicit function theorem, we can write M near 0 in the form

$$M = \{(z, w) : w - Q(z, \bar{z}, \overline{w}) = 0\} = \{(z, w) : \overline{w} - \overline{Q}(\bar{z}, z, w) = 0\},$$

where Q is a $\mathbb{C}^d$-valued holomorphic function defined in a neighborhood of 0 in $\mathbb{C}^{2n+d}$ and vanishing at 0. We now apply the definition of minimum degeneracy given in §2.3 to the (complex valued) defining function of M given by

$$\Theta(z,w,\overline{z},\overline{w}) := \overline{w} - \overline{Q}(\overline{z},z,w). \tag{3.1}$$

It can be easily checked that the identity (2.7) holds with $\rho(z,w,\overline{z},\overline{w})$ replaced by $\Theta(z,w,\overline{z},\overline{w})$ (even though here $\Theta(z,w,\overline{z},\overline{w})$ is complex valued). Consider the basis of $(0,1)$ vector fields on M given by

$$L_j := \frac{\partial}{\partial \overline{z}_j} + \sum_{i=1}^{d} \overline{Q}^i_{\overline{z}_j}(\overline{z},Z)\frac{\partial}{\partial \overline{w}_i}, \quad 1 \le j \le n,$$

where, as above, $Z = (z,w)$. Observe that since $\overline{Q}$ is independent of $\overline{w}$, for $\alpha \in \mathbb{Z}^n_+$ and $1 \le j \le d$,

$$L^\alpha \Theta^j_Z(Z,\overline{Z}) = -\overline{Q}^j_{Z,\overline{z}^\alpha}(\overline{z},Z).$$

Since M is of minimum degeneracy at 0, it follows that for Z in a neighborhood of 0 in M,

$$\dim \operatorname{span}_{\mathbb{C}}\{\overline{Q}^j_{Z,\overline{z}^\alpha}(\overline{z},Z) : \alpha \in \mathbb{Z}^n_+; 1 \le j \le d\} = N - r_2.$$

By a standard complexification argument (see, e.g., Lemma 11.5.8 in [4]), we conclude that for $\chi \in \mathbb{C}^n$ and $Z \in \mathbb{C}^N$ near the origin, we also have

$$\dim \operatorname{span}_{\mathbb{C}}\{\overline{Q}^j_{Z,\chi^\alpha}(\chi,Z) : \alpha \in \mathbb{Z}^n_+; 1 \le j \le d\} = N - r_2.$$

Hence there exists an integer $l \ge 0$ such that for $Z \in \mathbb{C}^N$ in a neighborhood of 0,

$$\dim \operatorname{span}_{\mathbb{C}}\{\overline{Q}^j_{Z,\chi^\alpha}(0,Z) : 0 \le |\alpha| \le l; 1 \le j \le d\} = N - r_2.$$

In particular, if K is d times the number of multi-indices $\alpha \in \mathbb{Z}^n_+$ with $0 \le |\alpha| \le l$, the map ψ given by

$$Z \mapsto \psi(Z) = (\overline{Q}^j_{\chi^\alpha}(0,Z))_{0 \le |\alpha| \le l, 1 \le j \le d} \in \mathbb{C}^K$$

is of constant rank equal to $N - r_2$ for Z in a neighborhood of 0 in $\mathbb{C}^N$. By the implicit function theorem, there exists a holomorphic change of coordinates $Z = \Phi(\widetilde{Z}^1, \widetilde{Z}^2)$ with $(\widetilde{Z}^1, \widetilde{Z}^2) \in \mathbb{C}^{N-r_2} \times \mathbb{C}^{r_2}$ such that $\psi(\Phi(\widetilde{Z}^1, \widetilde{Z}^2)) \equiv \psi(\Phi(\widetilde{Z}^1, 0))$. It follows that $\overline{Q}_{\chi^\alpha}(0, \Phi(\widetilde{Z}^1, \widetilde{Z}^2))$ is independent of $\widetilde{Z}^2$ for all α, $0 \le |\alpha| \le l$, and hence, by the choice of l, for all $\alpha \in \mathbb{Z}_+^n$. Therefore, if we write the complexification of (3.1) in the form

$$\Theta(Z, \zeta) = \tau - \overline{Q}(\chi, z, w), \quad Z = (z, w) \in \mathbb{C}^n \times \mathbb{C}^d,\ \zeta = (\chi, \tau) \in \mathbb{C}^n \times \mathbb{C}^d,$$

we conclude that $\Theta(\Phi(\widetilde{Z}^1, \widetilde{Z}^2), \zeta)$ is independent of $\widetilde{Z}^2$. Hence the (complex valued) function given by

$$\widetilde{\Theta}(\widetilde{Z}^1, \widetilde{Z}^2, \widetilde{\zeta}^1, \widetilde{\zeta}^2) := \Theta(\Phi(\widetilde{Z}^1, \widetilde{Z}^2), \overline{\Phi}(\widetilde{\zeta}^1, \widetilde{\zeta}^2))$$

is independent of $\widetilde{Z}^2$, and M is given by $\widetilde{\Theta}\big(\widetilde{Z}^1, \widetilde{Z}^2, \overline{\widetilde{Z}^1}, \overline{\widetilde{Z}^2}\big) = 0$. Thus all vector fields $\partial/\partial\widetilde{Z}^2_j$, $1 \le j \le r_2$, are tangent to M and hence so are the vector fields $\partial/\partial\overline{\widetilde{Z}^2_j}$. After a linear change of the coordinates $\widetilde{Z}^1 = (\widetilde{Z}^{11}, \widetilde{Z}^{12}) \in \mathbb{C}^{n-r_2} \times \mathbb{C}^d$ we can write M near 0 in the form

$$M = \left\{(\widetilde{Z}^{11}, \widetilde{Z}^{12}, \widetilde{Z}^2) : \mathrm{Im}\ \widetilde{Z}^{12} = \phi(\widetilde{Z}^{11}, \overline{\widetilde{Z}^{11}}, \mathrm{Re}\ \widetilde{Z}^{12})\right\},$$

where ϕ is a real-analytic, real vector valued function. Hence the submanifold $M_1 \subset \mathbb{C}^{N_1}$ given by $M_1 := M \cap \{\widetilde{Z}^2 = 0\}$ satisfies the required assumptions. q.e.d.

The following proposition gives the structure of a generic submanifold at a point of minimum orbit codimension. Recall that we have used the notation $r_3 = \min_{p \in M} r_3(p)$, where $r_3(p)$ is the orbit codimension of p.

Proposition 3.3. *Let $M \subset \mathbb{C}^N$ be a be a connected real-analytic generic submanifold and $p_0 \in M$. The following are equivalent:*

(i) $p_0 \in M \setminus V_3$.

(ii) *There is an open neighborhood U of p_0 in M and a real-analytic mapping*

$$h : U \to \mathbb{R}^{r_3}, \quad h(p_0) = 0,$$

of rank r_3, which extends holomorphically to an open neighborhood of U in $\mathbb{C}^N$, such that $h^{-1}(0)$ is a CR *manifold of finite type.*

(iii) *In addition to the assumptions of Condition (ii), for all u in a neighborhood of 0 in $\mathbb{R}^3$, $h^{-1}(u)$ is a* CR *manifold of finite type.*

Proof. Since M is generic, and hence CR, we can choose a frame $(L_1, \ldots, L_n)$ of real-analytic $(0,1)$ vector fields on M near p_0, spanning the space of all $(0,1)$ tangent vectors to M at every point near p_0. (Here $n = N - d$, where d is the codimension of M.) We write $L_j = X_j + \sqrt{-1}X_{j+n}$, where X_j, $1 \leq j \leq 2n$, are real valued vector fields. We prove first that (i) implies (iii). By the condition that M is of minimum orbit codimension r_3 at p_0, it follows that the collection of the vector fields X_j, $1 \leq j \leq 2n$, generates a Lie algebra, whose dimension at every point near p_0 is $2n+d-r_3$. Therefore, by the (real) Frobenius theorem, we conclude that there exist r_3 real-analytic real valued functions $h_1, \ldots, h_{r_3}$ with independent differentials, defined in a neighborhood of p_0, vanishing at p_0 and such that $L_j h_m \equiv 0$ (i.e., h_m is a CR function) for all $1 \leq j \leq n$ and $1 \leq m \leq r_3$. Moreover, the local orbits of the X_j, $1 \leq j \leq 2n$, are all of the form $M_u = \{p \in M : h(p) = u\}$ with $h = (h_1, \ldots, h_{r_3})$ and $u \in \mathbb{R}^{r_3}$ sufficiently small. By a theorem of Tomassini ([23], see also [4], Corollary 1.7.13), the functions $h_1, \ldots, h_{r_3}$ extend holomorphically to a full neighborhood of p_0 in $\mathbb{C}^N$. This proves that (i) implies (iii). For the proof that (ii) implies (i), we observe that since h extends holomorphically, we have $L_j h_m \equiv 0$ for all $1 \leq j \leq n$ and $1 \leq m \leq r_3$. By the reality of h_m it follows that $X_j h_m \equiv 0$ for all $1 \leq j \leq 2n$ and $1 \leq m \leq r_3$. Hence the set $M_0 := h^{-1}(0)$ is the CR orbit of M at p_0 and is of dimension r_3, which proves (i). Since the implication (iii) $\implies$ (ii) is trivial, the proof of the proposition is complete. q.e.d.

The following proposition gives useful local holomorphic coordinates for a generic submanifold around a point of minimum orbit codimension.

Proposition 3.4. *Let M be a connected generic real-analytic submanifold of $\mathbb{C}^N$ of codimension d and $p_0 \in M \setminus V_3$. Set $n := N - d$, $d_2 := r_3$ and $d_1 := d - r_3$. Then there exist holomorphic local coordinates $Z = (z, w, u) \in \mathbb{C}^N = \mathbb{C}^n \times \mathbb{C}^{d_1} \times \mathbb{C}^{d_2}$ vanishing at p_0, an open neighborhood $\mathcal{O} = \mathcal{O}_1 \times \mathcal{O}_2 \subset \mathbb{C}^{n+d_1} \times \mathbb{C}^{d_2}$ of p_0, and a holomorphic map Q from a neighborhood of 0 in $\mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^{d_1} \times \mathbb{C}^{d_2}$ to $\mathbb{C}^{d_1}$ satisfying*

$$Q(z, 0, \tau, u) \equiv Q(0, \chi, \tau, u) \equiv \tau \tag{3.2}$$

such that

$$M \cap \mathcal{O} = \left\{(z, w, u) \in \mathcal{O} : u \in \mathbb{R}^{d_2},\, w = Q(z, \overline{z}, \overline{w}, u)\right\},$$

and for every $u \in \mathbb{R}^{d_2}$ close to 0 the submanifold

$$M_u := \left\{(z, w) \in \mathcal{O}_1 : w = Q(z, \bar{z}, \overline{w}, u)\right\} \subset \mathbb{C}^{n+d_1} \tag{3.3}$$

is generic and of finite type.

Proof. We take normal coordinates $Z' = (z', w') \in \mathbb{C}^n \times \mathbb{C}^d$ vanishing at p_0 (see, e.g., [4], §4.2), i.e., we assume that M is given by $w' = Q'(z', \overline{z}', \overline{w}')$ near 0, where Q' is a germ at 0 in $\mathbb{C}^{2n+d}$ of a holomorphic $\mathbb{C}^d$-valued function satisfying

$$Q'(z', 0, \tau') \equiv Q'(0, \chi', \tau') \equiv \tau'. \tag{3.4}$$

We may choose a frame $(L_1, \dots, L_n)$ spanning the space of all $(0,1)$ vector fields on M of the form $L_j = \frac{\partial}{\partial \overline{z}'_j} + \sum_{i=1}^d Q'^i_{\overline{z}'_j}(z', \overline{z}', \overline{w}') \frac{\partial}{\partial \overline{w}'}$ for $1 \le j \le n$. In particular, $L_j(0) = \frac{\partial}{\partial \overline{z}'_j}$. Let $h = (h_1, \dots, h_{d_2})$ be the functions given by (iii) in Proposition 3.3. Since, for $1 \le m \le d_2$, the functions h_m are real and extend holomorphically, we conclude that $L_j h_m \equiv \overline{L}_j h_m \equiv 0$. We denote again the by $h_1, \dots, h_{d_2}$ the extended functions. By the choice of the coordinates, $\partial h_m / \partial z'_j(0) = 0$, $1 \le m \le d_2$, $1 \le j \le n$. By using the independence of the differentials of $h_1, \dots, h_{d_2}$ and reordering the components $w'_1, \dots, w'_d$ if necessary, we may assume that

$$\det \left(\frac{\partial h_m}{\partial w'_j}(0) \right)_{1 \le m \le d_2,\, d_1+1 \le j \le d} \ne 0.$$

We make the following change of holomorphic coordinates in $\mathbb{C}^N$ near 0:

$$\begin{aligned} z'' = z', \quad w''_j &= w'_j \ \text{ for } \ 1 \le j \le d_1, \\ w''_j &= h_{j-d_1}(z', w') \ \text{ for } \ d_1 + 1 \le j \le d. \end{aligned} \tag{3.5}$$

Note that on M, we have $w''_j = \overline{w}''_j$ for $d_1 + 1 \le j \le d$. The reader can check that the new coordinates $(z'', w'') \in \mathbb{C}^n \times \mathbb{C}^d$ are again normal for M. Indeed, M is given by $w'' = Q''(z'', \overline{z}'', \overline{w}'')$ where Q'' satisfies the analog of (3.4), with $Q''^j(z'', \overline{z}'', \overline{w}'') \equiv \overline{w}''_j$ for $d_1 + 1 \le j \le d$. The desired coordinates are obtained by taking $(z, w, u) := (z'', w'')$ i.e., $z = z''$ and $(w, u) = w''$ with $w \in \mathbb{C}^{d_1}$, $u \in \mathbb{C}^{d_2}$. We take $Q^j := Q''^j$ for $1 \le j \le d_1$. By the properties of the functions $h_1, \dots, h_{d_2}$, the submanifold M_u given by (3.3), with $u \in \mathbb{R}^{d_2}$ close to 0, is of finite type if $\mathcal{O}_1$ is a sufficiently small neighborhood of 0 in $\mathbb{C}^{n+d_1}$. This completes the proof of Proposition 3.4. q.e.d.

4. Properties of k-equivalences between germs of real submanifolds

We first observe that if (M, p) and (M', p') are two germs in $\mathbb{C}^N$ of real-analytic submanifolds at p and p' respectively, then for any formal k-equivalence H between (M, p) and (M', p'), the kth Taylor polynomial of H is a convergent k-equivalence. Therefore we may, and shall, assume all k-equivalences in the rest of this paper to be convergent. By a *local parametrization* $Z(x)$ of M at p we shall mean a real-analytic diffeomorphism $x \mapsto Z(x)$ between open neighborhoods of 0 in $\mathbb{R}^{\dim M}$ and of p in M satisfying $Z(0) = p$. We say that a function $f(x)$ in a neighborhood of 0 in $\mathbb{R}^m$ *vanishes of order* k at 0, if $f(x) = O(|x|^k)$.

One of the main results of this section is to show that k-equivalences, for sufficiently large k preserve the integers $r_j(p)$, $j = 1, 2, 3$ introduced in §2, and their minimality. For simplicity of notation we state the result for $p = p' = 0$.

Proposition 4.1. *Let $(M, 0)$ and $(M', 0)$ be two germs at 0 in $\mathbb{C}^N$ of real-analytic submanifolds which are k-equivalent for every k. Denote by $r_j(0)$ and $r'_j(0)$, $j = 1, 2, 3$, the integers given by (2.1), (2.5), and (2.6) for M and M' respectively. Then the following hold:*

(i) $r_1(0) = r'_1(0)$. *Also M is* CR *at* 0 *if and only if M' is* CR *at* 0.

(ii) *If M is* CR *at* 0 *then $r_2(0) = r'_2(0)$, and M is of minimum degeneracy at* 0 *if and only if M' is of minimum degeneracy at* 0.

(iii) *If M is* CR *at* 0 *then $r_3(0) = r'_3(0)$, and M is of minimum orbit codimension at* 0 *if and only if M' is of minimum orbit codimension at* 0.

Before proving Proposition 4.1, we shall need some preliminary results. The following useful but elementary lemma gives alternative definitions of k-equivalences.

Lemma 4.2. *Let $H\colon (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ be an invertible germ of a holomorphic map and $(M, 0)$ and $(M', 0)$ be two germs at* 0 *of real-analytic submanifolds of $\mathbb{C}^N$ of the same dimension. Then for any integer $k > 1$, the following are equivalent:*

(i) *H is a k-equivalence between $(M, 0)$ and $(M', 0)$.*

(ii) *There exist local parametrizations $Z(x)$ and $Z'(x)$ at* 0 *of M and M' respectively such that $Z'(x) = H(Z(x)) + O(|x|^k)$.*

(iii) *For every local parametrization $Z(x)$ of M at 0, there exists a local parametrization $Z'(x)$ of M' at 0 such that $Z'(x) = H(Z(x)) + O(|x|^k)$.*

(iv) *There exist local defining functions $\rho(Z, \overline{Z})$ and $\rho'(Z', \overline{Z'})$ of M and M' respectively near 0 such that $\rho'(H(Z), \overline{H}(\zeta)) = \rho(Z, \zeta) + O(|(Z, \zeta)|^k)$.*

(v) *For every local defining function $\rho(Z, \overline{Z})$ of M near 0, there exists a local defining function $\rho'(Z', \overline{Z'})$ of M' near 0 such that $\rho'(H(Z), \overline{H}(\zeta)) = \rho(Z, \zeta) + O(|(Z, \zeta)|^k)$.*

(vi) *For any local defining functions $\rho(Z, \overline{Z})$ and $\rho'(Z', \overline{Z'})$ of M and M' respectively near 0, there exists a holomorphic function $a(Z, \zeta)$ defined in a neighborhood of 0 in $\mathbb{C}^{2N}$ with values in the space of $d \times d$ invertible matrices (where d is the codimension of M) such that $\rho'(H(Z), \overline{H}(\zeta)) = a(Z, \zeta)\rho(Z, \zeta) + O(|(Z, \zeta)|^k)$.*

In particular, inverses and compositions of k-equivalences are also k-equivalences.

Since the proof of Lemma 4.2 is elementary, it is left to the reader. We shall also need the following two lemmas for the proof of Proposition 4.1.

Lemma 4.3. *Let $(v_\alpha(x))_{\alpha \in A}$ be a collection of real-analytic $\mathbb{C}^K$-valued functions in a neighborhood of 0 in $\mathbb{R}^m$. If the dimension of the span in $\mathbb{C}^K$ of the $v_\alpha(x)$, $\alpha \in A$, is not constant for x in any neighborhood of 0, then there exists an integer $\kappa > 1$ such that, for any other collection of real-analytic $\mathbb{C}^K$-valued functions $(v'_\alpha(x))_{\alpha \in A}$ in some neighborhood of 0 with $v'_\alpha(x) = v_\alpha(x) + O(|x|^\kappa)$, the dimension of the span in $\mathbb{C}^K$ of the $v'_\alpha(x)$ is also nonconstant in any neighborhood of 0.*

Proof. Denote by r the dimension of the span in $\mathbb{C}^K$ of the $v_\alpha(0)$, $\alpha \in A$. By the assumption, there exists an $(r+1) \times (r+1)$ minor $\Delta(x)$, extracted from the components of the $v_\alpha(x)$, which does not vanish identically. Note that $\Delta(0) = 0$. Let $\gamma \in \mathbb{Z}_+^m$, $|\gamma| \geq 1$, be such that $\partial^\gamma \Delta(0) \neq 0$. Then for $\kappa := |\gamma| + 1$ and $v'_\alpha(x)$ as in the lemma, it follows that $\partial^\gamma \Delta'(0) \neq 0$, where $\Delta'(x)$ is the corresponding minor with $v_\alpha(x)$ replaced by $v'_\alpha(x)$. On the other hand, the dimension of the span of the $v'_\alpha(0)$ is also r. Since $\Delta'(x)$ is an $(r+1) \times (r+1)$ minor that does not vanish identically, the proof of the lemma is complete. q.e.d.

Lemma 4.4. *Let $M_1, M_1' \subset \mathbb{C}^{N_1}$ be two generic real-analytic submanifolds through* 0*, of the same dimension. Let $M := M_1 \times \{0\}$ and $M' := M_1' \times \{0\}$, both contained in $\mathbb{C}^N = \mathbb{C}^{N_1} \times \mathbb{C}^{N_2}$, and H a k-equivalence between $(M, 0)$ and $(M', 0)$, with $k > 1$. Let $Z = (Z^1, Z^2)$ and $H = (H^1, H^2)$ be the corresponding decompositions for the components of Z and H. Then $H^2(Z^1, 0) = O(|(Z^1)|^k)$ and $Z^1 \mapsto H^1(Z^1, 0)$ is a k-equivalence between $(M_1, 0)$ and $(M_1', 0)$.*

Proof. We write $Z' = (Z'^1, Z'^2) \in \mathbb{C}^{N_1} \times \mathbb{C}^{N_2}$. Let $\rho_1'(Z'^1, \overline{Z'^1})$ be a local defining function for $M_1' \subset \mathbb{C}^{N_1}$. Then

$$\rho'(Z', \overline{Z'}) := (\rho_1'(Z'^1, \overline{Z'^1}), \mathrm{Re}\ Z'^2, \mathrm{Im}\ Z'^2) \tag{4.1}$$

is a local defining function for M' in a neighborhood of 0 in $\mathbb{C}^N$. By the definition of k-equivalence, we obtain

$$\rho_1'(H^1(Z(x)), \overline{H^1(Z(x))}) = O(|x|^k), \quad H^2(Z(x)) = O(|x|^k), \tag{4.2}$$

for any local parametrization $Z(x)$ of M at 0. By the second identity in (4.2), the holomorphic function $H^2(Z^1, 0)$ vanishes of order k at 0 on the submanifold M_1 which is generic in $\mathbb{C}^{N_1}$. This implies the first statement of the lemma. Since H is invertible and by the first statement of the lemma we have $H^2_Z(0) = 0$, the map $Z^1 \mapsto H^1(Z^1, 0)$ must be invertible at 0. Hence the first identity in (4.2) implies the second statement of the lemma. q.e.d.

Proof of Proposition 4.1. We first observe that every 2-equivalence between $(M, 0)$ and $(M', 0)$ induces a linear isomorphism between T_0M and T_0M'. Since $(M, 0)$ and $(M', 0)$ are k-equivalent for every k, this implies $r_1(0) = r_1'(0)$. To complete the proof of (i), we argue by contradiction. We assume that M' is CR at 0 but that M is not. If $\rho(Z, \overline{Z})$ is a local defining function for M and $Z(x)$ is a local parametrization of M at 0, we set $v^j(x) := \rho^j_Z(Z(x), \overline{Z(x)})$, $1 \leq j \leq d$. Since M is assumed not to be CR at 0, the collection of functions $v^j(x)$ satisfies the assumptions of Lemma 4.3. Let κ be the integer given by the lemma. We take $k \geq \kappa + 1$ and let H be a k-equivalence between $(M, 0)$ and $(M', 0)$. If we set $\widetilde{M} := H^{-1}(M')$, then the identity map is a k-equivalence between $(M, 0)$ and $(\widetilde{M}, 0)$. Hence, by Lemma 4.2 (iii,v), there exist a local parametrization $\widetilde{Z}(x)$ of $\widetilde{M}$ at 0 and a local defining function $\widetilde{\rho}$ for $\widetilde{M}$ near 0 such that $\widetilde{Z}(x) = Z(x) + O(|x|^k)$ and $\widetilde{\rho}(Z, \overline{Z}) = \rho(Z, \overline{Z}) + O(|Z|^k)$. We apply Lemma 4.3 for the collection

$v^j(x)$ defined above and $v'^j(x) := \widetilde{\rho}^j_Z(\widetilde{Z}(x), \overline{\widetilde{Z}(x)})$ and conclude that $\widetilde{M}$ is not CR at 0. Thus we have reached a contradiction, since $\widetilde{M}$ and M' are biholomorphically equivalent. This completes the proof of (i).

To prove (ii), suppose that M and M' are CR at 0. Since M and M' are CR and $r_1(0) = r_1'(0)$ by (i), we may assume that $M = M_1 \times \{0\}$ and $M' = M_1' \times \{0\}$, both contained in $\mathbb{C}^{N_1} \times \mathbb{C}^{N_2}$ with $N_1 := N - r_1(0)$, $N_2 := r_1(0)$ and M_1 and M_1' generic in $\mathbb{C}^{N_1}$ (cf. beginning of proof of Proposition 3.1). By Lemma 4.4, M_1 and M_1' are also k-equivalent for every $k > 1$. We observe that M is of minimum degeneracy at 0 if and only if M_1 is of minimum degeneracy at 0 and the degeneracies of M and M_1 at 0 are the same. Therefore, by replacing M by M_1 and M' by M_1', we may assume that M and M' are generic (i.e., $r_1(0) = r_1'(0) = 0$) in the rest of the proof.

We show first that $r_2(0) = r_2'(0)$. From the definition (2.5) of these numbers, there exists $l \geq 0$ such that

$$\dim_{\mathbb{C}} E_l(0) = r_2(0), \quad \dim_{\mathbb{C}} E_l'(0) = r_2'(0), \tag{4.3}$$

where, for $p \in M$,

$$E_l(p) := \operatorname{span}_{\mathbb{C}}\Big\{(\mathcal{L}_1 \dots \mathcal{L}_s \rho^j_Z)(p, \overline{p}) : 0 \leq s \leq l; \\ \mathcal{L}_1, \dots, \mathcal{L}_s \in \mathcal{T}^{0,1}_{M,p}; 1 \leq j \leq d\Big\},$$

with $\rho(Z, \overline{Z})$ being a defining function for M near 0, and $E_l'(p') \subset \mathbb{C}^N$ is the corresponding subspace for M'. We may choose holomorphic coordinates $Z = (z, w) \in \mathbb{C}^n \times \mathbb{C}^d$ vanishing at 0 such that the $d \times d$ matrix $\rho_w(0)$ is invertible. In these coordinates we take a basis of $(0, 1)$ vector fields on M in the form

$$L_j = \frac{\partial}{\partial \overline{z}_j} - {}^\tau\rho_{\overline{z}_j}\, {}^\tau(\rho_w^{-1}) \left(\frac{\partial}{\partial \overline{w}}\right), \quad 1 \leq j \leq n, \tag{4.4}$$

where we have used matrix notation so that $\left(\frac{\partial}{\partial \overline{w}}\right) = \left(\frac{\partial}{\partial \overline{w}_1}, \dots, \frac{\partial}{\partial \overline{w}_1}\right)$ is viewed as a $d \times 1$ matrix. We now choose a local parametrization $Z(x)$ of M at 0, and put

$$v^j_\alpha(x) := L^\alpha \rho^j_Z(Z(x), \overline{Z(x)}), \quad 1 \leq j \leq d, \ \alpha \in \mathbb{Z}^n_+, \ 0 \leq |\alpha| \leq l. \tag{4.5}$$

We then choose $k > l+1$ and H to be a k-equivalence between $(M, 0)$ and $(M', 0)$ which exists by the assumptions of the proposition. Replacing

M' by $H^{-1}(M')$ we may assume without loss of generality that H is the identity map of $\mathbb{C}^N$. By Lemma 4.2 (ii,v), we can find a local parametrization $Z'(x)$ of M' at 0 satisfying $Z'(x) = Z(x) + O(|x|^k)$ and a local defining function ρ' of M' near 0 satisfying $\rho'(Z,\overline{Z}) = \rho(Z,\overline{Z}) + O(|Z|^k)$. Denote by L'_j, $1 \le j \le n$, the local basis of $(0,1)$ vector fields on M' given by the analog of (4.4) with ρ replaced by ρ'. (Observe that $\rho'_w(0)$ coincides with $\rho_w(0)$ and hence is invertible). By the choice of ρ', we have $L'_j = L_j + R_j$ in a neighborhood of 0 in $\mathbb{C}^N$, where R_j is a vector field whose coefficients vanish of order $k-1$ at 0. We put

$$v'^j_\alpha(x) := L'^\alpha \rho'^j_Z(Z'(x), \overline{Z'(x)}),\ 1 \le j \le d,\ \alpha \in \mathbb{Z}^n_+,\ 0 \le |\alpha| \le l. \tag{4.6}$$

Then it follows from the construction that

$$v'^j_\alpha(x) = v^j_\alpha(x) + O(k-l-1) \tag{4.7}$$

and, in particular, $v'^j_\alpha(0) = v^j_\alpha(0)$ for all j and α as in (4.5). Hence, by making use of (4.3), we have $r_2(0) = r'_2(0)$, which proves the first part of (ii).

To prove the second part of (ii), assume that M' is of minimum degeneracy at 0 and that M is not. We shall reach a contradiction by again making use of Lemma 4.3. From the definition of minimum degeneracy there exists an integer $l' \ge 0$ such that

$$\dim E'_{l'}(p') \equiv \dim E'_{l'}(0), \quad \dim E_{l'}(p) \not\equiv \dim E_{l'}(0), \tag{4.8}$$

for $p \in M$, and $p' \in M'$ near 0. Hence the collection of real-analytic functions given by (4.5) with l replaced by l' satisfies the assumption of Lemma 4.3. We let $\kappa > 1$ be the integer given by that lemma and choose H to be a k-equivalence with k satisfying $\kappa = k - l' - 1$. As before, we may assume that H is the identity. Using again Lemma 4.2 (ii,v), we obtain the analogue of (4.7), with l replaced by l'. We conclude by Lemma 4.3 that the dimension of the span of the $v'^j_\alpha(x)$ given by (4.6), with l replaced by l', is not constant in any neighborhood of 0. This contradicts the first part of (4.8) and proves the second part of (ii).

The proof of (iii) is quite similar to that of (ii), and the details are left to the reader. The proof of Proposition 4.1 is complete. q.e.d.

5. Reduction of Theorem 1.1 to the case of generic, finitely nondegenerate submanifolds

In this section we reduce Theorem 1.1 to the case where M and M' are generic and M' is finitely nondegenerate. For this case, the precise statement is given in Theorem 5.1 below. After the reduction to this case, the rest of the paper will be devoted to the proof of Theorem 5.1.

Theorem 5.1 (Main Technical Theorem)**.** *Let $(M, 0)$ and $(M', 0)$ be two germs of generic real-analytic submanifolds of $\mathbb{C}^N$ of the same dimension. Assume that M is of minimum orbit codimension at 0 and that M' is finitely nondegenerate at 0. Then for any integer $\kappa > 1$, there exists an integer $k > 1$ such that if H is a k-equivalence between $(M, 0)$ and $(M', 0)$, then there exists a biholomorphic equivalence $\widehat{H}$ between $(M, 0)$ and $(M', 0)$ with $\widehat{H}(Z) = H(Z) + O(|Z|^\kappa)$.*

Remark 5.2. We should mention here that the proof of Theorem 5.1 is simpler when M and M' are hypersurfaces of $\mathbb{C}^N$. In fact in this case under the assumptions of the theorem, if H is a formal equivalence between $(M, 0)$ and $(M', 0)$, then if follows from Theorem 5 in [7] that H is convergent. Hence, in particular, the proof of the equivalence of (ii) and (iii) in Corollary 1.2 is simpler in the case of hypersurfaces.

In order to show that Theorem 1.1 is a consequence of Theorem 5.1, we shall need the following.

Lemma 5.3. *Let $M_1, M_1' \subset \mathbb{C}^{N_1}$ be generic real-analytic submanifolds through 0 and assume that M_1' is l-nondegenerate at 0. Let $M := M_1 \times \mathbb{C}^{N_2}$ and $M' := M_1' \times \mathbb{C}^{N_2}$ (both contained in $\mathbb{C}^N = \mathbb{C}^{N_1} \times \mathbb{C}^{N_2}$), and let H be a k-equivalence between $(M, 0)$ and $(M', 0)$ with $k > l + 1$. Let $Z = (Z^1, Z^2)$ and $H = (H^1, H^2)$ be the corresponding decompositions for the components of Z and H. Then the following hold:*

(i) $\left(\partial H^1/\partial Z^2\right)(Z) = O(|(Z)|^{k-l-1})$.

(ii) *$Z^1 \mapsto H^1(Z^1, 0)$ is a k-equivalence between $(M_1, 0)$ and $(M_1', 0)$.*

Proof. Observe that M and M' are generic submanifolds of $\mathbb{C}^N$. We write $Z' = (Z'^1, Z'^2) \in \mathbb{C}^{N_1} \times \mathbb{C}^{N_2}$. Let $\rho_1'(Z'^1, \overline{Z'^1})$ be a local defining function for $M_1' \subset \mathbb{C}^{N_1}$. Then $\rho'(Z', \overline{Z'}) := \rho_1'(Z'^1, \overline{Z'^1})$ is a local defining function for M' in a neighborhood of 0 in $\mathbb{C}^N$. By the definition of k-equivalence, we obtain

$$\rho_1'(H^1(Z(x)), \overline{H^1(Z(x))}) = O(|x|^k). \tag{5.1}$$

for any local parametrization $Z(x)$ of M at 0. We choose $Z(x)$ in the form

$$(5.2)\qquad \begin{aligned} \mathbb{R}^{\dim M_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_2} \ni x = (x^1, x^2, y^2) &\mapsto Z(x) \\ &= \big(Z^1(x^1), x^2 + iy^2\big) \in M, \end{aligned}$$

where $x \mapsto Z^1(x^1)$ is a local parametrization of M_1 at 0. Similarly, we choose a local defining function $\rho_1(Z^1, \overline{Z^1})$ for M_1 near 0 in $\mathbb{C}^{N_1}$ and put $\rho(Z, \overline{Z}) := \rho_1(Z^1, \overline{Z^1})$. Since H is a k-equivalence between $(M, 0)$ and $(M', 0)$, the identity map is a k-equivalence between $(M, 0)$ and $(\widetilde{M}, 0)$ with $\widetilde{M} := H^{-1}(M')$. We let L_j, $1 \le j \le n$, be the $(0,1)$ vector fields defined in a neighborhood of 0 in $\mathbb{C}^N$ given by (4.4) (after reordering coordinates in the form $Z = (z, w) \in \mathbb{C}^N$ with $\rho_w(0)$ invertible). Similarly we define $\widetilde{L}_j$ by an analogue of (4.4) with ρ replaced by $\widetilde{\rho}$, where $\widetilde{\rho}$ is the defining function of $\widetilde{M}$ given by Lemma 4.2 (v) for the identity map so that $\widetilde{\rho}(Z, \overline{Z}) = \rho(Z, \overline{Z}) + O(|Z|^k)$. (We may take the same decomposition $Z = (z, w)$ since $\widetilde{\rho}_w(0) = \rho_w(0)$). Hence $\widetilde{L}_j = L_j + R_j$ with R_j a $(0,1)$ vector field in a neighborhood of 0 in $\mathbb{C}^N$ whose coefficients vanish of order $k-1$ at 0. Observe that the vector fields $L'_j := H_* \widetilde{L}_j$ are tangent to M'.

By Lemma 4.2 (vi), there exists a $d \times d$ real-analytic matrix valued function $a(Z, \overline{Z})$ such that

$$(5.3)\qquad \rho'(H(Z), \overline{H(Z)}) = a(Z, \overline{Z})\rho(Z, \overline{Z}) + O(|Z|^k).$$

Differentiating (5.3) with respect to Z and applying L^α for $|\alpha| \le l$, we obtain

$$(5.4)\qquad \begin{aligned} &L^\alpha\big(\rho'_{Z'}(H(Z), \overline{H(Z)})H_Z(Z)\big) \\ &\qquad = a(Z, \overline{Z})(L^\alpha \rho_Z)(Z, \overline{Z}) + \sum_{0 \le |\beta| < |\alpha|} A_\beta(Z, \overline{Z})(L^\beta \rho_Z)(Z, \overline{Z}) \\ &\qquad\quad + L^\alpha\left(\sum_{j=1}^{d} \rho^j(Z, \overline{Z})B_j(Z, \overline{Z})\right) + O(|Z|^{k-l-1}), \end{aligned}$$

where $A_\beta(Z, \overline{Z})$ and $B_j(Z, \overline{Z})$ are real-analytic functions in a neighborhood of 0 in $\mathbb{C}^N$, valued in $d \times d$ and in $d \times N$ matrices respectively. Using the relation $\widetilde{L}_j = L_j + O(k-1)$ and the definition of L'_j given above, we conclude

$$
\begin{aligned}
(5.5)\quad & (L'^{\alpha}\rho'_{Z'})(H(Z),\overline{H(Z)})H_Z(Z) \\
&= a(Z,\overline{Z})(L^{\alpha}\rho_Z)(Z,\overline{Z}) + \sum_{0\le|\beta|<|\alpha|} A_{\beta}(Z,\overline{Z})(L^{\beta}\rho_Z)(Z,\overline{Z}) \\
&\quad + L^{\alpha}\left(\sum_{j=1}^{d}\rho^j(Z,\overline{Z})B_j(Z,\overline{Z})\right) + O(|Z|^{k-l-1}).
\end{aligned}
$$

We now choose a local parametrization $Z'(x)$ of M' at 0 given by Lemma 4.2 (iii), i.e., $Z'(x) = H(Z(x)) + O(|x|^k)$, with $Z(x)$ given by (5.2). Since the L_j are tangent to M, we conclude from (5.5) that

$$
\begin{aligned}
(5.6)\quad & (L'^{\alpha}\rho'_{Z'})\big(Z'(x),\overline{Z'(x)}\big)H_Z(Z(x)) \\
&= a(Z(x),\overline{Z(x)})(L^{\alpha}\rho_Z)(Z(x),\overline{Z(x)}) \\
&\quad + \sum_{0\le|\beta|<|\alpha|} A_{\beta}(Z(x),\overline{Z(x)})(L^{\beta}\rho_Z)(Z(x),\overline{Z(x)}) + O(|x|^{k-l-1}).
\end{aligned}
$$

By the choices of $\rho(Z,\overline{Z})$ and $\rho'(Z',\overline{Z}')$, we have the decompositions in $\mathbb{C}^{N_1}\times\mathbb{C}^{N_2}$

$$
(5.7)\quad L^{\alpha}\rho^j_Z = (L^{\alpha}\rho^j_{Z^1},0),\quad L'^{\alpha}\rho'^j_{Z'} = (L'^{\alpha}\rho'^j_{Z'^1},0),\quad j=1,\dots,d.
$$

We multiply both sides of (5.6) on the right by the $N\times N_2$ constant matrix $C=\binom{0}{I}$ with I being the $N_2\times N_2$ identity matrix. We conclude that

$$
(5.8)\quad (L'^{\alpha}\rho'_{Z'^1})(Z'(x),\overline{Z'(x)})H^1_Z(Z(x))C = O(|x|^{k-l-1}).
$$

We now use the assumption that M'_1 is l-nondegenerate. By this assumption, we can choose multi-indices $\alpha^1,\dots,\alpha^{N_1}$ and integers $j_1,\dots,j_{N_1}$, with $0\le|\alpha^{\mu}|\le l$, $1\le j_{\mu}\le d$, such that the $N_1\times N_1$ matrix given by

$$
B(x) := \Big(L'^{\alpha^{\mu}}\rho'^{j_{\mu}}_{Z'^1}(Z'(x),\overline{Z'(x)})\Big)_{1\le\mu\le N_1}
$$

is invertible for x near 0. Since $B(x)H^1_Z(Z(x))C = O(|x|^{k-l-1})$ by (5.8) and $H^1_{Z^2}(Z(x)) \equiv H^1_Z(Z(x))C$, we conclude that $H^1_{Z^2}(Z(x)) = O(|x|^{k-l-1})$. Since $M = M_1\times\mathbb{C}^{N_2}$ is generic in $\mathbb{C}^N = \mathbb{C}^{N_1}\times\mathbb{C}^{N_2}$, the statement (i) follows.

From (i) it follows in particular that $H^1_{Z^2}(0) = 0$. Since H is invertible, we conclude that $H^1_{Z^1}(0)$ is also invertible, and (ii) follows from (5.1) by taking $x^2 = y^2 = 0$. This completes the proof of Lemma 5.3.

q.e.d.

We now give the proof of Theorem 1.1 assuming that Theorem 5.1 has been proved. As mentioned in the beginning of this section, the proof of Theorem 5.1 will be given in the remaining sections.

Proof of Theorem 1.1. Set $V := V_1 \cup V_2 \cup V_3 \subset M$, where V_1, V_2, V_3 are defined by (2.2), (2.9) and (2.10) respectively. Let $p \in M \setminus V$, M' a real-analytic submanifold of $\mathbb{C}^N$ and $p' \in M'$. We may assume that M and M' have the same dimension, since otherwise there is nothing to prove. Let $\kappa > 1$ be fixed. If, for some integer $s > 1$, (M, p) and (M', p') are not s-equivalent, then we can take $k = s$ to satisfy the conclusion of Theorem 1.1.

Assume for the rest of the proof that (M, p) and (M', p') are k-equivalent for all $k > 1$. Without loss of generality we may assume $p = p' = 0$. We shall make use of Proposition 4.1. Since M is CR, of minimum degeneracy, and of minimum orbit codimension at 0, M' is also CR, of minimum degeneracy, and of minimum orbit codimension at 0. Furthermore, in the notation of Proposition 4.1, we have $r_j(0) = r'_j(0)$, $j = 1, 2, 3$. Hence we may apply Proposition 3.1 to both $(M, 0)$ and $(M', 0)$ with the same integers N_1, N_2, N_3 to obtain the decompositions

$$M = M_1 \times \mathbb{C}^{N_2} \times \{0\}, \qquad M' = M'_1 \times \mathbb{C}^{N_2} \times \{0\}, \tag{5.9}$$

where both decompositions are understood in the sense of germs at 0 in $\mathbb{C}^N$. Since M'_1 is finitely nondegenerate at 0, there exists an integer $l \geq 0$ such that M'_1 is l-nondegenerate at 0.

Assume first that M and M' are generic at 0, i.e., $N_3 = 0$. Then, for every $k > l + 1$, the conclusions of Lemma 5.3 hold. By conclusion (ii) of that lemma, for every k-equivalence $H = (H^1, H^2)$ between $(M, 0)$ and $(M', 0)$, the map

$$h \colon Z^1 \mapsto H^1(Z^1, 0) \tag{5.10}$$

is a k-equivalence between $(M_1, 0)$ and $(M'_1, 0)$. Furthermore, $(M_1, 0)$ and $(M'_1, 0)$ satisfy the assumptions of Theorem 5.1.

By Theorem 5.1, there exists a biholomorphic equivalence $\widehat{h}$ between $(M_1, 0)$ and $(M'_1, 0)$ with $\widehat{h}(Z^1) = h(Z^1) + O(|Z^1|^\kappa)$. As we mentioned in the beginning of §4, without loss of generality, we can assume that

H is convergent. Then we may define the germ $\widehat{H}\colon (\mathbb{C}^N,0)\to(\mathbb{C}^N,0)$ of a biholomorphism at the origin as follows:

$$\widehat{H}^1(Z^1,Z^2) := \widehat{h}(Z^1) \tag{5.11}$$

$$\widehat{H}^2(Z) := H^2(Z) \tag{5.12}$$

It is then a consequence of Lemma 5.3 that $\widehat{H}$ satisfies the conclusion of Theorem 1.1 if $k > \kappa + l + 1$.

We now return to the general case in which M and M' are not necessarily generic, and let $H = (H^1, H^2, H^3)$ be a k-equivalence corresponding to the decomposition given by (5.9) with $k > \kappa + l + 1$. By Lemma 4.4 the mapping $(Z^1, Z^2) \mapsto \big(H^1(Z^1,Z^2,0), H^2(Z^1,Z^2,0)\big)$ is a k-equivalence between the generic submanifolds $(M_1 \times \mathbb{C}^{N_2}, 0)$ and $(M_1' \times \mathbb{C}^{N_2}, 0)$ and $H^3(Z^1,Z^2,0) = O(|Z^1,Z^2|^k)$. It follows from the generic case, treated above, that there exists a biholomorphic equivalence $\widehat{h}(Z^1,Z^2)$ between $(M_1 \times \mathbb{C}^{N_2}, 0)$ and $(M_1' \times \mathbb{C}^{N_2}, 0)$ such that $\widehat{h}(Z^1,Z^2) = \big(H^1(Z^1,Z^2,0), H^2(Z^1,Z^2,0)\big) + O(|Z^1,Z^2|^\kappa)$. We write $\widehat{h} = (\widehat{h}^1, \widehat{h}^2)$ corresponding to the product $\mathbb{C}^{N_1} \times \mathbb{C}^{N_2}$. We may now define $\widehat{H}(Z^1,Z^2,Z^3)$ by

$$\widehat{H}^1(Z^1,Z^2,Z^3) := \widehat{h}^1(Z^1,Z^2) + H^1(Z) - H^1(Z^1,Z^2,0) \tag{5.13}$$

$$\widehat{H}^2(Z) := H^2(Z) \tag{5.14}$$

$$\widehat{H}^3(Z) := H^3(Z) - H^3(Z^1,Z^2,0) \tag{5.15}$$

and conclude that $\widehat{H}$ satisfies the desired conclusion of the theorem. This completes the proof of Theorem 1.1 (assuming Theorem 5.1). q.e.d.

6. Rings $\mathcal{R}(V, V_0)$ of germs of holomorphic functions

An important idea of the proof of Theorem 5.1 is to parametrize all k-equivalences between $(M,0)$ and $(M',0)$ by their jets in an expression of the form (10.2) below. For this we shall introduce some notation for certain rings of germs of holomorphic functions. If V is a finite dimensional complex vector space and $V_0 \subset V$ is a vector subspace, we define $\mathcal{R}(V,V_0)$ to be the ring of all germs of holomorphic functions f at V_0 in V such that the restrictions $\partial^\alpha f|_{V_0}$ of all partial derivatives are polynomial functions on V_0. Here ∂^α denotes the partial derivative with respect to the multiindex $\alpha \in \mathbb{Z}_+^{\dim V}$ and some linear coordinates $x \in V$. Recall that, if f and g are two functions holomorphic in some neighborhoods of V_0 in V, then f and g define the same germ of a holomorphic function at V_0 in V if they coincide in some (possibly smaller) neighborhood of V_0 in V. We shall identify such a germ with any representative of it. It is easy to see that the ring $\mathcal{R}(V, V_0)$ does not depend on the choice of linear coordinates in V and is invariant under partial differentiation with respect to these coordinates. In the following we fix a complement V_1 of V_0 in V so that we have the identification $V \cong V_0 \times V_1$ and fix linear coordinates $x = (x_0, x_1) \in V_0 \times V_1$. In terms of these coordinates we may write an element $f \in \mathcal{R}(V, V_0)$ in the form

$$f(x_0, x_1) = \sum_\beta p_\beta(x_0) x_1^\beta, \quad \beta \in \mathbb{Z}_+^{\dim V_1}, \tag{6.1}$$

where the $p_\beta(x_0)$ are polynomials in V_0 satisfying the estimates

$$|p_\beta(x_0)| \leq C(x_0)^{|\beta|+1} \quad \text{for all } \beta, \tag{6.2}$$

where $C(x_0)$ is a positive locally bounded function on V_0. Conversely, every power series of the form (6.1) satisfying (6.2) defines a unique element of $\mathcal{R}(V, V_0)$.

In the following we shall consider germs of holomorphic maps whose components are in $\mathcal{R}(V, V_0)$. If W is another finite dimensional complex vector space and $W_0 \subset W$ is a subspace, we shall write $\phi\colon (V, V_0) \to (W, W_0)$ to mean a germ at V_0 of a holomorphic map from V to W such that $\phi(V_0) \subset W_0$. It can be shown using the chain rule that a composition $f \circ \phi$ with ϕ as above with components in $\mathcal{R}(V, V_0)$ and $f \in \mathcal{R}(W, W_0)$ always belongs to $\mathcal{R}(V, V_0)$. We shall prove the analogue of this property for more general expressions which we shall need in the proof of Theorem 9.1 below.

Lemma 6.1. *Let V_0, V_1, $\widetilde{V}_0$, $\widetilde{V}_1$ be finite dimensional complex vector spaces with fixed bases and x_0, x_1, $\widetilde{x}_0$, $\widetilde{x}_1$ be the linear coordinates with respect to these bases. Let $q \in \mathbb{C}[x_0]$ and $\widetilde{q} \in \mathbb{C}[\widetilde{x}_0]$ be nontrivial polynomial functions on V_0 and $\widetilde{V}_0$ respectively, and let*

$$\phi = (\phi_0, \phi_1)\colon \big(\mathbb{C} \times V_0 \times V_1, \mathbb{C} \times V_0\big) \to \big(\widetilde{V}_0 \times \widetilde{V}_1, \widetilde{V}_0\big)$$

be a germ of a holomorphic map with components in the ring $\mathcal{R}\big(\mathbb{C} \times V_0 \times V_1, \mathbb{C} \times V_0\big)$ and satisfying

$$\widetilde{q}\left(\phi_0\left(\frac{1}{q(x_0)}, x_0, 0\right)\right) \not\equiv 0. \tag{6.3}$$

Then there exists a ring homomorphism

$$\mathcal{R}\big(\mathbb{C}\times\widetilde{V}_0\times\widetilde{V}_1,\mathbb{C}\times\widetilde{V}_0\big)\ni\widetilde{f}\mapsto f\in\mathcal{R}\big(\mathbb{C}\times V_0\times V_1,\mathbb{C}\times V_0\big)\tag{6.4}$$

such that

$$\widetilde{f}\left(\frac{1}{\widetilde{q}\big(\phi_0(\frac{1}{q(x_0)},x_0,x_1)\big)},\phi\left(\frac{1}{q(x_0)},x_0,x_1\right)\right)\equiv f\left(\frac{1}{p(x_0)},x_0,x_1\right),\tag{6.5}$$

with $p(x_0):=q(x_0)^{d_0+1}\widetilde{q}\big(\phi_0(\frac{1}{q(x_0)},x_0,0)\big)$, *where* d_0 *is the degree of the polynomial* $(\theta,x_0)\mapsto\widetilde{q}\big(\phi_0(\theta,x_0,0)\big)$ *with respect to* θ. *Furthermore,* f *vanishes on* $\mathbb{C}\times V_0$ *if* $\widetilde{f}$ *vanishes on* $\mathbb{C}\times\widetilde{V}_0$.

Proof. For $\widetilde{f}$ as above and $\theta',\theta''\in\mathbb{C}$, define a germ g at $\mathbb{C}\times\mathbb{C}\times V_0$ of a holomorphic function on $\mathbb{C}\times\mathbb{C}\times V_0\times V_1$ by

$$\begin{aligned}&g(\theta',\theta'',x_0,x_1):=\\&\quad\widetilde{f}\left(\frac{\theta'}{1+\theta'[\widetilde{q}\big(\phi_0(\theta'',x_0,x_1)\big)-\widetilde{q}(\phi_0(\theta'',x_0,0))]},\phi(\theta'',x_0,x_1)\right).\end{aligned}\tag{6.6}$$

We use the consequence of the chain rule that any partial derivative of a composition of two holomorphic maps can be written as a polynomial expression in the partial derivatives of the components. Then it follows from the assumptions of the lemma that g is in the ring $\mathcal{R}\big(\mathbb{C}\times\mathbb{C}\times V_0\times V_1,\mathbb{C}\times\mathbb{C}\times V_0\big)$. It is straightforward to see that, if f is given by

$$f(\theta,x_0,x_1):=g\left(\theta q(x_0)^{d_0+1},\theta q(x_0)^{d_0}\widetilde{q}\left(\phi_0\left(\frac{1}{q(x_0)},x_0,0\right)\right),x_0,x_1\right),$$

then (6.5) holds and the map $\widetilde{f}\mapsto f$ satisfies the conclusion of the lemma. q.e.d.

7. Jet spaces of mappings

For integers $r,m,l\geq 0$, we denote by $J^r_{m,l}$ the space of all jets at 0 of order r of holomorphic maps from $\mathbb{C}^m$ to $\mathbb{C}^l$. This is a complex vector space that can be identified with the space of $\mathbb{C}^l$-valued polynomials on $\mathbb{C}^m$ of degree at most r. We write such a polynomial in the form $\sum_{0\leq|\alpha|\leq r}(\lambda_\alpha/\alpha!)Z^\alpha$, $\lambda_\alpha\in\mathbb{C}^l$, and call $(\lambda_\alpha)_{0\leq|\alpha|\leq r}$ the standard linear coordinates in $J^r_{m,l}$. For fixed integers $n, d \geq 0$ and $N := n + d$, we introduce the complex vector spaces

$$E^r := J^r_{N,N} \times J^r_{n,d} \times \mathbb{C}^n, \quad E^r_0 := J^r_{N,N} \times \{(0,0)\}, \\ E^r_1 := \{0\} \times J^r_{n,d} \times \mathbb{C}^n \tag{7.1}$$

with $E^r_0, E^r_1 \subset E^r$. We use the identification $E^r \cong E^r_0 \times E^r_1$.

Let M and M' satisfy the assumptions of Theorem 5.1. According to Proposition 3.4 we write M near 0 in the form

$$M = \big\{(z, w, u) \in \mathbb{C}^n \times \mathbb{C}^{d_1} \times \mathbb{R}^{d_2} : w = Q(z, \overline{z}, \overline{w}, u)\big\}, \tag{7.2}$$

where Q is a germ at 0 in $\mathbb{C}^{2n+d}$ of a holomorphic $\mathbb{C}^{d_1}$-valued function satisfying conditions (3.2). We also choose normal coordinates for M' so that

$$M' = \big\{(z', w') \in \mathbb{C}^n \times \mathbb{C}^d : w' = Q'(z', \overline{z'}, \overline{w'})\big\},$$

where Q' is a germ at 0 in $\mathbb{C}^{2n+d}$ of a holomorphic $\mathbb{C}^d$-valued function satisfying

$$Q'(z', 0, \tau') \equiv Q'(0, \chi', \tau') \equiv \tau'. \tag{7.3}$$

In these coordinates (which will be fixed for the remainder of the paper), for every invertible germ of a holomorphic map $H\colon (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ we write $H(Z) = (F(Z), G(Z))$ with $z' = F(Z)$, $w' = G(Z)$ and $Z = (z, w, u)$. For $Z \in \mathbb{C}^N$ near the origin, we define

$$\mathcal{J}^r H(Z) := \left(\left(\frac{\partial^{|\alpha|} H}{\partial Z^\alpha}(Z) \right)_{0 \leq |\alpha| \leq r}, \left(\frac{\partial^{|\nu|} G}{\partial z^\nu}(Z) \right)_{0 \leq |\nu| \leq r}, F(Z) \right). \tag{7.4}$$

We think of $\mathcal{J}^r H$ as a germ at 0 of a holomorphic map from $\mathbb{C}^N$ into the vector space E^r defined by (7.1).

Now let $H = (F, G)$ be a k-equivalence between $(M, 0)$ and $(M', 0)$ with $k > 1$. By a standard complexification argument, H is a k-equivalence means that the identity

$$G(z, Q(z, \chi, \tau, u), u) \equiv Q'\big(F(z, Q(z, \chi, \tau, u), u), \overline{H}(\chi, \tau, u)\big) \\ + R(z, \chi, \tau, u) \tag{7.5}$$

holds for all $(z, \chi, \tau, u) \in \mathbb{C}^{2n+d}$ near the origin, where $R(z, \chi, \tau, u) = O(k)$. In particular, for $(\chi, \tau, u) = 0$ we obtain from (7.5) and (7.3) the identity

$$G(z, 0) = O(k) \tag{7.6}$$

and hence for $r < k$, we have $\mathcal{J}^r H(0) \in E_0^r$, where E_0^r is defined by (7.1).

8. The basic identity

We assume that the assumptions of Theorem 5.1 hold and that $\rho(Z,\overline{Z})$ is a defining function for M at 0. We begin by establishing a relation, called the *basic identity*, between two jets of a k-equivalence H at points Z and ζ in $\mathbb{C}^N$ satisfying $(Z,\zeta) \in \mathcal{M}$, i.e., $\rho(Z,\zeta) = 0$. We shall make use of the notation introduced in §6-7. In particular, we have normal coordinates $(z,w,u) \in \mathbb{C}^n \times \mathbb{C}^{d_1} \times \mathbb{C}^{d_2}$ for M and $(z',w') \in \mathbb{C}^n \times \mathbb{C}^d$ for M' and write $Z = (z,w,u)$, $\zeta = (\chi,\tau,u)$. Furthermore we use matrix notation and regard $F_z(Z)$ as an $n \times n$ matrix, $F_w(Z)$ as an $n \times d_1$ matrix, $G_z(Z)$ as a $d \times n$ matrix and $G_w(Z)$ as a $d \times d_1$ matrix. Similarly $Q_z(z,\chi,\tau,u)$ is regarded as a $d_1 \times n$ matrix.

To shorten the notation it will be convenient to write for r, m non-negative integers

$$\mathcal{R}_m^r := \mathcal{R}\big(\mathbb{C} \times E^r \times \mathbb{C}^m, \mathbb{C} \times E_0^r \times \{0\}\big), \tag{8.1}$$

where the rings $\mathcal{R}(V,V_0)$ are defined as in §6 and the vector spaces E^r and E_0^r are defined in (7.1). We can now state precisely the basic identity.

Theorem 8.1 (Basic Identity)**.** *Let $(M,0)$ and $(M',0)$ be two germs of generic real-analytic submanifolds of $\mathbb{C}^N$ satisfying the assumptions of Theorem 5.1. Assume that M' is l-nondegenerate at 0 (with $l \geq 0$) and that normal coordinates for M and M' are chosen as above. Then for every integer $r > 0$, there exists a germ of a holomorphic map*

$$\Psi^r \colon \big(\mathbb{C} \times E^{r+l} \times \mathbb{C}^{2N}, \mathbb{C} \times E_0^{r+l} \times \{0\}\big) \to (E^r, E_0^r) \tag{8.2}$$

and for $r = 0$, a germ $\Psi^0 \colon \big(\mathbb{C} \times E^l \times \mathbb{C}^{2N}, \mathbb{C} \times E_0^l \times \{0\}\big) \to (E^0, 0)$, such that the components of Ψ^r, $r \geq 0$, are in the ring $\mathcal{R}_{2N}^{r+l}$ and the following holds. For every k-equivalence $H = (F,G)$ between $(M,0)$ and $(M',0)$ with $k > r + l$, one has for (Z,ζ) near the origin in $\mathbb{C}^{2N}$,

$$\mathcal{J}^r H(Z) = \Psi^r\left(\frac{1}{\det(\overline{F}_\chi(\zeta))}, \mathcal{J}^{r+l}\overline{H}(\zeta), \zeta, Z\right) + R_H^r(Z,\zeta), \tag{8.3}$$

where $R_H^r(Z,\zeta)$ is a germ at 0 of a holomorphic map from $\mathbb{C}^{2N}$ into E^r whose restriction to $\mathcal{M}$ vanishes of order $k - r - l$ at 0.

Proof. For convenience we use the notation

$$\begin{aligned} \omega &:= (z,\zeta) = (z,\chi,\tau,u) \in \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^{d_1} \times \mathbb{C}^{d_2}, \\ Z(\omega) &:= (z, Q(\omega), u) \in \mathbb{C}^n \times \mathbb{C}^{d_1} \times \mathbb{C}^{d_2}, \end{aligned}$$

so that the equation of $\mathcal{M} \subset \mathbb{C}^{2N}$ near 0 is given by $w = Q(\omega)$, or equivalently, by $Z = Z(\omega)$. We first differentiate the identity (7.5) in $z \in \mathbb{C}^n$. Using the chain rule we obtain the identity in matrix notation

$$\begin{aligned} \text{(8.4)}\quad & G_z(Z(\omega)) + G_w(Z(\omega))Q_z(\omega) \\ & \equiv Q'_{z'}\left(F(Z(\omega)), \overline{H}(\zeta)\right)\left(F_z(Z(\omega)) + F_w(Z(\omega))Q_z(\omega)\right) + R_z(\omega), \end{aligned}$$

where $R_z(\omega) = O(|\omega|^{k-1})$. (Observe that R_z in (8.4) depends on the map H.) The invertibility of H implies the invertibility of $F_z(0)$ and hence of $F_z(Z(\omega)) + F_w(Z(\omega))Q_z(\omega)$ for ω near the origin (since $Q_z(0) = 0$ by (3.2)). Hence we conclude for ω sufficiently small,

$$\begin{aligned} \text{(8.5)}\quad & Q'_{z'}\left(F(Z(\omega)), \overline{H}(\zeta)\right) = \\ & \left(G_z(Z(\omega)) + G_w(Z(\omega))Q_z(\omega)\right)\left(F_z(Z(\omega)) + F_w(Z(\omega))Q_z(\omega)\right)^{-1} \\ & \quad + O(|\omega|^{k-1}). \end{aligned}$$

Our next goal will be to express the right-hand side of (8.5) and then its derivatives in terms of functions in $\mathcal{R}^r_{2n+d}$ that vanish on certain vector subspaces. For this we introduce the notation

$$\begin{aligned} \text{(8.6)}\qquad A^r := \mathbb{C} \times (J^r_{N,N} \times \{0\} \times \mathbb{C}^n) \times (\mathbb{C}^n \times \{0\}) \subset \\ \mathbb{C} \times (J^r_{N,N} \times J^r_{n,d} \times \mathbb{C}^n) \times (\mathbb{C}^n \times \mathbb{C}^{n+d}) \\ = \mathbb{C} \times E^r \times \mathbb{C}^{2n+d}. \end{aligned}$$

We have the following lemma.

Lemma 8.2. *With the notation above there exists a $d \times n$ matrix P, independent of H, with entries in $\mathcal{R}^1_{2n+d}$ such that, for ω in a neighborhood of 0 in $\mathbb{C}^{2n+d}$,*

$$\begin{aligned} \text{(8.7)}\quad & \left(G_z(Z(\omega)) + G_w(Z(\omega))Q_z(\omega)\right)\left(F_z(Z(\omega)) + F_w(Z(\omega))Q_z(\omega)\right)^{-1} \\ & \qquad \equiv P\Big(\frac{1}{\det F_z(Z(\omega))}, \mathcal{J}^1 H(Z(\omega)), \omega\Big) \end{aligned}$$

and P vanishes on the subspace $A^1 \subset \mathbb{C} \times E^1 \times \mathbb{C}^{2n+d}$ defined by (8.6).

Proof. For simplicity we drop the argument $Z(\omega)$ in G_z, G_w, F_z, F_w and $\mathcal{J}^1H$. We have

$$(8.8)\quad \big(G_z + G_w Q_z(\omega)\big)\big(F_z + F_w Q_z(\omega)\big)^{-1} \equiv \big(G_z + G_w Q_z(\omega)\big)\big(I + F_z^{-1}F_w Q_z(\omega)\big)^{-1}F_z^{-1}.$$

The first factor in the right-hand side of (8.8) can be expressed as a matrix valued polynomial in the entries of G_z and G_w with holomorphic coefficients in ω. We now think of the entries of G_z as variables in $J^1_{n,d}$ and those of G_w as part of the variables in $J^1_{N,N}$ and write

$$\big(G_z + G_w Q_z(\omega)\big) \equiv P_1\left(\frac{1}{\det F_z}, \mathcal{J}^1H, \omega\right)$$

with P_1 independent of the variable in the first factor $\mathbb{C}$ and having entries in $\mathcal{R}^1_{2n+d}$. Since $Q_z(z,0,0,0)\equiv 0$, P_1 vanishes on the subspace $A^1 \subset \mathbb{C}\times E^1 \times \mathbb{C}^{2n+d}$ defined by (8.6) with $r=1$. By the standard formula for the inverse of a matrix, the third factor in the right-hand side of (8.8) can be also written in the form $P_3\big(\frac{1}{\det F_z}, \mathcal{J}^1H, \omega\big)$, where P_3 is a matrix valued polynomial (with entries in $\mathcal{R}^1_{2n+d}$) depending only on part of the variables in $\mathbb{C}\times J^1_{N,N}$ and independent of the variables in $J^1_{n,d}\times\mathbb{C}^n$ and ω. The second factor in the right-hand side of (8.8) can also be written in the form $P_2\big(\frac{1}{\det F_z}, \mathcal{J}^1H, \omega\big)$ with the entries of P_2 in $\mathcal{R}^1_{2n+d}$. This can be shown by using the chain rule in addition to the arguments used for the first and third factors. The proof of the lemma is completed by taking $P := P_1P_2P_3$ and using the fact that $\mathcal{R}^1_{2n+d}$ is a ring. q.e.d.

For the sequel we shall need the following lemma, which is proved by repeated use of the chain rule, making use of the identities (7.6), $Q(z,0,0,0)\equiv 0$, and induction on $|\alpha|$. The details are left to the reader.

Lemma 8.3. *Let M and M' be as in Theorem 8.1. Then for every $f\in\mathcal{R}^r_{2n+d}$ with $r\geq 1$ and every $\alpha\in\mathbb{Z}^{2n+d}_+$, there exists $f^\alpha\in\mathcal{R}^{r+|\alpha|}_{2n+d}$ such that the following holds. For any k-equivalence $H=(F,G)$ between $(M,0)$ and $(M',0)$ with $k>r+|\alpha|$,*

$$(8.9)\quad \big(\partial^{|\alpha|}/\partial\omega^\alpha\big) f\left(\frac{1}{\det F_z(Z(\omega))}, \mathcal{J}^rH(Z(\omega)), \omega\right) \equiv f^\alpha\left(\frac{1}{\det F_z(Z(\omega))}, \mathcal{J}^{r+|\alpha|}H(Z(\omega)), \omega\right).$$

If in addition $\alpha \in \mathbb{Z}_+^n \times \{0\}$ (i.e., the differentiation in (8.9) is taken with respect to z only) and if f vanishes on the subspace $A^r \subset \mathbb{C} \times E^r \times \mathbb{C}^{2n+d}$ defined by (8.6), then f^α vanishes on the subspace $A^{r+|\alpha|} \subset \mathbb{C} \times E^{r+|\alpha|} \times \mathbb{C}^{2n+d}$.

We now return to the proof of Theorem 8.1. By making use of (8.5) and (8.7) we obtain the identity

$$Q'_{z'}\left(F(Z(\omega)), \overline{H}(\zeta)\right) = P\Big(\frac{1}{\det F_z(Z(\omega))}, \mathcal{J}^1 H(Z(\omega)), \omega\Big) + O(|\omega|^{k-1}), \tag{8.10}$$

where $\zeta = (\chi, \tau, u)$ as before and P is given by Lemma 8.2.

We claim that for every $\beta \in \mathbb{Z}_+^n$ with $0 \le |\beta| \le l$, there exists $P^\beta \in (\mathcal{R}^{|\beta|}_{2n+d})^d$, independent of H, vanishing on the subspace $A^{|\beta|} \subset \mathbb{C} \times E^{|\beta|} \times \mathbb{C}^{2n+d}$, and such that the following identity holds for ω in a neighborhood of 0 in $\mathbb{C}^{2n+d}$:

$$Q'_{z'^\beta}\left(F(Z(\omega)), \overline{H}(\zeta)\right) = P^\beta\Big(\frac{1}{\det F_z(Z(\omega))}, \mathcal{J}^{|\beta|} H(Z(\omega)), \omega\Big) + O(|\omega|^{k-|\beta|}). \tag{8.11}$$

Indeed, for $\beta = 0$, (8.11) follows directly from (7.5) and for $|\beta| = 1$, (8.11) is a reformulation of (8.10). For $|\beta| > 1$ we prove the claim by induction on $|\beta|$. Assume that (8.11) holds for some β. By differentiating (8.11) with respect to z we obtain in matrix notation the identity

$$\left(Q'_{z'^\beta}\right)_{z'}\left(F(Z(\omega)), \overline{H}(\zeta)\right)\left(F_z(Z(\omega)) + F_w(Z(\omega))Q_z(\omega)\right) = (\partial/\partial z)\, P^\beta\Big(\frac{1}{\det F_z(Z(\omega))}, \mathcal{J}^{|\beta|} H(Z(\omega)), \omega\Big) + O(|\omega|^{k-|\beta|-1}). \tag{8.12}$$

By Lemma 8.3 we have

$$(\partial/\partial z)\, P^\beta\Big(\frac{1}{\det F_z(Z(\omega))}, \mathcal{J}^{|\beta|} H(Z(\omega)), \omega\Big) \equiv S\Big(\frac{1}{\det F_z(Z(\omega))}, \mathcal{J}^{|\beta|+1} H(Z(\omega)), \omega\Big), \tag{8.13}$$

where S is a $d \times n$ matrix with entries in $\mathcal{R}^{|\beta|+1}_{2n+d}$, vanishing on the subspace $A^{|\beta|+1} \subset \mathbb{C} \times E^{|\beta|+1} \times \mathbb{C}^{2n+d}$. Since, as in the proof of Lemma 8.2, each entry of the matrix $\left(F_z(Z(\omega)) + F_w(Z(\omega))Q_z(\omega)\right)^{-1}$ can be written in the form $f\big(\frac{1}{\det F_z(Z(\omega))}, \mathcal{J}^1 H(Z(\omega)), \omega\big)$ with f in the ring $\mathcal{R}^1_{2n+d}$, the identity (8.11) for β replaced by any multiindex β' with $|\beta'| = |\beta| + 1$ follows from (8.12) and (8.13) by observing that the ring $\mathcal{R}^1_{2n+d}$ has a natural embedding into $\mathcal{R}^{|\beta|+1}_{2n+d}$. This completes the proof of the claim.

We now use the condition that M' is l-nondegenerate which is equivalent to

$$\operatorname{span}_{\mathbb{C}}\big\{{Q'}^j_{z'^\beta\chi'}(0,0,0) : 1 \le j \le d,\, 1 \le |\beta| \le l\big\} = \mathbb{C}^n$$

(see, e.g., [4], 11.2.14). From this, together with (7.3), we conclude that we can select a subsystem of N scalar identities from (8.11) from which $\overline{H}(\zeta)$ can be solved uniquely by the implicit function theorem. We obtain

$$\begin{aligned}(8.14)\quad \overline{H}(\zeta) = {} & \\ & T\bigg(F(Z(\omega)), P^\beta\Big(\frac{1}{\det F_z(Z(\omega))}, \mathcal{J}^{|\beta|} H(Z(\omega)), \omega\Big)_{0\le|\beta|\le l}\bigg) \\ & \quad + O(|\omega|^{k-l}),\end{aligned}$$

where T is a germ of a holomorphic map $T\colon (\mathbb{C}^n \times \mathbb{C}^m, 0) \to (\mathbb{C}^N, 0)$, with

$$m := d \times \#\{\beta \in \mathbb{Z}^n_+ : 0 \le |\beta| \le l\}. \tag{8.15}$$

Observe that the germ T depends only on Q' but not on H.

We claim that there exists $\Phi \in (\mathcal{R}^l_{2n+d})^N$, independent of H, such that

$$\overline{H}(\zeta) = \Phi\Big(\frac{1}{\det F_z(Z(\omega))}, \mathcal{J}^l H(Z(\omega)), \omega\Big) + O(|\omega|^{k-l}). \tag{8.16}$$

In order to prove the claim we use the notation $x_0 := (\theta, \Lambda) \in \mathbb{C} \times J^l_{N,N}$ and $x_1 := (\Lambda', z', \omega) \in J^l_{n,d} \times \mathbb{C}^n \times \mathbb{C}^{2n+d}$, and for $l \ge r$, we denote by $\pi^l_r : E^l \to E^r$ the natural projection from E^l onto E^r. We define Φ by

$$\Phi(\theta, \Lambda, \Lambda', z', \omega) := T\bigg(z', P^\beta\big(\theta, \pi^l_{|\beta|}(\Lambda, \Lambda', z'), \omega\big)_{0\le|\beta|\le l}\bigg). \tag{8.17}$$

To show that Φ is in $(\mathcal{R}^l_{2n+d})^N$, we must differentiate the right hand side of (8.17) with respect to $x_1 = (\Lambda', z', \omega)$ and evaluate at $x_1 = 0$.

By using the chain rule and the fact that each P^β is in $(\mathcal{R}^{|\beta|}_{2n+d})^d$ and vanishes when $x_1=0$, it is easy to check that for any multiindex α,

$$\frac{\partial^\alpha}{\partial x_1^\alpha} T\Big(z', P^\beta\big(\theta, \pi^l_{|\beta|}(\Lambda,\Lambda',z'),\omega\big)_{0\le|\beta|\le l}\Big)\Big|_{x_1=0}$$

is a polynomial in x_0. This proves the claim (8.16).

We now differentiate the identity (8.16) with respect to $\zeta=(\chi,\tau,u)$. By using Lemma 8.3 again, we find $\Phi^\beta \in (\mathcal{R}^{l+|\beta|}_{2n+d})^N$, independent of H, such that

$$\partial^\beta \overline{H}(\zeta) = \Phi^\beta\Big(\frac{1}{\det F_z(Z(\omega))}, \mathcal{J}^{l+|\beta|}H(Z(\omega)),\omega\Big) + O(|\omega|^{k-l-|\beta|}). \tag{8.18}$$

For any $\beta\in\mathbb{Z}^N_+$ we decompose $\Phi^\beta=(\Phi^\beta_1,\Phi^\beta_2)\in\mathbb{C}^n\times\mathbb{C}^d$ and set for $\theta,\Lambda,\Lambda',z',\omega$ as above and $Z=(z,w,v)\in\mathbb{C}^n\times\mathbb{C}^{d_1}\times\mathbb{C}^{d_2}$,

$$\widetilde{\Phi}^\beta(\theta,\Lambda,\Lambda',z',Z,\zeta) := \tag{8.19}$$
$$\begin{cases} \Phi^0(\theta,\Lambda,\Lambda',z',\omega)-\Phi^0(\theta,\Lambda,0,0,0) & \text{for } \beta=0,\\ \big(\Phi^\beta_1(\theta,\Lambda,\Lambda',z',\omega), \\ \quad \Phi^\beta_2(\theta,\Lambda,\Lambda',z',\omega)-\Phi^\beta_2(\theta,\Lambda,0,0,0)\big) & \text{for } \beta\in\mathbb{Z}^n_+\times\{0\}, \beta\neq 0\\ \Phi^\beta(\theta,\Lambda,\Lambda',z',\omega) & \text{otherwise .}\end{cases}$$

Clearly $\widetilde{\Phi}^\beta$ is in $(\mathcal{R}^{l+|\beta|}_{2N})^N$ and is independent of w and v. Since for any k-equivalence $H=(F,G)$ we have $\partial^\beta G(0)=0$ for $\beta\in\mathbb{Z}^n_+\times\{0\}$ with $|\beta|<k$ by (7.6), it follows from (8.18) that $\widetilde{\Phi}^\beta_2\big(\frac{1}{\det F_z(0)},\mathcal{J}^{l+|\beta|}H(0),0\big)=0$. Hence (8.18) implies

$$\partial^\beta\overline{H}(\zeta)=\widetilde{\Phi}^\beta\left(\frac{1}{\det F_z(Z)},\mathcal{J}^{l+|\beta|}H(Z),Z,\zeta\right)+\widetilde{R}^\beta_H(Z,\zeta), \tag{8.20}$$

where R^β_H is a germ at 0 of a holomorphic map from $\mathbb{C}^{2N}$ to $\mathbb{C}^N$ depending on H and whose restriction to $\mathcal{M}$ vanishes at 0 of order $k-l-|\beta|$. By taking complex conjugates of (8.20) for $0\le|\beta|\le r$, and using the fact that $(Z,\zeta)\in\mathcal{M}$ is equivalent to $(\overline{\zeta},\overline{Z})\in\mathcal{M}$, we obtain (8.3) with Ψ^r satisfying the conclusion of Theorem 8.1. q.e.d.

9. The iterated basic identity

In this section we apply the relation given by Theorem 8.1 to different points and iterate them, i.e., substitute one into the next and so on. Let $(M,0)$ and $(M',0)$ satisfy the assumptions of Theorem 8.1. If $\rho(Z,\overline{Z})$ is a defining function of M near 0 and $s \geq 1$ is an integer, we define a germ $\mathcal{M}^{2s}$ at 0 of a complex manifold of $\mathbb{C}^{(2s+1)N}$ by

$$\begin{aligned}(9.1)\quad \mathcal{M}^{2s} := \big\{&(Z,\zeta^1,Z^1,\dots,\zeta^s,Z^s) \in \mathbb{C}^{(2s+1)N} :\\ &\rho(Z,\zeta^1) = \cdots = \rho(Z^{s-1},\zeta^s) = \rho(Z^1,\zeta^1) = \cdots = \rho(Z^s,\zeta^s) = 0\big\}.\end{aligned}$$

Hence $\mathcal{M}^{2s}$ has codimension $2sd$ in $\mathbb{C}^{(2s+1)N}$, where d is the codimension of M in $\mathbb{C}^N$. (The iterated complexification $\mathcal{M}^{2s}$ was introduced by the third author in [25]. For Z_s fixed in (9.1), this corresponds to the Segre manifold of order $2s$ of M at Z_s in the terminology of [4].) For a k-equivalence H between $(M,0)$ and $(M',0)$, we use the notation $\mathcal{J}^r H(Z)$ introduced in (7.4). It will be also convenient to write

$$j^r H(Z) := \left(\frac{\partial^{|\alpha|} H}{\partial Z^\alpha}(Z)\right)_{0 \leq |\alpha| \leq r},$$

which is the first $J^r_{N,N}$-valued component of $\mathcal{J}^r H(Z)$. The main result of this section is the following:

Theorem 9.1. *Under the assumptions of Theorem 8.1, for all integers $r \geq 0$ and $s \geq 1$, there exists a polynomial q^r_s on $J^{r+2sl}_{N,N}$ and, for $r > 0$, a germ*

$$(9.2)\qquad \Psi^{r,s} \colon \big(\mathbb{C} \times E^{r+2sl} \times \mathbb{C}^{(2s+1)N}, \mathbb{C} \times E_0^{r+2sl} \times \{0\}\big) \mapsto (E^r, E^r_0)$$

and for $r = 0$, a germ $\Psi^{0,s} \colon \big(\mathbb{C} \times E^{2sl} \times \mathbb{C}^{(2s+1)N}, \mathbb{C} \times E_0^{2sl} \times \{0\}\big) \mapsto (E^0, 0)$, whose components are in the ring $\mathcal{R}^{r+2sl}_{(2s+1)N}$ such that, if $H = (F,G)$ is a k-equivalence between $(M,0)$ and $(M',0)$ with $k > 2sl + r$, the following holds:

$$(9.3)\qquad q^r_s(j^{r+2sl} H(0)) = \big(\det F_z(0)\big)^{a^r_s} \overline{\big(\det F_z(0)\big)}^{b^r_s}$$

for some $a^r_s, b^r_s \in \mathbb{Z}_+$,

$$\begin{aligned}(9.4)\quad &\mathcal{J}^r H(Z)\\ &\quad = \Psi^{r,s}\left(\frac{1}{q^r_s(j^{r+2sl}H(Z^s))}, \mathcal{J}^{r+2sl}H(Z^s), Z, \zeta^1, Z^1, \dots, \zeta^s, Z^s\right)\\ &\qquad + R^{r,s}_H(Z,\zeta^1,Z^1,\dots,\zeta^s,Z^s),\end{aligned}$$

where $R^{r,s}_H$ is a germ at 0 of a holomorphic map from $\mathbb{C}^{(2s+1)N}$ to E^r, depending on H, whose restriction to $\mathcal{M}^{2s}$ vanishes of order $k - r - 2sl$ at 0.

Note that since H is a k-equivalence, it follows that $\det F_z(0) \neq 0$, and hence the right hand side of (9.3) is necessarily nonvanishing.

Proof. We prove the theorem by induction on $s \geq 1$. We start first with the case $s = 1$ and assume that $H = (F, G)$ is a k-equivalence between $(M, 0)$ and $(M', 0)$ with $k > r + 2l$. By conjugating (8.3) with r replaced by $r + l$ we obtain

$$\tag{9.5} \begin{aligned} &\mathcal{J}^{r+l}\overline{H}(\zeta) \\ &\qquad \equiv \overline{\Psi^{r+l}}\left(\frac{1}{\det(F_z(Z^1))}, \mathcal{J}^{r+2l}H(Z^1), Z^1, \zeta\right) + \overline{R_H^{r+l}}(\zeta, Z^1) \end{aligned}$$

with Ψ^{r+l} and R_H^{r+l} as in Theorem 8.1. If we observe that $(Z, \zeta) \in \mathcal{M} \iff (\bar{\zeta}, \overline{Z}) \in \mathcal{M}$, we conclude that the second term on the right hand side of (9.5) vanishes at 0 of order $k - r - 2l$ when $(Z^1, \zeta) \in \mathcal{M}$. Our next goal will be to substitute (9.5) into (8.3) and to apply Lemma 6.1. For this, we define polynomials $q \in \mathbb{C}[\Lambda]$ and $\widetilde{q} \in \mathbb{C}[\widetilde{\Lambda}]$, for $\Lambda = x_0 \in V_0 := E_0^{r+2l} \cong J_{N,N}^{r+2l}$ and $\widetilde{\Lambda} = \widetilde{x}_0 \in \widetilde{V}_0 := E_0^{r+l} \cong J_{N,N}^{r+l}$, to be the determinants of the parts of the jets Λ and $\widetilde{\Lambda}$ obtained from the first n rows and first n columns of the linear terms of Λ and $\widetilde{\Lambda}$ respectively (i.e., corresponding to $\det F_z(Z)$ and to $\det \overline{F}_\chi(\zeta)$ for $\Lambda = j^{r+2l}H(Z)$ and $\widetilde{\Lambda} = j^{r+l}\overline{H}(\zeta)$ respectively). We also set $x_1 = (\Lambda', z', Z, \zeta, Z^1) \in V_1 := J_{n,d}^{r+2l} \times \mathbb{C}^n \times \mathbb{C}^N \times \mathbb{C}^N \times \mathbb{C}^N \cong E_1^{r+2l} \times \mathbb{C}^{3N}$, $\widetilde{V}_1 := E_1^{r+l} \times \mathbb{C}^{2N}$ and for $\theta \in \mathbb{C}$,

$$\begin{aligned} &\phi(\theta, \Lambda, \Lambda', z', Z, \zeta, Z^1) \\ &\qquad := \left(\overline{\Psi^{r+l}}(\theta, \Lambda, \Lambda', z', Z^1, \zeta), \zeta, Z\right) \in E^{r+l} \times \mathbb{C}^N \times \mathbb{C}^N. \end{aligned}$$

(Observe that $E^{r+l} \times \mathbb{C}^{2N} = \widetilde{V}_0 \times \widetilde{V}_1$ by the definition of $\widetilde{V}_0$ and $\widetilde{V}_1$ above.) Then ϕ satisfies the assumptions of Lemma 6.1, in particular, (6.3) holds since by (9.5) we have

$$\tag{9.6} \widetilde{q}\left(\overline{\Psi_0^{r+l}}\left(\frac{1}{q(j^{r+2l}H(0))}, \mathcal{J}^{r+2l}H(0), 0, 0\right)\right) = \det \overline{F}_\chi(0),$$

and the right hand side of (9.6) is nonvanishing whenever $H = (F, G)$ is a k-equivalence with $k > 1$.

From substituting (9.5) into (8.3) we obtain the identity

$$
(9.7)\quad \mathcal{J}^r H(Z) \equiv \Psi^r \left(\frac{1}{\widetilde{q}\Big(\overline{\Psi_0^{r+l}}\Big(\frac{1}{q(j^{r+2l}H(Z^1))}, \mathcal{J}^{r+2l}H(Z^1), Z^1, \zeta\Big)\Big)}, \right.
$$
$$
\left. \overline{\Psi^{r+l}}\left(\frac{1}{q(j^{r+2l}H(Z^1))}, \mathcal{J}^{r+2l}H(Z^1), Z^1, \zeta \right), \zeta, Z \right) + R_H^{r,1}(Z,\zeta,Z^1),
$$

where the restriction of $R_H^{r,1}$ to $\mathcal{M}^2 \subset \mathbb{C}^{3N}$ vanishes of order $k-r-2l$ at the origin. Then for $s=1$, (9.4) is a consequence of Lemma 6.1 with q_1^r being the polynomial p given by the lemma. The required property (9.3) follows from (9.6) and from the explicit formula for p in the lemma.

Now we assume that (9.3) and (9.4) hold for some fixed $s \geq 1$ and any $r \geq 0$ and shall prove them for $s+1$ and any $r \geq 0$. We replace the terms $j^{r+2sl}H(Z^s)$ and $\mathcal{J}^{r+2sl}H(Z^s)$ by using (9.4) with $s=1$ and r replaced by $r+2sl$. We obtain

$$
\begin{aligned}
(9.8)\quad & \mathcal{J}^r H(Z) \\
&\equiv \Psi^{r,s}\left(\frac{1}{q_s^r(\Psi_0^{r+2sl,1}(\alpha))}, \Psi^{r+2sl,1}(\alpha), Z, \zeta^1, Z^1, \dots, \zeta^s, Z^s \right) \\
&\quad + R_H^{r,s+1}(Z, \zeta^1, Z^1, \dots, \zeta^{s+1}, Z^{s+1})
\end{aligned}
$$

where

$$
\alpha = \left(\frac{1}{q_1^{r+2sl}(j^{r+2l(s+1)}H(Z^{s+1}))}, \mathcal{J}^{r+2l(s+1)}H(Z^{s+1}), Z^s, \zeta^{s+1}, Z^{s+1} \right)
$$

with the restriction of $R_H^{r,s+1}$ to $\mathcal{M}^{2(s+1)} \subset \mathbb{C}^{(2(s+1)+1)N}$ vanishing of order $k-r-2l(s+1)$. Similarly to the preceeding proof of (9.4) for $s=1$, the desired conclusion of the theorem follows by making use of Lemma 6.1. q.e.d.

10. Reducing the number of parameters

The expression in the right-hand side of (9.4) depends on $(2s+1)N$ complex variables. Our goal in this section will be to reduce the variables to only N variables, namely $Z=(z,w,u) \in \mathbb{C}^N$. The main result of this section is the following:

Theorem 10.1. *Under the assumptions of Theorem 8.1, there is an integer $s \geq 0$, a germ of a holomorphic map*

$$\Gamma\colon \big(\mathbb{C} \times E^{2sl} \times \mathbb{C}^N, \mathbb{C} \times E_0^{2sl} \times \{0\}\big) \to (\mathbb{C}^N, 0) \tag{10.1}$$

with components in the ring $\mathcal{R}_N^{2sl}$, and an integer $r \geq 1$ such that for every k-equivalence H between $(M,0)$ and $(M',0)$ with $k > 2sl$, one has for $Z = (z,w,u)$ sufficiently small,

$$H(Z) = \Gamma\left(\frac{1}{q(j^{2sl}H(0,0,u))}, \mathcal{J}^{2sl}H(0,0,u), Z\right) + O\left(\frac{k-2sl}{r}\right), \tag{10.2}$$

where q is the polynomial q_s^0 on $J_{N,N}^{2sl}$ given by Theorem 9.1.

Remark 10.2. The proof of Theorem 10.1 shows that the integer $s \geq 0$ in this theorem can be chosen to be the *Segre number* of M at 0 introduced in [4]. In particular, $s = 0$ if and only if M is totally real, in which case the conclusion of Theorem 10.1 is obvious since $Z = u$. In all other cases we have $s \geq 1$.

Before proving Theorem 10.1, we shall state the following corollary, which is of independent interest.

Corollary 10.3. *Under the assumptions of Theorem 8.1 a formal equivalence H between $(M,0)$ and $(M',0)$ is convergent if and only if the power series $j^{2sl}H(0,0,u)$ is convergent in $u \in \mathbb{C}^{d_2}$.*

Proof of Corollary 10.3. Suppose that H is a formal equivalence. If H is convergent, it is clear that $j^{2sl}H(0,0,u)$ is also convergent. Conversely, if $j^{2sl}H(0,0,u)$ is convergent, then the first term on the right hand side of (10.2) is a convergent power series in Z by composition. Since H is a k-equivalence for every k, the remainder term is 0, and hence $H(Z)$ is also convergent by (10.2). q.e.d.

For the proof of Theorem 10.1, we begin by defining inductively a sequence of germs of holomorphic maps

$$V^\kappa\colon (\mathbb{C}^{\kappa n} \times \mathbb{C}^{d_2}, 0) \to (\mathbb{C}^N, 0), \quad \kappa = 0, 1, \dots,$$

as follows. As before, we choose a holomorphic map Q from a neighborhood of 0 in $\mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^{d_1} \times \mathbb{C}^{d_2}$ to $\mathbb{C}^{d_1}$ satisfying (3.2) so that M is given near 0 by (7.2). We put $V^0(u) := (0,0,u) \in \mathbb{C}^n \times \mathbb{C}^{d_1} \times \mathbb{C}^{d_2}$ and

$$\begin{aligned} &V^{\kappa+1}(t^0, t^1, \dots, t^\kappa, u) \\ &\qquad := \big(t^0, Q(t^0, \overline{V^\kappa}(t^1, \dots, t^\kappa, u)), u\big) \in \mathbb{C}^n \times \mathbb{C}^{d_1} \times \mathbb{C}^{d_2} \end{aligned} \tag{10.3}$$

for $\kappa \geq 0$, $t^0, t^1, \ldots, t^\kappa \in \mathbb{C}^n$ and $u \in \mathbb{C}^{d_2}$. It is easy to check that for $\kappa \geq 0$,

$$\big(V^{\kappa+1}(t^0, t^1, \ldots, t^\kappa, u), \overline{V^\kappa}(t^1, \ldots, t^\kappa, u)\big) \in \mathcal{M}, \tag{10.4}$$

and hence also

$$\big(V^{\kappa}(t^1, \ldots, t^\kappa, u), \overline{V^{\kappa+1}}(t^0, t^1, \ldots, t^\kappa, u)\big) \in \mathcal{M}. \tag{10.5}$$

It will be convenient to introduce for every $s \geq 1$, the germ at 0 of a holomorphic map

$$\begin{aligned}\Xi^s(t^0, \ldots, t^{2s-1}, u) :=& \\ \big(V^{2s}(t^0, ..., t^{2s-1}, u), &\overline{V^{2s-1}}(t^1, ..., t^{2s-1}, u), ..., \overline{V^1}(t^{2s-1}, u), V^0(u)\big).\end{aligned} \tag{10.6}$$

Observe that the map

$$\begin{aligned}\mathbb{C}^{2sn} \times \mathbb{C}^{d_2} \ni (t^0, \ldots, t^{2s-1}, u)& \\ \mapsto \Xi^s(t^0, \ldots, t^{2s-1}, u) &\in \mathcal{M}^{2s} \subset \mathbb{C}^{(2s+1)N}\end{aligned} \tag{10.7}$$

parametrizes a germ at 0 of the submanifold of $\mathcal{M}^{2s}$ given by

$$\big\{(Z, \zeta^1, Z^1, \ldots, \zeta^s, Z^s) \in \mathcal{M}^{2s} : Z^s = (0, 0, u)\big\}.$$

In this notation we have the following consequence of Theorem 9.1.

Corollary 10.4. *Under the assumptions of Theorem 8.1, for any integer $s \geq 1$, there exists a germ of a holomorphic map*

$$\Phi^s \colon \big(\mathbb{C} \times E^{2sl} \times \mathbb{C}^{2sn+d_2}, \mathbb{C} \times E_0^{2sl} \times \{0\}\big) \to (\mathbb{C}^N, 0) \tag{10.8}$$

whose components are in the ring $\mathcal{R}^{2sl}_{2sn+d_2}$ such that, if H is a k-equivalence between $(M, 0)$ and $(M', 0)$ with $k > 2sl$, then

$$\begin{aligned}&H(V^{2s}(t^0, \ldots, t^{2s-1}, u)) \\ &\quad\equiv \Phi^s\Big(\frac{1}{q_s^0(j^{2sl}H(0,0,u))}, \mathcal{J}^{2sl}H(0,0,u), t^0, \ldots, t^{2s-1}, u\Big) \\ &\qquad + r_H^s(t^0, \ldots, t^{2s-1}, u),\end{aligned} \tag{10.9}$$

where q_s^0 is the polynomial given by Theorem 9.1 and r_H^s is a germ at 0 of a holomorphic map from $\mathbb{C}^{2ns+d_2}$ to $\mathbb{C}^N$ that vanishes of order $k-2sl$ at the origin.

Proof. We use (9.4) for $r=0$ and substitute $\Xi^s(t^0,\ldots,t^{2s-1},u)$ for $(Z,\zeta^1,Z^1,\ldots,\zeta^s,Z^s)$, where Ξ^s is given by (10.6). The corollary easily follows by taking

$$\Phi^s(\theta,\Lambda,\Lambda',z',t^0,\ldots,t^{2s-1},u) := \Psi^{0,s}\big(\theta,\Lambda,\Lambda',z',\Xi^s(t^0,\ldots,t^{2s-1},u)\big)$$

and $r_H^s := R^{0,s}\circ\Xi^s$. q.e.d.

We next define a sequence of germs v^κ at 0 of holomorphic maps from $\mathbb{C}^{\kappa n}$ to $\mathbb{C}^{n+d_1}$, $\kappa\geq 0$, by

$$V^s(t^0,\ldots,t^{\kappa-1},u)|_{u=0} = \big(v^\kappa(t^0,\ldots,t^{\kappa-1}),0\big)\in\mathbb{C}^{n+d_1}\times\mathbb{C}^{d_2}. \tag{10.10}$$

Recall that the submanifold $M_0\subset\mathbb{C}^{n+d_1}$ defined by (3.3) is of finite type at 0. The map v^κ defined above is the κth *Segre map* of M_0 in the sense of [5]. Hence by [5] (Theorem 3.1.9) the generic rank of v^κ equals $n+d_1$ for κ sufficiently large. (See also [3] for a different and simpler proof of this result.) As in [4] we call the smallest such κ the *Segre number* of M_0 at 0 and denote it by s. By [5] (Lemma 4.1.3) we have

$$\max_{(x^1,\ldots,x^s)\in\mathcal{O}} \operatorname{rk}\frac{\partial v^{2s}}{\partial(t^0,t^{s+1},t^{s+2},\ldots,t^{2s-1})}(\hat p) = n+d_1 \tag{10.11}$$

and

$$v^{2s}(\hat p)\equiv 0, \tag{10.12}$$

with

$$\hat p := (0,x^1,\ldots,x^{s-1},x^s,x^{s-1},\ldots,x^1),$$

where $\mathcal{O}$ is an arbitrary sufficiently small neighborhood of 0 in $\mathbb{C}^{sn}$. Note that in (10.11) we differentiate only with respect to the first vector t^0 and the last $s-1$ vectors $t^{s+1},\ldots,t^{2s-1}$.

For the proof of Theorem 10.1, we shall also need the following special case of Proposition 4.1.18 in [5].

Lemma 10.5. *Let*

$$V\colon(\mathbb{C}^{r_1}\times\mathbb{C}^{r_2},0)\to(\mathbb{C}^N,0),\quad r_2\geq N,$$

be a germ of a holomorphic map satisfying $V(x,\xi)|_{\xi=0}\equiv 0$, with $(x,\xi)\in\mathbb{C}^{r_1}\times\mathbb{C}^{r_2}$, and for any sufficiently small neighborhood $\mathcal{O}$ of 0 in $\mathbb{C}^{r_1}$

$$\max_{x\in\mathcal{O}}\left\{\operatorname{rk}\frac{\partial V}{\partial\xi}(x,0)\right\} = N. \tag{10.13}$$

Then there exist germs of holomorphic maps

$$\delta\colon (\mathbb{C}^{r_1},0)\to\mathbb{C},\quad \delta(x)\not\equiv 0,\quad \phi\colon (\mathbb{C}^{r_1}\times\mathbb{C}^N,0)\to(\mathbb{C}^{r_2},0) \tag{10.14}$$

satisfying

$$V\left(x,\phi\left(x,\frac{Z}{\delta(x)}\right)\right)\equiv Z \tag{10.15}$$

for all $(x,Z)\in\mathbb{C}^{r_1}\times\mathbb{C}^N$ *such that* $\delta(x)\neq 0$ *and both* x *and* $Z/\delta(x)$ *are sufficiently small.*

Proof of Theorem 10.1. We shall take s to be the Segre number of M_0 at 0. In the notation of Lemma 10.5 we take $x=(x^1,\dots,x^s)\in\mathbb{C}^{sn}$, $\xi=(y,u)=(y^0,y^1,\dots,y^{s-1},u)\in\mathbb{C}^{sn}\times\mathbb{C}^{d_2}$ and set

$$\begin{aligned} L(x,y,u) &:= (y^0,x^1,\dots,x^s,x^{s-1}+y^{s-1},\dots,x^1+y^1,u),\\ V(x,\xi)&=V(x,y,u):=V^{2s}(L(x,y,u)), \end{aligned} \tag{10.16}$$

where V^{2s} is defined by (10.3). Here $r_1:=sn$ and $r_2:=sn+d_2$. Observe that L is a linear automorphism of $\mathbb{C}^{2sn+d_2}$. It follows from (10.10) and (10.12) that $V(x,0)\equiv 0$. Furthermore it follows from (10.3), (10.10) and (10.11) that condition (10.13) also holds. Hence we can apply Lemma 10.5. Let

$$\delta\colon(\mathbb{C}^{sn},0)\to\mathbb{C},\quad \phi\colon(\mathbb{C}^{sn+N},0)\to(\mathbb{C}^{sn+d_2},0)$$

be given by the lemma, so that (10.15) holds. By Corollary 10.4, we obtain

$$\begin{aligned} &H(Z)\\ &\quad\equiv \Phi^s\left(\frac{1}{q_s^0(j^{2sl}H(0,0,u))},\mathcal{J}^{2sl}H(0,0,u),L\left(x,\phi\left(x,\frac{Z}{\delta(x)}\right)\right)\right)\\ &\qquad + r_H^s\left(L\left(x,\phi\left(x,\frac{Z}{\delta(x)}\right)\right)\right), \end{aligned} \tag{10.17}$$

with $Z=(z,w,u)$. By a simple change of Φ^s and r_H^s, we obtain from (10.17) the equivalent identity

$$\begin{aligned} H(Z)=\widetilde{\Phi}^s&\left(\frac{1}{q_s^0(j^{2sl}H(0,0,u))},\mathcal{J}^{2sl}H(0,0,u),\frac{Z}{\delta(x)},x\right)\\ &+\widetilde{r}_H^s\left(\frac{Z}{\delta(x)},x\right) \end{aligned} \tag{10.18}$$

for all $(x,Z)\in\mathbb{C}^{ns+N}$ such that $\delta(x)\neq 0$ and both x and $Z/\delta(x)$ are sufficiently small. Here δ and $\widetilde{\Phi}^s$ are independent of H, the components of $\widetilde{\Phi}^s$ are in the ring $\mathcal{R}^{2sl}_{sn+N}$ and $\widetilde{r}^s_H$ is a germ at 0 in $\mathbb{C}^{sn+N}$, depending on H and vanishing of order $k-2sl$ at 0.

Observe that the left-hand side of (10.18) is independent of the parameter $x\in\mathbb{C}^{ns}$, whereas the right-hand side contains this parameter. We choose $x_0\in\mathbb{C}^{ns}$ such that the function $\widetilde{\lambda}\mapsto\delta(\widetilde{\lambda}x_0)$ does not vanish identically for $\widetilde{\lambda}$ in a neighborhood of 0 in $\mathbb{C}$, and put $x=\widetilde{\lambda}x_0$ in (10.18). For convenience we consider a holomorphic change of variable $\lambda=h(\widetilde{\lambda})$ near the origin in $\mathbb{C}$, where h is determined by the identity $\delta(\widetilde{\lambda}x_0)=\lambda^m$ for an appropriate integer $m\geq 0$. By a further simple change of $\widetilde{\Phi}^s$ and $\widetilde{r}^s_H$, we conclude from (10.18) that the identity

$$H(Z)\equiv\widehat{\Phi}^s\left(\frac{1}{q^0_s(j^{2sl}H(0,0,u))},\mathcal{J}^{2sl}H(0,0,u),\frac{Z}{\lambda^m},\lambda\right)+\widehat{r}^s_H\left(\frac{Z}{\lambda^m},\lambda\right), \tag{10.19}$$

holds for all $(\lambda,Z)=(\lambda,z,w,u)\in\mathbb{C}^{1+N}$ such that $\lambda\neq 0$ and both λ and Z/λ^m are sufficiently small. Again $\widehat{\Phi}^s$ is independent of H and its components are in the ring $\mathcal{R}^{2sl}_{N+1}$ and $\widehat{r}^s_H$ is a germ at 0 in $\mathbb{C}^{N+1}$, depending on H and vanishing of order $k-2sl$ at 0.

We next expand both sides of (10.19) in Laurent series in λ and equate the constant terms. The required properties of those terms are established in the following lemma.

Lemma 10.6. *Let V_0 and V_1 be finite-dimensional vector spaces with fixed linear coordinates x_0 and x_1 respectively, and $P(x_0,x_1,\lambda)$ be in the ring $\mathcal{R}(V_0\times V_1\times\mathbb{C},V_0)$ with $P(x_0,0,0)\equiv 0$. For a fixed integer $m\geq 0$, consider the Laurent series expansion*

$$P\left(x_0,\frac{x_1}{\lambda^m},\lambda\right)=\sum_{\nu\in\mathbb{Z}}c_\nu(x_0,x_1)\lambda^\nu.$$

Then $c_0(x_0,0)\equiv 0$ and, for every $\nu\in\mathbb{Z}$, c_ν is in the ring $\mathcal{R}(V_0\times V_1,V_0)$. In addition, if $P=O(K)$ for some integer $K>0$, then $c_\nu=O\left(\frac{K-\nu}{m+1}\right)$ for all $\nu\in\mathbb{Z}$ such that $\nu\leq K$.

Proof. We expand P in power series of the form

$$P(x_0,x_1,\lambda)=\sum P_{\beta,\mu}(x_0)x_1^\beta\lambda^\mu=\sum P_{\alpha,\beta,\mu}x_0^\alpha x_1^\beta\lambda^\mu, \tag{10.20}$$

with $P_{0,0}(x_0) \equiv 0$, where $\alpha \in \mathbb{Z}_+^{\dim V_0}$, $\beta \in \mathbb{Z}_+^{\dim V_1}$, $\mu \in \mathbb{Z}_+$. Then $P_{\beta,\mu}$ is a polynomial in x_0 satisfying the estimates (6.2). Since

$$c_\nu(x_0,x_1) = \sum_{\substack{\beta,\mu \\ \mu-m|\beta|=\nu}} P_{\beta,\mu}(x_0)x_1^\beta = \sum_{\substack{\alpha,\beta,\mu \\ \mu-m|\beta|=\nu}} P_{\alpha,\beta,\mu}x_0^\alpha x_1^\beta, \tag{10.21}$$

we conclude that $c_0(x_0,0) \equiv P_{0,0}(x_0) \equiv 0$ and $c_\nu \in \mathcal{R}(V_0 \times V_1, V_0)$ for every $\nu \in \mathbb{Z}$. Now assume that $P = O(K)$. This means that $\mu + |\alpha| + |\beta| \geq K$ holds whenever $P_{\alpha,\beta,\mu} \neq 0$. For fixed $\nu \in \mathbb{Z}$, this inequality together with $\mu = \nu + m|\beta|$ implies, in particular, that $\nu + (m+1)(|\alpha| + |\beta|) \geq K$ in the last sum of (10.21) or, equivalently, $|\alpha| + |\beta| \geq \frac{K-\nu}{m+1}$ in that sum. This completes the proof of the lemma. q.e.d.

We now complete the proof of Theorem 10.1. We expand the right-hand side of (10.19) in Laurent series in λ. Since the left-hand side is independent of λ, we equate it to the constant term of the Laurent series. The required conclusion of Theorem 10.1 with $r := m+1$ follows by applying Lemma 10.6 for $\nu = 0$ to $\widehat{\Phi}^s$ with $V_0 := \mathbb{C} \times E_0^{2sl}$, $V_1 = E_1^{2sl} \times \mathbb{C}^N$ and to $\widehat{r}_H^s$ with $V_0 := 0$, $V_1 := \mathbb{C}^N$. The proof of Theorem 10.1 is now complete. q.e.d.

11. Equations in jet spaces

In Theorem 10.1 we showed that every k-equivalence H between $(M,0)$ and $(M',0)$ for k sufficiently large satisfies the identity (10.2), i.e., is parametrized up to the given order by the jet $j^{2sl}H(0,0,u)$. However, it will be more convenient to regard

$$\Theta_H(u) := \left(\frac{1}{q(j^{2sl}H(0,0,u))}, \mathcal{J}^{2sl}H(0,0,u) \right)$$

as the main parameter since the parametrization then becomes polynomial rather than rational. Our goal in this section is to give a set of equations such that any germ at 0 of a real-analytic map $\mathbb{R}^{d_2} \ni u \mapsto \Theta(u) \in \mathbb{C} \times E^{2sl}$ satisfies these equations if and only if the mapping $\mathbb{C}^N \ni (z,w,u) \mapsto \Gamma(\Theta(u),z,w,u) \in \mathbb{C}^N$ is a germ at 0 of a holomorphic self map of $\mathbb{C}^N$ sending M into M'; here Γ is the mapping given by (10.1).

We shall need a real analogue of the ring $\mathcal{R}(V,V_0)$ defined in §6. Given a finite dimensional *real* vector space W and a real vector subspace

$W_0 \subset W$, define $\mathcal{R}_{\mathbb{R}}(W, W_0)$ to be the ring of all germs of real valued real-analytic functions f at W_0 in W such that all partial derivatives $\partial^\alpha f|_{W_0}$ are real polynomial functions on W_0. In the following we shall consider the spaces $\mathbb{C}$ and E^{2sl} as real vector spaces and real-analytic functions on these spaces with respect to real and imaginary parts of vectors in these spaces.

Theorem 11.1. *Assume that the conditions of Theorem 10.1 are satisfied, and let s, q, and Γ be given by that theorem. Then there exist a finite collection of functions $f_j \in \mathcal{R}_{\mathbb{R}}\big(\mathbb{C} \times E^{2sl} \times \mathbb{R}^{d_2}, \mathbb{C} \times E_0^{2sl} \times \{0\}\big)$, $1 \le j \le j_0$, and positive real numbers a and b, with $b \ge 2sla$, such that the following hold:*

(i) *For every k-equivalence H between $(M, 0)$ and $(M', 0)$ with $k > b/a$, one has*

$$f_j\left(\frac{1}{q(j^{2sl}H(0,0,u))}, \mathcal{J}^{2sl}H(0,0,u), u\right) = O(|u|^{ak-b}),\ 1 \le j \le j_0. \tag{11.1}$$

(ii) *For every germ $\Theta\colon (\mathbb{R}^{d_2}, 0) \to (\mathbb{C} \times E^{2sl}, \mathbb{C} \times E_0^{2sl})$ of a real-analytic map satisfying*

$$f_j(\Theta(u), u) \equiv 0, \quad 1 \le j \le j_0, \tag{11.2}$$

the germ $\Gamma_\Theta\colon (\mathbb{C}^n \times \mathbb{C}^{d_1} \times \mathbb{R}^{d_2}, 0) \to (\mathbb{C}^N, 0)$ of the real-analytic map defined by

$$\Gamma_\Theta(z, w, u) := \Gamma(\Theta(u), z, w, u), \tag{11.3}$$

extends to a germ at 0 of a holomorphic map of $\mathbb{C}^N$ into itself sending $(M, 0)$ into $(M', 0)$.

Remark 11.2. It should be mentioned that the holomorphic extension of the germ Γ_Θ defined by (11.3) need not be invertible.

Before starting the proof of Theorem 11.1 we shall need a composition lemma for the rings $\mathcal{R}_{\mathbb{R}}(W, W_0)$ whose complex analogue is a special case of Lemma 6.1.

Lemma 11.3. *Let W_0, W_1 and $\widetilde{W}$ be finite-dimensional real vector spaces with fixed bases. Denote by x_0, x_1 and $\widetilde{x}$ the corresponding real linear coordinates in these spaces. Let $\phi\colon (W_0 \times W_1, W_0) \to (\widetilde{W}, 0)$ be* *a germ at W_0 of a real-analytic map whose components are in the ring $\mathcal{R}_{\mathbb{R}}(W_0 \times W_1, W_0)$. Then, for every germ $\widetilde{f}\colon (\widetilde{W}, 0) \to \mathbb{R}$ of a real-analytic map, there exists $f \in \mathcal{R}_{\mathbb{R}}(W_0 \times W_1, W_0)$ such that $f(x_0, x_1) = \widetilde{f}(\phi(x_0, x_1))$.*

Lemma 11.3 is a straightforward consequence of the chain rule and is left to you, gentle reader.

Proof of Theorem 11.1. We continue to work with normal coordinates near the origin $Z = (z, w, u)$ for M. We fix a local parametrization of M at 0 of the form

$$\mathbb{R}^{2n+d_1} \times \mathbb{R}^{d_2} \ni (t, u) \mapsto (z(t, u), w(t, u), u) \in M \subset \mathbb{C}^N.$$

Let $\rho'(Z', \overline{Z'}) = \big(\rho^{1\prime}(Z', \overline{Z'}), \dots, \rho^{d\prime}(Z', \overline{Z'})\big)$ be a defining function for M' near 0 and Γ be given by Theorem 10.1. For $1 \le i \le d$, $\alpha \in \mathbb{Z}_+^{2n+d_1}$ and $\Theta \in \mathbb{C} \times E^{2sl}$, we consider the functions

(11.4)
$$f_\alpha^i(\Theta, u) := \frac{\partial}{\partial t^\alpha} \rho^{i\prime}\Big(\Gamma\big(\Theta, z(t,u), w(t,u), u\big), \overline{\Gamma\big(\Theta, z(t,u), w(t,u), u\big)}\Big)\Big|_{t=0}.$$

It follows from the properties of Γ, the chain rule and Lemma 11.3 that the f_α^i are in the ring $\mathcal{R}_{\mathbb{R}}\big(\mathbb{C} \times E^{2sl} \times \mathbb{R}^{d_2}, \mathbb{C} \times E_0^{2sl} \times \{0\}\big)$. If follows from the definition that we can think of this ring as a subring of the following formal power series ring with polynomial coefficients

$$\mathbb{R}\big[\operatorname{Re}\theta, \operatorname{Im}\theta, \operatorname{Re}\Lambda, \operatorname{Im}\Lambda\big]\big[\big[\operatorname{Re}\Lambda', \operatorname{Im}\Lambda', \operatorname{Re}z', \operatorname{Im}z', u\big]\big], \tag{11.5}$$

where $(\theta, \Lambda, \Lambda', z')$ are complex coordinates in

$$\mathbb{C} \times J_{N,N}^{2sl} \times J_{n,d}^{2sl} \times \mathbb{C}^n = \mathbb{C} \times E^{2sl}.$$

It is a standard fact from commutative algebra that any formal power series ring with coefficients in a Noetherian ring is again Noetherian; in particular, the ring (11.5) is Noetherian. Hence there exists an integer $m_0 \ge 0$ such that the subset

$$\big\{f_\alpha^i : 1 \le i \le d, |\alpha| \le m_0\big\} \tag{11.6}$$

generates the same ideal in the ring (11.5) as all the f_α^i, $1 \le i \le d$, $\alpha \in \mathbb{Z}_+^{2n+d_1}$.

By the identity (10.2) and the definition of k-equivalence we have for $1 \le i \le d$ and $|\alpha| \le m_0$,

$$
\begin{aligned}
(11.7)\quad & f^i_\alpha\Big(\frac{1}{q(j^{2sl}H(0,0,u))}, \mathcal{J}^{2sl}H(0,0,u), u\Big) \\
&= \frac{\partial}{\partial t^\alpha}\rho^{i\prime}\Big(H\big(z(t,u),w(t,u),u\big), \overline{H\big(z(t,u),w(t,u),u\big)}\Big)\Big|_{t=0} \\
&\quad + O\Big(\frac{k-2sl}{r} - m_0\Big) \\
&= O(k-m_0) + O\Big(\frac{k-2sl}{r} - m_0\Big).
\end{aligned}
$$

Hence we proved (11.1) with the collection f_j, $1 \le j \le j_0$, being the set of functions given by (11.6) and $a := 1/r \le 1$, $b := m_0 + (2sl/r)$. This completes the proof of (i).

We shall now prove (ii). By the choice of the set (11.6), every germ $f^i_\alpha(\Theta, u)$ can be written in the form

$$
f^i_\alpha(\Theta,u) \equiv \sum_{j=1}^{j_0} c_j(\Theta,u) f_j(\Theta,u), \tag{11.8}
$$

where $c_j(\Theta,u)$ are in the ring given by (11.5). Since $\Theta(0) \in \mathbb{C} \times E_0^{2sl}$, the germ $\Theta(u)$ can be substituted for Θ in each $c_j(\Theta,u)$ to obtain a formal power series in $\mathbb{R}[[u]]$. From (11.8) and the assumption (11.2) on $\Theta(u)$ we obtain the following identities of convergent power series in u:

$$
f^i_\alpha\big(\Theta(u),u\big) \equiv 0, \quad 1 \le i \le d, \quad \alpha \in \mathbb{Z}_+^{2n+d_1}.
$$

In view of (11.4) we conclude that $\rho'(\Gamma_\Theta(z,w,u), \overline{\Gamma_\Theta(z,w,u)}) = 0$ for $(z,w,u) \in M$ near the origin. This completes the proof of (ii) and hence that of Theorem 11.1. q.e.d.

12. Artin and Wavrik theorems

We state two approximation results due to Artin [1] and Wavrik [24] which will be used (in conjunction with Theorem 11.1) in the proof of Theorem 5.1. We start by stating the result of Artin, which implies that any formal solution of a system of real-analytic equations may be approximated to any preassigned order by a convergent solution of that system. We use the superscripts f, c, a to denote formal, convergent and approximate solutions respectively.

Theorem 12.1 ([1]). *Let $g_j(t,u) \in \mathbb{R}\{t,u\}$, $1 \le j \le j_0$, be convergent power series in $t = (t_1, \dots, t_\delta)$ and $u = (u_1, \dots, u_\gamma)$. Then for any integer $\kappa \ge 1$ and any formal power series $t^f(u) \in \big(\mathbb{R}[[u]]\big)^\delta$, satisfying*

$$t^f(0) = 0, \quad g_j(t^f(u), u) \equiv 0, \quad 1 \le j \le j_0,$$

there exists a convergent power series $t^c(u) \in \big(\mathbb{R}\{u\}\big)^\delta$ satisfying

$$t^c(u) = t^f(u) + O(|u|^\kappa), \quad g_j(t^c(u), u) \equiv 0, \quad 1 \le j \le j_0.$$

We now turn to a result of Wavrik which states that an approximate formal solution of a system of formal equations of a certain type may be approximated by an exact formal solution of that system. (This result generalizes another result of Artin [2] which deals with more special systems of equations. See also Denef-Lipshitz [15] for related results.)

Theorem 12.2 ([24]). *Let $h_j(x,y,u) \in \mathbb{R}\big[x\big]\big[\big[y,u\big]\big]$, $1 \le j \le j_0$, be formal power series in $y = (y_1, \dots, y_\beta)$ and $u = (u_1, \dots, u_\gamma)$ with coefficients which are polynomials in $x = (x_1, \dots, x_\alpha)$. Then for any integer $\kappa \ge 1$, there exists an integer $\eta \ge 1$ such that, for any formal power series $x^a(u) \in \big(\mathbb{R}[[u]]\big)^\alpha$, $y^a(u) \in \big(\mathbb{R}[[u]]\big)^\beta$ satisfying*

$$y^a(0) = 0, \quad h_j(x^a(u), y^a(u), u) = O(|u|^\eta), \quad 1 \le j \le j_0,$$

there exist formal power series $x^f(u) \in \big(\mathbb{R}[[u]]\big)^\alpha$, $y^f(u) \in \big(\mathbb{R}[[u]]\big)^\beta$ satisfying

$$\begin{gathered} x^f(u) = x^a(u) + O(|u|^\kappa), \quad y^f(u) = y^a(u) + O(|u|^\kappa), \\ h_j(x^f(u), y^f(u), u) \equiv 0, \quad 1 \le j \le j_0. \end{gathered} \tag{12.1}$$

An immediate corollary of Theorems 12.1 and 12.2, which we shall need, is the following.

Corollary 12.3. *Let X, Y, U be real finite-dimensional vector spaces with fixed linear coordinates $x = (x_1, \dots, x_\alpha)$, $y = (y_1, \dots, y_\beta)$, $u = (u_1, \dots, u_\gamma)$ respectively and let $h_j(x,y,u)$, $1 \le j \le j_0$, be germs of functions in the ring $\mathcal{R}_\mathbb{R}(X \times Y \times U, X)$. Then for any integer $\kappa \ge 1$, there exists an integer $\eta \ge 1$ such that, for any germs at 0 of real-analytic maps $x^a \colon (U,0) \to X$, $y^a \colon (U,0) \to Y$ satisfying*

$$y^a(0) = 0, \quad h_j(x^a(u), y^a(u), u) = O(|u|^\eta), \quad 1 \le j \le j_0,$$

there exists germs at 0 *of real-analytic maps* $x^c : (U,0) \to X$, $y^c : (U,0) \to Y$ *satisfying*

$$x^c(u) = x^a(u) + O(|u|^\kappa), \quad y^c(u) = y^a(u) + O(|u|^\kappa),$$
$$h_j(x^c(u), y^c(u), u) \equiv 0, \quad 1 \le j \le j_0.$$

Proof. Let $h_j \in \mathcal{R}_{\mathbb{R}}(X \times Y \times U, X)$, $1 \le j \le j_0$, be given. It follows from the definition of the ring $\mathcal{R}_{\mathbb{R}}(X \times Y \times U, X)$ that h_j can be viewed as an element of $\mathbb{R}\big[x\big]\big[\big[y,u\big]\big]$. By Theorem 12.2, given $\kappa \ge 1$, there exists $\eta \ge 1$ such that if x^a and y^a are as in the corollary, there exist $x^f(u) \in \big(\mathbb{R}[[u]]\big)^\alpha$, $y^f(u) \in \big(\mathbb{R}[[u]]\big)^\beta$ satisfying (12.1). We may now apply Theorem 12.1 with $t = (x - x^a(0), y)$ (and hence $\delta = \alpha + \beta$) to conclude that there exists $t^c(u) = (x^c(u) - x^a(0), y^c(u))$ with x^c, y^c satisfying the conclusion of the corollary. q.e.d.

13. End of proof of Theorem 5.1

We keep the notation used in §11 and, in particular, that of Theorem 11.1. Let s, q, and Γ be given by Theorem 10.1, and let H be a k-equivalence between $(M,0)$ and $(M',0)$ with $k > 2sl$. Define the germ $\Theta_H \colon (\mathbb{R}^{d_2},0) \to (\mathbb{C} \times E^{2sl}, \mathbb{C} \times E_0^{2sl})$ of a real-analytic map by

$$\Theta_H(u) := \left(\frac{1}{q(j^{2sl}H(0,0,u))}, \mathcal{J}^{2sl}H(0,0,u) \right).$$

Recall that $q(j^{2sl}H(0,0,0)) \ne 0$ by (9.3). Let f_j, $1 \le j \le j_0$, be given by Theorem 11.1. By part (i) of that theorem we have for $k > b/a \ge 2sl$,

$$f_j(\Theta_H(u), u) = O(|u|^{ak-b}), \quad 1 \le j \le j_0. \tag{13.1}$$

We shall now apply Corollary 12.3 to the system of equations $f_j(\Theta(u), u) = 0$, $1 \le j \le j_0$. Indeed, if $\Theta \colon (\mathbb{R}^{d_2},0) \to \big(\mathbb{C} \times E^{2sl}, \mathbb{C} \times E_0^{2sl}\big)$ is a germ of a real-analytic map, we may write $\Theta(u) = (x(u), y(u)) \in (\mathbb{C} \times E_0^{2sl}) \times E_1^{2sl}$ so that the system of equations above becomes $f_j(x(u), y(u), u) = 0$, $1 \le j \le j_0$. Given $\kappa > 1$ we conclude by Corollary 12.3 that there exists $\eta \ge 1$ such that, if $ak - b \ge \eta$, the identity (13.1) implies the existence of a germ $\Theta^c \colon (\mathbb{R}^{d_2},0) \to \big(\mathbb{C} \times E^{2sl}, \mathbb{C} \times E_0^{2sl}\big)$ of a real-analytic map satisfying

$$\Theta^c(u) = \Theta_H(u) + O(|u|^\kappa), \quad f_j(\Theta^c(u), u) \equiv 0, \quad 1 \le j \le j_0. \tag{13.2}$$

Then, by Theorem 11.1 (ii), we conclude that Γ_{Θ^c} defined by (11.3) extends as a germ of a holomorphic map $(\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ sending $(M, 0)$ into $(M', 0)$. We take $\widehat{H}(Z) := \Gamma_{\Theta^c}(Z)$. The first identity in (13.2) implies

$$\widehat{H}(Z) = \Gamma_{\Theta_H}(Z) + O(|Z|^\kappa). \tag{13.3}$$

On the other hand, by Theorem 10.1 and, in particular, (10.2), we have for $k > 2sl$

$$H(Z) - \Gamma_{\Theta_H}(Z) = O\left(\frac{k - 2sl}{r}\right). \tag{13.4}$$

By increasing k if necessary, we can assume that $(k-2sl)/r \geq \kappa$ so that $\widehat{H}(Z) = H(Z) + O(|Z|^\kappa)$ by (13.3) and (13.4). Since H is invertible and $\kappa > 1$, it follows that $\widehat{H}$ is also invertible. The proof of Theorem 5.1 is now complete.

14. CR equivalences

If M and M' are real-analytic CR submanifolds of $\mathbb{C}^N$, with $p \in M$ and $p' \in M'$, and $h : (M, p) \to (M', p')$ is a germ of a mapping of class C^k, $1 \leq k \leq \infty$, recall that h is a *germ of a CR map of class C^k* if the differential of h sends any $(0, 1)$ vector on M to a $(0, 1)$ vector on M'. If, in addition, h is a diffeomorphism at p we shall say that h is a *CR equivalence of class C^k* between (M, p) and (M', p'). It is standard that the Taylor power series of any CR equivalence of class C^∞ between (M, p) and (M', p') induces a formal equivalence between (M, p) and (M', p'). Similarly, the kth Taylor polynomial of any CR equivalence of class C^k induces a k-equivalence (see, e.g., [4], Proposition 1.7.14). Hence Corollary 1.2 implies the following.

Corollary 14.1. *Let $M \subset \mathbb{C}^N$ be a connected real-analytic* CR *submanifold. Then there exists a closed, proper real-analytic subvariety $V \subset M$ such that for every $p \in M \setminus V$, every real-analytic* CR *submanifold $M' \subset \mathbb{C}^N$, and every $p' \in M'$, the following are equivalent:*

(i) *(M, p) and (M', p') are k-equivalent for all $k > 1$.*

(ii) *(M, p) and (M', p') are* CR *equivalent of class C^k for all finite $k > 1$.*

(iii) *(M, p) and (M', p') are formally equivalent.*

(iv) (M, p) *and* (M', p') *are* CR *equivalent of class* C^∞.

(v) (M, p) *and* (M', p') *are biholomorphically equivalent.*

References

[1] M. Artin, *On the solutions of analytic equations*, Invent. Math. **5** (1968) 277–291.

[2] M. Artin, *Algebraic approximation of structures over complete local rings*, Inst. Hautes Etudes Sci. Publ. Math. **36** (1969) 23–58.

[3] M.S. Baouendi, P. Ebenfelt, & L.P. Rothschild, *Dynamics of the Segre varieties of a real submanifold in complex space*, to appear in J. Alg. Geom.; preprint, math.CV/0008112, 2000.

[4] M.S. Baouendi, P. Ebenfelt, & L.P. Rothschild, *Real Submanifolds in Complex Space and Their Mappings*, Princeton Math. Series **47** (1999), Princeton Univ. Press.

[5] M.S. Baouendi, P. Ebenfelt, & L.P. Rothschild, *Rational dependence of smooth and analytic* CR *mappings on their jets*, Math. Ann. **315** (1999) 205–249.

[6] M.S. Baouendi, P. Ebenfelt, & L.P. Rothschild, *Convergence and finite determination of formal* CR *mappings*, J. Amer. Math. Soc. **13**(**4**) (2000) 697–723.

[7] M.S. Baouendi, P. Ebenfelt, & L.P. Rothschild, *Parametrization of local biholomorphisms of real-analytic hypersurfaces*, Asian J. Math. **1** (1997) 1–16.

[8] M.S. Baouendi, X. Huang, & L.P. Rothschild, *Regularity of* CR *mappings between algebraic hypersurfaces*, Invent. Math. **125** (1996) 13–36.

[9] V.K. Beloshapka, *The normal form of the germ of a four-dimensional real manifold in* C^5 *at an RC-singular point in general position* (Russian), Mat. Zametki **61** (1997) 931–934; English transl. in Math. Notes **61** (1997) 777–779.

[10] T. Bloom & I. Graham, *On type conditions for generic real submanifolds of* $\mathbb{C}^n$, Invent. Math. **40** (1977) 217–243.

[11] E. Cartan, *Sur la géométrie pseudo-conforme des hypersurfaces de deux variables complexes*, I, Ann. Math. Pura Appl. **11** (1932) 17–90; *Sur la géométrie pseudo-conforme des hypersurfaces de deux variables complexes*, II, Ann. Scoula Norm. Sup. Pisa **1** (1932) 333-354.

[12] H. Cartan, *Variétés analytiques réelles et variétés analytiques complexes*, Bull. Soc. Math. France **85** (1957) 77–99.

[13] S.-S. Chern & J.K. Moser, *Real hypersurfaces in complex manifolds*, Acta Math. **133** (1974) 219–271.

[14] A. Coffman, *Formal stability of the* CR *cross-cap*, Pacific J. of Math., to appear (1999).

[15] J. Denef & L. Lipshitz, *Power series solutions of algebraic differential equations*, Math. Ann. **267** (1984) 213–238.

[16] X. Gong, *Divergence for the normalization of real analytic glancing hypersurfaces*, Comm. Partial Differential Equations **19** (1994) 643–654.

[17] J.J. Kohn, *Boundary behavior of $\bar\partial$ on weakly pseudo-convex manifolds of dimension two*, J. Differential Geom. **6** (1972) 523–542.

[18] J. Moser & S.M. Webster, *Normal forms for real surfaces in $\mathbb{C}^2$ near complex tangents and hyperbolic surface transformations*, Acta Math. **150** (1983) 255–296.

[19] T. Nagano, *Linear differential systems with singularities and an application to transitive lie algebras*, J. Math. Soc. Japan **18** (1966) 398–404.

[20] K. Oka, *Sur les fonctions analytiques de plusieurs variables* VII: *Sur quelques notions arithmetiques*, Bull. Soc. Math. France **78** (1950) 1–27.

[21] H. Poincaré, *Les fonctions analytiques de deux variables et la représentation conforme*, Rend. Circ. Mat. Palermo, II **23** (1907) 185-220.

[22] N. Tanaka, *On the pseudo-conformal geometry of hypersurfaces of the space of n complex variables*, J. Math. Soc. Japan **14**, (1962) 397–429.

[23] G. Tomassini, *Tracce delle functional olomorfe sulle sotto varieta analitiche reali d'una varieta complessa*, Ann. Scuola Norm. Sup. Pisa **20** (1966) 31–43.

[24] J.J. Wavrik, *A theorem on solutions of analytic equations with applications to deformations of complex structures*, Math. Ann. **216** (1975) 127–142.

[25] D. Zaitsev, *Germs of local automorphisms of real-analytic* CR *structures and analytic dependence on k-jets*, Math. Res. Lett. **4** (1997) 823–842.

University of California at San Diego
La Jolla, CA 92093-0112

Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen, Germany