J. DIFFERENTIAL GEOMETRY
57 (2001) 509-578

STABLE MORPHISMS TO SINGULAR SCHEMES AND RELATIVE STABLE MORPHISMS

JUN LI

Abstract

Let W/C be a degeneration of smooth varieties so that the special fiber has normal crossing singularity. In this paper, we first construct the stack of expanded degenerations of W. We then construct the moduli space of stable morphisms to this stack, which provides a degeneration of the moduli spaces of stable morphisms associated to W/C. Using a similar technique, for a pair (Z, D) of smooth variety and a smooth divisor, we construct the stack of expanded relative pairs and then the moduli spaces of relative stable morphisms to (Z, D). This is the algebro-geometric analogue of Donaldson-Floer theory in gauge theory. The construction of relative Gromov-Witten invariants and the degeneration formula of Gromov-Witten invariants will be treated in the subsequent paper.

0. Introduction

The goal of this paper is to develop a theory of relative Gromov-Witten invariants and prove a degeneration formula relating the Gromov-Witten invariants of smooth varieties with the relative Gromov-Witten invariants of pairs.

Let C be a connected smooth curve, $0 \in C$ be a fixed closed point and let $W \to C$ be a family of projective schemes so that the fibers W_t of W over $t \neq 0 \in C$ are smooth and the central fiber W_0 is the union of two smooth varieties Y_1 and Y_2 intersecting transversally along a smooth irreducible divisor. We denote by $D_i \subset Y_i$ the divisor $Y_1 \cap Y_2 \subset Y_i$. We will call (Y_i, D_i) the decomposition pairs of W_0. As for $t \neq 0$, since W_t are members of a connected family of smooth varieties, the Gromov-Witten invariants of W_t are all equivalent. The question is how to relate

Received May 7, 2001. The author was supported partially by NSF grants.

the Gromov-Witten invariants of W_t with those of the pairs (Y_i, D_i). In this project, we will construct the relative Gromov-Witten invariants of the pairs (Y_i, D_i). We then show that the Gromov-Witten invariants of W_t can be recovered from the relative Gromov-Witten invariants of the pairs (Y_1, D_1) and (Y_2, D_2), via a formula of the form

$$GW(W_t) = GW(Y_1, D_1) * GW(Y_2, D_2).$$

Here we use $GW(W_t)$ to denote the full GW-invariants of W_t and use $GW(Y_i, D_i)$ to denote the full relative GW-invariants of (Y_i, D_i). The operator $*$ is an involution type product linear in both arguments.

Gromov-Witten invariants are essentially a virtual enumeration of algebraic curves in algebraic varieties (or pseudo-holomorphic curves in symplectic manifolds). It is a topological theory. Since their introduction, the Gromov-Witten invariants have become an invaluable tool in studying many mathematical problems, not to mention their significance to mathematical physics. A challenging problem of GW-invariants is the understanding of GW-invariants of singular varieties and the behavior of GW-invariants under the degeneration of target varieties. This line of research has been pursued by several groups. In [39], Tian first studied the Gromov-Witten invariants of symplectic sums for semi-positive symplectic manifolds and derived the decomposition formula of the Gromov-Witten invariants in this setting. Later, A. Li-Ruan [28, 37] worked out a version of degeneration formula of Gromov-Witten invariants in the general setting, along the line of Donaldson-Floer theory. A parallel theory was developed by Ionel-Parker around the same time [17, 18, 19]. A part of the SFT theory of Eliashberg-Givental-Hofer [5] can also be interpreted as research along this line. As is now well understood, one approach to degeneration of moduli spaces (in differential geometry) is to follow Floer's original idea in his cohomology theory. This is the case for the Donaldson-Floer theory, and is the case for Gromov-Witten invariants as demonstrated by the works mentioned.

However, the algebro-geometric analogue of this approach to degeneration of moduli spaces has eluded algebraic geometers up until now. This in part explains why the Donaldson-Floer theory was never developed fully in algebraic geometry. This project fills in this gap. We have found the algebro-geometric approach to Floer's original idea in treating degenerate objects in moduli spaces. Though this project is mainly about moduli of stable morphisms and Gromov-Witten invariants, the technique developed can be applied to many other moduli problems, including the moduli of stable sheaves.

Out approach is geometric. We first construct a degeneration of the moduli spaces of stable morphisms, associated to the family W/C. Recall that to define the Gromov-Witten invariants of a projective scheme Z, one needs to apply the machinery of virtual moduli cycles, first developed by G. Tian and the author [24, 25]. (An alternative construction was achieved by K. Behrend and B. Fantechi [1, 2]. For related work using analytic techniques, see [7, 26, 35, 38].) It turns out that the moduli of stable morphisms to W, which can be viewed as a degeneration, is not suitable for studying Gromov-Witten invariants. Thus the first task is to construct a new degeneration in line with the construction of virtual moduli cycles. This is achieved in the present paper. In this paper, we first construct a stack $\mathfrak{W}$ of expanded degenerations of W/C. We then investigate the moduli of stable morphisms to this stack $\mathfrak{W}$ of expanded degenerations. Similarly, to each pair (Y, D) of a smooth divisor in a smooth variety, we construct the stack of expanded relative pairs $\mathfrak{Y}_{\rm rel}$ and the moduli space of relative stable morphisms to the stack of expanded relative pairs $\mathfrak{Y}_{\rm rel}$. The main results of this paper are the following existence theorems.

Theorem 0.1. *The moduli functor $\mathfrak{M}(\mathfrak{W}, \Gamma)$ of stable morphisms to $\mathfrak{W}$ of topological type Γ is a separated and proper Deligne-Mumford stack over C.*

Theorem 0.2. *The moduli functor $\mathfrak{M}(\mathfrak{Z}^{rel}, \Gamma)$ of relative stable morphisms to $\mathfrak{Z}^{rel}$ of topological type Γ is a separated and proper Deligne-Mumford stack.*

In the subsequent paper [23], we will develop the obstruction theory of the moduli stacks $\mathfrak{M}(\mathfrak{W}, \Gamma)$ and $\mathfrak{M}(\mathfrak{Z}^{\rm rel}, \Gamma)$, we will show that they admit perfect-obstruction theories and thus have canonical virtual moduli cycles. We will construct the relative Gromov-Witten invariants of the pair (Y, D) and prove the degeneration formula of the Gromov-Witten invariants mentioned at the beginning of the introduction. We will address the application of this machinery in our future research.

As indicated and/or shown by the works of [16, 17, 29, 36], the degeneration machinery developed in this project will be useful in enumerating curves in varieties, including algebraic surfaces and Calabi-Yau manifolds. Some of these problems are crucial to the research in mathematical physics. The moduli of relative stable morphisms and the relative GW-invariants will help answering several outstanding conjectures raised by mathematical physicists [14]. The techniques developed

in this project can be applied to the study of other moduli problems, including stable sheaves over algebraic varieties. Finally, we point out that based on this project the localization technique can be applied to the moduli of relative stable morphisms, which should be useful as suggested by [14]. We mention that the moduli space of relative stable morphisms and relative GW-invariants were constructed for a restricted class of varieties for genus 0 curves [9].

We now briefly describe the main idea of this paper. We let $W \to C$ be a projective family of schemes with general smooth fibers W_t for $t \neq 0 \in C$ and singular fiber W_0 with normal crossing singularity, in the situation we mentioned at the beginning of the introduction. Let g, n be two integers and let b be a homology class. We consider the moduli space (stack) $\mathfrak{M}_{g,k}(W_t, b)$ of stable morphisms from k-pointed arithmetic genus g curves to W_t of degree b. The union

$$\mathfrak{M}_{g,k}(W_{C^\circ}, b) = \cup_{t\in C^\circ}\mathfrak{M}_{g,k}(W_t, b), \quad C^\circ = C - 0,$$

is a proper family over C°. Inserting the central fiber $\mathfrak{M}_{g,k}(W_0, b)$ gives one extension (which we will call a degeneration of moduli spaces). However, the natural obstruction theory of this new family is no longer perfect near degenerate stable morphisms. Here we say a stable morphism $f : X \to W_0$ is degenerate if some irreducible components of X are mapped entirely to the singular locus of W_0. Usually, it is impossible to avoid degenerate stable morphisms if the extension family is proper. This makes this choice of degeneration inappropriate to study Gromov-Witten invariants. In this paper, we will work out a new degeneration of $\mathfrak{M}_{g,k}(W_{C^\circ}, b)$ that will allow us to apply the machinery of virtual moduli cycles.

As a warming up to our construction, let us look at a simple case where degenerate stable morphisms arise. Let $\pi : S \to C$ be a flat morphism from a smooth curve to C with $\pi^{-1}(0)$ consists of a single point s. We let $f : \mathcal{X} \to W$ be a flat family of stable morphisms over S with f a C-morphism so that $f_s : \mathcal{X}_s \to W_0$ is degenerate. We now let $W^\circ = W \times_C C^\circ$, let $S^\circ = S - s$, let $\mathcal{X}^\circ = \mathcal{X} \times_{C^\circ} S^\circ$ and let $f^\circ = f|_{\mathcal{X}^\circ}$ be the restriction of f to over $\mathcal{X}^\circ$. Here we view W as a degeneration (extension) of the family W° to C and view f as an extension of f°. The fact that the family f is degenerate means that the degeneration W is not a suitable choice to extend f°. There are other degenerations of W°. For instance, we let $(\widetilde{0}, \widetilde{C}) \to (0, C)$ be a base change ramified at $\widetilde{0} \in \widetilde{C}$ with ramification index r. We let $\widetilde{W}$ be the desingularization of the fiber

product $W \times_C \widetilde{C}$. Note that the central fiber of $\widetilde{W}$ over $\widetilde{0}$ has $(r+1)$ irreducible components: Y_1, Y_2 and $(r-1)$ copies of the ruled variety $\mathbf{P}_{D_1}(\mathbf{1}_{D_1} \oplus N_{D_1/Y_1})$ over the divisor D_1. We now consider the new family $\widetilde{f}^\circ : \widetilde{\mathcal{X}}^\circ \to \widetilde{W}$ over $\widetilde{S}^\circ = S \times_C \widetilde{C}$, where $\widetilde{\mathcal{X}}^\circ = \mathcal{X}^\circ \times_C \widetilde{C}$. We can extend it to a family of stable morphisms $\widetilde{f} : \widetilde{\mathcal{X}} \to \widetilde{W}$ over $S \times_C \widetilde{C}$. It is conceivable that for some choice of r this new extension $\widetilde{f}$ will be nondegenerate. It is less obvious, but will be proven, that there is a minimal choice of r so that $\widetilde{f}$ is nondegenerate. Thus to avoid degenerate stable morphisms we will replace the original family W by some expanded degenerations of W°. The exact choice of the expanded degeneration, after imposing the minimality condition, depends uniquely on the individual family.

Using expanded degenerations to study degenerate stable morphisms is certainly not new. However, to obtain a moduli space that contains stable morphisms to various expanded degenerations, namely to $\mathfrak{W}[r]_0$ for various r, we need to give the space of all expanded degenerations an algebraic structure. The new ingredient of this paper is that we can naturally give such space the structure of an (Artin) stack. This is the stack $\mathfrak{W}$ mentioned before the statement of Theorem 0.1. The moduli space of stable morphisms to all expanded degenerations of W is then defined to be the moduli space of *stable morphisms to this Artin stack* $\mathfrak{W}$. With appropriately defined stability, this moduli space is a proper Deligne-Mumford stack.

It turns out that this is not sufficient to construct the desired degeneration of moduli spaces. We need to build the notion of *predeformable* morphisms into the definition of the stable morphisms to $\mathfrak{W}$. Let $f : X \to W_0$ be a nondegenerate stable morphism so that there is a smooth point $p \in X$ that is mapped to the singular locus D of W_0. Then a simple argument shows that there is no small deformation of f that moves it away from the singular fiber W_0. In other words, there is a local obstruction to deforming such morphisms away from the singular fiber W_0. As we are interested in the degeneration of the moduli spaces, we should exclude such morphisms from our total moduli space. To accommodate this we require all stable morphisms to $\mathfrak{W}$ to have vanishing local obstructions to deforming to smooth fibers of W/C. Those morphisms that have vanishing local obstructions will be called *predeformable* morphisms.

In the end, we define stable morphisms to $\mathfrak{W}$ of given topological type[1] Γ to be those that are nondegenerate, are minimal among all possible target schemes $W[n]_0$ and are predeformable. The main theorem of this paper is that the moduli of such stable morphisms form a separate and proper Deligne-Mumford stack over C, as stated in Theorem 0.1.

As I mentioned earlier in the introduction, this construction can be viewed as an algebro-geometric adaptation of Floer's pioneering work [6]. I should point out that in early 1980's, Gieseker [10] (see also Gieseker-Morrison [13]) constructed a degeneration of the moduli spaces of stable vector bundles over a family of smooth curves degenerating to a nodal curve. The moduli space of vector bundles over the singular curve D consists of the stable vector bundles over D and the stable vector bundles over $\widetilde{D}$ that are the result of replacing the node of D by a rational curve. Bundles over $\widetilde{D}$ are exactly the substitute of those stable non-locally free sheaves over D. The current work can be viewed as a realization of Gieseker's construction to the general cases. I should also mention that the notion of admissible cover introduced by Harris-Mumford [15] in early 80's utilized similar idea to deal with singular objects. A more recent work related to this construction is Caporaso's construction of the universal Picard scheme over $\overline{M}_g$ [3].

This technique can be applied to construct the *relative stable morphisms* of pairs. Let (D, Z) be a pair consisting of a smooth divisor D in a smooth projective variety Z. We consider the moduli of stable morphisms $f: X \to Z$. We call a stable morphism $f: X \to Z$ nondegenerate (relative to the divisor D) if $f^{-1}(D)$ is a proper divisor in X and is away from the nodal and the marked points of X. As before, the issue is to construct a complete moduli space that contains no degenerate stable morphisms relative the to D. As for W, we should consider stable morphisms to expanded relative pairs. An expanded relative pair of (Z, D) is a reducible scheme that is the union of Z with several copies of the ruled variety $\mathbf{P}_D(1_D \oplus N_{D/Z})$. To define the moduli of relative stable morphisms we first construct the (Artin) stack $\mathfrak{Z}^{\mathrm{rel}}$ of all expanded relative pairs. We then introduce the notion of relative stable morphisms to $\mathfrak{Z}^{\mathrm{rel}}$, similar to the case of stable morphisms to $\mathfrak{W}$. After that we can define the moduli functor of relative stable morphisms to $\mathfrak{Z}^{\mathrm{rel}}$ of topological type Γ. In the end we prove the existence Theorem 0.2.

The paper is organized as follows. In Section 1 we will construct

[1] Γ contains all the topological restrictions on the stable morphisms, including the genus, the number of marked points and the degree, etc.

the stack $\mathfrak{W}$ of all expanded degenerations of W. The notion of predeformable morphisms and related properties will be worked out in Section 2. In Section 3 we will define the notion of stable morphisms to $\mathfrak{W}$ and prove that the moduli functor of stable morphisms to $\mathfrak{W}$ is a separated and proper algebraic stack over C. Section 4 is devoted to study the moduli of relative stable morphisms $\mathfrak{M}(\mathfrak{Z}^{\mathrm{rel}}, \Gamma)$. The relation of stable morphisms to $\mathfrak{W}_0 = \mathfrak{W} \times_C 0$ and relative stable morphisms to pairs (Y_i, D) will be investigated there as well.

0.1 Conventions

We provide a selected list of conventions used throughout this paper.

$W[n]$. It is the expanded degeneration (of W/C) constructed in Section 2.

$C[n]$. $C[n] = C \times_{\mathbf{A}^1} \mathbf{A}^{n+1}$. It has two tautological morphisms $C[n] \to \mathbf{A}^1$ and $C[n] \to \mathbf{A}^{n+1}$. It is defined in Section 2.

$G[n]$. It is $\cong G_m \times \cdots \times G_m$, n copies. For notation related to this group see (1.2) and (1.3).

(Y_i, D_i). They are the pairs (of irreducible components with singular locus) derived from the decomposition of W_0.

$X_1 \sqcup X_2$. This is the gluing of two schemes X_1 and X_2. Let $A_1 \subset X_1$ and $A_2 \subset X_2$ be two pairs of closed subschemes in schemes. Assume A_1 is isomorphic to A_2. Then we can define a new scheme, denoted by $X_1 \sqcup X_2$, that is the result of gluing X_1 and X_2 along A_1 and A_2. $X_1 \sqcup X_2$ is the scheme that satisfies the universal property of the push-out.

$C[n]_{[l^\vee]}$. $\mathbf{H}_l^{n+1}$ is the coordinate hyperplane given by the vanishing of the l-th coordinate axis of $\mathbf{A}^{n+1}$. We will call it the l-th coordinate hyperplane. $C[n]_{[l^\vee]} = C[n] \times_{\mathbf{A}^{n+1}} \mathbf{H}_l^{n+1}$.

$\mathbf{t}$. $\mathbf{t} = (t_1, \ldots, t_{n+1})$ and hence $\mathbf{k}[\mathbf{t}] = \mathbf{k}[t_1, \ldots, t_{n+1}]$.

$[n]$. This is the set of all integers between 1 and n.

1. The stack of expanded degeneration

We first fix the notation that will be used throughout this paper. In this paper, we will work with an algebraically closed field $\mathbf{k}$ with characteristic 0. We let C be a smooth irreducible curve, $0 \in C$ a closed point and $\pi : W \to C$ a flat and projective family of schemes satisfying the following condition: The morphism π is smooth away from the central fiber $W_0 = W \times_C 0$ and the central fiber W_0 is reducible with normal crossing singularity and has two smooth irreducible components Y_1 and Y_2 intersecting along a smooth divisor $D \subset W_0$. When we view D as a divisor in Y_i, we will denote it by $D_i \subset Y_i$.

We now construct the class of expanded degenerations of W mentioned in the introduction of this paper. We let Δ be the projective bundle over D:

$$\Delta = \mathbf{P}(\mathbf{1}_D \oplus N_{D_2/Y_2}),$$

where $\mathbf{1}_D$ is the trivial holomorphic line bundle on D and N_{D_2/Y_2} is the normal bundle of D_2 in Y_2, viewed as a line bundle on D. Δ has two distinguished divisors

$$D_- = \mathbf{P}(\mathbf{1}_D \oplus 0) \quad \text{and} \quad D_+ = \mathbf{P}(0 \oplus N_{D_2/Y_2}).$$

For convenience, we call D_- the left distinguished and D_+ the right distinguished divisor. Of course both D_- and D_+ are canonically isomorphic to D. Note that

$$N_{D_-/\Delta} \cong N_{D_2/Y_2} \quad \text{and} \quad N_{D_+/\Delta} \cong N_{D_1/Y_1} \cong N^{-1}_{D_2/Y_2}.$$

Using this identification, we can *glue*[2] an ordered chain of n Δ's by identifying the right distinguished divisor (i.e., D_+) in the k-th Δ with the left distinguished divisor (i.e., D_-) of the $(k+1)$-th Δ, for $k = 1, \ldots, n-1$. We denote the resulting scheme by $\Delta[n]$. It is connected, has normal crossing singularity and has n irreducible components all isomorphic to Δ. We keep the ordering of these n Δ's. We then *glue* Y_1 to $\Delta[n]$ by identifying D_1 in Y_1 with the left distinguished divisor D_- in the first Δ of $\Delta[n]$, and then *glue* Y_2 to this scheme by identifying the right distinguished divisor D_+ of the last Δ in $\Delta[n]$ with D_2 in Y_2. We denote the resulting scheme by $W[n]_0$. Note that $W[n]_0$ has $(n+2)$ irreducible components. These $(n+2)$ components form a chain ordered from left to right according to their intersection pattern:

$$W[n]_0 = Y_1 \sqcup \Delta \sqcup \cdots \sqcup \Delta \sqcup Y_2.$$

[2]See the convention in the Introduction.

Often, we will denote these $n+2$ components by $\Delta_1, \dots, \Delta_{n+2}$, according to this ordering. As a result, $W[n]_0$ has $(n+1)$ nodal divisors, ordered so that the k-th nodal divisor is $D_k \triangleq \Delta_k \cap \Delta_{k+1}$. Note that $D_k \subset \Delta_k$ (resp. $D_k \subset \Delta_{k+1}$) is the right (resp. left) distinguished divisor. We agree $W[0]_0$ is W_0, the fiber of W over $0 \in C$.

The purpose of this paper is to construct a stack $\mathfrak{W}$ representing all degenerations $\widetilde{W} \to \widetilde{C}$, where $\rho : \widetilde{C} \to C$ are base changes, so that the fiber of $\widetilde{W}$ over t is $W_{\rho(t)}$ in case $\rho(t) \neq 0$ and is one of $W[n]_0$ when $\rho(t) = 0$.

1.1 Construction of the standard model

We begin with the construction of the standard models $W[n]$. By replacing C with an open neighborhood of $0 \in C$ we can assume that there is an étale morphism $C \to \mathbf{A}^1$ so that $0 \in C$ is mapped to $0_{\mathbf{A}^1} \in \mathbf{A}^1$ and $0 \in C$ is the only point in C lies over $0_{\mathbf{A}^1} \in \mathbf{A}^1$. We fix such a map $C \to \mathbf{A}^1$ once and for all. We let G_m be the general linear group $\mathrm{GL}(1)$ and let $G[n]$ be the group scheme

$$G[n] = G_m \times \cdots \times G_m \qquad n \text{ copies} \tag{1.1}$$

which acts on $\mathbf{A}^{n+1}$ via

$$\mathbf{t}^\sigma = (\sigma_1 t_1, \sigma_1^{-1}\sigma_2 t_2, \dots, \sigma_{n-1}^{-1}\sigma_n t_n, \sigma_n^{-1} t_{n+1}), \tag{1.2}$$

where $\mathbf{t} = (t_1, \dots, t_{n+1})$ is the standard coordinate of $\mathbf{A}^{n+1}$. (Here and in the following we will use superscripts to denote the result of the group action.) The convention throughout this paper is that for $\sigma \in G[n]$ we will denote by σ_i its i-th component via isomorphism (1.1). For convenience, we introduce

$$\bar{\sigma}_i = \sigma_i / \sigma_{i-1} \quad \text{with} \quad \sigma_0 = \sigma_{n+1} = 1. \tag{1.3}$$

In this way, (1.2) can be rewritten as $\mathbf{t}^\sigma = (\bar{\sigma}_1 t_1, \dots, \bar{\sigma}_{n+1} t_{n+1})$. Note that if we let

$$\mathbf{p} : \mathbf{A}^{n+1} \to \mathbf{A}^1, \quad \mathbf{p}(\mathbf{t}) = t_1 \times \cdots \times t_{n+1},$$

then $\mathbf{p}$ is $G[n]$-equivariant with the trivial $G[n]$-action on $\mathbf{A}^1$. We define

$$C[n] = C \times_{\mathbf{A}^1} \mathbf{A}^{n+1}$$

with the (unique) distinguished point $\mathbf{0} = 0 \times_{\mathbf{A}^1} 0_{\mathbf{A}^{n+1}} \in C[n]$. We view $C[n]$ as a C-scheme via the first projection. $C[n]$ is a $G[n]$-scheme as well.

The standard model $W[n]$ will be constructed as a desingularization of

$$W \times_{\mathbf{A}^1} \mathbf{A}^{n+1} = W \times_C C[n].$$

We begin with some more notation. For any integer n, we denote by $[n]$ the set of integers between 1 and n. Let $I \subset [n+1]$ be any subset with cardinality $|I| = m+1$. The subset I defines a unique increasing map $[m+1] \to [n+1]$ whose image is I. By abuse of notation, we will denote this map by I. Hence $I(k)$ is the k-th element in I. Given any such I there is a standard embedding $\mathbf{A}^{m+1} \to \mathbf{A}^{n+1}$ that sends $z \in \mathbf{A}^{m+1}$ to $w \in \mathbf{A}^{n+1}$ via $w_{I(k)} = z_k$ for $k \in I$ and $w_l = 1$ for $l \notin I$. It follows that it induces an immersion

$$\underline{\gamma_I} : C[m] \to C[n]. \tag{1.4}$$

We call this the *standard embedding* associated to $I \subset [n+1]$. There is another embedding $\mathbf{A}^{m+1} \to \mathbf{A}^{n+1}$ via the rule $w_{I(k)} = z_k$ for $k \in I$ and $w_l = 0$ for $l \notin I$. This induces an immersion $\mathbf{A}^{m+1} \to C[n]$ whose image lies in a coordinate plane. We call this the *coordinate plane* associated to I and denoted it by $C[n]_I$. Lastly, when l is an integer in $[n+1]$, we denote by l the set of one element with $l \to [n+1]$ the inclusion. Hence $C[n]_l$ is the l-th coordinate axis of $C[n]$. Similarly, we let $[l^\vee]$ be the complement of l in $[n]$. Hence $C[n]_{[l^\vee]}$ is the coordinate hyperplane spanned by all coordinate axes except the l-th axis.

Now we construct $W[n]$ by induction on n. For $n = 0$, $W[0] = W$. For $n = 1$, $W_1 = W \times_C C[1]$ is smooth except along the locus $D \times_C \mathbf{0}$, which is a smooth codimension 3 subscheme in W_1 and the formal completions (the germs) of its normal slices in W_1 is isomorphic to the formal completion of

$$X = \{z_1 z_2 = t_1 t_2\} \subset \mathbf{A}^4 \tag{1.5}$$

along its origin. Here we use (z_1, z_2, t_1, t_2) to denote the coordinate of $\mathbf{A}^4$. After blowing up W_1 along $D \times_C \mathbf{0}$ we obtain a smooth scheme $\widetilde{W_1}$ whose exceptional divisor is a $\mathbf{P}^1 \times \mathbf{P}^1$ bundle over D. In the following, we will show how to contract one of these two $\mathbf{P}^1$-factors from this divisor to obtain the desired scheme $W[1]$.

We first work out the a desingularization of (1.5).

Lemma 1.1. *There is a desingularization Z of* (1.5) *so that its exceptional locus Σ is a $\mathbf{P}^1$ and the proper transforms of the z_1 and the t_2-axis (resp. the z_2 and the t_1-axis) in Z intersect in Σ.*

Proof. We first blow up the singular point (the origin) of the threefold X in (1.5). The resulting threefold, denoted by $\widetilde{X}$, is isomorphic to the total space of the restriction of the tautological line bundle on $\mathbf{P}^3$ to the surface $\{w_1w_2 = w_3w_4\} \subset \mathbf{P}^3$. The exceptional divisor $B \subset \widetilde{X}$ is isomorphic to the zero section of this line bundle, which is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$. An isomorphism

$$\mathbf{P}^1 \times \mathbf{P}^1 \longrightarrow B = \{w_1w_2 = w_3w_4\} \subset \mathbf{P}^3$$

is given by

$$([a_0, a_1], [b_0, b_1]) \mapsto [a_0b_1, a_1b_0, a_0b_0, a_1b_1]. \tag{1.6}$$

The normal bundle $N_{B/\widetilde{X}}$ is isomorphic to the restriction of the tautological line bundle to B, which is $\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(-1,-1)$, under the above isomorphism. Hence we can contract either of the $\mathbf{P}^1$-factors in B to obtain a smooth threefold. In the following we will show that we can choose the contraction so that the proper transforms of the t_1 and the z_2-axes intersect and the proper transforms of the t_2 and the z_1-axes intersect.

Let $\widetilde{p}\colon \widetilde{X} \to X$ be the obvious projection. Then under the isomorphism (1.6) the map $\widetilde{p}$ is of the form

$$\widetilde{p}\colon ([a_0, a_1], [b_0, b_1], \zeta) \mapsto (z_1, z_2, t_1, t_2) = (a_0b_1\zeta, a_1b_0\zeta, a_0b_0\zeta, a_1b_1\zeta),$$

where ζ is the variable representing points on the total space of the tautological line bundle. Note that ζ has homogeneous degree $(-1,-1)$, as a_i and b_i have homogeneous degree $(1,0)$ and $(0,1)$ respectively.[3] Hence the proper transforms of the z_1, z_2, t_1 and the t_2-axes intersect B at

$$([1,0],[1,0]),\ ([0,1],[0,1]),\ ([1,0],[0,1]) \text{ and } ([0,1],[1,0]) \in \mathbf{P}^1 \times \mathbf{P}^1,$$

respectively. Therefore if we contract the first $\mathbf{P}^1$-factor the resulting threefold Z satisfies the required property. q.e.d.

For later application, we now work out an atlas of Z. By out construction, Z is the total space of the vector bundle associated to the

[3]By this we mean $([a_0, a_1], [b_0, b_1], \zeta) \sim ([\lambda a_0, \lambda a_1], [\mu b_0, \mu b_1], \lambda^{-1}\mu^{-1}\zeta)$.

locally free sheaf $\mathcal{O}_{\mathbf{P}^1}(-1)^{\oplus 2}$. Since Z is derived from $\widetilde{X}$ by contracting the first $\mathbf{P}^1$ factor, we will use $[b_0, b_1]$ as the homogeneous coordinate of the exceptional locus of $p: Z \to X$. In this way, $p: Z \to X$ is of the form

$$p : ([b_0, b_1], \eta_1, \eta_2) \mapsto (b_1\eta_1, b_0\eta_2, b_0\eta_1, b_1\eta_2) \tag{1.7}$$

and the contraction $\widetilde{X} \to Z$ is given by

$$([a_0, a_1], [b_0, b_1], \zeta) \mapsto ([b_0, b_1], a_0\zeta, a_1\zeta). \tag{1.8}$$

Hence the projection $Z \to \mathbf{A}^2$ is given by

$$\varphi : ([b_0, b_1], \eta_1, \eta_2) \mapsto (t_1, t_2) = (b_0\eta_1, b_1\eta_2). \tag{1.9}$$

From this we see that $Z \times_{\mathbf{A}^2} 0_{\mathbf{A}^2}$ is a nodal curve obtained by adjoining an $\mathbf{A}^1$ (with η_1 variable) to $[0, 1]$ in $\mathbf{P}^1$ and an $\mathbf{A}^1$ (with η_2 variable) to $[1, 0]$ in $\mathbf{P}^1$. Further if we let $\mathbf{A}_1$ (resp. $\mathbf{A}_2$; resp. S) be the t_1-axis of $\mathbf{A}^2$ (resp. the t_2-axis; resp. a curve in $\mathbf{A}^2$ passing though the origin but not entirely contained in the two coordinate lines) then $Z \times_{\mathbf{A}^2} \mathbf{A}_1$ (resp. $Z \times_{\mathbf{A}^2} \mathbf{A}_2$; resp. $A \times_{\mathbf{A}^2} S$) is a smoothing of the nodal point $[0, 1]$ (resp. the nodal point $[1, 0]$; resp. both nodal points) of $Z \times_{\mathbf{A}^2} 0_{\mathbf{A}^2}$. We also mention that near $([0, 1], 0, 0)$ (resp. $([1, 0], 0, 0)$) the projection $Z \to \mathbf{A}^2$ has the form $(b_0, \eta_1, \eta_2) \mapsto (b_0\eta_1, \eta_2)$ (resp. $(b_1, \eta_1, \eta_2) \mapsto (\eta_1, b_1\eta_2)$).

We continue our construction of $W[1]$. After blowing up the subscheme $D \times_C \mathbf{0}$ in $W \times_C C[1]$ we obtain the scheme $\widetilde{W}_1$ whose exceptional divisor is isomorphic to a $\mathbf{P}^1 \times \mathbf{P}^1$-bundle over D. By Lemma 1.1, we can contract either factor of this $\mathbf{P}^1 \times \mathbf{P}^1$-bundle to obtain a smooth scheme. We now specify the choice of the $\mathbf{P}^1$-factor to be contracted. Let $W[1]$ be the resulting scheme after contracting one such $\mathbf{P}^1$-factor. Then $W[1]$ has an induced projection $\pi: W[1] \to C[1]$ and the fiber of π over the origin $\mathbf{0} \in C[1]$, denoted by $W[1]_0$, is reduced with normal crossing singularity whose irreducible components are Y_1, Y_2 and a ruled variety $\widetilde{\Delta}$ over D. The singular locus of $W[1]_0$ consists of two disjoint divisors $Y_1 \cap \widetilde{\Delta}$ and $Y_2 \cap \widetilde{\Delta}$, both isomorphic to D. Following the proof of Lemma 1.1 and the description afterwards, we can contract one of the $\mathbf{P}^1$ factors so that the family $W[1] \times_{C[1]} C[1]_1$ (resp. $W[1] \times_{C[1]} C[1]_2$) over C is a smoothing of the nodal divisor $Y_1 \cap \widetilde{\Delta}$ (resp. the nodal divisor $Y_2 \cap \widetilde{\Delta}$) of $W[1] \times_{\mathbf{A}^2} 0_{\mathbf{A}^2}$. Here $C[1]_i \subset C[1]$ is the i-th coordinate line defined in the beginning of this subsection. It is direct to check that $\widetilde{\Delta}$ is the ruled variety $\mathbf{P}_D(\mathbf{1}_D \oplus N_{D_1/Y_1})$ mentioned at the beginning of this section.

We let $\pi_2 : W[1] \to \mathbf{A}^1$ be the composite of $W[1] \to C[1] \to \mathbf{A}^2$ with the second projection $\mathbf{A}^2 \to \mathbf{A}^1$. It is easy to see that the singular locus of π_2 is the proper transform of $D \times \mathbf{A}^1 \subset W \times_{\mathbf{A}^1} \mathbf{A}^2$ where $D \times \mathbf{A}^1 \to W \times_{\mathbf{A}^1} \mathbf{A}^2$ is the morphism induced by $D \to W$ and $\mathbf{A}^1 \to \mathbf{A}^2$ as the t_1-axis. We denote the singular locus of π_2 by $\mathbf{D}_2$. Further, at each $p \in \mathbf{D}_2$, there is a formal coordinate chart[4] $(z_1, \dots)$ of p so that $\pi^* : \Gamma(\mathbf{A}^2) \to \hat{\mathcal{O}}_{W[1],p}$ satisfies $t_1 \mapsto z_1$ and $t_2 \mapsto z_2 z_3$.

We now construct $W[2]$. We let W_2 be the fiber product $W[1] \times_{\mathbf{A}^2} \mathbf{A}^3$ with $\mathbf{A}^3 \to \mathbf{A}^2$ the morphism $(t_1, t_2, t_3) \mapsto (t_1, t_2 t_3)$. Then W_2 has singularity along $\mathbf{D}_2 \times_{\mathbf{A}^2} \mathbf{A}^3_{[23^\vee]} \subset W[1] \times_{\mathbf{A}^2} \mathbf{A}^3$, where following our convention $\mathbf{A}^3_{[23^\vee]}$ is the codimension two coordinate plane (line) spanned by all but the second and the third axes in $\mathbf{A}^3$. Note that the singularity type of W_2 along $\mathbf{D}_2 \times_{\mathbf{A}^2} \mathbf{A}^3_{[23^\vee]}$ is identical to the one of W_2 along $D \times_{\mathbf{A}^1} 0_{\mathbf{A}^1}$. We now blow up W_2 along its singular locus and then contract one copy of the $\mathbf{P}^1$-bundle in the resulting exceptional divisor. We will contract the $\mathbf{P}^1$-bundle so that the resulting scheme, denoted by $W[2] \to C[2]$, has the property that its fibers over the l-th axis $C[2]_l \subset C[2]$ is a smoothing of the l-th nodal divisor of $W[2]_0 \triangleq W[2] \times_{C[2]} \mathbf{0}$. The proof of this is a word by word repetition of our earlier construction of $W[1]$ from $W \times_{\mathbf{A}^1} \mathbf{A}^2$.

The model $W[n]/C[n]$ is constructed by induction. Assume $W[n-1]/C[n-1]$ was constructed, we then form $W_n = W[n-1] \times_{\mathbf{A}^n} \mathbf{A}^{n+1}$, where $\mathbf{A}^{n+1} \to \mathbf{A}^n$ is defined by $(\dots, t_{n+2}) \mapsto (\dots, t_n, t_{n+1} t_{n+2})$. We then resolve the singularity of W_n by a small resolution. We require that the resulting family $W[n]/C[n]$ be so that $W[n] \times_{C[n]} \mathbf{0} \cong W[n]_0$ and the fibers of $W[n]$ over the l-th factor $C[n]_l \subset C[n]$ is a smoothing of the l-th nodal divisor of $W[n]_0$. Again the details to work this out are exactly the same as in the construction of $W[1]$.

Throughout this paper, we will use $W[n]_0$ to denote the fiber of $W[n]$ over $\mathbf{0} \in C[n]$ and denote by $\Delta_1, \dots, \Delta_{n+2}$ the $n+2$ irreducible components of $W[n]_0$, ordered as specified at the beginning of this section.

1.2 Construction of the stack of expanded degenerations

In this part, we will construct the stack $\mathfrak{W}$ of the expanded degenerations of W.

We begin with an investigation of the $G[n]$-group action on $W[n]$.

[4] Namely they generate the maximal ideal of the formal ring $\hat{\mathcal{O}}_{W[1],p}$.

Clearly, since $\mathbf{A}^{n+1} \to \mathbf{A}^1$ is $G[n]$-equivariant, its action lifts to $C[n] \to C$. To show that it lifts to $W[n]$, we need to understand the corresponding G_m-action on the desingularization Z of X, constructed in Subsection 1.1. Following the notation there, G_m acts on X via

$$(z_1, z_2, t_1, t_2)^\sigma = (z_1, z_2, \sigma t_1, \sigma^{-1} t_2), \quad \sigma \in G_m$$

and its lifting to Z is

$$([b_0, 1], \eta_1, \eta_2)^\sigma = ([\sigma b_0, 1], \eta_1, \sigma^{-1}\eta_2). \tag{1.10}$$

Based on this, it is easy see that the $G[n]$-action on $C[n]$ lifts uniquely to $W[n]$.

For later application, we now give a precise description of the $G[n]$-action near the nodal divisor of $W[n]_0$. This can best be done by looking at the simplest case where $W = \mathbf{A}^2$, $C = \mathbf{A}^1$ and $W \to C$ is given by $t = u_1 u_2$, where (u_1, u_2) and t are the standard coordinates of $\mathbf{A}^2$ and $\mathbf{A}^1$. With this choice of $W \to C$, we can construct the corresponding $W[n] \to C[n]$. To distinguish this from the general case, we will denote this special degeneration by $\Gamma[n] \to \mathbf{A}^{n+1}$.

Lemma 1.2. *Let the notation be as before. Then $\Gamma[n]_0$ is a chain of $n+2$ curves of which the first and the last are isomorphic to $\mathbf{A}^1$ and the remainders are isomorphic to $\mathbf{P}^1$. Further, we can cover $\Gamma[n]$ by $G[n]$-invariant affine open subsets $U_1, \dots, U_{n+1}$, each isomorphic to $\mathbf{A}^{n+2}$, so that if we denote by $(u_1^l, \dots, u_{n+2}^l)$ the standard coordinates of $U_l \cong \mathbf{A}^{n+2}$, then:*

(i) *The restriction of $\pi : \Gamma[n] \to \mathbf{A}^{n+1}$ to U_l is given by*

$$\pi^l : (u_1^l, \dots, u_{n+2}^l) \mapsto (\dots, u_{l-2}^l, u_{l-1}^l, u_l^l u_{l+1}^l, u_{l+2}^l, u_{l+3}^l, \dots).$$

(ii) *The $G[n]$-action on U_l is given by*

$$\begin{aligned}&(u_1^l, \dots, u_{n+2}^l)^\sigma \\ &= (\dots, \bar\sigma_{l-2} u_{l-2}^l, \bar\sigma_{l-1} u_{l-1}^l, \sigma_{l-1}^{-1} u_l^l, \sigma_l u_{l+1}^l, \bar\sigma_{l+1} u_{l+2}^l, \bar\sigma_{l+2} u_{l+3}^l, \dots),\end{aligned}$$

where $\bar\sigma_i = \sigma_i/\sigma_{i-1}$ with $\sigma_i = 1$ for $i = 0, n+1$.

(iii) *The transition function from $(u_\cdot^l)$ to $(u_\cdot^{l+1})$ over $U_l \cap U_{l+1}$ is given by*

$$\begin{aligned}&(u_1^{l+1}, \dots, u_{n+2}^{l+1}) \\ &\qquad = (\dots, u_{l-2}^l, u_{l-1}^l, u_l^l u_{l+1}^l, 1/u_{l+1}^l, u_{l+2}^l u_{l+1}^l, u_{l+3}^l, \dots).\end{aligned}$$

Proof. We prove this Lemma by induction on n. For $n = 0$, there is nothing to prove. We now assume that the Lemma holds for $\Gamma[n-1]$. Namely, we have found coverings $U_1, \ldots, U_n$ of $\Gamma[n-1]$ that satisfy the required properties. Following the construction, $\Gamma[n]$ is a small resolution of $\Gamma[n-1] \times_{\mathbf{A}^n} \mathbf{A}^{n+1}$. Using the explicit description of $\pi|_{U_l}: U_l \to \mathbf{A}^n$, we see that

$$V_l \triangleq U_l \times_{\mathbf{A}^n} \mathbf{A}^{n+1} \cong \mathbf{A}^{n+2}, \quad \text{where } l < n,$$

are smooth $G[n]$-invariant open subsets of $\Gamma[n]$. Clearly, we can choose a unique coordinate $(v_\cdot^l)$ of V_l so that the projection $V_l \to U_l$ is given by

$$u_i^l = v_i^l, \quad i \le n; \quad u_{n+1}^l = v_{n+1}^l v_{n+2}^l,$$

and the projection $\pi^l : V_i \to \mathbf{A}^{n+1}$ is given by the formula (i) in the statement of the Lemma. A routine check shows that for $l < n$ the $G[n]$-action on V_l and the transition function from V_{l-1} to V_l is exactly as shown in the statement of the Lemma.

It remains to check the case where $l = n$ and $n+1$. We now look at the chart U_n. Let $\varphi : \Gamma[n] \to \Gamma[n-1]$ be the composite of the desingularization $\Gamma[n] \to \Gamma[n-1] \times_{\mathbf{A}^n} \mathbf{A}^{n+1}$ with the obvious projection. Then

$$\varphi^{-1}(U_n) \cong \mathbf{A}^{n-1} \times \mathbf{L}^{\oplus 2},$$

where L is the tautological line bundle (degree -1) on $\mathbf{P}^1$ and $\mathbf{L}^{\oplus 2}$ is the total space of $L^{\oplus 2}$. We now cover $\mathbf{P}^1$ by $B_0 = \{[b_0, 1]\}$ and $B_1 = \{[1, b_1]\} \subset \mathbf{P}^1$. We choose trivializations $\mathbf{L}^{\oplus 2}|_{B_0} \cong B_0 \times \mathbf{A}^2$ and $\mathbf{L}^{\oplus 2}|_{B_1} \cong B_1 \times \mathbf{A}^2$ so that the transition function from $B_0 \times \mathbf{A}^2$ to $B_1 \times \mathbf{A}^2$ is given by

$$(b_0, \zeta_1, \zeta_2) \longrightarrow (b_1, \zeta_1', \zeta_2') = (1/b_0, \zeta_1 b_0, \zeta_2 b_0).$$

We let V_n be $\mathbf{A}^{n-1} \times \mathbf{L}^{\oplus 2}|_{B_0}$ and let $V_{n+1} = \mathbf{A}^{n-1} \times \mathbf{L}^{\oplus 2}|_{B_1} \subset \mathbf{A}^{n-1} \times \mathbf{L}^{\oplus 2}$. Then $V_1, \ldots, V_{n+1}$ form a covering of $\Gamma[n]$. Next for V_n we choose coordinates

$$(v_1^n, \ldots, v_{n+2}^n) = (z_1, \ldots, z_{n-1}, \zeta_1, b_0, \zeta_2)$$

and for V_{n+1} we choose

$$(v_1^{n+1}, \ldots, v_{n+2}^{n+1}) = (z_1, \ldots, z_{n-1}, \zeta_1', b_1, \zeta_2').$$

We now check that this choice of coordinates satisfies the requirements of the Lemma.

We first check the transition functions. The transition function from V_n to V_{n+1} is determined by the transition function from $\mathbf{L}^{\oplus 2}|_{B_0}$ to $\mathbf{L}^{\oplus 2}|_{B_1}$. Thus we have

$$\begin{aligned}(v_1^{n+1}, \dots, v_{n+2}^{n+1}) &= (z_1, \dots, z_{n-1}, \zeta_1', b_1, \zeta_2') \\ &= (z_1, \dots, z_{n-1}, b_0\zeta_1, 1/b_0, \zeta_2 b_0) \\ &= (v_1^n, \dots, v_{n-1}^n, v_n^n v_{n+1}^n, 1/v_{n+1}^n, v_{n+1}^n v_{n+2}^n).\end{aligned}$$

This is the requirement in (iii) for $l = n$.

It remains to work out the transition function from V_{n-1} to V_n. We let $\mathbf{v}^{n-1} = (v_1^{n-1}, \dots, v_{n+2}^{n-1})$ be a general point and let $\mathbf{v}^n = (v_1^n, \dots, v_{n+2}^n)$ be the point defined by the relation (iii) with u replaced by v and l replaced by $n-1$. Then

$$v_{n+2}^n = v_{n+2}^{n-1},\ v_{n+1}^n = v_{n+1}^{n-1} v_n^{n-1},\ v_n^n = 1/v_n^{n-1},\ v_{n-1}^n = v_{n-1}^{n-1} v_n^{n-1}.$$

Clearly, to prove (iii) we only need to check that the images of $\mathbf{v}^{n-1}$ and $\mathbf{v}^n$ in U_n coincide and their images in $\mathbf{A}^{n+1}$ also coincide. By definition, the image of $\mathbf{v}^{n-1}$ in $\mathbf{A}^{n+1}$ is

$$(v_1^{n-1}, \dots, v_{n-1}^{n-1} v_n^{n-1}, v_{n+1}^{n-1}, v_{n+2}^{n-1})$$

and the image of $\mathbf{v}^n$ in $\mathbf{A}^{n+1}$ is

$$(v_1^n, \dots, v_{n-1}^n, v_n^n v_{n+1}^n, v_{n+2}^n).$$

They coincide using the relations mentioned above. On the other hand the image of $\mathbf{v}^{n-1}$ in U_{n-1} is $(\dots, v_n^{n-1}, v_{n+1}^{n-1} v_{n+2}^{n-1})$ and its image in U_n, using (iii) and the induction hypothesis, is

$$(v_1^{n-1}, \dots, v_{n-1}^{n-1} v_n^{n-1}, 1/v_n^{n-1}, v_n^{n-1} v_{n+1}^{n-1} v_{n+2}^{n-1}).$$

The image of $\mathbf{v}^n$ in U_n is

$$(v_1^n, \dots, v_{n-1}^n, v_n^n, v_{n+1}^n v_{n+2}^n).$$

They coincide as well. This proves the relation (iii). The part (ii) concerning the group action can be checked directly and will be omitted.

q.e.d.

To apply this Lemma to the general case $W[n]$, we have the following useful observation. Let $\hat{D}$ be the formal completion of W along D. For any open $V \subset D$, we let $\hat{V}$ be $V \subset \hat{D}$ endowed with the open subscheme

structure. We choose V so that $\mathcal{O}_{\hat{V}} \cong \mathcal{O}_V[\![w_1, w_2]\!]$ and the induced morphism $\hat{V} \to \mathbf{A}^1$ is given by $t \mapsto w_1 w_2$. We let $\widetilde{V} \subset W[n]$ be an open subset so that $\widetilde{V} \cap (W[n] \times_W D) = W[n] \times_W V$ and let $\hat{W[n]}$ be the formal completion of $\widetilde{V}$ along $W[n] \times_W V$. Similarly we let $U \subset \Gamma[n]$ be the preimage of $0_{\mathbf{A}^2} \in \mathbf{A}^2$ under $\Gamma[n] \to \Gamma[1] = \mathbf{A}^2$ and let $\hat{\Gamma}[n]$ be the formal completion of $\Gamma[n]$ along U. Then we have a $G[n]$-equivariant isomorphism

$$\hat{W[n]} \cong V \times \hat{\Gamma}[n]. \tag{1.11}$$

We also denote by

$$\underline{\Psi} : G[n] \times C[n] \longrightarrow C[n] \quad \text{and} \quad \Psi : G[n] \times W[n] \longrightarrow W[n] \tag{1.12}$$

the morphisms of $G[n]$-actions.

Corollary 1.3. *Let $G[n]_l$ be the subgroup of $G[n]$ that is the l-th copy of G_m in $G[n]$, using the isomorphism* (1.1)*. Then the $G[n]_l$ action on Δ_{l+1} is induced by the linear action of G_m on $\mathbf{1}_D$ and N_{D_2/Y_2} of weight* 0 *and* 1*, respectively. It acts trivially on all other Δ_i's.*

Corollary 1.4. *Let $h : W[n]_0 \to W[n]_0$ be an isomorphism commuting with $W[n]_0 \to W_0$. Then there is a $\sigma \in G[n]$ so that h is identical to the automorphism of $W[n]_0$ induced by the action of the σ. Namely $h(w) = \Psi(\sigma, w)$ for and $w \in W[n]_0$.*

Corollary 1.5. *Let $s \in C[n]$ be a closed point and let $W[n]_s$ be the fiber of $W[n]$ over s. Then the stabilizer of the $W[n]_s$ (which is the subgroup of $G[n]$ that fixes $W[n]_s$) is the trivial subgroup of $G[n]$.*

We now fix the notation for the group action. Let G be any group scheme acting on an S-scheme X via $\Psi : G \times X \to X$. Let $f : T \to X$ be any S-morphism of schemes and let $\lambda : G' \to G$ be a homomorphism of groups. Then we define the induced group action morphism

$$\Psi_f^\lambda \triangleq \Psi \circ (\lambda \times f) : G' \times T \longrightarrow X.$$

In case $G = G'$ and $\lambda = 1_G$, Ψ_f^λ will be shortened to Ψ_f; In case $T = X$ and $f = 1_X$, then Ψ_f^λ will be shortened to Ψ^λ. Now let $\rho : T \to G$ and $f : T \to X$ be two morphisms. We define

$$f^\rho \triangleq \Psi \circ (\rho, f) : T \longrightarrow X, \tag{1.13}$$

in case the group action Ψ is clear from the context. Later, we will encounter the situation where X is a scheme over a scheme Y and ρ

is a morphism $Y \to G$. By abuse of notation we will use ρ to denote the composite morphism $X \to Y \to G$ as well and use f^ρ to denote the morphism induced by the group action, as in (1.13).

Let $I \subset [n+1]$ be a subset of $l+1$ elements. It associates an embedding $\underline{\gamma_I}: C[l] \to C[n]$ (see (1.4)). From the construction of $W[n]$, it is clear that there is a canonical W-isomorphism

$$\gamma_I : W[l] \cong W[n] \times_{C[n]} C[l].$$

Now let J be the complement of I in $[n+1]$ with entries $j_1 \leq \ldots \leq j_{n-l}$. We let $\lambda_J : G[n-l] \to G[n]$ be the homomorphism defined so that the j_i-th component of $\lambda_J(\sigma)$ is σ_i (σ_i is the i-th component of σ) and all other components of $\lambda_J(\sigma)$ are 1. Here the components of $G[n]$ are defined via the isomorphism (1.1). Then by the explicit $G[n]$-action on $\mathbf{A}^{n+1}$, we see immediately that

$$\underline{\Psi}^{\lambda_J}_{\underline{\gamma_I}} : G[n-l] \times C[l] \longrightarrow C[n]$$

defined by the $G[n]$-action $\underline{\Psi}$ is an open immersion. We denote the image open subscheme by $C[\Sigma I]$ and denote $W[n] \times_{C[n]} C[\Sigma I]$ by $W[\Sigma I]$. Clearly, $W[\Sigma I]$ is $G[n-l]$-invariant under the induced action

$$\Psi^{\lambda_J} : G[n-l] \times W[n] \longrightarrow W[n].$$

Lemma 1.6. *The scheme $W[\Sigma I]$ (resp. $C[\Sigma I]$) is a principal fiber bundle over $W[l]$ (resp. $C[l]$) with group $G[n-l]$ via the action Ψ^{λ_J} (resp. $\underline{\Psi}^{\lambda_J}$) and the morphism γ_I (resp. $\underline{\gamma}_I$) is a section of this principal fiber bundle.*

The proof is based on the parallel result concerning $\mathbf{A}^{l+1} \to \mathbf{A}^{n+1}$, which is obvious. We shall omit the proof here.

The two principal fiber bundles are compatible in the sense that the following diagram is commutative:

$$\begin{array}{ccc} G[n-l] \times W[\Sigma I] & \xrightarrow{\Psi^{\lambda_J}} & W[\Sigma I] \\ \downarrow & & \downarrow \\ G[n-l] \times C[\Sigma I] & \xrightarrow{\underline{\Psi}^{\lambda_J}} & C[\Sigma I]. \end{array}$$

Often, we have several morphisms $r : S \to C[n]$ and like to compare the corresponding fiber products $W[n] \times_{C[n]} S$. Adopting the convention in groupoid, we shall view $W[n] \times_{C[n]} S$ as the pullback of $W[n] \to C[n]$ via $r : S \to C[n]$. Accordingly, we will denote such a fiber product by $r^* W[n]$.

Corollary 1.7. *Let (s, S) be a pointed scheme, $\widetilde{\iota}: S \to C[n]$ be a morphism such that $\widetilde{\iota}(s) \in C[\Sigma I]$ for some subset $I \subset [n+1]$ with $|I| = l+1$. Let $S_0 = \widetilde{\iota}^{-1}(C[\Sigma I])$, which is an open neighborhood of s in S. Then there is a morphism $(\rho, \iota): S_0 \to G[n-l] \times C[l]$ such that the restriction of $\widetilde{\iota}$ to $S_0 \subset S$ is identical to $\underline{\Psi}^{\lambda_J}_{\gamma_I} \circ (\rho, \iota)$. Further it induces a canonical isomorphism*

$$\widetilde{\iota}^* W[n] \times_S S_0 \cong \iota^* W[l],$$

compatible to the projections to $W \times_C S_0$, using the principal fiber bundle structure $W[\Sigma I] \to W[l]$ and the morphism γ_I.

We now introduce the notion of the expanded degenerations of W. Let S be any C-scheme. An *effective degeneration* over S is a C-morphism $\xi: S \to C[n]$ for some n. Note that such a morphism comes with an S-family

$$\mathcal{W} = W[n] \times_{C[n]} S \longrightarrow S$$

and the tautological projection $\mathcal{W} \to W \times_C S$. (For convenience, we will not distinguish the family $\mathcal{W}$ from the morphism ξ, and vice versa if no confusion arises. As a convention, we will call $\mathcal{W}$ the associated family of ξ and call ξ the associated morphism of $\mathcal{W}$.) Let $\xi_1: S \to C[n_1]$ and $\xi_2: S \to C[n_2]$ be two effective degenerations. An *effective arrow* $r: \xi_1 \to \xi_2$ consists of a standard embedding $\iota: C[n_1] \to C[n_2]$ and a morphism $\rho: S \to G[n_2]$ so that $(\iota \circ \xi_1)^\rho = \xi_2$. By Corollary 1.7, this identity defines a canonical S-isomorphisms $\xi_1^* W[n_1] \cong \xi_2^* W[n_2]$ compatible with their projections to $W \times_C S$. We call this the associated isomorphism of the arrow $r: \xi_1 \to \xi_2$. By abuse of notation, we will denote this isomorphism by r as well. Now let ξ_1 and ξ_2 be any two effective degenerations over S. We say ξ_1 is equivariant to ξ_2 via an effective arrow if there is either an effective arrow $\xi_1 \to \xi_2$ or an effective arrow $\xi_2 \to \xi_1$. We say ξ_1 and ξ_2 are equivalent via a sequence of effective arrows if there is a sequence of effective degenerations $\eta_0, \ldots, \eta_m$ so that $\eta_0 = \xi_1$, $\eta_m = \xi_2$ and η_i is equivariant to η_{i+1} via an effective arrow for all i. Note that once such a sequence of effective arrows is given, then there is a unique induced S-isomorphism $\xi_1^* W[n_1] \cong \xi_2^* W[n_2]$ compatible with their projections to $W \times_C S$.

Now let $\mathcal{W}_1$ and $\mathcal{W}_2$ be two effective degenerations over S. We say $\mathcal{W}_1$ is *isomorphic* to $\mathcal{W}_2$ if $\mathcal{W}_1$ is S-isomorphic to $\mathcal{W}_2$ and the isomorphism is compatible to their tautological projections to $W \times_C S$. The following Lemma says that such isomorphisms are locally generated by a sequence of arrows.

Lemma 1.8. *Let ξ_1 and ξ_2 be two effective degenerations over S so that their associated families $\mathcal{W}_1$ and $\mathcal{W}_2$ are isomorphic. Then to each $p \in S$ there is an open neighborhood S_0 of $p \in S$ so that the (induced) isomorphism between $\mathcal{W}_1 \times_S S_0$ and $\mathcal{W}_2 \times_S S_0$ is induced by a sequence of effective arrows between $\xi_1 \times_S S_0$ and $\xi_2 \times_S S_0$.*

Proof. In case p lies over $C - 0$, then there is nothing to prove. Now assume p lies over $0 \in C$. Let n be the integer so that $\mathcal{W}_1 \times_S p$ has $n+2$ irreducible components. Then by Corollary 1.7, there is an open neighborhood S_0 of $p \in S$ and two morphisms $\xi_1', \xi_2' : S_0 \to C[n]$ so that $\xi_i \times_S S_0$ is equivalent to ξ_i' via an arrow. Thus to prove the Lemma it suffices to consider the case where $n_1 = n_2 = n$. Now we assume, $\xi_i : S \to C[n]$ and $\xi_i(p) = \mathbf{0} \in C[n]$. Let $\varphi : \mathcal{W}_1 \longrightarrow \mathcal{W}_2$ be the W_S-isomorphism and let $\phi_i : \mathcal{W}_i \longrightarrow W[n]$ be the associated morphisms. We define a functor $\mathfrak{Isom}[S]$ from the category of S-schemes to sets that associates to any S-scheme T the set of all morphisms $\rho : T \to G[n]$ such that with $\iota : \mathcal{W}_1 \times_S T \to \mathcal{W}_1$ the induced morphism (i.e., the first projection) then

$$(\phi_1 \circ \iota)^\rho = \phi_2 \circ \varphi \circ \iota : \mathcal{W}_1 \times_S T \longrightarrow W[n].$$

(For the notation here see the formula (1.13).) By the existence theorem of Hilbert schemes developed by Grothendieck coupled with the trivialization of the stabilizers (see Corollary 1.5), $\mathfrak{Isom}[S]$ is represented by a subscheme R of $S \times G[n]$.

We now show that R is nonempty and dominates a neighborhood of $p \in S$. Since $\xi_1(p) = \xi_2(p) = 0 \in C[n]$, the restriction of the isomorphism φ to fibers over p induces an isomorphism $W[n]_0 \cong W[n]_0$ that commutes with $W[n]_0 \to W_0$. Then it is induced by an element $g_0 \in G[n]$ by Corollary 1.4. Hence $(p, g_0) \in R$. Let $\mathfrak{m}$ be the maximal ideal of the local ring $\mathcal{O}_{S,p}$. By the previous discussion, we have $\rho_0 : p \to G[n]$ so that $(i_0, \rho_0) : p \to S \times G[n]$ factor through $R \subset S \times G[n]$, where $i_0 : p \to S$ is the inclusion. Let $B_k = \operatorname{Spec} \mathcal{O}_{S,p}/\mathfrak{m}^k$ with $i_k : B_k \to S$ the inclusion morphism. Suppose for some $k \geq 1$ we have

$$(i_{k-1}, \rho_{k-1}) : B_{k-1} \longrightarrow S \times G[n],$$

extending (i_0, ρ_0) that factor through $R \subset S \times G[n]$. We will show that ρ_{k-1} extends to ρ_k satisfying similar condition. Since $G[n]$ is smooth, we can extend ρ_{k-1} to $h_k : B_k \to G[n]$. Let $\iota_k : \mathcal{W}_1 \times_S B_k \to \mathcal{W}_1$ be the canonical immersion. We consider the problem of extending the

W-morphism

$$(\phi_1 \circ \iota_{k-1})^{h_{k-1}} = \phi_2 \circ \varphi \circ \iota_{k-1} : \mathcal{W}_1 \times_S B_{k-1} \longrightarrow W[n]$$

to a W-morphism

$$\mathcal{W}_1 \times_S B_k \longrightarrow W[n].$$

This is a typical extension problem. Since $(\phi_1 \circ \iota_k)^{h_k}$ and $\phi_2 \circ \varphi \circ \iota_k$ are two such extensions, by deformation theory they are related by an element in $\mathfrak{m}^k/\mathfrak{m}^{k+1} \otimes K$, where

$$K = \ker \Big\{ \operatorname{Hom}_{W[n]_0}(\Omega_{W[n]}, \mathcal{O}_{W[n]_0}) \longrightarrow \operatorname{Hom}_{W[n]_0}(\Omega_W \otimes_{\mathcal{O}_W} \mathcal{O}_{W[n]_0}, \mathcal{O}_{W[n]_0}) \Big\}.$$

We claim that K is isomorphic to $\mathbf{k}^{\oplus n}$ and is generated by the group action of $G[n]$. We let $i : W[n]_0 \to W[n]$ be the immersion and $j : W[n]_0 \to C[n]$ be the composite of i with the projection. Then we have the exact sequence

$$0 \longrightarrow j^*\Omega_{C[n]} \longrightarrow i^*\Omega_{W[n]} \longrightarrow \Omega_{W[n]_0} \longrightarrow 0 \tag{1.14}$$

and its induced exact sequence

$$\operatorname{Hom}(i^*\Omega_{W[n]}, \mathcal{O}_{W[n]_0}) \xrightarrow{\beta} \operatorname{Hom}(j^*\Omega_{C[n]}, \mathcal{O}_{W[n]_0}) \xrightarrow{\delta} \operatorname{Ext}^1(\Omega_{W[n]_0}, \mathcal{O}_{W[n]_0}).$$

At the origin $\mathbf{0} \in C[n]$, we have a canonical isomorphism $T_{\mathbf{0}}C[n] \cong T_0\mathbf{A}^{n+1}$. We let $v_l : \mathbf{k} \to \mathbf{k}^{\oplus n+1} \equiv T_0\mathbf{A}^{n+1}$ be the l-th component. The dual of v_l, which is $v_l^\vee : \Omega_{C[n]} \otimes_{\mathcal{O}_{C[n]}} \mathbf{k}_0 \to \mathbf{k}$, defines a homomorphism $\mathbf{v}_l^\vee : j^*\Omega_{C[n]} \to \mathcal{O}_{W[n]_0}$. Now let

$$0 \neq h \in \operatorname{Im}(\beta) \subset \operatorname{Hom}(j^*\Omega_{C[n]}, \mathcal{O}_{W[n]_0}),$$

say given by a homomorphism $j^*\Omega_{C[n]} \to \mathcal{O}_{W[n]_0}$. Then it must be a linear combination of $\mathbf{v}_l^\vee$, say equal to $a_1\mathbf{v}_1^\vee + \cdots + a_{n+1}\mathbf{v}_{n+1}^\vee$. Now let c be the extension class in $\operatorname{Ext}^1(\Omega_{W[n]_0}, j^*\Omega_{C[n]})$ of the exact sequence (1.14) and let

$$h^* : \operatorname{Ext}^1(\Omega_{W[n]_0}, j^*\Omega_{C[n]}) \longrightarrow \operatorname{Ext}^1(\Omega_{W[n]_0}, \mathcal{O}_{W[n]_0})$$

be the homomorphism induced by h. We claim that $h^*(c) \neq 0$. Indeed, let $T_2 = \operatorname{Spec} \mathbf{k}[t]/(t^2) \to C[n]$ be the immersion representing the tangent vector $\mathbf{v} = a_1\partial_{z_1} + \cdots + a_{n+1}\partial_{z_{n+1}} \in T_0C[n]$. Then

$$W[n] \times_{C[n]} \operatorname{Spec} \mathbf{k}[t]/(t^2) \longrightarrow \operatorname{Spec} \mathbf{k}[t]/(t^2)$$

is an infinitesimal smoothing of the l-th nodal divisor of $W[n]_0$ whenever $a_l \neq 0$, and consequently the exact sequence

(1.15)
$$0 \longrightarrow \mathcal{O}_{W[n]_0} \longrightarrow \Omega_{W[n]\times_{C[n]}T_2} \otimes_{W[n]\times_{C[n]}T_2} \mathcal{O}_{W[n]_0} \longrightarrow \Omega_{W[n]_0} \longrightarrow 0$$

does not split since $h \neq 0$. On the other hand, this exact sequence is exactly defined by the extension class $\delta(h)$. In case $h \in \operatorname{Im}(\beta)$, then $\delta(h) = 0$, violating the nonsplitness of (1.15). This proves $\beta = 0$. As a consequence

$$\operatorname{Hom}(\Omega_{W[n]_0}, \mathcal{O}_{W[n]_0}) \cong \operatorname{Hom}(i^*\Omega_{W[n]}, \mathcal{O}_{W[n]_0}).$$

Now let α be any element in K. Then $\alpha \in \operatorname{Hom}(\Omega_{W[n]_0}, \mathcal{O}_{W[n]_0})$ by the above identity. Hence the restriction of α to Δ_l, denoted by α_l, is a vector field of Δ_l. Since α lies in the kernel, we have $\alpha_1 = \alpha_{n+2} = 0$. Furthermore, the restriction of α_l to the two (or one if $l = 1$ or $n + 2$) distinguished divisors is tangential to these divisors and must satisfy the compatibility condition $\alpha_l|_{D_-} = \alpha_{l+1}|_{D_+}$. Now let V_l be the subspace of $H^0(T_{\Delta_l})$ consisting of the vector fields whose restrictions to the two distinguished divisors are tangential to the two divisors. Then V_l is canonically isomorphic to the direct sum $H^0(T_D) \oplus \mathbb{C}$, where $\mathbb{C}$ is generated by the vector field generated by the group action $G[n]_l$. Let $e(\alpha_l)$ be the component of α_l in $H^0(T_D)$. Then the compatibility condition translates to $e(\alpha_1) = \cdots = e(\alpha_{n+2})$. Adding the fact that $\alpha_1 = 0$, we conclude that $e(\alpha_l) = 0$ for $2 \leq l \leq n + 1$. Therefore, α_l must be a vector field tangential to the fibers of $\Delta_l \to D$ and vanishes on the two distinguished divisors. By Corollary 1.4, such α is generated by the group action $G[n]$.

If we apply this to the two extensions $(\phi_1 \circ \iota_k)^{h_k}$ and $\phi_2 \circ \varphi \circ \iota_k$, we see that there is a morphism $h_k^+ : B_k \to G[n]$ with $h_k^+|_{B_{k-1}} : B_{k-1} \to G[n]$ factor through $\{e\} \subset G[n]$ such that with $\rho_k = h_k^+ \cdot \iota_k$,

$$(\phi_1 \circ \iota_k)^{\rho_k} = \phi_2 \circ \varphi \circ \iota_k : \mathcal{W}_1 \times_S B_k \longrightarrow W[n].$$

Since k is arbitrary, this implies that the projection $R \to S$ dominates a neighborhood of $p \in S$.

We let $\pi : R \to S$ be the projection. To complete the proof of the Lemma, we need to find a neighborhood R_0 of $(p, \rho_0) \in R$, so that $\pi|_{R_0} : R_0 \to S$ is an open neighborhood of $p \in S$. First, by Corollary 1.5, for any $s \in S$ the set $\pi^{-1}(s)$ has at most one point. Applying Corollary 1.5 again, we see that the choice of ρ_k in the proof is unique. Hence the formal completion of R along (p, g_0) is isomorphic to the formal completion of S along p, under π. Hence $R \to S$ is étale over a neighborhood of $p \in S$. Combining the one-to-one and étale property, we conclude that there is a neighborhood $R_0 \subset R$ of (p, g_0) so that $\pi|_{R_0} : R_0 \to S$ is an open immersion. Let $\rho : R_0 \to G[n]$ be the morphism given by the definition of the functor $\mathfrak{Isom}[S]$ and let $\eta : R_0 \to S$ be the inclusion. Then ρ defines an effective arrow between $\xi_1 \times_S R_0$ and $\xi_2 \times_S R_0$. This proves Lemma 1.8. q.e.d.

We now define the stack $\mathfrak{W}$ of the expanded degenerations of W.

Definition 1.9. Let S be any C-scheme. An expanded degeneration of W over S is a pair $(\mathcal{W}, \rho)$, where $\mathcal{W}$ is a family over S and $\rho : \mathcal{W} \to W \times_C S$ is an S-projection, so that there is an open covering S_α of S such that for each S_α the restriction pair $(\mathcal{W} \times_S S_\alpha, \rho|_{\mathcal{W} \times_S S_\alpha})$ is isomorphic to an effective expanded degeneration of W. Let $(\mathcal{W}, \rho)$ and $(\mathcal{W}', \rho')$ be two degenerations over S and S' respectively. An *arrow* $\mathcal{W} \to \mathcal{W}'$ consists of a C-morphism $S \to S'$ and an S-isomorphism $\mathcal{W} \to \mathcal{W}' \times_{S'} S$ compatible to their projections to $W \times_C S$.

We define the groupoid of the expanded degenerations of W to be the category $\mathfrak{W}$ whose objects are expanded degenerations of W over S, where S is a C-scheme. In the following, we will call expanded degenerations of W simply degenerations if its meaning is clear from the context. The functor $\mathfrak{p} : \mathfrak{W} \to (\mathrm{Sch}/C)$ is the one that sends degenerations over S to S. Arrows between two objects are those defined in the above Definition. Obviously, if $\xi \in \mathfrak{W}$ with $\mathfrak{p}(\xi) = S$ and $i : T \to S$ is a morphism, then there is a unique pullback family $i^*\xi \in \mathfrak{W}$ with an arrow $i^*\xi \to \xi$. In this way, $(\mathfrak{W}, \mathfrak{p})$ forms a groupoid. For any $S \in (\mathrm{Sch}/C)$, we define $\mathfrak{W}(S)$ to be the category of all degenerations over S.

Proposition 1.10. *The groupoid $\mathfrak{W}$ is a stack.*

Proof. This is straightforward and will be left to the readers. q.e.d.

2. Predeformable morphisms

Recall that a prestable curve is a connected complete curve with

at most nodal singularity. (Since the marked points will not affect our discussion of predeformable morphisms, we will discuss curves without marked points in this section.) A morphism f whose domain is a prestable curve is called a prestable morphism. Let $f: X \to W[n]$ be a prestable morphism so that $f(X) \subset W[n]_0$. We say f is nondegenerate if no irreducible component of X is mapped entirely to the singular locus of $W[n]_0$. It is well-known that in general f may not be deformable to a prestable morphism to $W[n]$ over a general point $t \in C[n]$. One set of conditions is purely local in nature. It reflects the fact that $W[n]_0$ has nodal singularity. In this section, we will investigate such condition in details.

2.1 Morphisms of pure contact

We will investigate the following situation in this subsection. Let $\mathbf{k}[t] \to \mathbf{k}[w_1, w_2]$ be the homomorphism defined by $t \mapsto w_1 w_2$ and let $\phi: \mathbf{k}[\![s]\!] \to \mathbf{k}[\![z_1, z_2]\!]$ be the homomorphism defined by $\phi(s) = z_1 z_2$ or $\phi(s) = z_1$. Note that the former corresponds to a smoothing of the node and the latter corresponds to a family of smooth curves. We let A be a $\mathbf{k}[\![s]\!]$-algebra and let $R = \mathbf{k}[\![z_1, z_2]\!] \otimes_{\mathbf{k}[\![s]\!]} A$. We assume A is (s)-adically complete and let $\hat{R}$ be the (z_1, z_2)-adic completion of R. In this part, we will investigate the situation where there is a homomorphism $\psi : \mathbf{k}[t] \to A$ with induced homomorphism $\widetilde{\psi}: \mathbf{k}[t] \to \hat{R}$ via $\widetilde{\psi}(t) = 1 \otimes \psi(t)$ and a $\mathbf{k}[t]$-homomorphism

$$\varphi : \mathbf{k}[w_1, w_2] \longrightarrow \hat{R} \tag{2.1}$$

satisfying the following nondegeneracy condition: *The ideals* (z_1, z_2) *and* $(\varphi(w_1), \varphi(w_2))$ *in* $\hat{R}$ *satisfy*

$$(z_1, z_2)^m \subset (\varphi(w_1), \varphi(w_2)) \subset (z_1, z_2)$$

for some integer m.

Geometrically, this states that there are no irreducible components of the fibers of $\operatorname{Spec} \hat{R} \to \operatorname{Spec} A$ that are mapped entirely to the node in $\operatorname{Spec} \mathbf{k}[w_1, w_2]/(w_1 w_2)$.

We now state and prove our first technical result which explains the notion of curves of pure contact. We begin with the notion of normal forms of elements in $\hat{R}$.

Lemma 2.1. *Let $\hat{R}$ be as above. Then any $\alpha \in \hat{R}$ has a unique expression, called its normal form, as*

$$\alpha = a_0 + \sum_{i>0} a_i z_1^i + \sum_{i>0} b_i z_2^i, \quad a_i, b_i \in A. \tag{2.2}$$

We call such expansion the normal form of α.

Proof. Clearly, any α can be expressed as above. We now show that such expression is unique. Let $I_n \subset \hat{R}$ be the A-submodule spanned by all $z_1^i z_2^j$ satisfying $|i-j| > n$. Clearly, the quotient A-module $\hat{R}/I_n$ is a free A-module with basis $z_1^n, \ldots, z_1, 1, z_2, \ldots, z_2^n$. Let $\pi_n : \hat{R} \longrightarrow \hat{R}/I_n$ be the quotient homomorphism of A-modules. Assume $\alpha = 0$ has an expression as in (2.2), then $0 = \pi_n(\alpha) = a_0 + \sum_{j\geq 1}^{n} a_j z_1^j + b_j z_2^j$ in $\hat{R}/I_n$. Since $\hat{R}/I_n$ is a free A-module and $z_1^n \ldots, z_2^n$ is a basis, all a_j and b_j are zero for $j \leq n$. Since n is arbitrary, this implies that all a_j and b_j are zero. This proves the Lemma. q.e.d.

Proposition 2.2. *Let the notation be as above. We assume further that A is a local ring and is flat over $\mathbf{k}[t]$. Then of the two choices of $\phi : \mathbf{k}[\![s]\!] \to \mathbf{k}[\![z_1, z_2]\!]$ given before, only $\phi(s) = z_1 z_2$ is possible. Further, possibly after exchanging z_1 and z_2, the homomorphisms φ and ψ must be of the forms*

$$\varphi(w_1) = z_1^n \beta_1, \quad \varphi(w_2) = z_2^n \beta_2 \quad \text{and} \quad \psi(t) = s^n \epsilon$$

for some integer n, units β_1 and $\beta_2 \in \hat{R}$ and a unit $\epsilon \in A$ such that $\beta_1 \beta_2 = \epsilon$ in $\hat{R}$.

Proof. Let $\mathfrak{m} \subset A$ be the maximal idea and let

$$\varphi(w_i) = \sum_{j\geq 0} a_{i,j} z_1^j + \sum_{j\geq 0} b_{i,j} z_2^j, \qquad a_{i,0} = b_{i,0} \tag{2.3}$$

be the normal form of $\varphi(w_i)$. Since φ is nondegenerate, the homomorphism

$$\varphi_0 : \mathbf{k}[w_1, w_2] \longrightarrow \mathbf{k}[\![z_1, z_2]\!]/(z_1 z_2)$$

induced by φ and $\mathbf{k}[\![z_1, z_2]\!]/(z_1, z_2) \cong \hat{R}/\mathfrak{m}\hat{R}$ must be of the form (possibly after exchanging z_1 and z_2)

$$\begin{aligned} \varphi_0(w_1) &= c_1 z_1^{n_1} \mod z_1^{n_1+1} \qquad \text{and} \\ \varphi_0(w_2) &= c_2 z_2^{n_2} \mod z_2^{n_2+1}, \quad c_1, c_2 \neq 0, \end{aligned}$$

for some positive integers n_1 and n_2. Hence in (2.3), all $a_{1,j}$ for $j < n_1$ and $b_{2,j}$ for $j < n_2$ are in the maximal ideal $\mathfrak{m} \subset A$. We let

$$I = \big(\{a_{1,j}s^j, b_{1,j}, a_{2,j}, b_{2,j}s^j | j \geq 0\}\big) \subset A$$

be the ideal generated by the listed terms for all $j \geq 0$.

We first show that $I \subset \mathfrak{m}$. After expressing $\varphi(w_1)\varphi(w_2)$ in its normal form $\varphi(w_1)\varphi(w_2) = c_0 + \sum_{j>0} c_j z_1^j + d_j z_2^j$ we have

$$c_k = \sum_{i+j=k} a_{1,i}a_{2,j} + \sum_{i\geq 0} a_{1,k+i}b_{2,i}s^i + \sum_{i\geq 0} a_{2,k+i}b_{1,i}s^i, \quad k > 0$$

and

$$d_k = \sum_{i+j=k} b_{1,i}b_{2,j} + \sum_{i\geq 0} b_{1,k+i}a_{2,i}s^i + \sum_{i\geq 0} b_{2,k+i}a_{1,i}s^i, \quad k > 0.$$

Because $\varphi(w_1)\varphi(w_2) = \varphi(w_1w_2)$ is the image of t in A, all c_k and d_k for $k > 0$ are zero. Now let l be the largest integer so that $b_{1,j} \in \mathfrak{m}$ for $j < l$. Then by considering the relation $d_{l+n_2} = 0$, we obtain

$$\sum_{i+j=l+n_2} b_{1,i}b_{2,j} + b_{1,n_2+l}a_{2,0} + b_{2,n_2+l}a_{1,0} \in \mathfrak{m}.$$

By assumption, all terms in the above expression except $b_{1,l}b_{2,n_2}$ are in $\mathfrak{m}$. Hence $b_{i,l}b_{2,n_2} \in \mathfrak{m}$. Since $b_{2,n_2} \notin \mathfrak{m}$, $b_{1,l} \in \mathfrak{m}$. This shows that there is no such l and hence all $b_{1,j}$ are in $\mathfrak{m}$. Similarly, by considering the relation $c_k = 0$ we can show that all $a_{2,j} \in \mathfrak{m}$.

Now let $n = \min(n_1, n_2)$ and let $\bar{I}$ be the associated ideal of I in the quotient ring $A/(s^n)$. Let $\overline{\mathfrak{m}}$ be the maximal ideal of $A/(s^n)$. We now show that $\bar{I} \subset \overline{\mathfrak{m}} \cdot \bar{I}$. Note that in the expression of c_k and d_k, terms $a_{2,k+i}b_{1,i}s^i$ and $b_{1,k+i}a_{2,i}s^i$ are in $\mathfrak{m} \cdot I$ for all $k, i \geq 0$. We let $J = \mathfrak{m} \cdot I + (s^n)$. Note that $b_{2,k}s^k \in J$ for $k \geq n$. Now let k be the smallest integer so that $b_{2,k+i}s^{k+i} \in J$ for all $i > 0$. Hence $k < n$. Using $c_{n_1-k} = 0$, we obtain

$$\sum_{n_1-k+i\geq 0} a_{1,n_1-k+i}b_{2,i}s^i + \sum_{i+j=n_1-k} a_{1,i}a_{2,j} \in \mathfrak{m} \cdot I,$$

which implies

$$\begin{aligned} &a_{1,n_1}b_{2,k}s^k + \sum_{k-n_1<i<k} a_{1,n_1-k+i}b_{2,i}s^i + \sum_{k<i} a_{1,n_1-k+i}b_{2,i}s^i \\ &\equiv a_{1,n_1}b_{2,k}t^k \mod J. \end{aligned}$$

Hence $b_{2,k}s^k \in J$. By minimality of k we have $b_{2,j}s^j \in J$ for all $j \geq 0$. For similar reason, we can prove $a_{1,k}s^k \in J$ for all $k \geq 0$. To show $a_{2,k} \in J$, we use the relation $c_{n_1+k} = 0$, which combined with $a_{2,k+i}b_{1,i} \in J$ and $b_{2,j}s^j \in J$ implies $\sum_{i+j=n_1+k} a_{1,i}a_{2,j} \in J$. Now assume $a_{2,j} \in J$ for $j < k$, then the above inclusion implies

$$a_{1,n_1}a_{2,k} + \sum_{i>0} a_{1,n_1-i}a_{2,k+i} + \sum_{i>0} a_{1,n_1+i}a_{2,k-i} \in J.$$

Since $a_{1,n_1-i} \in \mathfrak{m}$ and $a_{2,k+i} \in I$, we have $a_{1,n_1}a_{2,k} \in J$ and hence $a_{2,k} \in J$. For similar reason, $b_{1,k} \in J$ for all $k > 0$. Combined, this proves that $\bar{I} \subset \bar{\mathfrak{m}} \cdot \bar{I}$. Since A is Noetherian, this is possible only if $\bar{I} = 0$. This proves $I \subset (s^n)$.

We next show $n_1 = n_2$. Assume $n = n_1 < n_2$. Then $d_{n_2-n_1} = 0$ implies that

$$\sum_{i+j=n_2-n_1} b_{1,i}b_{2,j} + \sum_{i\geq 0} b_{1,n_2-n_1+i}a_{2,i}s^i + \sum_{i\geq 0} b_{2,n_2-n_1+i}a_{1,i}s^i = 0.$$

Note that all terms in this expression except $b_{2,n_2}a_{1,n_1}s^{n_1}$ belong to $\mathfrak{m} \cdot (s^n)$. This is impossible. Hence $n_1 = n_2$.

We now look at the normal form of $\varphi(w_1)$. Since $b_{1,j} \in (s^n)$ for all $j \geq 0$, there are $b'_{1,j} \in A$ so that $b_{1,j} = b'_{1,j}s^n$. On the other hand, since $a_{1,j}s^j \in (s^n)$, there are $a'_{1,j}$ so that $a_{1,j} = a'_{1,j}s^{n-j}$ for $j < n$. Using $s = z_1z_2$, we see that there are β_1 and β_2 in $\hat{R}$ so that $\varphi(w_i) = z_i^n\beta_i$. Further by our choice of n both β_1 and β_2 are units in $\hat{R}$. It remains to show that there is a unit $\epsilon \in A$ so that $\psi(t) = s^n\epsilon$ and $\beta_1\beta_2 = \epsilon$. First, since $\psi(t) = \varphi(w_1w_2)$, which is in the ideal (s^n), there is an $\epsilon \in A$ so that $\psi(t) = s^n\epsilon$. Hence $s^n(\beta_1\beta_2 - \epsilon) = 0$ in $\hat{R}$. Because $\hat{R}$ is flat over $\mathbf{k}[t]$, this implies that $\beta_1\beta_2 - \epsilon = 0$. Hence ϵ is a unit in A. This completes the proof of Proposition 2.2. q.e.d.

This Proposition inspires the notion of homomorphisms of pure contacts as follows.

Definition 2.3. Let the notation be as stated in the beginning of this subsection with A only assumed to be (s)-adically complete. We say φ has pure contact if, possibly after exchanging z_1 and z_2, there is a unit β in $\hat{R}$, a unit ϵ in A and an integer n such that $\varphi(w_1) = \beta z_1^n$ and $\varphi(w_2) = \epsilon\beta^{-1}z_2^n$.

The integer n is called the order of contact. Sometimes to emphasize this order we will say φ is of pure n-contact. The following is the main existence Lemma of this section.

Lemma 2.4. *Let n be any positive integer. Suppose $a_{1,n}$ and $b_{2,n}$ in the normal forms of $\varphi(w_1)$ and $\varphi(w_2)$ in (2.3) respectively are units in A. Then there is an ideal $I_A \subset A$ so that for any $A \to T$ the induced homomorphism*

$$\varphi_T : \mathbf{k}[w_1, w_2] \longrightarrow \hat{R} \otimes_A T$$

has pure n-contact if and only if $A \to T$ factors through $A/I_A \to T$. The ideal I_A satisfies the following base change property: Let $A \to A'$ be a homomorphism of rings and let $\varphi' : \mathbf{k}[w_1, w_2] \to (\mathbf{k}[\![z_1, z_2]\!] \otimes_{\mathbf{k}[\![s]\!]} A')\hat{}$, where $(\cdot)\hat{}$ is the (z_1, z_2)-adic completion of the respective ring, be the induced $\mathbf{k}[t]$-homomorphism. Then $I_{A'} = I_A \cdot A'$.

Proof. We let $\zeta, \epsilon, \xi_j, \eta_j$, where j runs through all positive integers, be indeterminants. We consider the ring $B = A[\![\zeta, \zeta^{-1}, \epsilon, \epsilon^{-1}, \boldsymbol{\xi}, \boldsymbol{\eta}]\!]$, where $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ mean $(\xi_1, \xi_2 \cdots)$ and $(\eta_1, \eta_2, \dots)$ respectively. We let C be the (z_1, z_2)-adic completion of $\mathbf{k}[\![z_1, z_2]\!] \otimes_{\mathbf{k}[\![s]\!]} B$. We let $\gamma \in C$ and its inverse be

$$\gamma = \zeta \left(1 + \sum_{j>0} \xi_i z_1^i + \eta_j z_2^j\right) \quad \text{and} \quad \gamma^{-1} = \zeta^{-1} \left(1 + \sum_{j>0} \widetilde{\xi}_i z_1^i + \widetilde{\eta}_j z_2^j\right),$$

where $\widetilde{\xi}_j$ and $\widetilde{\eta}_j$ are elements in B. We consider the ideal $J \subset C$ that is generated by all coefficients of z_i^j in the normal forms of

$$\Phi_1 = \varphi(w_1) - z_1^n \zeta \left(1 + \sum_{j>0} \xi_j z_1^j + \sum_{j>0} \eta_j z_2^j\right) \tag{2.4}$$

and

$$\Phi_2 = \varphi(w_2) - z_2^n \zeta^{-1} \left(1 + \sum_{j>0} \widetilde{\xi}_j z_1^j + \sum_{j>0} \widetilde{\eta}_j z_2^j\right) \epsilon. \tag{2.5}$$

We now investigate these generators of J, or equivalently the relations in C/J. First note that the coefficient of z_1^{n+k} in Φ_1 is $a_{1,k+n} - \xi_k$. Hence we obtain elements $a_{1,k+n} - \xi_k$ in J for all $k \geq 1$. Similarly, the coefficient of z_1^n in Φ_1 is $a_{1,n} - \zeta$, which by definition is in J. In the following we will call them the canonical relations of ξ_k and η, respectively. We let J_1 be the ideal generated by $a_{1,n} - \zeta$ and all $a_{1,k+n} - \zeta_k$. We next find relations of η_j. First, by expanding $1/\gamma$ in power series and using

$s = z_1 z_2$, we can express $\widetilde{\eta}_k$ canonically as $\widetilde{\eta}_k = -(\eta_k + \sum_{j\geq 1} c_{k,j} s^j)$, where $c_{k,j} \in \mathbf{k}[\![\boldsymbol{\xi}, \boldsymbol{\eta}]\!]$. Then the coefficient of z_2^{k+n} in Φ_2 becomes

$$b_{2,n+k} + \zeta^{-1}\eta_k \epsilon + \sum_{j\geq 1} \zeta^{-1} c_{k,j} s^j \epsilon.$$

We let J_2 be the ideal generated by J_1 and this set of elements for all $k > 0$. Since $c_{k,j} \in \mathbf{k}[\![\boldsymbol{\xi}, \boldsymbol{\eta}]\!]$, modulo J_2 we can substitute ζ and all ξ_l by elements in A using relations in J_1 and then substitute all η_l in $c_{k,j}$ by $-\zeta\epsilon^{-1} b_{2,n+k} - \sum c_{k,j} s^j$. This way we obtain

$$\eta_k - \Big(d_{1,k} + \sum_{j\geq 2} \widetilde{c}_{k,j} s^j \Big) \in J_2,$$

where $d_{k,1} \in A$ and $\widetilde{c}_{k,1} \in A[\![\boldsymbol{\eta}, \epsilon^{-1}]\!]$. Note that here we have used the fact that A is complete with respect to the ideal (s). Repeating this procedure, we eventually obtain an element $\eta_k - f_k(\epsilon^{-1}) \in J_2$, where $f_k(\epsilon^{-1}) \in A[\![\epsilon^{-1}]\!]$. It remains to find relation in ϵ. By using the coefficient of z_2^n in Φ_2, we obtain an element

$$b_{2,n} - \epsilon\zeta^{-1}\Big(1 + \sum_{j\geq 1} h_j s^j\Big) \in J,$$

where $h_j \in \mathbf{k}[\![\boldsymbol{\xi}, \boldsymbol{\eta}]\!]$. We then let $J_0 \subset C$ be the ideal generated by J_2 and this element. Then again modulo J_0, which amounts to repeatedly replacing ζ, ξ_k and η_k by relations in J_0, we obtain an element $-1 + \zeta\epsilon^{-1} b_{2,n} + s\phi(\epsilon^{-1}) \in J_0$, where $\phi(\epsilon^{-1}) \in A[\![\epsilon^{-1}]\!]$. By iteration and taking the limit, we obtain a canonical element $\epsilon - h \in J$ for some $h \in A$. It is clear from this construction that h is a unit since $b_{2,n}$ is a unit. This way we have found elements $c_{1,k}$ and $c_{2,k} \in A$ for $k \geq 1$ and a unit $h \in A$ so that

$$J_0 = \Big(\{\xi_k - c_{1,k},\ \eta_k - c_{2,k} \mid k > 0\};\ \zeta - a_{1,n},\ \epsilon - h \Big) \subset C. \tag{2.6}$$

From this set of generators, we see that the canonical homomorphism $A \to C/J_0$ is an isomorphism. Since J_0 is a subideal of J, we have quotient homomorphism $C/J_0 \to C/J$. We let $I_A \subset A$ be the kernel of $A \to C/J_0 \to C/J$. We will show that the ideal I_A satisfies the property specified in the statement of the Lemma. First, from the construction it is direct to check that I_A satisfies the base change property. Now let $A \to T$ and φ_T be as in the Lemma so that φ_T has pure n-contact.

Then it follows from our construction that the ideal $I_T = \{0\}$. Therefore $A \to T$ factors through $A/I_A \to T$ since $I_T = I_A \cdot T$. This proves one direction of the Lemma. Now assume $A \to T$ is a homomorphism so that it factors through $A/I_A \to T$. Then by the base change property we have $I_T = \{0\}$. It follows from the construction of I_T that φ_T does have pure n-contact. This completes the proof of Lemma 2.4. q.e.d.

2.2 Families of predeformable morphisms

In this subsection, we will introduce the notion of predeformable morphisms. We will then prove that for any family of nondegenerate stable morphisms $f:\mathcal{X} \to W[n]$ over S, the locus $S_{pd} \subset S$ of all predeformable morphisms admits a natural closed subscheme structure.

We begin with some notation that will be used throughout this section. We let l be any integer in $[n+1]$. Following our convention in Section 1, we let $l^\vee$ be the complement of l in $[n+1]$ and $C[n]_{[l^\vee]}$ the smooth divisor $C \times_{\mathbf{A}^1} \mathbf{H}_l^{n+1} \subset C[n]$, where $\mathbf{H}_l^{n+1} \subset \mathbf{A}^{n+1}$ is the l-th coordinate hyperplane. We define

$$C_0[n] \triangleq C[n] \times_C 0 = \cup_{l=1}^{n+1} C[n]_{[l^\vee]}.$$

We denote by $\mathbf{D}_l$ the nodal divisor of $W[n] \times_{C[n]} C[n]_{[l^\vee]}$ and by Δ_l^- and Δ_l^+ its two smooth irreducible components. According to our ordering of the irreducible components of $W[n]_0$, Δ_l^- is on the left of Δ_l^+. Then $W[n] \times_{C[n]} C[n]_{[l^\vee]}$ is the union of Δ_l^- and Δ_l^+ intersecting transversely along $\mathbf{D}_l$. Let $\pi_l : C[n] \to \mathbf{A}^1$ be induced by the l-th projection $\mathbf{A}^{n+1} \to \mathbf{A}^1$. For any point $q \in \mathbf{D}_l$ there is a neighborhood $\mathcal{W}$ of $q \in W[n]$ with two regular functions w_1 and w_2 so that $w_1 = 0$ (resp. $w_2 = 0$) defines the divisor $\mathcal{W} \cap \Delta_l^+$ (resp. $\mathcal{W} \cap \Delta_l^-$). By adjusting w_2 if necessary, we can assume $w_1 w_2$ is the image of t_l via $\Gamma(\mathbf{A}^{n+1}) \to \Gamma(\mathcal{W})$. Hence by shrinking $\mathcal{W}$ if necessary, we obtain a smooth morphism

$$\psi : \mathcal{W} \longrightarrow \Theta_l \triangleq \operatorname{Spec} \mathbf{k}[w_1, w_2] \otimes_{\mathbf{k}[t_l]} \mathbf{k}[\mathbf{t}]. \tag{2.7}$$

Here $\mathbf{k}[t_l] \to \mathbf{k}[w_1, w_2]$ is defined by $t_l \mapsto w_1 w_2$ and $\mathbf{k}[\mathbf{t}] = \mathbf{k}[t_1, \ldots, t_{n+1}]$. Later we will call $\mathcal{W}$ with ψ implicitly understood an admissible chart of $W[n]$ at $q \in \mathbf{D}_l$.

Let $f : X \to W[n]$ be a prestable morphism. We say f is nondegenerate if no irreducible component of X is mapped entirely to any of $\mathbf{D}_1, \ldots, \mathbf{D}_{n+1}$. Now assume f is nondegenerate and let $p \in f^{-1}(\mathbf{D}_l)$. We define the notion of upper and lower contact order of f along p.

Let $\hat{X}$ be the formal completion of X along p and let $\hat{\Delta}_l^-$ (resp. $\hat{\Delta}_l^+$; resp. $\hat{\mathbf{D}}_l$) be the formal completion of Δ_l^- (resp. Δ_l^+; resp. $\mathbf{D}_l$) along $f(p)$. For any irreducible component $\hat{V}$ of $\hat{X}$, we can define the contact order of f along $\hat{V}$ as follows. Assume $f(\hat{V}) \subset \hat{\Delta}_l^-$. We let z be a generator of the maximal ideal of $\mathcal{O}_{\hat{V}}$ and let w be a generator of the ideal $\mathcal{I}_{\hat{\mathbf{D}}_l \subset \hat{\Delta}_l^-} \subset \mathcal{O}_{\hat{\Delta}_l^-}$. Let $f^* : \mathcal{O}_{\hat{\Delta}_l^-} \to \mathcal{O}_{\hat{V}}$ be the induced homomorphism. Then $f^*(w) = z^\delta \beta$ for a positive integer δ and a unit $\beta \in \mathcal{O}_{\hat{V}}$. The integer δ is the contact order of f along $\hat{V}$. The contact order of f along $\hat{V}$ when $f(\hat{V}) \subset \hat{\Delta}_l^+$ is defined similarly. We define the left contact order (resp. right contact order) of f at p, denoted $\delta(f,p)^-$ (resp. $\delta(f,p)^+$), to be the sum of the contact orders of f along all irreducible components of $\hat{X}$ that are mapped to $\hat{\Delta}_l^-$ (resp. $\hat{\Delta}_l^+$).

Definition 2.5. Let $f : X \to W[n]$ be a nondegenerate prestable morphism. We say f is predeformable along $\mathbf{D}_l$ if $\delta(f,p)^- = \delta(f,p)^+$ for all $p \in f^{-1}(\mathbf{D}_l)$. We say f is predeformable if f is predeformable along all $\mathbf{D}_l$.

Note that if $f^{-1}(\mathbf{D}_l) = \emptyset$, then f is automatically predeformable along $\mathbf{D}_l$.

In the remainder part of this section, we fix a scheme S over C and a family

$$f : \mathcal{X} \to W[n] \tag{2.8}$$

of nondegenerate prestable morphisms over S. Our goal is to construct a maximal closed subscheme $S_{\mathrm{pd}} \subset S$ so that it parameterizes all predeformable morphisms in S.

We begin our investigation with the decomposition of f along $\mathbf{D}_l$. We let S_l be $S \times_{C[n]} C[n]_{[l^\vee]}$ endowed with the reduced scheme structure. We have the following Decomposition Lemma whose proof follows from the standard deformation theory of nodal singularity of curves.

Lemma 2.6. *Let the notation be as above. We let $\mathcal{X}_l = \mathcal{X} \times_S S_l$ and let f_l be the restriction of f to $\mathcal{X}_l$. Then there are two flat families of curves (not necessarily connected) $\mathcal{Y}_-$ and $\mathcal{Y}_+$ over S_l, a pair of S_l-morphisms $g_\pm : \mathcal{Y}_\pm \to \Delta_l^\pm$, a scheme Σ finite and étale over S_l and a pair of S_l-immersions $\iota_\pm : \Sigma \to \mathcal{Y}_\pm$ of which the followings hold: The original family $\mathcal{X}_l$ is the gluing of $\mathcal{Y}_-$ and $\mathcal{Y}_+$ along $\iota_-(\Sigma) \subset \mathcal{Y}_-$ and $\iota_+(\Sigma) \subset \mathcal{Y}_+$, and the restriction of f_l to $\mathcal{Y}_\pm \subset \mathcal{X}_l$ is $g_\pm : \mathcal{Y}_\pm \to \Delta_l^\pm \subset W[n]$.*

Later, we will write $\mathcal{X} = \mathcal{Y}_- \sqcup \mathcal{Y}_+$ and $f = g_- \sqcup g_+$ to indicate that $(\mathcal{Y}_\pm, g_\pm)$ are the decomposition of $(f, \mathcal{X})$.

Lemma 2.7. *Let $S_{\mathrm{pd},l} \subset S$ be the set of all closed points $\xi \in S$ so that f_ξ is predeformable along $\mathbf{D}_l$. Then $S_{\mathrm{pd},l}$ is closed in S.*

Proof. It suffices to prove the case where S is reduced and irreducible, which we assume now. Let $S \to \mathbf{A}^1$ be induced by $S \to C[n] \xrightarrow{\pi_l} \mathbf{A}^{n+1}_l \cong \mathbf{A}^1$. In case S is dominant over $\mathbf{A}^1$, then Proposition 2.2 implies that $S_{\mathrm{pd},l} = S$ as topological spaces and hence the Lemma holds. Now assume $S \to \mathbf{A}^1$ factor through $0 \in \mathbf{A}^1$. Since S is reduced, $S \to C[n]$ factors though $C[n]_{[l^\vee]} \subset C[n]$ and then $S = S_l$ as schemes. Let $\iota_\pm : \Sigma \to \mathcal{Y}_\pm$ and $g_\pm : \mathcal{Y}_\pm \to \Delta^\pm_l$ be the decomposition of f provided by Lemma 2.6. Let $\pi : \Sigma \to S$ be the projection. We let $\iota : \Sigma \to \mathcal{X}$ be the composite $\iota_\pm : \Sigma \to \mathcal{Y}_\pm \to \mathcal{X}$. As before, for any $\xi \in S$ we denote by f_ξ the restriction of f to the fiber over ξ and denote by Σ_ξ the fiber of Σ over ξ. Then the left and the right contact order of f along $\iota(\Sigma)$ define two functions

$$\delta(f_{\pi(\cdot)}, \iota(\cdot))^\mp : \Sigma \longrightarrow \mathbb{Z}. \tag{2.9}$$

It follows from the definition that $\xi \in S_{\mathrm{pd},l}$ if the following two conditions hold:

(i) *For any $x \in \Sigma_\xi$, $\delta(f_\xi, \iota(x))^+ = \delta(f_\xi, \iota(x))^-$.*

(ii) *The preimage $f_\xi^{-1}(\mathbf{D}_l)$ is identical to $\iota(\Sigma_\xi)$.*

Applying the usual semi-continuity theorem, we see that both $\delta(f_{\pi(\cdot)}, \iota(\cdot))^\pm$ are upper semicontinuous. Hence (i) and (ii) combined implies that $S_{\mathrm{pd},l}$ is a constructible set. Therefore, to show that $S_{\mathrm{pd},l}$ is closed it suffices to show that if R is a discrete valuation domain with residue field $\mathbf{k}$ and quotient field K, and if $\operatorname{Spec} K \to S_{\mathrm{pd},l}$ extends to $\operatorname{Spec} R \to S$, then the extension $\operatorname{Spec} R \to S$ factor through $S_{\mathrm{pd},l} \subset S$.

Now we let $\operatorname{Spec} R \to S$ be a morphism so that $\operatorname{Spec} K$ factors through $S_{\mathrm{pd},l} \subset S$. We let $f_R : \mathcal{X}_R \to W[n]$ be the pull back family over $\operatorname{Spec} R$ via $\operatorname{Spec} R \to S$. By abuse of notation, we still denote by $\mathcal{Y}_\pm$, $g_\pm$ and $(\iota_\pm, \Sigma)$ the decomposition data of the R-curve $(f_R, \mathcal{X}_R)$. Let $\eta = \operatorname{Spec} K$ be the generic point of $\operatorname{Spec} R$. Then by assumption, Conditions (i) and (ii) hold for $\xi = \eta$. Now let $\xi \in \operatorname{Spec} R$ be any element. As before, for $\alpha = -$ or $+$ we let $g_{\alpha,\xi}$ and $\mathcal{Y}_{\alpha,\xi}$ be the restrictions of the respective families to the fiber over ξ. We let $g^*_{\alpha,\xi}(\mathbf{D}_l)$ be the pullback divisor where $\mathbf{D}_l$ is viewed as a divisor in Δ^α_l. Since f_R is nondegenerate,

$$\deg(g^*_{\alpha,\xi}(\mathbf{D}_l), \mathcal{Y}_{\alpha,\xi}) \geq \sum_{x \in \iota(\Sigma) \cap \mathcal{X}_\xi} \delta(f_\xi, \iota(x))^\alpha, \quad \alpha = - \text{ and } +. \tag{2.10}$$

Clearly, the equality in (2.10) holds for α if $g^*_{\alpha,\xi}(\mathbf{D}_l) = \iota_\alpha(\Sigma) \cap \mathcal{X}_\xi$. Hence Condition (ii) is equivalent to the assumption that the equalities in the above two inequalities hold. Because the function of contact order is upper semicontinuous, for the closed point $\eta_0 \in \operatorname{Spec} R$,

$$\sum_{x \in \iota(\Sigma) \cap \mathcal{X}_{\eta_0}} \delta(f_{\eta_0}, \iota(x))^- \geq \sum_{x \in \iota(\Sigma) \cap \mathcal{X}_{\eta}} \delta(f_{\eta}, \iota(x))^-.$$

On the other hand, since f is a flat family of nondegenerate morphisms, we have

$$\deg(g^*_{-,\eta_0}(\mathbf{D}_l), \mathcal{Y}_{-,\eta_0}) = \deg(g^*_{-,\eta}(\mathbf{D}_l), \mathcal{Y}_{-,\eta}).$$

Combined with the inequality in (2.10) for $\alpha = -$ and $\xi = \eta_0$ and the equality in (2.10) for $\alpha = -$ and $\xi = \eta$, we see that the equality in (2.10) for $\alpha = -$ and $\xi = \eta_0$ must also hold. For similar reasons, the equality in (2.10) also holds for $\alpha = +$ and $\xi = \eta_0$. Hence the two contact order functions in (2.9) must be constant functions. Therefore, (i) and (ii) hold at η_0 and hence $\operatorname{Spec} R \to S$ factors through $S_{\mathrm{pd},l} \subset S$. This shows that $S_{\mathrm{pd},l}$ is closed in S. q.e.d.

Let S_{pd} be the subset of S consisting of $\xi \in S$ so that f_ξ is predeformable. Clearly, $S_{\mathrm{pd}} = \cap_{l=1}^{n+1} S_{\mathrm{pd},l}$. Hence S_{pd} is closed in S. In the remainder of this section, we will give S_{pd} a canonical closed scheme structure.

We begin with the notion of parameterizations of the formal neighborhood of nodes of prestable curves. Let $\pi : \mathcal{X} \to S$ be a family of prestable curves and let $\mathcal{X}_{\mathrm{node}} \subset \mathcal{X}$ be the locus of all nodes of the fibers of $\mathcal{X}$ over S, endowed with the reduced scheme structure. Let $p \in \mathcal{X}_{\mathrm{node}}$ be any closed point and let $V \subset \mathcal{X}_{\mathrm{node}}$ be an affine neighborhood of $p \in \mathcal{X}_{\mathrm{node}}$. Then by shrinking V if necessary we can find a scheme U containing V as its closed subscheme so that there is an étale $r : U \to S$ extending $\pi|_V : V \to S$.[5] We let $\hat{V}$ be the formal completion of U along V. We now argue that $\hat{V}$ is independent of the choice of U. Indeed, in case $r' : U' \to S$ is another such U and $\hat{V}'$ is the formal completion of U' along V. Then since both $\hat{U} \to S$ and $\hat{U}' \to S$ are étale, by the topological invariance of étale morphisms [31, p. 30][6] , the scheme $\hat{V}$ is canonically isomorphic to $\hat{V}'$. Thus $\hat{V}$ is independent of

[5]Such (U, r) can be constructed by elementary method. For instance, this follows from the proof of Lemma 1.5 in [23].

[6]In [31] this theorem was stated for the case $V \subset \hat{V}$ is defined by a nilpotent ideal. However, the proof can be adopted without change to cover our case.

the choice of U. Now let $\mathcal{X}_{\hat{V}}$ be $\mathcal{X}\times_S \hat{V}$. Clearly, $V\subset \mathcal{X}_{\rm node}$ is a closed subscheme of $\mathcal{X}_{\hat{V}}$. We let $\hat{\mathcal{X}}_{\hat{V}}$ be the formal completion of $\mathcal{X}_{\hat{V}}$ along $V\subset \mathcal{X}_{\hat{V}}$.

Lemma 2.8. *Let $\pi:\mathcal{X}\to S$ be a family of prestable morphisms as before and let $p\in\mathcal{X}_{\rm node}$ be any closed point. Then there is an affine neighborhood $V\subset\mathcal{X}_{\rm node}$ of p so that with $\hat{V}$ and $\hat{\mathcal{X}}_{\hat{V}}$ constructed as before, we have a morphism $\underline{\phi}:\hat{V}\to \operatorname{Spec}\mathbf{k}[\![s]\!]$ and a $\hat{V}$-isomorphism*

$$\phi:\hat{\mathcal{X}}_{\hat{V}}\longrightarrow \operatorname{Spec}\big(\mathbf{k}[\![z_1,z_2]\!]\otimes_{\mathbf{k}[\![s]\!]}\Gamma(\mathcal{O}_{\hat{V}})\big)\hat{}\,,$$

where $(\cdot)\hat{}$ is the (z_1,z_2)-adic completion of the respective ring. We will call ϕ a parameterization of the formal neighborhood of the nodes of $\mathcal{X}$ along V. Furthermore, all such parameterizations satisfy the following uniqueness property. Let ϕ' be another parameterization of $\mathcal{X}$ along V. Then there is a unit u in $\big(\mathbf{k}[\![z_1,z_2]\!]\otimes_{\mathbf{k}[\![s]\!]}\Gamma(\mathcal{O}_{\hat{V}})\big)\hat{}$ and a unit ϵ in $\Gamma(\mathcal{O}_{\hat{V}})$ such that $\underline{\phi'}^*(s)=\epsilon\underline{\phi}^*(s)$ and that the $\Gamma(\mathcal{O}_{\hat{V}})$-automorphism $(\phi'\circ\phi^{-1})^*$ of $\hat{R}$ is defined by

$$(\phi'\circ\phi^{-1})^*(z_1)=z_1u\quad\text{and}\quad(\phi'\circ\phi^{-1})^*(z_2)=z_2u^{-1}\epsilon.$$

Finally, the choice of such (u,ϵ) is unique.

Proof. The existence of such a parameterization follows immediately from the deformation of nodes [4]. This also follows from the local parameterization constructed in [23, Section 1]. Now let ϕ and ϕ' be two such parameterizations of $\mathcal{X}$ along V. Because $\underline{\phi'}^*(s)$ and $\underline{\phi}^*(s)$ define the same divisor in $\hat{U}$ corresponding to the subscheme where the nodes are not smoothed, there is a unit $\epsilon\in\Gamma(\mathcal{O}_{\hat{V}})$ such that $\underline{\phi'}^*(s)=\epsilon\underline{\phi}^*(s)$. Hence $\phi'^*(z_1z_2)=\epsilon\phi^*(z_1z_2)$ and hence

$$(\phi'\circ\phi^{-1})^*(z_1z_2)=(\phi^{*-1}\circ\phi'^*)(z_1z_2)=\epsilon z_1z_2. \tag{2.11}$$

It follows from the deformation of nodes that there are units u_1 and u_2 in $\hat{R}$ such that

$$(\phi'\circ\phi^{-1})^*(z_1)=z_1u_1\quad\text{and}\quad(\phi'\circ\phi^{-1})^*(z_2)=z_2u_2. \tag{2.12}$$

Using the above relation, we obtain $su_1u_2=s\epsilon$. Hence $u_2=\epsilon(u_1^{-1}+e)$ for some $e\in\hat{R}$ satisfying $se=0$. We now demonstrate that if we choose u_1, u_2 and ϵ appropriately, then e will be zero. We first write e in its normal form

$$e=\alpha_0+\sum_{i\ge1}z_1^i\alpha_i+z_2^i\beta_i,\quad \alpha_i,\beta_i\in\Gamma(\hat{U}).$$

Since $se = 0$, by the uniqueness of the normal form we have $z_1\beta_i = 0$ and $z_2\alpha_i = 0$ for $i \geq 1$ while $s\alpha_0 = 0$. We let $u_1 = \gamma_0 + \sum z_1^i\gamma_i + z_2^i\delta_i$ be its normal form. Since u_1 is a unit, γ_0 must be a unit. We now let $\epsilon' = \epsilon + \gamma_0\alpha_0$. Then since $s\alpha_0 = 0$, we can replace ϵ in (2.11) by ϵ'. Then the new error term $e' = u_2 - u_1^{-1}\epsilon'$, which obeys $u_2 = \epsilon'(u_1^{-1} + e')$, belongs to the ideal (z_1, z_2). Hence without loss of generality we can assume the ϵ, u_1 and u_2 are chosen a priori so that the α_0 in the normal form of e is zero. On the other hand, since $z_2\alpha_i = 0$ for $i > 0$, we can replace u_2 by $u_2' = u_2 - \sum_{i>0} z_1^i\alpha_i$ so that (2.11) and (2.12) still hold. Hence we can assume a priori that all $\alpha_i = 0$. Lastly, we let $u_1' = u_1(1 + eu_1\epsilon^{-1})^{-1}$. Clearly, $z_1u_1 = z_1u_1'$ and $u_2 = \epsilon u_1'^{-1}$ since $u_2 = u_1^{-1}\epsilon + e$. Thus (ϵ, u) with $u = u_1'$ is what we want.

The uniqueness of (u, ϵ) can be proved following same strategy. Since a more general version will be proved in [23, Section 1], we will omit the proof here. q.e.d.

We keep the integer $l \in [n]$ and continue to use the notation developed so far. Let $\pi : \Sigma \to S_l$ be the projection and let $p \in \Sigma$ be any closed point so that $\pi(p) \in S_{\mathrm{pd},l}$. We pick a connected neighborhood V of $p \in \Sigma$ and a local parameterization ϕ of the formal neighborhood of the nodes of $\mathcal{X}$ along V. Let $A = \Gamma(\mathcal{O}_{\hat{V}})$ and let $\hat{V} = \operatorname{Spec} A$. Let $\hat{R}$ be the (z_1, z_2)-adic completion of $\mathbf{k}[\![z_1, z_2]\!] \otimes_{\mathbf{k}[\![s]\!]} A$. Without loss of generality, we can assume that the image $f(V)$ is contained in an admissible chart $(\mathcal{W}, \psi)$ of $W[n]$ constructed in the beginning of this subsection. Let

$$\varphi : \mathbf{k}[w_1, w_2] \longrightarrow \hat{R} \tag{2.13}$$

be the morphism induced by f, the parameterization ϕ and ψ, and the projection $\Theta_l \to \operatorname{Spec}\mathbf{k}[w_1, w_2]$. Let $\mathfrak{m}$ be the ideal of the point $\xi \in V$, where $\xi = \pi(p)$. Since f_ξ is a predeformable morphism, the left and the right contact order of f_ξ at p coincide, which we denote by n. Hence possibly after exchanging z_1 and z_2, near p we have $f_\xi^*(w_i) = c_i z_i^n$ for some units c_i in $\mathbf{k}[\![z_1, z_2]\!]$. Hence in the normal form $\varphi(w_i) = \sum a_{i,j}z_1^j + b_{i,j}z_2^j$, as in (2.3), both $a_{1,n}$ and $b_{2,n}$ are not in $\mathfrak{m}$. Hence for sufficiently small neighborhood V_0 of $p \in V$ both $a_{1,n}$ and $b_{2,n}$ are units in $B = \Gamma(\mathcal{O}_{V_0})$. Therefore we can apply Lemma 2.4 to obtain an ideal $I_B \subset B$ for such B.

Definition 2.9. Let the notation be as before. Let $V \subset \Sigma$ be an affine neighborhood of $p \in \Sigma$ so that $f(V)$ is contained in an admissible neighborhood $(\mathcal{W}, \psi)$ of $f(p) \in W[n]$. We say f is predeformable near

p if for some neighborhood V_0 of $p \in V$ both $a_{1,n}$ and $b_{2,n}$ are units in $B = \Gamma(\mathcal{O}_{V_0})$ and further the ideal $I_B \subset B$ just constructed is the zero ideal in B. The family f is said to be predeformable along $\mathbf{D}_l$ if it is predeformable at every $p \in f^{-1}(\mathbf{D}_l)$. We say f is predeformable if it is predeformable along all divisors $\mathbf{D}_l$.

A remark about this definition is in order. First, in defining the notion of predeformable at p we need to choose a parameterization of the formal neighborhood of the nodal singularity of $\mathcal{X}$ along V. Because of the second part of Lemma 2.8, this definition does not depend on this choice. For the same reason, this definition is also independent of the choice of $(\mathcal{W}, \psi)$.

We still denote by f a family of nondegenerate morphisms to $W[n]$ over S. We will conclude this section by showing that there is a canonical closed subscheme structure on $S_{\mathrm{pd}} \subset S$ making it the closed subscheme parameterizing all predeformable morphisms over S.

Definition 2.10. Let f be a family of nondegenerate morphisms to $W[n]$ over S. We define a functor $\mathfrak{F}_{\mathrm{pd},l}[f, S]$ (resp. $\mathfrak{F}_{\mathrm{pd}}[f, S]$) that associates to any scheme T over C the set of morphisms $T \to S$ so that the pullback family $f_T : \mathcal{X}_T \to W[n]$ is predeformable along the divisor $\mathbf{D}_l$ (resp. all divisors $\mathbf{D}_l$).

When the choice of f is apparent from the context, we will omit the reference f and abbreviate $\mathfrak{F}_{\mathrm{pd},l}[f, S]$ to $\mathfrak{F}_{\mathrm{pd},l}[S]$.

Theorem 2.11. *Let $f : \mathcal{X} \to W[n]$ be a family of nondegenerate stable morphisms. Then the functor $\mathfrak{F}_{\mathrm{pd},l}[S]$ and $\mathfrak{F}_{\mathrm{pd}}[S]$ are represented by closed subschemes of S.*

Proof. Clearly, should $\mathfrak{F}_{\mathrm{pd},l}[S]$ be represented by a subscheme in S, then set-theoretically it must be $S_{\mathrm{pd},l}$. Hence it suffices to give $S_{\mathrm{pd},l}$ a closed scheme structure so that it represents the functor $\mathfrak{F}_{\mathrm{pd},l}[S]$. As before, we let S_l be $S \times_{\mathbf{A}^{n+1}} \mathbf{H}_l^{n+1}$ endowed with the reduced scheme structure and $\hat{S}_l$ be the formal completion of S along S_l. We let $\hat{V}$ be any connected component of $\hat{S}_l$ endowed with the open scheme structure and let V be $\hat{V}$ endowed with the reduced scheme structure. Our first objective is to show that $S_{\mathrm{pd},l} \cap \hat{V}$ admits a canonical scheme structure (as closed scheme of $\hat{V}$) so that it represents the functor $\mathfrak{F}_{\mathrm{pd},l}[\hat{V}]$. We continue to use the notation developed so far. For instance $(\mathcal{Y}_\pm, g_\pm)$ and $(\iota_\pm, \Sigma)$ are the decomposition data of the restriction of f to $V \subset S$ along the divisor $\mathbf{D}_l$. We let $\hat{\Sigma} \to \hat{V}$ be the étale morphism so that $\hat{\Sigma} \times_{\hat{V}} V \cong \Sigma$. We let $\pi : \Sigma \to V$ and $\hat{\pi} : \hat{\Sigma} \to \hat{V}$ be the projections.

Let $\xi \in S_{\mathrm{pd},l} \cap V$ be any closed point and let $p_1, \dots, p_k$ be the set of points in $\pi^{-1}(\xi)$. Note that $p_1, \dots, p_k$ are exactly the preimage set $f_\xi^{-1}(\mathbf{D}_l)$. Hence each p_α is associated with an integer n_α that is the contact order of f_ξ at p_α. For each α between 1 and k, we pick an affine open neighborhood $\hat{U}_\alpha$ of $p_\alpha \in \hat{\Sigma}$. We let $\mathcal{X}_\alpha = \mathcal{X} \times_S \hat{U}_\alpha$ and let $f_\alpha : \mathcal{X}_\alpha \to W[n]$ be the pullback of f to $\mathcal{X}_\alpha$. We let U_α be $\hat{U}_\alpha \cap \Sigma$. Then the multi-section $\iota : \Sigma \to \mathcal{X}$ defines a section $\iota_\alpha : U_\alpha \to \mathcal{X} \times_{\hat{U}_\alpha} U_\alpha$. Without loss of generality, we can assume $\mathcal{X}_\alpha$ admits a parameterization of the formal neighborhood of its nodes along $\iota_\alpha(U_\alpha)$. By shrinking U_α if necessary, we can assume that there is an admissible chart $(\mathcal{W}_\alpha, \psi_\alpha)$ of $W[n]$ near $f_\xi(p_\alpha)$ so that $f_\alpha(\iota_\alpha(U_\alpha))$ is entirely contained in $\mathcal{W}_\alpha$. Therefore, following the construction associated to Definition 2.9, by shrinking U_α if necessary we have an ideal sheaf $\mathcal{I}_\alpha \subset \mathcal{O}_{\hat{U}_\alpha}$ so that it defines the subscheme of $\hat{U}_\alpha$ that parameterizes all morphisms in the family $f_\alpha : \mathcal{X}_\alpha \to W[n]$ that are predeformable and have contact order n_α along the nodes $\iota_\alpha(U_\alpha) \subset \mathcal{X}_\alpha$.

To obtain the ideal sheaf of $\mathcal{O}_{\hat{V}}$ that defines the desired closed subscheme structure of $S_{\mathrm{pd},l} \cap \hat{V}$ we need to descend the "intersection" of $\mathcal{I}_\alpha$ since $S_{\mathrm{pd},l}$ is defined by the requirement that all nodes are predeformable. As before, we denote by k the number of elements in $\pi^{-1}(\xi)$ for $\xi \in V$. Since V is connected and $\Sigma \to V$ is étale and finite, k is independent of the choice of $\xi \in V$. We consider the fiber product, denoted $\hat{\Sigma}^{\times k}$, of k-copies of $\hat{\Sigma}$ over $\hat{V}$. We denote by $\Sigma^{\times k}$ accordingly the fiber product of k copies of Σ over V. Note that points in $\Sigma^{\times k}$ are $(a_1, \dots, a_k)$ with not necessarily distinct $a_1, \dots, a_k \in \pi^{-1}(\xi)$ for some $\xi \in V$. We let $\Gamma \subset \Sigma^{\times k}$ be the subset of those points $(a_1, \dots, a_k) \in \Sigma^{\times k}$ so that all $a_1, \dots, a_k$ are distinct. Since $\Sigma \to V$ is finite and étale, and that each fiber of $\pi : \Sigma \to V$ has cardinal k, Γ is an open subset of $\Sigma^{\times k}$. Then since $\Sigma^{\times k}$ is homeomorphic to $\hat{\Sigma}^{\times k}$, Γ is an open subset of $\hat{\Sigma}^{\times k}$. We let $\hat{\Gamma}$ be $\Gamma \subset \hat{\Sigma}^{\times k}$ endowed with the open scheme structure. Then the obvious projection $\hat{\Gamma} \to \hat{V}$ is étale and finite. Now let $\hat{\mathrm{pr}}_\alpha : \hat{\Gamma} \to \hat{\Sigma}$ be the morphism induced by the α-th projection of $\hat{\Sigma}^{\times k}$, which is étale.

Now consider any point $\eta = (p_1, \dots, p_k) \in \hat{\Gamma}$. To each α between 1 and k we let $\hat{U}_\alpha \subset \hat{\Sigma}$ be the neighborhood of $p_\alpha \in \hat{\Sigma}$ and $\mathcal{I}_\alpha \subset \mathcal{O}_{\hat{U}_\alpha}$ the ideal sheaf just constructed. We let $\hat{\mathcal{U}} \subset \hat{\Gamma}$ be an open neighborhood of η so that $\hat{\mathrm{pr}}_\alpha(\hat{\mathcal{U}}) \subset \hat{U}_\alpha$ for each α. We then let $\widetilde{\mathcal{I}}_\alpha$ be the ideal sheaf of $\mathcal{O}_{\hat{\mathcal{U}}}$ that is the pullback of $\mathcal{I}_\alpha \subset \mathcal{O}_{\hat{U}_\alpha}$ under $\hat{\mathrm{pr}}_\alpha|_{\hat{\mathcal{U}}} : \hat{\mathcal{U}} \to \hat{U}_\alpha$, and let $\mathcal{J} \subset \mathcal{O}_{\hat{\mathcal{U}}}$ be the intersection $\mathcal{J} = \widetilde{\mathcal{I}}_1 \cap \cdots \cap \widetilde{\mathcal{I}}_l$. This ideal sheaf defines a close subscheme of $\hat{\mathcal{U}}$, denoted by $Z(\mathcal{J})$. Now let $\Phi : \hat{\Gamma} \to \hat{S}$ be the

composite $\hat{\Gamma} \to \hat{V} \to \hat{S}$. It may happens that $Z(\mathcal{J})$ and $\Phi^{-1}(S_{\mathrm{pd},l})$ are different as subsets of $\hat{\Gamma}$. However, by the proof of the closedness of $S_{\mathrm{pd},l}$, the connected component of $Z(\mathcal{J})$ containing η coincides with the connected component of $\Phi^{-1}(S_{\mathrm{pd},l})$ containing η. Hence by shrinking $\hat{\mathcal{U}}$ if necessary, we can assume $Z(\mathcal{J})$ is connected. Now let $f_{\hat{\mathcal{U}}}$ be the family over $\hat{\mathcal{U}}$ that is the pullback of the family $f\colon \mathcal{X} \to W[n]$ via $\hat{\Gamma} \xrightarrow{\Phi} \hat{S} \to S$. Since for $\xi \in \hat{\mathcal{U}}$ the preimage $f_\xi^{-1}(\mathbf{D}_l)$ has exactly k points, it is direct to check that the subscheme $Z(\mathcal{J})$ represents the functor $\mathfrak{F}_{\mathrm{pd},l}[\hat{\mathcal{U}}]$.

Now by repeating this procedure, we can cover $\hat{\Gamma}$ by open subsets $\hat{\mathcal{U}}_\beta$ with ideal sheaves $\mathcal{J}_\beta \subset \mathcal{O}_{\hat{\mathcal{U}}_\beta}$, for $\beta = 1, \ldots, L$, such that the closed subscheme $Z(\mathcal{J}_\beta)$ defined by the ideal sheaf $\mathcal{J}_\beta$ represents the functor $\mathfrak{F}_{\mathrm{pd},l}[\hat{\mathcal{U}}_\beta]$. By the universal property of these schemes, whenever $\hat{\mathcal{U}}_\alpha \cap \hat{\mathcal{U}}_\beta \neq \emptyset$ the ideal sheaves $\mathcal{J}_\alpha$ and $\mathcal{J}_\beta$ coincide over $\hat{\mathcal{U}}_\alpha \cap \hat{\mathcal{U}}_\beta$. Thus they patch together to form an ideal sheaf $\mathcal{J} \subset \mathcal{O}_{\hat{\Gamma}}$. Again, by the universal property, this ideal sheaf forms a descent data for the faithfully flat étale morphism $\hat{\Gamma} \to \hat{V}$. Hence it descends to an ideal sheaf $\mathcal{I} \subset \mathcal{O}_{\hat{V}}$ which defines a closed subscheme $Z(\mathcal{I})$. It follows from the previous construction that $Z(\mathcal{I})$ represents the functor $\mathfrak{F}_{\mathrm{pd},l}[\hat{V}]$.

We now show that the functor $\mathfrak{F}_{\mathrm{pd},l}[S]$ is represented by a closed subscheme. Since each connected component $\hat{V}$ of $\hat{S}_l$ has a closed scheme structure that represents the functor $\hat{V}$, the functor $\mathfrak{F}_{\mathrm{pd},l}[\hat{S}_l]$ is represented by a closed subscheme of $\hat{S}_l$. We denote this subscheme by $\hat{S}_{\mathrm{pd},l}$. It remains to show that there is a closed scheme structure on $S_{\mathrm{pd},l}$ so that it represents the functor $\mathfrak{F}_{\mathrm{pd},l}[S]$. As before, we let t_l be the l-th standard parameter of $\mathbf{A}^n$. We let $\mathcal{K} \subset \mathcal{O}_S$ be the subsheaf of elements annihilated by some power of t_l. Since S is noetherian, there is an integer m_0 so that $t_l^{m_0}\mathcal{K} \subset \mathcal{O}_S$ is zero. Similarly, we let $\hat{\mathcal{K}}$ be the subsheaf of $\mathcal{O}_{\hat{S}}$ consisting of elements annihilated by some power of t_l. Then $\hat{\mathcal{K}}$ is canonically isomorphic to $\mathcal{K}$. As an ideal sheaf, $\hat{\mathcal{K}}$ defines a closed subscheme $\hat{S}_{fl} \subset \hat{S}_l$. Clearly, $\hat{S}_{fl}$ is flat over $\operatorname{Spec}\mathbf{k}[t_l]$ near 0. Hence by Proposition 2.2, the restriction of the family f to $\hat{S}_{fl}$ is predeformable along $\mathbf{D}_l$. Consequently, the ideal sheaf of $\hat{S}_{\mathrm{pd},l} \subset \hat{S}_l$ is contained in $\hat{\mathcal{K}}$. Using the natural isomorphism $\hat{\mathcal{K}} \cong \mathcal{K}$, this ideal sheaf defines an ideal sheaf J of $\mathcal{O}_S$. By our construction the support of the subscheme defined by J coincides with $S_{\mathrm{pd},l}$. We define the closed scheme structure of $S_{\mathrm{pd},l}$ to be the one defined by J. It is straightforward to check that with this closed scheme structure, $S_{\mathrm{pd},l}$ represents the functor $\mathfrak{F}_{\mathrm{pd},l}[S]$.

As to $\mathfrak{F}_{\mathrm{pd}}[S]$, it is clear that it is represented by the intersection of the

closed subschemes $\cap_{l=1}^{n} S_{\mathrm{pd},l}$. This completes the proof of Theorem 2.11. q.e.d.

3. Moduli stack of stable morphisms

In this section, we will introduce the notion of stable morphisms to the stack $\mathfrak{W}$. We will then show that the moduli of such stable morphisms form an algebraic stack (i.e., a Deligne-Mumford stack).

3.1 Stable morphisms

We first define the notion of stable morphisms in our setting. Let S be any C-scheme. A prestable morphism to $\mathfrak{W}(S)$ consists of a triple $(f, \mathcal{X}, \mathcal{W})$ where $\mathcal{X}$ is a family[7] of marked connected, prestable curves over S, $\mathcal{W}$ is a member in $\mathfrak{W}(S)$ and f is an S-morphism $f : \mathcal{X} \to \mathcal{W}$ that is stable as a family of morphisms to $\mathcal{W}$. When there is no confusion, we will abbreviate the triple $(f, \mathcal{X}, \mathcal{W})$ to f. Let $(f', \mathcal{X}', \mathcal{W}')$ be another prestable morphism over S. An isomorphism from f to f' consists of an S-isomorphism $r_1 : \mathcal{X} \to \mathcal{X}'$ (as pointed curves) and an arrow $r_2 : \mathcal{W} \to \mathcal{W}'$ so that $r_2 \circ f = f' \circ r_1$. When $f \equiv f'$, then such isomorphisms are called the automorphisms of f. To define the group scheme of the automorphisms of f, we follow the usual procedure. Let $\mathfrak{Aut}_{\mathfrak{W}}(f)$ be the functor that associates any S-scheme T the set of all automorphisms of $f \times_S T$. $\mathfrak{Aut}_{\mathfrak{W}}(f)$ is represented by a group scheme over S, denoted $\mathrm{Aut}_{\mathfrak{W}}(f)$ and called the automorphism group of f.

Let f be a map $f : X \to W[n]$ with X proper and reduced. We fix an ample line bundle H on W. Using $W[n] \to W$, H pulls back to a line bundle over $W[n]$, which we denote by H if there is no confusion. We define the degree of f to be the degree of f^*H over X. We fix a triple of integers $\Gamma = (b, g, k)$ where k will be the number of marked points of the domain curve, g will be the arithmetic genus of the domain curve and d will be the degree of the map. We say $f : X \to W[n]$ is a prestable map of topological type Γ if $f : X \to W[n]$ is an ordinary stable morphism whose domain is a connected, k-pointed arithmetic genus g nodal curve and f has degree b. We fix such an H and Γ once and for all in this section.

Definition 3.1. Let S be any C-scheme and $f : \mathcal{X} \to \mathcal{W}$ be a family of prestable morphisms to $\mathfrak{W}(S)$. Suppose $\mathcal{W}$ is an effective

[7] All families are flat families unless otherwise is mentioned.

degeneration, say given by a C-morphism $r : S \to C[n]$. Then we say f is a family of predeformable morphisms to $\mathfrak{W}(S)$ if the associated morphism $\widetilde{f} : \mathcal{X} \to W[n]$ is a family of predeformable morphisms; We say f is a family of stable morphisms to $\mathfrak{W}(S)$ if in addition to f being a family of predeformable morphisms we have that for every point $\xi \in S$ the automorphism group $\mathrm{Aut}_{\mathfrak{W}}(f_\xi)$ of $f_\xi : \mathcal{X}_\xi \to W[n]$ is finite. We say f is a family of topological type Γ if for every $\xi \in S$ the restriction of f to the fiber of $\mathcal{X}$ over ξ is of topological type Γ. In general, given a family of prestable morphisms f to $\mathfrak{W}(S)$, we say that f is a family of predeformable (resp. stable, resp. of topological type Γ) morphisms to $\mathfrak{W}(S)$ if there is an open covering S_α of S such that to each α the family $\mathcal{W} \times_S S_\alpha$ is an effective family over S_α and the restriction of f to $\mathcal{X} \times_S S_\alpha$ is a family of predeformable (resp. stable, resp. of topological type Γ) morphisms to $\mathfrak{W}(S_\alpha)$.

Clearly, the definition of stability does not depend on the choice of the local representatives of the family $\mathcal{W}$. The goal of this section is to construct the moduli stack of all marked stable morphisms to $\mathfrak{W}$ of given topological type Γ. In the sequel of this paper, we will use this moduli stack to study the degeneration of GW-invariants.

We begin with the definition of the groupoid of stable morphisms to $\mathfrak{W}$. We fix the relative ample line bundle H on W and the triple Γ. We define $\mathfrak{M}(\mathfrak{W}, \Gamma)$ to be the category whose objects are all families of stable morphisms to $\mathfrak{W}$ of topological type Γ. We define $\mathfrak{p} : \mathfrak{M}(\mathfrak{W}, \Gamma) \to (\mathrm{Sch}/C)$ to be the morphism that sends families over S to S. Let $\xi = \{f : \mathcal{X} \to \mathcal{W}\} \in \mathfrak{M}(\mathfrak{W}, \Gamma)(S)$, $\xi' = \{f' : \mathcal{X}' \to \mathcal{W}'\} \in \mathfrak{M}(\mathfrak{W}, \Gamma)(S')$ and let $h : S' \to S$ be an arrow (a morphism). An arrow $r : \xi' \to \xi$ with $\mathfrak{p}(r) = h$ consists of an arrow $\mathcal{W}' \to \mathcal{W}$ covering h (i.e., a $W_{S'}$-isomorphism $\mathcal{W} \times_{S'} S' \cong \mathcal{W}'$) and an S'-isomorphism $\mathcal{X} \times_S S' \cong \mathcal{X}'$ (as marked curves) so that

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{f'} & \mathcal{W}' \\ \downarrow\cong & & \downarrow\cong \\ \mathcal{X} \times_S S' & \xrightarrow{f \times_S S'} & \mathcal{W} \times_S S' \end{array}$$

is commutative. In such case, we call ξ' the pullback of ξ via $h : S' \to S$. Obviously, this makes $(\mathfrak{M}(\mathfrak{W}, \Gamma), \mathfrak{p})$ a groupoid.

Before proceeding, we need to study the structure of stable morphisms to $\mathfrak{W}$. Let $f : X \to W[n]$ be a predeformable morphism with $B \subset X$ the divisor of the marked points. We assume that f factor

through $W[n]_0 \subset W[n]$. By the decomposition Lemma, f splits X, according to the nodal divisors of $W[n]_0$, into components $X_1, \ldots, X_{n+2}$ (some of them may be empty), marked points $\iota_i^+ : \Sigma_i \to X_{i+1}$ and $\iota_i^- : \Sigma_i \to X_i$, where Σ_i is a finite set, so that $f(X_i) \subset \Delta_i \subset W[n]_0$ and that X is the result of successively gluing X_i and X_{i+1} along $\iota_i^+ : \Sigma_i \to X_{i+1}$ and $\iota_i^- : \Sigma_i \to X_i$. Namely,

$$X = X_1 \sqcup \cdots \sqcup X_{n+2} \quad \text{and} \quad f = f_1 \sqcup \cdots \sqcup f_{n+2}.$$

Now let A be any purely one-dimensional closed subset of X. We define the weight of f along A to be

$$\omega(f, A) = \deg(A, f^*H) + 2g(A) - 2 + \#(B \cap A) + \#\tau(A),$$

where $\tau(A)$ is the set of smooth points of A that are nodes of X. (We agree $\omega(f, A) = 0$ if $A = \emptyset$.) Note that if A_1 and A_2 are two such purely one-dimensional closed subsets with no common irreducible components, then

$$\omega(f, A_1 \cup A_2) = \omega(f, A_1) + \omega(f, A_2).$$

Lemma 3.2. *The predeformable morphism $f : X \to W[n]_0$ is a stable morphism in $\mathfrak{M}(\mathfrak{W}, \Gamma)(\mathbf{k})$ if and only if $\omega(f, X_i) > 0$ for all $i = 2, \ldots, n+1$.* [8]

Proof. Let i be any integer between 2 and $n+1$. We let

$$\Psi_i : G_m \times \Delta_i \longrightarrow \Delta_i$$

be the group action given in Corollary 1.3. For any closed purely one-dimensional subset A of X_i we define the relative automorphism group $\mathrm{Aut}_{\mathfrak{W}}(f, A)_{\mathrm{rel}}$ to be the set of pairs (a, b) where $a : A \to A$ is an automorphism leaving $B \cap A$ and $\tau(A)$ fixed and b is an element in G_m such that $\Psi_i(b, f_i(w)) = f_i(a(w))$ for all $w \in A$. Let $\sigma \in \mathrm{Aut}_{\mathfrak{W}}(f)$ be any automorphism. Then σ induces a permutation of Σ_i for each i. As a result, it defines a homomorphism

$$\mathrm{Aut}_{\mathfrak{W}}(f) \longrightarrow S_{\Sigma_1} \times \cdots \times S_{\Sigma_{n+1}},$$

where S_{Σ_i} is the permutation group of Σ_i. Clearly, its kernel is

$$\mathrm{Aut}_{\mathfrak{W}}(f, X_1)_{\mathrm{rel}} \times \cdots \times \mathrm{Aut}_{\mathfrak{W}}(f, X_{n+2})_{\mathrm{rel}}.$$

[8] For simplicity we agree $\mathfrak{M}(\mathfrak{W}, \Gamma)(\mathrm{Spec}\,\mathbf{k}) = \mathfrak{M}(\mathfrak{W}, \Gamma)(\mathbf{k})$.

Since f is an ordinary stable morphism, $\mathrm{Aut}_{\mathfrak{W}}(f, X_1)_{\mathrm{rel}}$ and $\mathrm{Aut}_{\mathfrak{W}}(f, X_{n+2})_{\mathrm{rel}}$ are finite. Hence $\mathrm{Aut}_{\mathfrak{W}}(f)$ is finite if and only if $\mathrm{Aut}_{\mathfrak{W}}(f, X_i)_{\mathrm{rel}}$ are finite for $i = 2, \ldots, n+1$.

Now assume A is a connected component of X_i so that $f(A)$ is not a single point set and that $\mathrm{Aut}_{\mathfrak{W}}(f, A)_{\mathrm{rel}}$ is infinite. Then the following must hold: $f(A)$ must be contained in a fiber of $\Delta_i \to D$; A must be a smooth rational curve; A contains no marked points and $\tau(A)$ consists of exactly two nodal points of X, one lies in $\iota_{i-1}^{+}(\Sigma_{i-1})$ and the other lies in $\iota_i^{-}(\Sigma_i)$. Further if we let $\mathbf{P}^1 \subset \Delta_i$ be the image of A then $f|_A : A \to \mathbf{P}^1$ is a branched covering ramified at $\tau(A)$. We call such connected components *trivial components*. Clearly, if A is not a trivial component, then $\mathrm{Aut}_{\mathfrak{W}}(f, A)_{\mathrm{rel}}$ is finite. On the other hand, if A is a trivial component then $\mathrm{Aut}_{\mathfrak{W}}(f, A)_{\mathrm{rel}} \to G_m$ is surjective. Hence $\mathrm{Aut}_{\mathfrak{W}}(f, X_i)_{\mathrm{rel}}$ is finite if and only if not all connected components of X_i are trivial.

To complete the proof of the Lemma, it suffices to show that $\omega(f, X_i) = 0$ if and only if all connected components of X_i are trivial. Let A be any connected component of X_i. In case $\deg(A, f^*H) > 0$, then since H is very ample we can assume without loss of generality that $\deg(A, f^*H) > 3$, and hence $\omega(f, A) > 0$. In case $\deg(A, f^*H) = 0$, then $f(A)$ must be contained in a fiber of $\Delta_i \to D$. Since $\deg(X, f^*H) > 0$ and X is connected, $f(A)$ can not be a single point. Hence $f(A)$ is a fiber of $\Delta_i \to D$. Because f is predeformable, A contains at least two nodal points of X, one lies in $f^{-1}(D_{i-1})$ and the other lies in $f^{-1}(D_i)$. Hence $\omega(f, A) \geq 0$. Moreover, the equality holds if and only if $g(A) = 0$, A contains no marked points and contains exactly two nodal points of X. Again since f is predeformable, it is easy to see that this is possible only if A is a trivial component. This shows that any connected component A of X_i has $\omega(f, A) \geq 0$ and the equality holds if and only if A is a trivial component.

Now we assume f is stable. Then to each i between 2 and $n+1$ at least one connected component of X_i is nontrivial. Then combined with the fact that the weight of any connected components of X_i are nonnegative, we have $\omega(f, X_i) > 0$. Conversely, if for some i between 2 and $n+1$ the weight $\omega(f, X_i) = 0$, then all connected components of X_i are trivial. Since to each component A of X_i the automorphism group $\mathrm{Aut}_{\mathfrak{W}}(f, A)_{\mathrm{rel}}$ surjects onto G_m, $\mathrm{Aut}_{\mathfrak{W}}(f, X_i)_{\mathrm{rel}}$ will surjects onto G_m as well. Thus $\mathrm{Aut}_{\mathfrak{W}}(f, X)_{\mathrm{rel}}$, and hence $\mathrm{Aut}_{\mathfrak{W}}(f)$ is infinite. This proves Lemma 3.2. q.e.d.

For convenience we define the norm of the triple $\Gamma = (b, g, k)$ to be $|\Gamma| = b + 2g - 2 + k$.

Lemma 3.3. *Let $f : X \to W[n]_0$ be a stable morphism in $\mathfrak{M}(\mathfrak{W}, \Gamma)(\mathbf{k})$ and let $X_1, \ldots, X_{n+2}$ be the splitting components of X. Then*

$$|\Gamma| = \omega(f, X_1) + \cdots + \omega(f, X_{n+2}).$$

We need a more general form of this Lemma. Let $S = \operatorname{Spec} R$ be a C-scheme where R is a discrete valuation domain. Let $f : \mathcal{X} \to \mathcal{W}$ be a family of stable morphisms in $\mathfrak{M}(\mathfrak{W}, \Gamma)(S)$ and $\iota : S \to C[n]$ be the morphism associated to $\mathcal{W}$. Let $\eta_0 \in S$ be the closed point and $\eta \in S$ be the generic point. Let $\Delta_1^0, \ldots, \Delta_{n_0+2}^0$ be the $(n_0 + 2)$-irreducible components of $W[n] \times_{C[n]} \iota(\eta_0)$ and $\Delta_1, \ldots, \Delta_{n_1+2}$ be the $(n_1 + 2)$-irreducible components of $W[n] \times_{C[n]} \iota(\eta)$. We say $\Delta_i^0 \preceq \Delta_j$ if Δ_i^0 is contained in the closure of Δ_j.

Lemma 3.4. *Let the notation be as before and let X_i^0 (resp. X_j) be the splitting components of $\mathcal{X} \times_X \eta_0$ (resp. $\mathcal{X} \times_S \eta$). Then for each $j \in [1, n_1 + 2]$,*

$$\omega(f_\eta, X_j) = \sum_{\Delta_i^0 \preceq \Delta_j} \omega(f_{\eta_0}, X_i^0).$$

Proof. The proofs of these two Lemmas are standard and will be omitted. q.e.d.

We next introduce the standard choice of ample line bundle on the domain and the target of a stable morphism in $\mathfrak{M}(\mathfrak{W}, \Gamma)$. We begin with line bundles on $W[n]$. As before, we fix the relative ample line bundle H on W/C and its pullback on $W[n]$, still denoted by H. For $l \in [n]$, the divisor $W[n] \times_{C[n]} C[n]_{[l^\vee]} \subset W[n]$ is the union of two smooth irreducible components, one *left* and one *right* according to our convention of ordering. We denote the *right* component by Ξ_l. (Note that $\Xi_l \times_{C[n]} \mathbf{0}$ contains the last irreducible component of $W[n]_0$.) Then associating to any sequence of integers $[a] = (a_1, \ldots, a_n)$ we define a line bundle

$$H_{[a]} = H(a_1\Xi_1 + \cdots + a_{n+1}\Xi_{n+1})$$

on $W[n]$. Since $W[n] \times_C (C - 0) \to W \times_C (C - 0)$ is $G[n]$-equivariant and since restricting to $W[n] \times_C (C - 0)$ the line bundle $H_{[a]}$ is the pullback of H from $W \times_C (C - 0)$, the trivial $G[n]$-linearization of H over W pullback to a unique $G[n]$-linearization of $H_{[a]}$ over $W[n] \times_C (C - 0)$.

Further, this linearization extends to a $G[n]$-linearization of $H_{[a]}$ over $W[n]$. We call this the canonical $G[n]$-linearization of $H_{[a]}$.

Let $\Delta \to D$ be the ruled variety that is one of the middle components of $W[n]_0$, where $n > 1$. We denote by H_Δ the pullback of the ample line bundle H via $\Delta \to D \to W$. Without loss of generality, we can assume H on W is sufficiently ample so that for any integer $0 \le a < b \le |\Gamma|$ the divisor $H_\Delta(aD_+ + bD_-)$ is ample on Δ, where $D_\pm$ are the two distinguished divisors of Δ. We now choose a standard choice of ample line bundle on the target of $f : X \to W[n]_0$. Let $X_1, \dots, X_{n+2}$ be the splitting components of X. We define

$$a_i(f) = \sum_{j=1}^{i} \omega(f, X_j),$$

and define the standard line bundle on $W[n]_0$ (associated to f) to be

$$H_{[f]} = H(a_1(f)\Xi_1 + \cdots + a_{n+1}(f)\Xi_{n+1}).$$

Because $\{a_i(f)\}$ is a strictly increasing sequence, $H_{[f]}$ is ample on $W[n]_0$.

Let S be any scheme and $f : \mathcal{X} \to \mathcal{W}$ be a family in $\mathfrak{M}(\mathfrak{W}, \Gamma)(S)$. We now extend this construction to the family $\mathcal{W}/S$. Let $\xi \in S$ be any closed point and let U be an open neighborhood of $\xi \in S$ so that $\mathcal{W} \times_S U$ is isomorphic to the pullback $\rho^* W[n]$ for a $\rho : U \to C[n]$. Without loss of generality, we can assume $\rho(\xi) = 0 \in C[n]$. Let $\Xi_1, \dots, \Xi_n$ be the divisors of $W[n]$ mentioned before. Let $f_\xi : \mathcal{X}_\xi \to W[n]_0$ be the restriction of f to the fiber over ξ. We then form the line bundle

$$H_{[f_\xi]} = H(a_1(f_\xi)\Xi_1 + \cdots + a_{n+1}(f_\xi)\Xi_{n+1})$$

on $W[n]$. Because of Lemma 3.4, after shrinking U if necessary, for any $\eta \in U$ the restriction of $H_{[f_\xi]}$ to $\mathcal{W}_\eta = W[n] \times_{C[n]} \rho(\eta)$ is the standard choice of ample line bundle associated to f_η. We define the standard choice of ample line bundle on $\mathcal{W} \times_S U$ to be $\rho^* H_{[f_\xi]}$ and denote it by $H_{[f,U]}$. Repeating this procedure, we can find an open covering U_α of S so that each $\mathcal{W} \times_S U_\alpha$ is isomorphic to $\rho_\alpha^* W[n_\alpha]$ for some $\rho_\alpha : U_\alpha \to C[n_\alpha]$ and over $\mathcal{W} \times_S U_\alpha$ we have the standard choice of ample line bundle $H_{[f,U_\alpha]}$. By Lemma 1.8 we can assume that whenever $U_{\alpha\beta} = U_\alpha \cap U_\beta \neq \emptyset$ then there is an arrow $\varphi_{\alpha\beta}$ making the diagram

$$\begin{array}{ccc} \rho_\alpha^* W[n_\alpha] \times_{U_\alpha} U_{\alpha\beta} & \xrightarrow{\varphi_{\beta\alpha}} & \rho_\beta^* W[n_\beta] \times_{U_\beta} U_{\alpha\beta} \\ \downarrow & & \downarrow \\ \mathcal{X} \times_S U_{\alpha\beta} & = & \mathcal{X} \times_S U_{\alpha\beta} \end{array}$$

commute. Using the $G[n]$-linearization, such an arrow defines a canonical isomorphism

$$\psi_{\beta\alpha} : \varphi_{\beta\alpha}^* H_{[f,U_\beta]} \cong H_{[f,U_\alpha]}.$$

Since $\psi_{\beta\alpha}$ is canonical, the collection $\{H_{[f,U_\alpha]}\}$ forms a line bundle on $\mathcal{W}$. In the following, we will call the collection $H_{[f,U_\alpha]}$ the standard choice of relative ample line bundle on $\mathcal{W}$ and denote it by $H_{[f,\mathcal{W}]}$.

The standard choice of the ample line bundle on the domain of f is

$$H_{[f,\mathcal{X}]} = f^* H_{[f,\mathcal{W}]}(5B) \otimes \omega_{\mathcal{X}/S}^{\otimes 5},$$

where $\omega_{\mathcal{X}/S}$ is the relative dualizing sheaf and $B \subset \mathcal{X}$ is the divisor of the marked points.

3.2 Basic Properties of $\mathfrak{M}(\mathfrak{W}, \Gamma)$

In this subsection, among other things, we will show that $\mathfrak{M}(\mathfrak{W}, \Gamma)$ is a separated and proper algebraic stack over C.

Proposition 3.5. *The groupoid $\mathfrak{M}(\mathfrak{W}, \Gamma)$ over C is a stack.*

Proof. Following the definition of stacks (cf. [40]), it suffices to show the following:

(i) For any scheme S over C and two stable morphisms ξ_1 and $\xi_2 \in \mathfrak{M}(\mathfrak{W}, \Gamma)(S)$, the functor

$$\mathfrak{Isom}_S(\xi_1, \xi_2) : (\mathrm{Sch}/S) \longrightarrow (\mathrm{Sets})$$

which associates to any morphism $\varphi : T \to S$ the set of isomorphisms in $\mathfrak{M}(\mathfrak{W}, \Gamma)(T)$ between $\varphi^*\xi_1$ and $\varphi^*\xi_2$ is a sheaf in the étale topology.

(ii) Let $\{S_i \to S\}$ be a covering of S (over C) in the étale topology. Let $\xi_i \in \mathfrak{M}(\mathfrak{W}, \Gamma)(S_i)$ and let $\varphi_{ij} : \xi_i|_{S_i \times_S S_j} \to \xi_j|_{S_i \times_S S_j}$ be isomorphisms in $\mathfrak{M}(\mathfrak{W}, \Gamma)(S_i \times_S S_j)$ satisfying the cocycle condition. Then there is a $\xi \in \mathfrak{M}(\mathfrak{W}, \Gamma)(S)$ with isomorphism $\psi_i : \xi|_{S_i} \to \xi_i$ so that

$$\varphi_{ij} = (\psi_i|_{S_i \times_S S_j}) \circ (\psi_j|_{S_i \times_S S_j})^{-1}.$$

We first prove (i). Let ξ_i, $i = 1$ and 2, be represented by $f_i : \mathcal{X}_i \to \mathcal{W}_i$. To each morphism $\rho : U \to S$ we define $\mathfrak{Isom}_S(\xi_1, \xi_2)(U)$ to be the set of all arrows from $\rho^*\xi_1$ to $\rho^*\xi_2$. This defines a functor

$$\mathfrak{Isom}_S(\xi_1, \xi_2) : (\text{étale opens in S}) \longrightarrow (\mathrm{Sets}).$$

Clearly it is a presheaf. To show that this is a sheaf it suffices to show that if $U = \cup_{j\in J} U_j$ is an étale open covering of U, then

$$\mathfrak{Isom}_S(\xi_1,\xi_2)(U) \xrightarrow{e} \prod_{j\in J} \mathfrak{Isom}_S(\xi_1,\xi_2)(U_j) \xrightarrow{p_1,p_2} \prod_{i,j\in J} \mathfrak{Isom}_S(\xi_1,\xi_2)(U_i \times_S U_j)$$

is an equalizer diagram. This means that if $\bar{r} = \{r_j | j \in J\}$ is an element in the middle term above, then

$$r_i|_{U_i\times_S U_j} \equiv r_j|_{U_i\times_S U_j} \tag{3.1}$$

for all $i, j \in J$ if and only if there is an $r \in \mathfrak{Isom}_S(\xi_1,\xi_2)(U)$ so that its restriction to U_i is r_i. By definition, each r_i consists of two isomorphisms $r_{i,1}$ and $r_{i,2}$ and a commutative diagram

$$\begin{array}{ccc} \mathcal{X}_1|_{U_i} & \xrightarrow{f_1|_{U_i}} & \mathcal{W}_1|_{U_i} \\ {\scriptstyle r_{i,1}}\downarrow & & \downarrow{\scriptstyle r_{i,2}} \\ \mathcal{X}_2|_{U_i} & \xrightarrow{f_2|_{U_i}} & \mathcal{W}_2|_{U_i}. \end{array}$$

The condition (3.1) makes $\{r_{i,1} | i \in J\}$ a descent data for an isomorphism between $\mathcal{X}_1$ and $\mathcal{X}_2$ and makes $\{r_{i,2} | i \in J\}$ a descent data for an isomorphism between $\mathcal{W}_1$ and $\mathcal{W}_2$. Since $\coprod U_i \to U$ is an étale covering, by Descent Lemma there is an isomorphism $r_1 : \mathcal{X}_1 \to \mathcal{X}_2$ and $r_2 : \mathcal{W}_1 \to \mathcal{W}_2$ restricting to $r_{i,1}$ and $r_{i,2}$ respectively. Hence $r = (r_1, r_2) \in \mathfrak{Isom}_S(\xi_1,\xi_2)(U)$ is the element so that $e(r) = \bar{r}$. This proves (i).

The proof of (ii) is similar. It amounts to construct a family $\mathcal{X}$ over S, a family $\mathcal{W}$ in $\mathfrak{W}(S)$ and an S-morphism $f : \mathcal{X} \to \mathcal{W}$, which form an element $\xi \in \mathfrak{M}(\mathfrak{W}, \Gamma)(S)$, so that $\xi_i = \xi|_{S_i}$ and the identity map is φ_{ij}. This is again a standard application of Descent Lemma and will be omitted. q.e.d.

Lemma 3.6. *Let S be any C-scheme and let $\xi_1, \xi_2 \in \mathfrak{M}(\mathfrak{W}, \Gamma)(S)$ be any two families over S. Then the functor $\mathfrak{Isom}_S(\xi_1,\xi_2)$ is represented by a scheme quasiprojective over S.*

Proof. We let ξ_i be represented by $f_i : \mathcal{X}_i \to \mathcal{W}_i$. Let $H_{[f_i,\mathcal{W}_i]}$ and $H_{[f_i,\mathcal{X}_i]}$ be the standard choices of relative ample line bundles on $\mathcal{W}_i$

and on $\mathcal{X}_i$. It follows from the uniqueness of these line bundles that whenever $T \to S$ is a morphism and (r_1, r_2) is a pair of isomorphisms shown below that makes the diagram

$$\begin{array}{ccc} \mathcal{X}_1 \times_S T & \xrightarrow{r_1} & \mathcal{X}_2 \times_S T \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow \\ \mathcal{W}_1 \times_S T & \xrightarrow{r_2} & \mathcal{W}_2 \times_S T. \end{array}$$

commutative then

$$r_1^* H_{[f_2, \mathcal{X}_2]} \cong H_{[f_1, \mathcal{X}_1]} \quad \text{and} \quad r_2^* H_{[f_2, \mathcal{W}_2]} \cong H_{[f_1, \mathcal{W}_1]}.$$

Hence $\mathfrak{Isom}_S(\xi_1, \xi_2)$ is isomorphic to the functor associating to $T \to S$ the set of pairs of isomorphisms of polarized projective schemes over T satisfying the obvious compatibility condition. It is a routine application of Grothendieck's results on the representability of the Hilbert scheme and related functor [33], see also [4, p. 84], that the functor $\mathfrak{Isom}_S(\xi_1, \xi_2)$ is represented by a scheme quasiprojective over S. This completes the proof of Lemma 3.6. q.e.d.

In the following, we will denote the scheme representing the functor $\mathfrak{Isom}_S(\xi_1, \xi_2)$ by $\mathrm{Isom}_S(\xi_1, \xi_2)$.

Lemma 3.7. *The scheme* $\mathrm{Isom}_S(\xi_1, \xi_2)$ *is quasifinite and unramified over* S.

Proof. To prove that $\mathrm{Isom}_S(\xi_1, \xi_2)$ is unramified over S, we only need to check the case where S is itself a closed point. Namely $S = \operatorname{Spec} \mathbf{k}$, in which case $\mathrm{Isom}_S(\xi_1, \xi_2)$ is either empty or is isomorphic to $\mathrm{Aut}_{\mathfrak{W}}(\xi_1)$. Since $\operatorname{char} \mathbf{k} = 0$, it is well-known that $\mathrm{Aut}_{\mathfrak{W}}(\xi_1)$ is unramified over $\mathbf{k}$ if there are no vector fields v_1 of $\mathcal{X}_1$ and vector fields v_2 of $\mathcal{W}_1$ (whose push-forward to W is trivial) so that they are compatible via $f_1 : \mathcal{X}_1 \to \mathcal{W}_1$. Such vector field does not exist because ξ_1 is stable. Hence $\mathrm{Isom}_S(\xi_1, \xi_2)$ is unramified over S. Further, since $\mathrm{Isom}_S(\xi_1, \xi_2)$ is of finite type over S, it is quasifinite over S. This proves the Lemma. q.e.d.

Lemma 3.8. *Let* $S = \operatorname{Spec} R$ *be a* C*-scheme where* R *is a discrete valuation domain with residue field* $\mathbf{k}$ *and quotient field* K. *We denote by* η_0 *and* η *the closed and the generic point of* S. *Let* $\xi_1, \xi_2 \in \mathfrak{M}(\mathfrak{W}, \Gamma)(S)$ *be two families over* S *so that there is an arrow* r_K *from* $\xi_1 \times_S \eta$ *to* $\xi_2 \times_S \eta$. *Then possibly after a base change* r_K *extends to an arrow* r_R *from* ξ_1 *to* ξ_2.

Proof. Without loss of generality we can assume that ξ_i is represented by $f_i : \mathcal{X}_i \to W[n_i]$ with $\iota_i : S \to C[n_i]$ the associated morphism so that $\iota_i(\eta_0) = \mathbf{0} \in C[n_i]$. We will prove the case where $S \to C$ does not factor through $0 \in C$. The other case is similar and will be left to readers.

We let η and η_0 be the generic and the closed point of S, as stated in the Lemma. Let $\mathcal{X}_K = \mathcal{X}_1 \times_S \eta$, which is isomorphic to $\mathcal{X}_2 \times_S \eta$ by assumption. Let $\bar{f}_K : \mathcal{X}_K \to W$ be the composite $\mathcal{X}_1 \times_S \eta \to W[n_1] \to W$. Since $S \to C$ does not factor through $0 \in C$, $\bar{f}_K$ is an ordinary stable morphism. By the property of stable morphisms, possibly after a base change $\widetilde{S} \to S$ the morphism $\bar{f}_K \times_S \widetilde{S}$ extends to a family of stable morphisms $\bar{f} : \mathcal{X} \to W$ over $\widetilde{S}$. For simplicity by replacing S with $\widetilde{S}$ we can assume $\widetilde{S} = S$. We consider the induced morphism

$$p_i \circ f_i : \mathcal{X}_i \longrightarrow W,$$

where $p_i : W[n_i] \to W$ is the canonical projection. By our assumption, the restriction of $p_i \circ f_i$ to $\mathcal{X}_i \times_S \eta$ is isomorphic to $\bar{f}_K$. Therefore by the property of stable morphisms $\bar{f} : \mathcal{X} \to W$ is the stabilization of $p_i \circ f_i : \mathcal{X}_i \to W$ for $i = 1, 2$. In particular, there is a unique contraction morphism $q_i : \mathcal{X}_i \to \mathcal{X}$ so that

$$p_i \circ f_i = \bar{f} \circ q_i : \mathcal{X}_i \longrightarrow W.$$

Now let $X_i = \mathcal{X}_i \times_S \eta_0$ and let $A \subset X_1$ be any irreducible component contracted under q_1. Following the same argument in the proof of Lemma 3.2, one can show that $\deg(f_1^* H, A) = 0$, $f_1(A)$ is contained in $\Delta_l \subset W[n_1]_0$ for some $2 \le l \le n_1$ and $f_1(A)$ is contained in a fiber of $\Delta_l \to D$. There are two cases we need to distinguish. One is when $f_1(A)$ is a point. Then that A is contracted by q_1 implies A contains at least two nodal points of X_1. On the other hand, when $f_1(A)$ is not a point, then $f_1(A)$ is a fiber of $\Delta \to D$ and hence because $f_1 \times_S \eta_0$ is stable and hence predeformable, the component A contains at least two nodal points of X_1 as well. Because this is true for every irreducible components in X_1 contracted under q_1, by the property of stable contraction, an irreducible component A' of X_1 is contracted under q_1 if and only if it is isomorphic to $\mathbf{P}^1$, it contains exactly two nodal points but no marked points and $\deg(f_1^* H, A') = 0$. Then since $f_1 \times_S \eta_0$ is stable, such A' must be a trivial component of X_1. Conversely, any trivial component of X_1 obviously will be contracted under q_1. This shows that the morphism $q_i : \mathcal{X}_i \to \mathcal{X}$ contracts exactly all trivial components of X_i.

Consequently, if we let $E_i \subset X_i$ be the union of all trivial components in X_i, then the contractions q_1 and q_2 define an isomorphism

$$\varphi : \mathcal{X}_1 - E_1 \xrightarrow{\cong} \mathcal{X}_2 - E_2 \tag{3.2}$$

extending the isomorphism $\mathcal{X}_1 \times_S \eta \cong \mathcal{X}_2 \times_S \eta$.

To proceed, we assume $n_1 \geq n_2$ without loss of generality. Note that $W[n_1]$ is a resolution of $W[n_2] \times_{\mathbf{A}^{n_2+1}} \mathbf{A}^{n_1+1}$, [9] which induces a morphism $q : W[n_1] \to W[n_2]$. We consider

$$q \circ f_1 : \mathcal{X}_1 \longrightarrow W[n_2].$$

Clearly no irreducible components of X_1 are mapped entirely to the nodal divisor of $W[n_2]_0$ unless $n_1 > n_2$ and in which case some nontrivial components of X_1 will be mapped to and only to the n_2-th nodal divisor of $W[n_2]_0$. Let $\widetilde{\iota}_1 : S \to C[n_2]$ be the morphism induced by $q \circ f_1$. Because $S \to C$ does not factor through $0 \in C$, there are isomorphisms $r_{1,\eta}$ and $r_{2,\eta}$ fitting into the commutative diagram

$$\begin{array}{ccc} \mathcal{X}_1 \times_S \eta & \xrightarrow{q \circ f_1} & W[n_2] \times_{C[n_2]} \widetilde{\iota}_1(\eta) \\ \downarrow{\scriptstyle r_{1,\eta}} & & \downarrow{\scriptstyle r_{2,\eta}} \\ \mathcal{X}_2 \times_S \eta & \xrightarrow{f_2} & W[n_2] \times_{C[n_2]} \iota_2(\eta) \end{array} \tag{3.3}$$

By Lemma 1.8 and Corollary 1.7, $r_{2,\eta}$ is induced by the $G[n_2]$-action on $W[n_2]$ via a morphism $\rho : \operatorname{Spec} K \to G[n_2]$. Let v be the uniformizing parameter of R and let ρ be defined by

$$\rho^*(\sigma_1) = c_1 v^{a_1}, \ldots, \rho^*(\sigma_{n_2}) = c_{n_2} v^{a_{n_2}},$$

where all c_i are units in R and a_i are integers. Here we follow the convention

$$G[n_2] = (G_m)^{\times n_2} \quad \text{and} \quad \Gamma(G[n_2]) = \mathbf{k}[\sigma_1, \sigma_1^{-1}, \ldots, \sigma_{n_2}, \sigma_{n_2}^{-1}].$$

Now let A be an irreducible component of X_1 whose image under f_1 is contained in the first irreducible component of $W[n_1]_0$. Then $p_1 \circ f_1(A)$ is not contained in $D \subset W$, A is a nontrivial component and $f_2 \circ \varphi(A)$ is also contained in the first component of $W[n_2]_0$. Now let A be a nontrivial irreducible component of X_1 whose image under

[9] $\mathbf{A}^{n_1+1} \to \mathbf{A}^{n_2+1}$ is defined by $(t_1, \ldots, t_{n_1+1}) \mapsto (t_1, \ldots, t_{n_2}, \Pi_{j=n_2+1}^{n_1+1} t_j)$.

f_1 is contained in the second irreducible component of $W[n_1]_0$. (Here we assume $n_1 \geq 1$ since otherwise there is nothing to prove.) We let $h : (\widetilde{\eta}_0, T) \to \mathcal{X}_1 - E_1$, where E_1 is defined in (3.2), be a curve so that $h(\widetilde{\eta}_0)$ is a closed point in A in general position and the composite $T \xrightarrow{h} \mathcal{X}_1 \to S$ is a branched covering ramified at $\widetilde{\eta}_0$. We pick an affine open $V \subset W$ so that the image of $h(\widetilde{\eta}_0)$ under $p_1 \circ f_1$ is contained in $V \cap D$. Let $\hat{T}$ be the formal completion of T along $\widetilde{\eta}_0$, let $\hat{V}$ be the formal completion of V along $D \cap V$ and let $\hat{h}{:}\hat{T} \to \mathcal{X}_1$ be the induced morphism. Then the morphism

$$q \circ f_1 \circ \hat{h} : \hat{T} \to W[n_2]$$

factor through $W[n_2] \times_W \hat{V} \to W[n_2]$, which we denote by

$$\hat{h}_1 : \hat{T} \longrightarrow W[n_2] \times_W \hat{V} \cong (V \cap D) \times \hat{\Gamma}[n_2].$$

Here $\hat{\Gamma}[n_2]$ is the scheme defined before (1.11). We let

$$\hat{h}_2 : \hat{T} \longrightarrow W[n_2] \times_W \hat{V} \cong (V \cap D) \times \hat{\Gamma}[n_2]$$

be the morphism induced by $f_2 \circ \varphi \circ h{:}T \to W[n_2]$. Because of (3.3), if we let pr_2 be the second projection of $(V \cap D) \times \hat{\Gamma}[n_2]$ and let

$$\Phi_\Gamma{:}G[n_2] \times \hat{\Gamma}[n_2] \to \hat{\Gamma}[n_2]$$

be the group action morphism, then

$$(\mathrm{pr}_2 \circ \hat{h}_1)^\rho = \mathrm{pr}_2 \circ \hat{h}_2 : \hat{T} \longrightarrow \hat{\Gamma}[n_2].$$

To utilize this identity, we use the open subset U_1 of $\Gamma[n_2]$ constructed in Lemma 1.2. Since $\mathrm{pr}_2 \circ \hat{h}_1(\widetilde{\eta}_0)$ is contained in the second irreducible component of $\hat{\Gamma}[n_2]$, if we let (u_i^1) be the coordinate chart of U_1 introduced in Lemma 1.2, then

$$(\mathrm{pr}_2 \circ \hat{h}_1)^*(u_j^1) = \alpha_j \in \mathcal{O}_{\hat{T}}, \quad j = 1, \ldots, n_2.$$

Because $\mathrm{pr}_2 \circ \hat{h}_1(\widetilde{\eta}_0)$ is in the smooth locus of $\Gamma[n_2]_0$, all α_j are units in $\mathcal{O}_{\hat{T}}$ except α_1, which is in the maximal ideal $\mathfrak{m}_{\hat{T}}$ of $\mathcal{O}_{\hat{T}}$. On the other hand, because $T \to S$ is flat, there is a $0 \neq c \in \mathfrak{m}_{\hat{T}}$ so that $R \to \mathcal{O}_{\hat{T}}$ is given by $v \mapsto c$. Therefore using the formula in Lemma 1.2,

$$\big((\mathrm{pr}_2 \circ \hat{h}_1)^\rho\big)^*(u_i^1) = (\alpha_1, c_1 c^{a_1} \alpha_2, w_3, \ldots, w_{n_2+2}) \tag{3.4}$$

for some $w_3, \dots, w_{n_2+2}$ in the quotient field of $\mathcal{O}_{\hat{T}}$. Hence if we let $\pi : \Gamma(n_2) \to \mathbf{A}^{n_2+2}$ be the projection, then $((\pi \circ (\mathrm{pr}_2 \circ \hat{h}_1)^{\rho})^*$ maps

$$t_1 \mapsto c_1 c^{a_1} \alpha_1 \alpha_2,\ t_2 \mapsto w_3, \dots, t_{n_2+1} \mapsto w_{n_2+2}.$$

Because this composition specializes to $0_{\mathbf{A}^{n_2+1}} \in \mathbf{A}^{n_2+1}$ by assumption, all w_i are regular and are in the maximal ideal $\mathfrak{m}_{\hat{T}}$. Furthermore, we know $\mathrm{pr}_2 \circ \hat{h}_2$ does not specialize to the first nodal point of $\Gamma[n_2]_0$, therefore $c_1 c^{a_1} \alpha_2$ is not in $\mathfrak{m}_{\hat{T}}$. Since α_2 is a unit in $\mathcal{O}_{\hat{T}}$, c_1 is a unit in R and c is in the maximal ideal $\mathfrak{m}_{\hat{T}} \subset \mathcal{O}_{\hat{T}}$, this implies $a_1 \leq 0$. Similarly, we can pick a nontrivial irreducible component A of X_2 so that its image under f_2 is contained in the second component of $W[n_2]_0$. Then because $f_1 \circ \varphi^{-1}(A)$ is not contained in the first nodal divisor of $W[n_2]_0$, a parallel argument shows that $a_1 \geq 0$. Therefore, $a_1 = 0$.

Repeating this argument, using the fact that no irreducible components of X_1 are mapped entirely to the first n_2 nodal divisors of $W[n_2]_0$ under $q \circ f_1$, we can show that all $a_1, \dots, a_{n_2}$ are zero. Therefore, $\rho : \mathrm{Spec}\, K \to G[n_2]$ extends to $\rho_R : S \to G[n_2]$.

Now we prove that the arrow r_K extends to an arrow r_R from ξ_1 to ξ_2. Let $\widetilde{f}_1 : \widetilde{\mathcal{X}}_1 \to W[n_2]$ be the stable contraction of $q \circ f_1 : \mathcal{X}_1 \to W[n_2]$ as a morphism to $W[n_2]$. Then $\widetilde{\mathcal{X}}_1 \times_S \eta$ is $\mathcal{X}_1 \times_S \eta$, which is isomorphic to $\mathcal{X}_2 \times_S \eta$ via the arrow r_K. We consider the morphism

$$(\widetilde{f}_1)^{\rho_R} : \widetilde{\mathcal{X}}_1 \longrightarrow W[n_2].$$

By assumption, its restriction to $\widetilde{\mathcal{X}}_1 \times_S \eta$ is identical to the restriction of f_2 to $\mathcal{X}_2 \times_S \eta$ via $\widetilde{\mathcal{X}}_1 \times_S \eta \cong \mathcal{X}_2 \times_S \eta$. On the other hand, both $(\widetilde{f}_1)^{\rho_R}$ and f_2 are stable extensions of a stable morphism from $\mathcal{X}_1 \times_S \eta \cong \mathcal{X}_2 \times_S \eta$ to $W[n_2]$. Hence by the uniqueness of the extensions of ordinary stable morphisms, the isomorphism $\mathcal{X}_1 \times_S \eta \cong \mathcal{X}_2 \times_S \eta$ extends to an isomorphism $\widetilde{\mathcal{X}}_1 \cong \mathcal{X}_2$ and the morphism $(\widetilde{f}_1)^{\rho_R}$ is identical to f_2. This shows that $(\widetilde{f}_1)^{\rho_R}$ maps no irreducible components of $\widetilde{\mathcal{X}}_1 \times_S \eta_0$ to the nodal divisors of $W[n_2]_0$. We claim that n_1 must be equal to n_2. Otherwise, all irreducible components of X_1 that are mapped to $\Delta_{n_2+1} \subset W[n_1]_0$ (under f_1) are trivial components, contradicting to Lemma 3.2. Once we have $n_1 = n_2$, then $\widetilde{f}_1 = f$ and hence $\widetilde{\mathcal{X}}_1 = \mathcal{X}_1$ and $f_2 = f_1^{\rho_R}$ follows immediately. This defines an arrow r_R that is an extension of r_K. This proves Lemma 3.8. q.e.d.

Lemma 3.9. *Let $S = \mathrm{Spec}\, R$ be a C-scheme where R is a discrete valuation domain as before and let η be the generic point of S. Let*

$\xi_K \in \mathfrak{M}(\mathfrak{W},\Gamma)(\eta)$ be a family of stable morphisms over $\operatorname{Spec} K$*. Then possibly after a base change $\widetilde{S} \to S$, $\xi_K \times_S \widetilde{S}$ extends to a family $\xi \in \mathfrak{M}(\mathfrak{W},\Gamma)(\widetilde{S})$.*

Proof. Let ξ_K be represented by $f_K : \mathcal{X}_K \to W[n]$. We will prove the case where $\operatorname{Spec} K \to C$ does not factor though $0 \in C$ and leave the other case to the readers.

Since $\operatorname{Spec} K \to C$ does not factor through $0 \in C$, we can assume without loss of generality that f_K is represented by $f_K : \mathcal{X}_K \to W$. Then f_K is an ordinary stable morphism over $\operatorname{Spec} K$. By the property of stable morphism, possibly after a base change $\widetilde{S} \to S$, the morphism $f_K \times_S \widetilde{S}$ extends to an $\widetilde{S}$-family of ordinary stable morphism $f_1 : \mathcal{X}_1 \to W$. Again by replacing S with $\widetilde{S}$ we can assume $S = \widetilde{S}$. Let η_0 and η be the closed and the generic points of S. In case $S \to C$ sends η_0 to $C - 0$, then f_1 is already a family of stable morphisms in $\mathfrak{M}(\mathfrak{W},\Gamma)(S)$. Now assume η_0 is mapped to $0 \in C$. Let $\iota : S \to C[n]$ be any C-morphism. Then since $W[n] \times_{C[n]} \eta$ is canonically isomorphic to $W \times_C \eta$, $f_K : \mathcal{X}_K \to W$ induces a morphism $\widetilde{f}_K : \mathcal{X}_K \to W[n]$. We say f_K admits a partial extension to $W[n]$ if after a base change $\widetilde{S} \to S$ the morphism $\widetilde{f}_K \times_S \widetilde{S}$ extends to a family of quasistable morphisms over $\widetilde{S}$

$$f_n : \mathcal{X}_n \longrightarrow W[n] \tag{3.5}$$

so that the associated $\iota : \widetilde{S} \to C[n]$ maps the closed point of $\widetilde{S}$ to $\mathbf{0}$. Here we say f_n is quasistable if $\operatorname{Aut}_{\mathfrak{W}}(f_{n,\eta_0})$ is finite, where the automorphism group is the set of pairs (a, b) such that a is an automorphism of the domain of f_{n,η_0} and $b \in G[n]$ such that $f_n \circ a = (f_{n,\eta_0})^b$. (As before, we let f_{n,η_0} be the restriction of f_n to the fiber $\mathcal{X}_{n,\eta_0} = \mathcal{X}_n \times_S \eta_0$.) It is clear that the extension f_1 is a quasistable extension to $W = W[1]$. On the other hand, by the proof of Lemma 3.2 any quasistable extension f_n satisfies $n \le |\delta|$.

Now we let n be the largest possible integer so that there is a quasistable extension f_n. We show that f_n defines an extension of f_K. Let f_n be a quasistable extension as in (3.5) via $\iota : S \to C[n]$. If

$$f_{n,\eta_0} : \mathcal{X}_{n,\eta_0} \longrightarrow W[n]_0$$

is nondegenerate, then Proposition 2.2 implies that f_n is predeformable. Hence $f_n \in \mathfrak{M}(\mathfrak{W},\Gamma)(S)$ as desired. Now assume f_{n,η_0} is degenerate. Let A be an irreducible component of $\mathcal{X}_{n,\eta_0}$ so that $f_{n,\eta_0}(A)$ is contained, say, in the l-th nodal divisor of $W[n]_0$. We distinguish the case where

$l \leq n$ from the case where $l = n+1$. We first consider the case $l \leq n$. We let v be a uniformizing parameter of R and let $\rho : \operatorname{Spec} K \to G_m$ be defined by $\rho^*(\sigma) = v^\alpha$ for an integer α. Let $\lambda_l : G_m \cong G[n]_l \to G[n]$ be the l-th one-parameter subgroup of $G[n]$. We consider

$$(f_{n,\eta})^{\lambda_l \circ \rho} : \mathcal{X}_{n,\eta} \longrightarrow W[n]. \tag{3.6}$$

As in the proof of Lemma 3.8, we can pick a curve $h : (\widetilde{\eta}_0, T) \to \mathcal{X}_n$ covering $\underline{h} : (\widetilde{\eta}_0, T) \to (0, C)$ so that $\underline{h}$ is flat, $h(\widetilde{\eta}_0) \in A$ and is in general position of A. Further we can find an integer α so that

$$(f_n \circ h)^{\lambda_l \circ \rho \circ \underline{h}} : T \longrightarrow W[n]$$

specializes to a point in the $(l+1)$-th irreducible component of $W[n]_0$ and is away from the nodal divisors of $W[n]_0$. A simple analysis using the covering U_l of $\Gamma[n]$ shows that if we let

$$\widetilde{f}_n : \widetilde{\mathcal{X}}_n \to W[n]$$

be the extension of (3.6) to a family of stable morphisms, then $\widetilde{f}_n$ is still quasistable. Further, some irreducible component of $\widetilde{\mathcal{X}}_{n,\eta_0}$ is mapped entirely to the $(l+1)$-th nodal divisor of $W[n]_0$. This shows that we can assume without loss of generality that f_n maps an irreducible component A of $\mathcal{X}_{n,\eta_0}$ to $D_{n+1} \subset W[n]_0$.

We now show that there is a quasistable extension of f_K to $W[n+1]$. Let f_n and $A \subset \mathcal{X}_{n,\eta_0}$ be as before so that $f_n(A) \subset D_n$. We let $C[n] \subset C[n+1]$ and $W[n] \to W[n+1]$ be the standard embedding associated to $[n] \subset [n+1]$. Namely, they are induced by the embedding $\mathbf{A}^{n+1} \to \mathbf{A}^{n+2}$ that keep the last coordinate 1. Let $\widetilde{f}_n : \mathcal{X}_n \to W[n+1]$ be the composite of f_n with $W[n] \to W[n+1]$. Then the same technique as before shows that we can find an integer α so that if we let $\rho : \operatorname{Spec} K \to G_m$ be defined by $\rho^*(\sigma) = v^\alpha$, then the extension (possibly after a base change $\widetilde{S} \to S$) of

$$(\widetilde{f}_n)^{\lambda_n \circ \rho} : \mathcal{X}_{n,\eta} \longrightarrow W[n+1]$$

to a family of ordinary stable morphisms

$$f_{n+1} : \mathcal{X}_{n+1} \longrightarrow W[n+1]$$

is quasistable. This contradicts to the maximal assumption of n. Hence f_n must already be a family of nondegenerate morphism. Since $\widetilde{S} \to C$ is flat, it must be predeformable and thus in $\mathfrak{M}(\mathfrak{W}, \Gamma)(\widetilde{S})$, extending $\xi_K \times_S \widetilde{S} \in \mathfrak{M}(\mathfrak{W}, \Gamma)(\eta)$. This proves Lemma 3.9. q.e.d.

Theorem 3.10. *The moduli stack $\mathfrak{M}(\mathfrak{W},\Gamma)$ of stable morphisms to $\mathfrak{W}$ of topological type Γ is separated and proper over C. Furthermore, it is a Deligne-Mumford stack.*

Proof. The fact that the moduli stack $\mathfrak{M}(\mathfrak{W},\Gamma)$ is separate over C follows from Lemma 3.8 and that it is proper over C follows from Lemma 3.9. It remains to show that it is algebraic. Namely, $\mathfrak{M}(\mathfrak{W},\Gamma)$ admits an étale cover by a scheme of finite type. We now show that it admits an étale covering by a quasiprojective scheme.

Let $\mathfrak{M}(W[n],\Gamma)$ be the moduli stack of stable morphisms from pointed curves to $W[n]$ of type Γ, and let $\mathfrak{M}(W[n],\Gamma)^{st}$ be the substack of $\mathfrak{M}(W[n],\Gamma)$ consisting of all predeformable morphisms that are stable in the sense of Definition 3.1. By Theorem 2.11, $\mathfrak{M}(W[n],\Gamma)^{st}$ is a locally closed substack of $\mathfrak{M}(W[n],\Gamma)$. Since members of $\mathfrak{M}(W[n],\Gamma)^{st}$ are stable morphisms to $\mathfrak{W}$, there is a natural morphism $\mathfrak{M}(W[n],\Gamma)^{st}\to\mathfrak{M}(\mathfrak{W},\Gamma)$. Because of Lemma 3.3, there is an integer $\overline{n}$ so that $\mathfrak{M}(W[\overline{n}],\Gamma)^{st}\longrightarrow\mathfrak{M}(\mathfrak{W},\Gamma)$ is surjective. Further, since $W[\overline{n}]$ is a $G[\overline{n}]$-variety, $\mathfrak{M}(W[\overline{n}],\Gamma)$ admits a natural $G[\overline{n}]$-action. By the definition of stability (cf. Definition 3.1), $\mathfrak{M}(W[\overline{n}],\Gamma)^{st}$ is $G[\overline{n}]$-invariant and the $G[\overline{n}]$ action on $\mathfrak{M}(W[\overline{n}],\Gamma)^{st}$ has only finite stabilizer. Hence $\mathfrak{M}(W[\overline{n}],\Gamma)^{st}/G[\overline{n}]$ is a DM-stack. Finally, it is clear that the induced morphism

$$\mathfrak{M}(W[\overline{n}],\Gamma)^{st}/G[\overline{n}]\longrightarrow\mathfrak{M}(\mathfrak{W},\Gamma)$$

is finite and étale. This proves that $\mathfrak{M}(\mathfrak{W},\Gamma)$ is an algebraic (i.e., Deligne-Mumford) stack. q.e.d.

4. Stacks of relative stable morphisms

Let $D\subset Z$ be a smooth connected divisor (called the distinguished divisor) in a smooth projective variety. In this section, we will construct the moduli stack of relative stable morphisms with prescribed contact with D. We will prove some basic property of this moduli stack and describe its relation with the stack of stable morphisms constructed in the previous section. This construction can easily be generalized to the case where D is smooth but not necessarily connected.

4.1 Standard models

We first construct the standard models $(Z[n],D[n])$ which are the building blocks of the stack of expanded relative pairs of (Z,D). For $n=0$,

we let $(Z[0], D[0]) = (Z, D)$. For $n = 1$, $Z[1]$ is the blowing-up of $Z \times \mathbf{A}^1$ along $D \times \{0\} \subset Z \times \mathbf{A}^1$. Its distinguished divisor $D[1]$ is the proper transform of $D \times \mathbf{A}^1 \subset Z \times \mathbf{A}^1$ in $Z[1]$. We now let $W \to \mathbf{A}^1$ be $Z[1] \to \mathbf{A}^1$. We shall view the ruled variety $\Delta = \mathbf{P}(\mathbf{1}_D \oplus N_{D/Z})$ and Z in $W_0 (= W \times_{\mathbf{A}^1} 0)$ as its top (left) and its bottom (right) components. With this understanding, we can construct a sequence of varieties $W[n]$ over $\mathbf{A}^{n+1}$ for all $n \geq 1$ as in Section 1 with the projection $\pi : W[n] \to Z \times \mathbf{A}^{n+1}$. We define $Z[n] = W[n-1]$ and define its distinguished divisor $D[n]$ to be the proper transform of $D \times \mathbf{A}^n \subset Z \times \mathbf{A}^n$ in $Z[n]$. We call $(Z[n], D[n])$ with the tautological projection $Z[n] \to Z \times \mathbf{A}^n$ an expanded relative pair of (Z, D). Note that under this projection $D[n]$ is isomorphic to $D \times \mathbf{A}^n \subset Z \times \mathbf{A}^n$. We denote the projection $Z[n] \to \mathbf{A}^n$ by π.

Clearly, the fiber of $Z[n]$ over $0 \in \mathbf{A}^n$, denoted $Z[n]_0$, is reduced with normal crossing singularity. It consists of a chain of smooth varieties, one Z and n ruled varieties Δ. We give these n components an ordering so that the first component of $Z[n]_0$ contains the distinguished divisor $D[n]_0$ and the remainder components are ordered according to their intersection chain structure. Namely, the k-th component will intersect the $(k+1)$-th component. We call this the standard ordering of $Z[n]_0$. We continue to denote by $G[n]$ the product of n copies of G_m. Since the G_m-action on $Z \times \mathbf{A}^1$ defined via $(z, t)^\sigma = (z, \sigma^{-1} t)$ leaves $D \times \{0\} \subset Z \times \mathbf{A}^1$ fixed, it lifts to an action on $Z[1] \to \mathbf{A}^1$. In general, we let $G[n]$ acts on $\mathbf{A}^n$ via

$$(t_1, \dots, t_n)^\sigma = (\bar{\sigma}_2 t_1, \bar{\sigma}_3 t_2, \dots, \bar{\sigma}_{n+1} t_n). \tag{4.1}$$

(Recall $\bar{\sigma}_i = \sigma_i / \sigma_{i-1}$, see (1.3).) Note that if we embed $\mathbf{A}^n$ in $\mathbf{A}^{n+1}$ in the standard way as the $n+1$-th coordinate hyperplane $\mathbf{H}^{n+1}_{n+1}$ and let $G[n]$ acts on $\mathbf{A}^{n+1}$ as defined in Section 1, then the embedding $\mathbf{A}^n \subset \mathbf{A}^{n+1}$ is $G[n]$-equivariant. It is direct to check that there is a unique $G[n]$-action on $Z[n]$ that makes $Z[n] \to Z \times \mathbf{A}^{n-1}$ $G[n]$-equivariant, where the $G[n]$ action on $Z \times \mathbf{A}^{n-1}$ is $(z, t)^\sigma = (z, t^\sigma)$ defined in (4.1).

For later application, we need to consider $Z[n]$ with the reversed ordering of its components. In the following for any n we define $\mathbf{1}^o_{\mathbf{A}^n} : \mathbf{A}^n \to \mathbf{A}^n$ be the morphism $(t_1, \dots, t_n) \mapsto (t_n, \dots, t_1)$. We let $Z[n]^o$ be $Z[n]$ with the tautological projection $\pi^o \equiv \mathbf{1}^o_{\mathbf{A}^n} \circ \pi : Z[n] \to \mathbf{A}^n$. At the same time we reverse the order of the components of $Z[n]_0$ so that the first irreducible component of $Z[n]^o_0$ is Z while its last component contains the distinguished divisor. Note that the restriction of $Z[n]^o_0$ to

the l-th coordinate line is still a smoothing of the l-th nodal divisor of $Z[n]_0^o$. Accordingly, we let the $G[n]$-action on $\mathbf{A}^n$ be

$$(t_1, \dots, t_n)^{\sigma^o} = (\overline{\sigma}_1 t_1, \dots, \overline{\sigma}_n t_n). \tag{4.2}$$

We next describe the connection between the standard model constructed in Section 1 with the relative standard models so constructed. Let $W \to C$ be the pair in Section 1 with $D_1 \subset Y_1$ and $D_2 \subset Y_2$ the two irreducible components of W_0 with their distinguished divisors. Using the pair (Y_i, D_i) we can construct the relative pair $(Y_i[n], D_i[n])$ with the associated projection $Y_i[n] \to Y_i \times \mathbf{A}^n$. Now let l be any integer in $[n]$. We consider the pairs

$$D_1[l] \times \mathbf{A}^{n-l} \subset Y_1[l]^o \times \mathbf{A}^{n-l} \quad \text{and} \quad \mathbf{A}^l \times D_2[n-l] \subset \mathbf{A}^l \times Y_2[n-l].$$

We have the canonical isomorphisms

$$D_1[l] \times \mathbf{A}^{n-l} \xrightarrow{\cong} (D_1 \times \mathbf{A}^l) \times \mathbf{A}^{n-l} \xrightarrow{\cong} D_1 \times \mathbf{A}^l \times \mathbf{A}^{n-l}$$

and canonical isomorphisms

$$\mathbf{A}^l \times D_2[n-l] \xrightarrow{\cong} \mathbf{A}^l \times (D_2 \times \mathbf{A}^{n-l}) \xrightarrow{\cong} D_2 \times \mathbf{A}^l \times \mathbf{A}^{n-l},$$

where the second arrow is induced by exchanging the factor $\mathbf{A}^{l-1}$ and D_2. Combined with the canonical isomorphism $D_1 \cong D_2$, we have

$$D_1[l] \times \mathbf{A}^{n-l} \cong \mathbf{A}^l \times D_2[n-l]. \tag{4.3}$$

Hence we can glue $Y_1[l]^o \times \mathbf{A}^{n-l}$ with $\mathbf{A}^l \times Y_2[n-l]$ along

$$D_1[l] \times \mathbf{A}^{n-l} \subset Y_1[l]^o \times \mathbf{A}^{n-l} \quad \text{and} \quad \mathbf{A}^l \times D_2[n-l] \subset \mathbf{A}^l \times Y_2[n-l]$$

according to the isomorphism (4.3) to obtain a new scheme, denoted by

$$Y_1[l]^o \times \mathbf{A}^{n-l} \sqcup \mathbf{A}^l \times Y_2[n-l]. \tag{4.4}$$

The group $G[n]$ acts on (4.4) as follows: We let $\mathbf{A}^l \times \mathbf{A}^{n-l} \to \mathbf{A}^{n+1}$ be the embedding as coordinate plane defined by $(r, s) \mapsto (r, 0, s)$. Then the $G[n]$-action on $\mathbf{A}^{n+1}$ (see (1.2)) lifts to an action on $\mathbf{A}^l \times \mathbf{A}^{n-l}$. Namely for $\sigma = (\sigma_1, \dots, \sigma_{n-1}) \in G[n]$,

$$\begin{aligned} (r_1, \dots, r_l)^\sigma &= (\overline{\sigma}_1 r_1, \dots, \overline{\sigma}_l r_l) \quad \text{and} \\ (s_1, \dots, s_{n-l})^\sigma &= (\overline{\sigma}_{l+2} s_1, \dots, \overline{\sigma}_{n+1} s_{n-l}). \end{aligned}$$

Let $\rho_-: G[n] \to G[l]$ and $\rho_+: G[n] \to G[n-l]$ be the homomorphism

$$\rho_-(\sigma) = (\sigma_1, \dots, \sigma_l) \quad \text{and} \quad \rho_+(\sigma) = (\sigma_{l+1}, \dots, \sigma_n).$$

Thus $r^\sigma = r^{\rho_-(\sigma)^o}$ and $s^\sigma = s^{\rho_+(\sigma)}$, where r^σ and s^σ are the actions defined above and $r^{\rho_-(\sigma)^o}$ and $s^{\rho_+(\sigma)}$ are the actions defined in (4.1) and (4.2). We define the $G[n]$-action on $Y_1[l]^o \times \mathbf{A}^{n-l}$ and on $\mathbf{A}^l \times Y_2[n-l]$ be

$$(x,s)^\sigma = (x^{\rho_-(\sigma)^o}, s^{\rho_+(\sigma)}) \quad \text{and} \quad (r,y)^\sigma = (r^{\rho_-(\sigma)^o}, y^{\rho_+(\sigma)}).$$

It is direct to check that the isomorphism (4.3) is $G[n]$-equivariant, and hence (4.4) is a $G[n]$-scheme.

Proposition 4.1. *Let $\pi: W[n] \to \mathbf{A}^{n+1}$ be the associated morphism and $\pi_l: W[n] \to \mathbf{A}^1$ be the composite of π with the l-th projection of $\mathbf{A}^{n+1}$. Then we have a canonical isomorphism*

$$Y_1[l]^o \times \mathbf{A}^{n-l} \sqcup \mathbf{A}^l \times Y_2[n-l] \cong W[n] \times_{\pi_l} 0_{\mathbf{A}^1} \subset W[n].^{10}$$

Further the above inclusion and isomorphism are $G[n]$-equivariant.

Proof. The proof is straightforward and will be omitted. q.e.d.

Now let $(\mathcal{Y}_i, \mathcal{D}_i)$, $i = 1$ and 2, be effective relative pairs over S associated to $\tau_i: S \to \mathbf{A}^{l_i}$ defined by

$$(\mathcal{Y}_i, \mathcal{D}_i) = (Y_i[l_i], D_i[l_i]) \times_{\mathbf{A}^{l_i}} S.$$

As before, we define $\tau_1^o = \mathbf{1}^o_{\mathbf{A}^{l_1}} \circ \tau_1 : S \to \mathbf{A}^{l_1}$ and let $\mathcal{Y}_1^o$ be $Y_1[l_1]^o \times_{\tau_1^o} S$. Obviously $\mathcal{Y}_1^o = \mathcal{Y}_1$ with only the ordering of the components of its fibers reversed. Since $\mathcal{D}_1$ (resp $\mathcal{D}_2$) is canonically isomorphic to $D_1 \times S$ (resp. $D_2 \times S$), we can construct a new scheme over S by gluing $\mathcal{Y}_1^o$ and $\mathcal{Y}_2$ along $\mathcal{D}_1$ and $\mathcal{D}_2$ using $\mathcal{D}_1 \cong \mathcal{D}_2$. Since both $\mathcal{D}_1$ and $\mathcal{D}_2$ are smooth over S, the resulting scheme $\mathcal{Y}_1^o \sqcup \mathcal{Y}_2$ is a flat family of schemes with normal crossing singularity relative to S. Let $\mathbf{A}^n \cong \mathbf{A}^{l_1} \times \mathbf{A}^1 \times \mathbf{A}^{l_2}$, $n = l_1 + l_2 + 1$, be the canonical isomorphism keeping the order of each copies of $\mathbf{A}^1$ in $\mathbf{A}^{n+1}$. Then we have an embedding $\mathbf{A}^{l_1} \times \mathbf{A}^{l_2} \to \mathbf{A}^{n+1}$ sending $(r,s) \in \mathbf{A}^{l_1} \times \mathbf{A}^{l_2}$ to $(r,0,s) \in \mathbf{A}^{n+1}$. Let $\tau: S \to \mathbf{A}^{n+1}$ be the composite

$$\tau : S \xrightarrow{(\tau_1^o, \tau_2)} \mathbf{A}^{l_1} \times \mathbf{A}^{l_2} \longrightarrow \mathbf{A}^{n+1}. \tag{4.5}$$

[10] By $W[n] \times_{\pi_l} 0_{\mathbf{A}^1}$ we mean $W[n] \times_{\mathbf{A}^1} 0_{\mathbf{A}^1}$ with $W[n] \to \mathbf{A}^1$ given by π_l.

Corollary 4.2. *Let the notation be as above and let $\tau : S \to \mathbf{A}^n$ be defined in* (4.5)*. Then the associated family $\mathcal{W} = \tau^* W[n]$ is canonically isomorphic to $\mathcal{Y}_1^o \sqcup \mathcal{Y}_2$.*

Proof. Clearly, the composite of $\tau : S \to \mathbf{A}^{n+1}$ with $\mathrm{pr}_{l_1} : \mathbf{A}^{n+1} \to \mathbf{A}^1$ is trivial. Hence by Proposition 4.1

$$\begin{aligned} W[n] \times_{\mathbf{A}^{n+1}} S &\cong \big(Y_1[l_1]^o \times \mathbf{A}^{l_2} \sqcup \mathbf{A}^{l_1} \times Y_2[l_2]\big) \times_{\mathbf{A}^{l_1} \times \mathbf{A}^{l_2}} S \\ &\cong Y_1[l_1]^o \times_{\mathbf{A}^{l_1}} S \underset{D_1[l_1] \times_{\mathbf{A}^{l_1}} S = D_2[l_2] \times_{\mathbf{A}^{l_2}} S}{\sqcup} Y_2[l_2] \times_{\mathbf{A}^{l_2}} S, \end{aligned}$$

which is isomorphic to $\mathcal{Y}_1^o \sqcup \mathcal{Y}_2$. This proves the Corollary. q.e.d.

We now define the stack of expanded relative pairs of (Z, D). Let S be any scheme. An *effective* family of expanded *relative pair* of (Z, D) (in short relative pair) is the associated family $(\mathcal{Z}, \mathcal{D})$ of a morphism $\tau : S \to \mathbf{A}^n$ for some n defined by

$$\mathcal{Z} = \tau^* Z[n] = Z[n] \times_{\mathbf{A}^n} S \quad \text{and} \quad \mathcal{D} = \tau^* D[n] = D[n] \times_{\mathbf{A}^n} S.$$

We call $\mathcal{D}$ the distinguished divisor of $\mathcal{Z}$. Now let

$$\underline{\Phi} : G[n] \times \mathbf{A}^n \longrightarrow \mathbf{A}^n \quad \text{and} \quad \Phi : G[n] \times Z[n] \longrightarrow Z[n]$$

be the $G[n]$-action on $\mathbf{A}^n$ and on $Z[n]$ defined earlier in this section. Then for any $\tau : S \to \mathbf{A}^n$ and $\rho : S \to G[n]$, the group action $\underline{\Phi}$ defines a new morphism $\tau^\rho : S \to \mathbf{A}^n$ and thus a new family of relative pair $(\tau^{\rho *} Z[n], \tau^{\rho *} D[n])$. Using the group action of $G[n]$ on $Z[n]$, there is a canonical isomorphism between the pair $(\tau^* Z[n], \tau^* D[n])$ and $(\tau^{\rho *} Z[n], \tau^{\rho *} D[n])$. As in the case for $W[n]$, given two effective relative pairs $\xi_1 = (\mathcal{Z}_1, \mathcal{D}_1)$ and $\xi_2 = (\mathcal{Z}_2, \mathcal{D}_2)$ over S associated to morphisms $\tau_1 : S \to \mathbf{A}^{n_1}$ and $\tau_2 : S \to \mathbf{A}^{n_2}$ respectively, an effective arrow from ξ_1 to ξ_2 consists of a standard embedding $\iota : \mathbf{A}^{n_1} \to \mathbf{A}^{n_2}$ (associated to an increasing map $[n_1] \to [n_2]$, see discussion at the beginning of Subsection 1.1) and a morphism $\rho : S \to G[n_2]$ such that $(\iota \circ \tau_1)^\rho = \tau_2$. Similar to the discussion before Lemma 1.8, this identity defines a canonical S-isomorphism of pairs

$$(\tau_1^* Z[n_1], \tau_1^* D[n_1]) \cong (\tau_2^* Z[n_2], \tau_2^* D[n_2]),$$

compatible to their projections to $Z \times S$. Along the same line, we can define an effective arrow between ξ_1 and ξ_2 to be either an effective

arrow from ξ_1 to ξ_2 or an effective arrow from ξ_2 to ξ_1. We say ξ_1 and ξ_2 are equivalent via a sequence of effective arrows if there is a sequence of effective families $\eta_0, \ldots, \eta_m$ so that $\xi_1 = \eta_0$, $\xi_2 = \eta_m$ and that η_i is equivalent to η_{i+1} via an effective arrow. Note that this sequence of effective arrows induces a canonical isomorphism of $(\mathcal{Z}_1, \mathcal{D}_1)$ and $(\mathcal{Z}_2, \mathcal{D}_2)$. We have a partial inverse to this, similar to Lemma 1.8.

Lemma 4.3. *Let $\xi_i = (\mathcal{Z}_i, \mathcal{D}_i)$, $i = 1$ and 2, be two effective relative pairs over S associated to $\tau_i : S \to \mathbf{A}^{n_i}$. Suppose there is an isomorphism $(\mathcal{Z}_1, \mathcal{D}_1) \cong (\mathcal{Z}_2, \mathcal{D}_2)$ compatible to their projections to $Z \times S$. Then to each $p \in S$ there is an open neighborhood S_0 of $p \in S$ such that over S_0 the induced isomorphism $\mathcal{Z}_1 \times_S S_0 \cong \mathcal{Z}_2 \times_S S_0$ is induced by a sequence of effective arrows between $\xi_1 \times_S S_0$ and $\xi_2 \times_S S_0$.*

Proof. The proof is similar to that of Lemma 1.8 and will be omitted. q.e.d.

We define a family of relative pairs over S to be a pair $(\mathcal{Z}, \mathcal{D})$, where $\mathcal{Z}$ is an S-family with an S-projection $\mathcal{Z} \to Z \times S$ and $\mathcal{D}$ is a Cartier divisor of $\mathcal{Z}$, such that there is an open covering S_α of S so that each $(\mathcal{Z} \times_S S_\alpha, \mathcal{D} \times_S S_\alpha)$ is isomorphic to an effective relative pair associated to some $\tau_\alpha : S_\alpha \to \mathbf{A}^{n_\alpha}$ and the isomorphism $\mathcal{Z} \times_S S_\alpha \cong \tau_\alpha^* Z[n_\alpha]$ is compatible to their projections to $Z \times S_\alpha$. Note that then $\mathcal{D}$ is isomorphic to $D \times S \subset Z \times S$ under the projection $\mathcal{Z} \to Z \times S$. We call $\mathcal{Z} \times_S S_\alpha \cong Z[n_\alpha] \times_{\mathbf{A}^{n_\alpha}} S_\alpha$ with its associated data a local representative of $(\mathcal{Z}, \mathcal{D})$. We say two such families $(\mathcal{Z}_1, \mathcal{D}_1)$ and $(\mathcal{Z}_2, \mathcal{D}_2)$ are isomorphic if there is an S-isomorphism $(\mathcal{Z}_1, \mathcal{D}_1) \to (\mathcal{Z}_2, \mathcal{D}_2)$ compatible to the projections $\mathcal{Z}_1 \to Z \times S$ and $\mathcal{Z}_2 \to Z \times S$. Clearly, if $(\mathcal{Z}, \mathcal{D})$ is a relative pair over S and $\rho : S' \to S$ is a morphism, then $(\mathcal{Z} \times_S S', \mathcal{D} \times_S S')$ coupled with the obvious projection $\mathcal{Z} \times_S S' \to Z \times S'$ is a relative pair over S', called the pullback relative pair.

Definition 4.4. We define $\mathfrak{Z}^{\mathrm{rel}}$ to be the category whose objects are families of expanded relative pairs of (Z, D) over S for some scheme S. Let S_i, $i = 1$ and 2, be any two schemes and $\xi_i \in \mathfrak{Z}^{\mathrm{rel}}(S_i)$ be two objects in $\mathfrak{Z}^{\mathrm{rel}}$. An arrow from ξ_1 to ξ_2 consists of an arrow (a morphism) $\rho : S_1 \to S_2$ and an isomorphism of relative pairs $\xi_1 \cong \rho^* \xi_2$. We define $\mathfrak{p} : \mathfrak{Z}^{\mathrm{rel}} \to (\mathrm{Sch})$ to be the functor that send families over S to S. The pair $(\mathfrak{Z}^{\mathrm{rel}}, \mathfrak{p})$ forms a groupoid.

Proposition 4.5. *The groupoid $(\mathfrak{Z}^{rel}, \mathfrak{p})$ is a stack.*

Proof. The proof is straightforward and will be omitted. q.e.d.

In the following, we will call $\mathfrak{Z}^{\rm rel}$ the stack of expanded relative pairs of (Z, D).

Now let (Y_1, D_1) and (Y_2, D_2) be the two relative pairs associated to the family W/C and let $\mathfrak{Y}_1^{\rm rel}$ and $\mathfrak{Y}_2^{\rm rel}$ be the associated stacks of expanded relative pairs of (Y_1, D_1) and (Y_2, D_2), respectively. Then the correspondence $(\mathcal{Y}_1, \mathcal{Y}_2) \mapsto \mathcal{Y}_1^o \sqcup \mathcal{Y}_2$ defined in Corollary 4.2 induces a canonical morphism of stacks

$$\mathfrak{Y}_1^{\rm rel} \times \mathfrak{Y}_2^{\rm rel} \longrightarrow \mathfrak{W} \times_C 0 \subset \mathfrak{W}. \tag{4.6}$$

4.2 Relative stable morphisms

Let (Z, D) be the pair and $\mathfrak{Z}^{\rm rel}$ the stack of expanded relative pairs of (Z, D). In this part we define the notion of relative stable morphisms to $\mathfrak{Z}^{\rm rel}$. We fix an ample line bundle H on Z, called a polarization of (Z, D). We first introduce the notion of admissible weighted graphs (in short admissible graphs).

In this paper, by a graph Γ we mean a finite collection of vertices, edges, legs and roots. Here an edge is as usual a line segment with both ends attached to vertices of Γ. A leg or a root is a line segment with only one end attached to a vertex of Γ. We will denote by $V(\Gamma)$ the set of vertices of Γ.

Definition 4.6. An admissible weighted graph Γ is a graph without edges coupled with the following additional data:

1. An ordered collection of legs, an ordered collection of weighted roots and two weight functions $g : \mathrm{V}(\Gamma) \to \mathbb{Z}_{\geq 0}$ and $b : V(\Gamma) \to A_2 Z/\sim_{\rm alg}$.
2. The graph is relatively connected in the sense that either $\mathrm{V}(\Gamma)$ consists of a single element or each vertex in $\mathrm{V}(\Gamma)$ has at least one root attached to it.

In this section, we will reserve the integer k to denote the number of legs and the integer r to denote the number of roots of Γ. We will denote by $\mu_1, \ldots, \mu_r$ the weights of the r roots.

Given two admissible graphs Γ_1 and Γ_2, an isomorphism $\Gamma_1 \cong \Gamma_2$ is an isomorphism between their respective vertices, legs and roots (order preserving in the later two cases) so that it preserves the weights of the roots and the two sets of weights of the vertices.

We now define the relative stable morphisms to $(Z[n], D[n])$ of type Γ.

Definition 4.7. Let S be an $\mathbf{A}^n$-scheme and Γ be an admissible graph with r roots, k legs and l vertices $v_1, \dots, v_l$. An S-family of relative stable morphisms to $(Z[n], D[n])$ of type Γ is a quadruple $(f, \mathcal{X}, q_i, p_j)$ as follows:

1. $\mathcal{X}$ is a disjoint union of $\mathcal{X}_1, \dots, \mathcal{X}_l$ such that each $\mathcal{X}_i$ is a flat family of prestable curves[11] over S of arithmetic genus $g(v_i)$.

2. $q_i : S \to \mathcal{X}$, $i = 1, \dots, r$ and $p_j : S \to \mathcal{X}$, $j = 1, \dots, k$, are disjoint sections away from the singular locus of the fibers of $\mathcal{X}/S$ so that $q_i(S) \subset \mathcal{X}_j$ (resp. $p_i(S) \subset \mathcal{X}_j$) if the i-th root (resp. i-th leg) is attached to the j-th vertex of Γ.

3. $f : \mathcal{X} \to Z[n]$ is an $\mathbf{A}^n$-morphism (i.e., an S-morphism $\mathcal{X} \to Z[n] \times_{\mathbf{A}^n} S$) so that as Cartier divisors $f^{-1}(D[n]) = \sum_{i=1}^{r} \mu(x) q_i(S)$, and that to each closed $s \in S$ the class $\varphi_*(f(\mathcal{X}_s))$ is algebraically equivalent to $b(v)$. Here $\varphi : Z[n] \to Z$ be the tautological projection.

4. Finally, each morphism $f|_{\mathcal{X}_i}$, considered as a morphism whose domain is $\mathcal{X}_i$ with all marked sections in $\mathcal{X}_i$, is a family of stable morphisms to $Z[n]$.

Note that a necessary condition for the existence of relative stable morphisms of type Γ is that to each $v \in V(\Gamma)$, $\sum_{i \prec v} \mu_i = b(v) \cdot D$, where $i \prec v$ means that the i-th root is attached to the vertex v. We will call q_i the distinguished marked sections and call p_j the ordinary marked sections.

We now fix an admissible graph Γ. Let $f : \mathcal{X} \to Z[n]$ be a relative stable morphism to $(Z[n], D[n])$ of type Γ. Let l be any integer in $[n]$. Then $Z[n] \times_{\mathbf{A}^n} \mathbf{H}_l^n$ has normal crossing singularity whose singular locus is isomorphic to $D \times \mathbf{A}^{n-1}$. We denote this nodal divisor by $\mathbf{B}_l$. As in Section 2, we call f nondegenerate if to each $s \in S$ there are no irreducible components of $\mathcal{X}_s$ mapped entirely to the union of $\mathbf{B}_1, \dots, \mathbf{B}_n$ under f. We call f predeformable if f is predeformable along $\mathbf{B}_1, \dots, \mathbf{B}_n$ as defined in Section 2.

We now change our view point of f. Since S is an $\mathbf{A}^n$-scheme,

$$(Z[n] \times_{\mathbf{A}^n} S, D[n] \times_{\mathbf{A}^n} S) \in Ob(\mathfrak{Z}^{\mathrm{rel}}(S)),$$

[11] Recall by prestable we mean that $\mathcal{X}/S$ is a family of connected, complete and nodal curves.

and hence f can be viewed as an S-family of morphisms to $\mathfrak{Z}^{\mathrm{rel}}$. We say f, considered as an S-family of morphisms to $\mathfrak{Z}^{\mathrm{rel}}$, is relative prestable (resp. predeformable; resp. of type Γ) if f, considered as an S-family of morphisms to $Z[n]$, is relative stable (resp. predeformable; resp. or type Γ). As before, we define $\mathrm{Aut}_{\mathfrak{Z}}(f)$ to be the functor associating to each S-scheme T the set of all (a,b), where a is an automorphism of $\mathcal{X}\times_S T$ as pointed nodal curve over T, namely a leaves all marked (distinguished and ordinary) sections of $\mathcal{X}\times_S T$ fixed, and b is a morphism $T\to G[n]$ such that if we let f_T be the morphism $\mathcal{X}\times_S T\to Z[n]$ induced by f then

$$(f_T)^b = f_T\circ a : \mathcal{X}\times_S T\to Z[n].$$

Here $(f_T)^b$ is the morphism resulting from the $G[n]$-action on $Z[n]$ associated to $b\colon T\to G[n]$ (see (1.13)). Obviously, this functor is represented by a group scheme over S, called the automorphism group of f and denoted by $\mathrm{Aut}_{\mathfrak{Z}}(f)$.

Let S be any scheme and $(\mathcal{Z},\mathcal{D})\in Ob(\mathfrak{Z}^{\mathrm{rel}}(S))$ be any object. Then S has an open covering by $\{S_\alpha\}$ so that over each S_α we have isomorphism

$$(\mathcal{Z},\mathcal{D})\times_S S_\alpha \cong (Z[n_\alpha]\times_{\mathbf{A}^{n_\alpha}} S_\alpha, D[n_\alpha]\times_{\mathbf{A}^{n_\alpha}} S_\alpha)$$

for some $S_\alpha\to\mathbf{A}^{n_\alpha}$. For any S-family of morphisms $f\colon\mathcal{X}\to\mathcal{Z}$, we will call the induced morphisms $f_\alpha : \mathcal{X}\times_S S_\alpha\longrightarrow Z[n_\alpha]$ local representatives of f.

Definition 4.8. An S-family of relative prestable morphisms to $\mathfrak{Z}^{\mathrm{rel}}$ of type Γ is a family $f:\mathcal{X}\to\mathcal{Z}$, where $(\mathcal{Z},\mathcal{D})\in Ob(\mathfrak{Z}^{\mathrm{rel}}(S))$, so that all its local representative f_α are relative stable (as morphisms to $\mathcal{Z}_\alpha$) of type Γ. We say f is stable if in addition to it being prestable, all its local representatives are predeformable and that for all closed $s\in S$ the automorphism groups $\mathrm{Aut}_{\mathfrak{Z}}(f_s)$ are finite.

It is routine to check that the definition of $\mathrm{Aut}_{\mathfrak{Z}}(f_s)$ and the definition of relative stable morphisms to $\mathfrak{Z}^{\mathrm{rel}}$ is independent of the choices of local representatives.

Definition 4.9. Let (Z,D) be as before and let Γ be an admissible graph. We define $\mathfrak{M}(\mathfrak{Z}^{\mathrm{rel}},\Gamma)$ to be the category whose objects are families of relative stable morphisms to $\mathfrak{Z}^{\mathrm{rel}}$ of type Γ. As usual, $\mathfrak{M}(\mathfrak{Z}^{\mathrm{rel}},\Gamma)(S)$ is the subcategory of families over S. Let ξ_1 and ξ_2 be objects in $\mathfrak{M}(\mathfrak{Z}^{\mathrm{rel}},\Gamma)(S)$ represented by relative stable morphisms $f_1\colon\mathcal{X}_1\to\mathcal{Z}_1$ and $f_2\colon\mathcal{X}_2\to\mathcal{Z}_2$. An arrow $\xi_1\to\xi_2$ in $\mathfrak{M}(\mathfrak{Z}^{\mathrm{rel}},\Gamma)(S)$ covering $1_S\colon S\to S$ consists of a pair (φ_1,φ_2), where φ_1 is an S-isomorphism

$\mathcal{X}_1 \to \mathcal{X}_2$ preserving all ordinary marked points and φ_2 is an arrow $\mathcal{Z}_1 \to \mathcal{Z}_2$ in $\mathfrak{Z}^{\mathrm{rel}}(S)$ covering 1_S so that $f_2 \circ \varphi_1 = \varphi_2 \circ f_1$. In case $\xi_1 \in \mathfrak{M}(\mathfrak{Z}^{\mathrm{rel}}, \Gamma)(S)$ and $\xi_2 \in \mathfrak{M}(\mathfrak{Z}^{\mathrm{rel}}, \Gamma)(T)$, then an arrow $\xi_1 \to \xi_2$ consists of a morphism $\sigma : S \to T$ and an arrow $\xi_1 \to \sigma^* \xi_2$ in $\mathfrak{M}(\mathfrak{Z}^{\mathrm{rel}}, \Gamma)(S)$ covering 1_S. Let $\mathfrak{p} : \mathfrak{M}(\mathfrak{Z}^{\mathrm{rel}}, \Gamma) \to (\mathrm{Sch})$ be the functor that sends families over S to S. The category $\mathfrak{M}(\mathfrak{Z}^{\mathrm{rel}}, \Gamma)$ coupled with the functor $\mathfrak{p}$ form a groupoid, called the groupoid of the relative stable morphisms to $\mathfrak{Z}^{\mathrm{rel}}$ of type Γ.

Theorem 4.10. *The groupoid $\mathfrak{M}(\mathfrak{Z}^{rel}, \Gamma)$ is an algebraic stack. It is separated and proper over* $\mathbf{k}$.

Proof. The proof is parallel to that of $\mathfrak{M}(\mathfrak{W}, \Gamma)$ and will be omitted. q.e.d.

We now move back to the family W defined before. We fix an ample line bundle H on W as before and let H_{Y_i} be the restriction of H to Y_i. We let (Y_1, D_1) and (Y_2, D_2) be the two relative pairs from decomposing W_0. We let $\mathfrak{Y}_1^{\mathrm{rel}}$ and $\mathfrak{Y}_2^{\mathrm{rel}}$ be the stack of expanded relative pairs of (Y_1, D_1) and (Y_2, D_2) respectively. In the remainder part of this section, we will investigate the relation of the stacks $\mathfrak{M}(\mathfrak{Y}_1^{\mathrm{rel}}, \Gamma_1)$, $\mathfrak{M}(\mathfrak{Y}_2^{\mathrm{rel}}, \Gamma_2)$ and $\mathfrak{M}(\mathfrak{W}, \Gamma)$. We begin with a discussion of gluing a pair of relative stable morphisms

$$(f_1, f_2) \in \mathfrak{M}(\mathfrak{Y}_1^{\mathrm{rel}}, \Gamma_1) \times \mathfrak{M}(\mathfrak{Y}_2^{\mathrm{rel}}, \Gamma_2)$$

to form a stable morphism in $\mathfrak{M}(\mathfrak{W}, \Gamma)$. Here as usual the domain of the new morphism will be the gluing of the domains of f_1 and f_2. In terms of their associated graphs, this amounts to connecting the roots in the graph Γ_1 to the roots in the graph Γ_2 to form a new graph with edges.[12]

Definition 4.11. Let Γ_1 and Γ_2 be two admissible graphs with identical numbers of roots and k_1 and k_2 legs respectively. We let r be the number of roots of Γ_1, let $k = k_1 + k_2$ and let $I \subset [k]$ be a subset of k_1 elements. We say that the triple (Γ_1, Γ_2, I) is an admissible triple if the following two conditions hold:

1. The weight assignments $\mu_{1,i}$ and $\mu_{2,i}$ of the roots of Γ_1 and Γ_2 coincide in the sense that $\mu_{1,i} = \mu_{2,i}$ for all $i = 1, \ldots, r$.

[12]Namely we identify the free end of a root with the free end of another root to form an edge.

2. After connecting the i-th root in Γ_1 with the i-th root in Γ_2 for all i, we obtain a new graph with k-edges (but without roots). We demand that this graph is connected.

Let $\eta = (\Gamma_1, \Gamma_2, I)$ be an admissible triple. Using the subset $I \subset [k]$ we obtain a unique bijection $[k_1] \cup [k_2] \to [k]$ so that it preserves the orders of $[k_1]$ and $[k_2]$ and the image of $[k_1]$ is $I \subset [k]$. Using this bijection, we get a unique ordering of the legs of this new graph. The genus of this graph is

$$g(\eta) = r + 1 - |V(\Gamma)| + \sum_{v \in \mathrm{V}(\Gamma_1) \cup \mathrm{V}(\Gamma_2)} g(v)$$

and the degree of this graph is

$$b(\eta) = \sum_{i=1}^{k_1} b_{\Gamma_1}(i) \cdot c_1(H_{Y_1}) + \sum_{i=1}^{k_2} b_{\Gamma_2}(i) \cdot c_1(H_{Y_2}).$$

For simplicity, we denote by $|\eta|$ the triple

$$|\eta| = (g(\eta), b(\eta), k).$$

Now let $f_1 : \mathcal{X}_1 \to \mathcal{Y}_1$ be a relative stable morphism in $\mathfrak{M}(\mathfrak{Y}_1^{\mathrm{rel}}, \Gamma_1)(S)$, where $(\mathcal{Y}_1, \mathcal{D}_1) \in \mathfrak{Y}_1^{\mathrm{rel}}(S)$, and let $q_{1,i} : S \to \mathcal{X}_1$ be the i-th distinguished section. Then each $f_1 \circ q_{1,i} : S \to \mathcal{Y}_1$ factor through $\mathcal{D}_1 \subset \mathcal{Y}_1$, which composed with the first projection pr_D of $\mathcal{D} \cong D \times S$ defines a morphism $S \to D$. This way we obtain an evaluation morphism

$$\mathbf{q}(f_1) = \big(\mathrm{pr}_D \circ f_1 \circ q_{1,1}, \ldots, \mathrm{pr}_D \circ f_1 \circ q_{1,r}\big) : S \to D^r. \tag{4.7}$$

Similarly, let $f_2 : \mathcal{X}_2 \to \mathcal{Y}_2$ be in $\mathfrak{M}(\mathfrak{Y}_2^{\mathrm{rel}}, \Gamma_2)(S)$ with distinguished sections $q_{2,i}$. Then we have a similarly defined evaluation morphism $\mathbf{q}(f_2) : S \to D^r$. Now we construct a new family of morphisms over S. We glue the i-th sections $q_{1,i}(S) \subset \mathcal{X}_1$ with the i-th section $q_{2,i}(S) \subset \mathcal{X}_2$ for all $i = 1, \ldots, r$ to obtain an S-family of nodal curves $\mathcal{X}_1 \sqcup \mathcal{X}_2$. Since $\eta = (\Gamma_1, \Gamma_2, I)$ is admissible, the new family $\mathcal{X}_1 \sqcup \mathcal{X}_2 / S$ has connected fibers. Also the data $I \subset [k]$ defines a natural ordering of the k ordinary marked sections of $\mathcal{X}_1 \sqcup \mathcal{X}_2$. Now let $\mathcal{Y}_1^o \sqcup \mathcal{Y}_2$ be the S-scheme constructed in Corollary 4.2, then the pair of morphisms (f_1, f_2) defines a morphism

$$f_1 \sqcup f_2 : \mathcal{X}_1 \sqcup \mathcal{X}_2 \longrightarrow \mathcal{Y}_1^o \sqcup \mathcal{Y}_2 \tag{4.8}$$

if and only if

$$\mathbf{q}(f_1) \equiv \mathbf{q}(f_2) : S \to D^r.$$

We state it as a Proposition.

Proposition 4.12. *Let $\eta = (\Gamma_1, \Gamma_2, I)$ be an admissible triple and let f_i be relative stable morphisms in $\mathfrak{M}(\mathfrak{Y}_i^{rel}, \Gamma_i)(S)$, $i = 1, 2$. Suppose $\mathbf{q}(f_1) \equiv \mathbf{q}(f_2)$, then the morphism $f_1 \sqcup f_2$ is an S-family of stable morphisms in $\mathfrak{M}(\mathfrak{W}, \Gamma)(S)$ with $\Gamma = |\eta|$.*

Proof. The proof is straightforward and will be omitted. q.e.d.

Let $f \in \mathfrak{M}(\mathfrak{W}, \Gamma)(S)$ be an S-family of stable morphisms to $\mathcal{W}$. In case there is an admissible triple $\eta = (\Gamma_1, \Gamma_2, I)$ and families $f_i \in \mathfrak{M}(\mathfrak{Y}_i^{\rm rel}, \Gamma_i)(S)$ so that $f = f_1 \sqcup f_2$, then we say f is η-decomposable or f is η-decomposed into families f_1 and f_2.

We let $\mathfrak{M}(\mathfrak{Y}_i^{\rm rel}, \Gamma_i) \to D^r$ be the morphisms induced by $\mathbf{q}$ in (4.7). Then this Proposition implies that we have a natural morphism of stacks

$$\mathfrak{M}(\mathfrak{Y}_1^{\rm rel}, \Gamma_1) \times_{D^r} \mathfrak{M}(\mathfrak{Y}_2^{\rm rel}, \Gamma_2) \longrightarrow \mathfrak{M}(\mathfrak{W}, \Gamma). \tag{4.9}$$

Note that if $r = 0$, then η is admissible implies that one of Γ_i is empty. Then the fiber product is understood to be either $\mathfrak{M}(\mathfrak{Y}_2^{\rm rel}, \Gamma_2)$ in case $\Gamma_1 = \emptyset$ or the otherwise.

Clearly, the above morphism factor through the substack $\mathfrak{M}(\mathfrak{W}, \Gamma) \times_C 0$, and is finite and representable. We let $\mathfrak{M}(\mathfrak{Y}_1^{\rm rel} \sqcup \mathfrak{Y}_2^{\rm rel}, \eta)$ be the image stack in $\mathfrak{M}(\mathfrak{W}, \Gamma)$. We denote the induced morphism by

$$\Phi_\eta : \mathfrak{M}(\mathfrak{Y}_1^{\rm rel}, \Gamma_1) \times_{D^r} \mathfrak{M}(\mathfrak{Y}_2^{\rm rel}, \Gamma_2) \longrightarrow \mathfrak{M}(\mathfrak{Y}_1^{\rm rel} \sqcup \mathfrak{Y}_2^{\rm rel}, \eta). \tag{4.10}$$

We now investigate the degree of Φ_η. Let $f : X \to W[n]_0$ be a stable morphism that lies in the image of Φ_η. By definition, there is an integer $l \in [n+1]$ so that we can η-decompose f along the divisor $\mathbf{D}_l \cap W[n]_0$. Namely, we have a pair of relative stable morphisms $f_i : X_i \to Y_i[n_i]$, where $n_1 = l - 1$ and $n_2 = n + 1 - l$, an ordered finite set Σ and embeddings $\Sigma \subset X_i$ so that X is the result of gluing X_1 and X_2 along $\Sigma \subset X_1$ and $\Sigma \subset X_2$, and the restriction of f to X_i is f_i using the gluing $Y_1[n_1]^o \sqcup Y_2[n_2] = W[n]_0$. Further, if we view $\Sigma \subset X_i$ as its ordered distinguished marked points, then the topological type of f_i with the induced ordering on its ordinary marked points is exactly the Γ_i in $\eta = (\Gamma_1, \Gamma_2, I)$. Here $I \subset [k]$ is the subset of those marked points that are in X_1. We fix such a ordering on Σ.

Now let $\sigma \subset S_r$ be any permutation. Then σ induces a new ordering on Σ, which we denote by Σ^σ. Accordingly, for the Γ_i in $\eta = (\Gamma_1, \Gamma_2, I)$, we define Γ_i^σ to be the graph Γ_i with its roots reordered according to σ. Namely, the j-th root of Γ_i^σ is the $\sigma^{-1}(j)$-root of Γ_i. We define $\eta^\sigma = (\Gamma_1^\sigma, \Gamma_2^\sigma, I)$. For any two admissible triples η_1 and η_2 we say $\eta_1 \sim \eta_2$

(called equivalent) if there is a $\sigma \in S_r$ so that η_1 is isomorphic to η_2^σ. We define $\mathrm{Eq}(\eta)$ to be the subgroup of $\sigma \in S_r$ so that $\eta \sim \eta^\sigma$. Note that $\eta \sim \eta^\sigma$ implies that whenever $\sigma(j_1) = j_2$ then the j_1-th and the j_2-th roots of Γ_i have identical weights $\mu_{j_1} = \mu_{j_2}$ and are attached to the same vertex of Γ_i.

Proposition 4.13. *The morphism Φ_η is finite and étale. It has pure degree*[13] $|\mathrm{Eq}(\eta)|$.

Proof. The proof that the morphism Φ_η is finite and étale is straightforward, and will be left to readers. We now check that the degree of Φ_η is as stated. Let $\xi \in \mathrm{Im}(\Phi_\eta)$ be any element. It follows from the discussion preceding to the statement of this Proposition that there is an $l \in [n+1]$ so that the decomposition of f along the divisor $\mathbf{D}_l \cap W[n]_0$ is the η-decomposition of f. Because f is stable, for all other $l' \in [n+1]-l$ the decomposition types of f along $\mathbf{D}_{l'} \cap W[n]_0$ will be different from η (i.e., $\not\sim$). We now let ξ_i be represented by $f_i : X_i \to Y_i[n_i]$, after fixing an ordering on $\Sigma = f^{-1}(\mathbf{D}_l)$ so that the topological type of ξ is Γ_i. Thus $f_i \in \mathfrak{M}(\mathfrak{Y}_i^{\mathrm{rel}}, \Gamma_i)$ and $f = f_1 \sqcup f_2$. Now let $\sigma \in S_r$ be any permutation. We let f_i^σ be the same morphism $f_i : X_i \to Y_i[n_i]$ except that the distinguished marked points of X_i are reordered according to σ. This way

$$(f_i^\sigma, f_2^\sigma) \in \mathfrak{M}(\mathfrak{Y}_1^{\mathrm{rel}}, \Gamma_1^\sigma) \times_{D^r} \mathfrak{M}(\mathfrak{Y}_1^{\mathrm{rel}}, \Gamma_1^\sigma).$$

Clearly, it is in the domain of Φ_η if and only if $\eta \sim \eta^\sigma$.

To derive the degree formula, we need to investigate the automorphism group $\mathrm{Aut}(\xi)$ of $\xi \in \mathfrak{M}(\mathfrak{W}, \Gamma)$ and the automorphism group $\mathrm{Aut}(\xi_i)$ of $\mathfrak{M}(\mathfrak{Y}_i^{\mathrm{rel}}, \Gamma_i)$. First, the automorphism group $\mathrm{Aut}((\xi_1, \xi_2))$ is naturally isomorphic to $\mathrm{Aut}(\xi_1) \times \mathrm{Aut}(\xi_2)$. Because f is derived from gluing f_1 and f_2 along $\Sigma \subset X_1$ and $\Sigma \subset X_2$, elements in $\mathrm{Eq}(\eta)$ induce permutations of Σ. Hence we obtain a natural homomorphism of groups $h : \mathrm{Aut}(\xi) \to S_r$ whose kernel is isomorphic to $\mathrm{Aut}(\xi_1) \times \mathrm{Aut}(\xi_2)$. Further the image of h lies in the subgroup $\mathrm{Eq}(\eta) \subset S_r$ and the coset $\mathrm{Eq}(\eta)/\mathrm{Im}(\eta)$ is exactly the set $\Phi_\eta^{-1}(\xi)$. This shows that $\Phi_\eta^{-1}(\xi)$ consists of exactly $|\mathrm{Eq}(\eta)|/|\mathrm{Im}(h)|$ distinct elements. Hence the degree of Φ_η is

$$|\mathrm{Eq}(\eta)|/|\mathrm{Im}(\eta)| \cdot |\mathrm{Aut}(\xi)| \cdot 1/|\mathrm{Aut}(\xi_1) \times \mathrm{Aut}(\xi_2)| = |\mathrm{Eq}(\eta)|.$$

This proves Proposition 4.13. q.e.d.

[13] Here we say $f : X \to Y$ has pure degree d if for any integral $A \subset Y$ the restriction $X \times_Y A \to A$ has degree d.

5. Some Comments

It is interesting to work out the analogous construction for the moduli spaces of stable sheaves over schemes. Before the discovery of SW-invariants, there were a lot of research work on using degeneration methods to study the Donaldson invariants. The differential geometric approach to this is well understood. But the algebro-geometric approach to this is still missing. It is clear that one should consider the moduli of stable sheaves over the stack $\mathfrak{W}$ and the stack $\mathfrak{Z}^{\mathrm{rel}}$. It is interesting to work this out in detail. One technical issue is that after using the expanded degenerations one should be able to avoid non-locally freeness (of sheaves) along the nodal divisor. This is the case for curves, following the pioneering work of [13]. It is true for surfaces, which is essentially proved by Gieseker and the author in [12].

A more challenging degeneration problem is to workout a degeneration of GW-invariants for the families $W \to C$ whose singular fibers are only assumed to have normal crossing singularities. Recall that the family W/C studied in this paper only deal with the case where W_0 has two irreducible components that intersect along a connected smooth divisor. It is easy to see that this construction can be generalized to the case where W_0 is only assumed to have normal crossing singularity and its singular locus is smooth. When the singular locus of W_0 is not smooth, it is not clear what constitutes the class of expanded degenerations of W/C. The progress along this line is important to the study of degeneration of moduli spaces.

References

[1] K. Behrend, *Gromov-Witten invariants in algebraic geometry*, Invent. Math. **127**(**3**) (1997) 601–617.

[2] K. Behrend & B. Fantechi, *The intrinsic normal cone*, Invent. Math. **128**(**1**) (1997) 45–88.

[3] L. Caporaso, *A compactification of the universal Picard variety over the moduli space of stable curves*, J. Amer. Math. Soc. **7**(**3**) (1994) 589–660.

[4] P. Deligne & D. Mumford, *The irreducibility of the space of curves of given genus*, Publ. I.H.E.S. **45** (1969) 101–145.

[5] Y. Eliashberg, A. Givental & H. Hofer, *Introduction to symplectic field theory*, preprint.

[6] A. Floer, *Morse theory for Lagrangian intersections*, J. Differential Geom. **28**(**3**) (1988) 513–547.

[7] K. Fukaya & K. Ono, *Arnold conjecture and Gromov-Witten invariant for general symplectic manifolds*, The Arnoldfest (Toronto, ON, 1997), 173–190, Fields Inst. Commun. **24**, Amer. Math. Soc., Providence, RI, 1999.

[8] W. Fulton & R. Pandharipande, *Notes on stable maps and quantum cohomology*, Algebraic geometry—Santa Cruz 1995, 45–96, Proc. Sympos. Pure Math., **62**, Part 2, Amer. Math. Soc., Providence, RI, 1997.

[9] A. Gathmann, *Absolute and relative Gromov-Witten invariants of very ample hypersurfaces*, math.AG/9908054.

[10] D. Gieseker, *A degeneration of the moduli space of stable bundles*, J. Differential Geom. **19**(**1**) (1984) 173–206.

[11] P. Griffith & J. Harris, *Principle of Algebraic Geometry*, John Wiley & Sons, New York, 1987.

[12] D. Gieseker & J. Li, *Irreducibility of moduli of rank two vector bundles*, J. Differential Geom. **40** (1994) 23–104.

[13] D. Gieseker & I. Morrison, *Hilbert stability of rank-two bundles on curves*, J. Differential Geom. **19**(**1**) (1984) 1–29.

[14] T. Graber & R. Vakil, *Hodge integrals and Hurwitz numbers via virtual localization*, math.AG/0003028.

[15] J. Harris & D. Mumford, *The Kodaira dimension of the moduli space of curves of genus* ≥ 23, Invent. Math. **90**(**2**) (1987) 359–387.

[16] E-N. Ionel, *Topological recursive relations in* $H^{2g}(M_{g,n})$, math.AG/9908060.

[17] E-N. Ionel & T. Parker, *Gromov-Witten Invariants of Symplectic Sums*, Math. Res. Lett. **5**(**5**) (1998) 563–576.

[18] E-N. Ionel & T. Parker, *Relative Gromov-Witten Invariants*, math.SG/9907155.

[19] E-N. Ionel & T. Parker, *The Symplectic Sum Formula for Gromov-Witten Invariants*, math.SG/0010217.

[20] M. Kontsevich & Y. Manin, *Gromov-Witten classes, quantum cohomology, and enumerative geometry*, Comm. Math. Phys. **164**(**3**) (1994) 525–562.

[21] J. Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics 32. Springer-Verlag, Berlin, 1996

[22] M. Kontsevich, *Enumeration of rational curves via torus actions*, The Moduli Space of Curves, edited by R.Dijkgraaf, C.Faber, G. van der Geer, Progress in Mathematics, **129**, Birkhauser, 1995.

[23] J. Li, *A degeneration formula of GW-invariants*, math.AG/0110113.

[24] J. Li & G. Tian, *Algebraic and symplectic geometry of virtual moduli cycles*, Algebraic geometry—Santa Cruz 1995, 143–170, Proc. Sympos. Pure Math., **62**, Part 2, Amer. Math. Soc., Providence, RI, 1997.

[25] J. Li & G. Tian, *Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties*, J. Amer. Math. **11**(**1**) (1998) 119–174.

[26] J. Li & G. Tian, *Virtual moduli cycles and Gromov-Witten invariants of general symplectic manifolds*, Topics in symplectic 4-manifolds (Irvine, CA, 1996), 47–83, First Int. Press Lect. Ser., I, Internat. Press, Cambridge, MA, 199.

[27] J. Li & G. Tian, *Comparison of algebraic and symplectic Gromov-Witten invariants*, Asian J. Math. **3**(**3**) (1999) 689–728.

[28] A-M. Li & Y. Ruan, *Symplectic surgery and Gromov-Witten invariants of Calabi-Yau 3-folds I*, math.AG/9803036.

[29] A-M. Li, G. Zhao & Q. Zheng, *The number of ramified covering of a Riemann surface by Riemann surface*, math.AG/9906053.

[30] Matumura, *Commutative ring theory*, Cambridge University Press, 1986.

[31] J. Milne, *Étale cohomology*, Princeton University Press, Princeton, New Jersey, 1980.

[32] Fogarty and D. Mumford, *Geometric Invariant theory*, Springer-Verlag, 1982.

[33] A. Grothendieck, *Techniques de construction et théorèmes d'existence en géométrie algébrique IV*, les schémas de Hilbert, Séminaire Bourbaki, 221, 1960-61.

[34] Y.B. Ruan & G. Tian, *A mathematical theory of quantum cohomology*, J. Differential Geom. **42**(**2**) (1995) 259–367.

[35] Y. Ruan, *Topological Sigma model and Donaldson type invariants in Gromov theory*, Duke Math. J. **83**(**2**) (1996) 461–500.

[36] Y. Ruan, *Virtual neighborhoods and pseudo-holomorphic curves*, Proceedings of 6th G okova Geometry-Topology Conference, Turkish J. Math. **23**(**1**) (1999) 161–231.

[37] Y. Ruan, *Quantum cohomology and birational geometry*, Northern California Symplectic Geometry Seminar, 183–198, Amer. Math. Soc. Transl. Ser. 2, 196, Amer. Math. Soc., Providence, RI, 1999.

[38] B. Siebert, *Symplectic Gromov-Witten invariants*, New trends in algebraic geometry (Warwick, 1996), 375–424, London Math. Soc. Lecture Note Ser., 264, Cambridge Univ. Press, Cambridge, 1999.

[39] G. Tian, *Quantum cohomology and its associativity*, Current developments in mathematics, 1995 (Cambridge, MA), 361–401, Internat. Press, Cambridge, MA, 1994.

[40] A. Vistoli, *Intersection theory on algebraic stacks and on their moduli spaces*, Invent. Math. **97**(**3**) (1989) 613-670.

[41] E. Witten, *Topological sigma models*, Comm. Math. Phys. **118**(1988);
Two dimensional gravity and intersection theory on moduli space, Surveys in Differential Geom. **1** (1991) 243–310.

Stanford University, Stanford, CA 94305