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L_p AFFINE ISOPERIMETRIC INEQUALITIES

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Abstract

The L_p analogues of the Petty projection inequality and the Busemann-Petty centroid inequality are established.

An affine isoperimetric inequality compares two functionals associated with convex (or more general) bodies, where the ratio of the functionals is invariant under non-degenerate linear transformations. These isoperimetric inequalities are more powerful than their better-known Euclidean relatives.

This article deals with affine isoperimetric inequalities for centroid and projection bodies. Centroid bodies were attributed by Blaschke to Dupin (see e.g., the books of Schneider [32] and Leichtweiß [17] for references). If K is an origin-symmetric convex body in Euclidean n-space, $\mathbb{R}^n$, then the centroid body of K is the body whose boundary consists of the locus of the centroids of the halves of K formed when K is cut by codimension 1 subspaces. Blaschke (see Schneider [32] for references) conjectured that the ratio of the volume of a body to that of its centroid body attains its maximum precisely for ellipsoids. This conjecture was proven by Petty [27] who also extended the definition of centroid bodies and gave centroid bodies their name. When written as an inequality, Blaschke's conjecture is known as the Busemann-Petty centroid inequality. Busemann's name is attached to the inequality because Petty showed that Busemann's random simplex inequality ([5]) could be reinterpreted as what would become known as the Busemann-Petty centroid inequality. In recent times, centroid bodies (and their

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associated inequalities) have attracted increased attention (see e.g., Milman and Pajor [25, 26]). In retrospect, it can be seen that much if not all of this recent interest was inspired by Petty's seminal work [27].

Projection bodies are of newer vintage. They were introduced at the turn of the previous century by Minkowski. He showed that corresponding to each convex body K in $\mathbb{R}^n$ is a unique origin-symmetric convex body ΠK, the projection body of K, which can be defined (up to dilation) by the amazing fact that the following ratio is independent of the choice of 1-dimensional subspace l of $\mathbb{R}^n$: the length of the image of the orthogonal projection of ΠK onto l, to the $(n-1)$-dimensional volume of the image of the orthogonal projection of K onto the codimension 1 subspace $l^{\perp}$. Interest in projection bodies was rekindled by three highly influential articles, which appeared in the latter half of the 60's, by Bolker [2], Petty [28], and Schneider [31]. Projection bodies have been the objects of intense investigation during the past three decades (see, e.g., Bourgain and Lindenstrauss [3], Schneider and Weil [33], Goodey and Weil [10], and the books of Schneider [32], Gardner [8], Leichtweiß [17], and Thompson [34]).

The fundamental inequality for projection bodies is the Petty projection inequality: Of all convex bodies of fixed (say, unit) volume, the ones whose polar projection bodies have maximal volume are precisely the ellipsoids. The inequality that states that simplices are precisely the bodies that minimize this volume is known as the Zhang projection inequality [35]. Petty [29] established the Petty projection inequality as a consequence of the Busemann-Petty centroid inequality. It was shown in [19] that this process could be reversed: the Busemann-Petty centroid inequality can be derived as a direct consequence of the Petty projection inequality. Both the Petty projection inequality and the Busemann-Petty centroid inequality have come to be recognized as fundamental affine inequalities.

All centroid and projection bodies belong to the class $\mathcal{Z}^n$, of zonoids in $\mathbb{R}^n$. Zonoids can be defined as limits, with respect to the Hausdorff metric, of (Minkowski) sums of ellipsoids. The class $\mathcal{Z}^n$ arises naturally in various guises. For example, zonoids are the ranges of non-atomic $\mathbb{R}^n$-valued measures. They are also the polars of the unit balls of n-dimensional subspaces of $L_1([0,1])$.

To be more specific, this article concerns L_p-analogues of centroid and projection bodies. The L_p-analogues of centroid bodies have already appeared elsewhere. For example, the L_2-analogues of centroid bodies is an ellipsoid (called the Legendre ellipsoid) that appears in classical

mechanics. However, the L_p-analogues of projection bodies are new. In order to correctly define them one needs the recently introduced (in [21] [22]) notion of L_p-curvature. Both the L_p-analogues of centroid bodies and the L_p-analogues of projection bodies belong to the class $\mathcal{Z}_p^n$ of L_p-zonoids. While less well known than the class of zonoids, the class $\mathcal{Z}_p^n$ is not new (see e.g., Schneider and Weil [33], and Goodey and Weil [10]). The bodies in $\mathcal{Z}_p^n$ are just the polars of the unit balls of n-dimensional subspaces of $L_p([0,1])$.

We shall derive the exact L_p-analogues of both the Busemann-Petty centroid inequality and the Petty projection inequality (as well as their equality conditions).

Let S^{n-1} denote the unit sphere in Euclidean n-space, $\mathbb{R}^n$. Let B denote the origin-centered standard unit ball in $\mathbb{R}^n$, and write ω_n for $V(B)$, the n-dimensional volume of B. Note that

$$\omega_n = \pi^{n/2}/\Gamma(1+\tfrac{n}{2})$$

defines ω_n for all non-negative real n (not just the positive integers). For real $p \geq 1$, define $c_{n,p}$ by

$$c_{n,p} = \frac{\omega_{n+p}}{\omega_2 \omega_n \omega_{p-1}}.$$

For each compact star-shaped about the origin $K \subset \mathbb{R}^n$, and each p such that $1 \leq p \leq \infty$, let the norm $\|\cdot\|_{\Gamma_p^* K}$ on $\mathbb{R}^n$ be defined by

$$\|x\|_{\Gamma_p^* K} = \left\{ \frac{1}{c_{n,p} V(K)} \int_K |x \cdot y|^p dy \right\}^{1/p},$$

where $x \cdot y$ denotes the standard inner product of x and y, and $V(K)$ denotes the volume of K. For the case $p = \infty$, this definition is to be interpreted as the limit as $p \to \infty$. The unit ball of the resulting n-dimensional L_p-space is denoted by $\Gamma_p^* K$, and called the *polar L_p-centroid body of K*. The (unusual) normalization above is chosen so that for the standard unit ball B in $\mathbb{R}^n$, we have $\Gamma_p^* B = B$.

In [24] the following centro–affine inequality involving the volumes of K and its polar L_p-centroid body, $\Gamma_p^* K$, was established:

If K is a star-shaped (about the origin) subset of $\mathbb{R}^n$, then for $1 \leq p \leq \infty$,

$$(*) \qquad V(K)\, V(\Gamma_p^* K) \leq \omega_n^2,$$

with equality if and only if K is an ellipsoid centered at the origin.

If K is an origin-symmetric convex body, then $\Gamma_\infty^* K$ is just the polar, K^*, of K where

$$K^* = \{x \in \mathbb{R}^n \,:\, x \cdot y \leq 1, \quad \text{for all } y \in K\}.$$

Thus, inequality $(*)$, for $p = \infty$, reduces to:

$$V(K)\, V(K^*) \leq \omega_n^2,$$

with equality if and only if K is an ellipsoid.

This is the well known Blaschke-Santaló inequality.

In light of the Blaschke-Santaló inequality, a stronger inequality than $(*)$ was conjectured in [24]. This stronger inequality is the inequality of our first theorem:

Theorem 1. *If K is a star body (about the origin) in $\mathbb{R}^n$, then for $1 \leq p < \infty$,*

$$V(\Gamma_p K) \geq V(K),$$

with equality if and only if K is an ellipsoid centered at the origin.

Here $\Gamma_p K$, the L_p-centroid body of K, is just the polar of $\Gamma_p^* K$.

For the case $p = 1$ the inequality of Theorem 1 is known as the Busemann-Petty centroid inequality [27] (see also the books of Schneider [32], Gardner [8], and Leichtweiß [17]). The case $p = 2$ is also well known and goes back to at least, to Blaschke [1] (see also Lindenstrauss and Milman [18], Milman and Pajor [25] [26], Petty [27], and also [23]). For all other values of p the inequality of Theorem 1 is new.

The inequality closely related to the Busemann-Petty centroid inequality is known as the Petty projection inequality [29] (see also the books of Schneider [32], Gardner [8], and Leichtweiß [17]). The L_p-version of the Petty projection inequality will also be established in this article.

It will be convenient throughout to restrict our attention to only those convex (and star-shaped) bodies which contain the origin in their interiors. This assumption will tacitly be made throughout.

1. The L_p version of the Petty Projection Inequality

If K is a convex body (i.e., a compact, convex subset containing the origin in its interior) in $\mathbb{R}^n$, then its support function, $h_K = h(K, \cdot\,) : \mathbb{R}^n \to [0, \infty)$, can be defined for $x \in \mathbb{R}^n$ by $h(K, x) = \max\{x \cdot y : y \in K\}$.

For $p \geq 1$, convex bodies K, L, and $\varepsilon > 0$, the Minkowski-Firey L_p-combination $K\!+\!\varepsilon\cdot\! L$ is defined as the convex body whose support function is given by

$$h(K\!+\!\varepsilon\cdot\! L, \cdot\,)^p = h(K, \cdot\,)^p + \varepsilon h(L, \cdot\,)^p. \tag{1}$$

Although Minkowski-Firey addition and scalar multiplication depend on p, our notation does not reflect this fact. Minkowski-Firey combinations of convex bodies were defined and studied by Firey [6] (who called them p–means of convex bodies).

For $p \geq 1$, the L_p–mixed volume, $V_p(K, L)$, of the convex bodies K, L was defined in [21] by:

$$\frac{n}{p} V_p(K, L) = \lim_{\varepsilon \to 0^+} \frac{V(K\!+\!\varepsilon\cdot\! L) - V(K)}{\varepsilon}. \tag{2}$$

That this limit exists was demonstrated in [21].

It was shown in [21], that corresponding to each convex body, K, that contains the origin in its interior, there is a positive Borel measure, $S_p(K, \cdot\,)$, on S^{n-1} such that

$$V_p(K, Q) = \frac{1}{n} \int_{S^{n-1}} h(Q, u)^p \, dS_p(K, u), \tag{3}$$

for each convex body Q. The measure $S_1(K, \cdot)$ is just the classical surface area measure of K. This measure is usually denoted by $S(K, \cdot)$ or S_K.

For positive real p, let C_p denote the spherical L_p-cosine transform on S^{n-1}; i.e., for each positive Borel measure, μ, on S^{n-1}, let $\mathrm{C}_p\,\mu$ be the continuous function on S^{n-1} defined by

$$(\mathrm{C}_p\,\mu)(u) = \left\{ \frac{1}{n\omega_n c_{n-2,p}} \int_{S^{n-1}} |u\cdot v|^p d\mu(v) \right\}^{1/p},$$

for each $u \in S^{n-1}$. The unusual normalization above was chosen so that for Lebesgue measure, S, on S^{n-1}, we have $\mathrm{C}_p\, S = 1$.

For $p = 1$ the spherical L_p-cosine transform is just the well-known spherical cosine transform which is closely related to the spherical Radon transform (see e.g., Goodey and Weil [9], [10]). The operator C_1 will be written simply as C. For $p > 1$ the spherical L_p-cosine transform is also well known (see e.g., Koldobsky [11], [12], [13], [14], [15], [16], and Rubin [30] for applications and references).

For each convex body K, define the L_p-projection body, $\Pi_p K$, of K to be the origin-symmetric convex body whose support function is given by

$$h(\Pi_p K, \cdot) = \mathrm{C}_p\, S_p(K, \cdot\,). \tag{4}$$

The unusual normalization above is chosen so that for the unit ball, B, we have $\Pi_p B = B$.

Just as $\Gamma_p^* K$, rather than $(\Gamma_p K)^*$, is used to denote the polar body of $\Gamma_p K$, we will denote the polar of the body $\Pi_p K$ by $\Pi_p^* K$, rather than $(\Pi_p K)^*$.

A $p = 1$ is often suppressed. The convex body ΠK is known simply as the projection body of K. Note again that we have adopted a normalization that differs from the classical in that ΠB is simply B (rather than the classical $\omega_{n-1} B$). We note again that in order to define the L_p-projection body of a convex body, for $p > 1$, the notion of an L_p-curvature measure (or function) is critical.

One of the classical affine isoperimetric inequalities is the Petty projection inequality [29]. It states that for each convex body K in $\mathbb{R}^n$,

$$V(K)^{n-1} V(\Pi^* K) \leq \omega_n^n,$$

with equality if and only if K is an ellipsoid.

The Petty projection inequality is the statement that the quantity $V(K)^{n-1} V(\Pi^* K)$ is maximized precisely when the body K is an ellipsoid. The Zhang projection inequality [35] states that this quantity is minimized precisely by simplices.

We will establish the L_p-analogue of the Petty projection inequality:

Theorem 2. *If K is a convex body in $\mathbb{R}^n$, then for $1 < p < \infty$,*

$$V(K)^{(n-p)/p} V(\Pi_p^* K) \leq \omega_n^{n/p},$$

with equality if and only if K is an ellipsoid centered at the origin.

The special case $p = 2$ of Theorem 2, and its reverse inequality, can be found in [23].

2. Outline of proof

For real $p \geq 1$ define the L_p-Petty projection product of a convex body K by

$$\mathrm{p}_p(K) = \omega_n^{-n/p} V(K)^{(n-p)/p} V(\Pi_p^* K).$$

For real $p \geq 1$ define the L_p-Busemann-Petty ratio of the star body K by

$$\mathrm{b}_p(K) = V(K)/V(\Gamma_p K).$$

Note that while p_p is defined only for convex bodies, b_p is defined for all star bodies.

From the definition of L_p-projection body, it follows immediately that for $\lambda > 0$ we have $\Pi_p \lambda K = \lambda^{(n-p)/p} \Pi_p K$, where $\lambda K = \{\lambda x : x \in K\}$ is the dilate of K by a factor of λ. Thus $\Pi_p^* \lambda K = \lambda^{(p-n)/p} \Pi_p^* K$ which shows that the functional p_p is dilation invariant. In the next section we prove (Lemma 2) that p_p is in fact a $\mathrm{GL}(n)$-invariant functional: For each convex body K,

$$\mathrm{p}_p(\phi K) = \mathrm{p}_p(K), \qquad \text{for all } \phi \in \mathrm{GL}(n),$$

where $\phi K = \{\phi x : x \in K\}$ is the image of K under ϕ.

In order to demonstrate the existence of a convex body at which p_p attains a maximum, proceed as follows: Let $\hat{\mathrm{p}}_p$ denote the supremum of the functional p_p taken over all convex bodies. Let K_i denote a maximizing sequence for p_p; i.e., K_i is a sequence of convex bodies such that

$$\lim_{i \to \infty} \mathrm{p}_p(K_i) = \hat{\mathrm{p}}_p.$$

In the next section we will use a *class reduction* technique to show (Lemma 7) that, unless the body K_i is origin-symmetric, there exists an origin-symmetric body $\overline{K}_i$ such that $\mathrm{p}_p(K_i) < \mathrm{p}_p(\overline{K}_i)$. Thus it may be assumed that the original maximizing sequence consists solely of bodies that are origin-symmetric. A classical theorem of John (see e.g., Thompson [34]) yields the existence of a sequence of origin-symmetric ellipsoids E_i such that $E_i \subset K_i \subset \sqrt{n} E_i$. But since p_p is a $\mathrm{GL}(n)$-invariant functional, we may assume that the maximizing sequence K_i is such that, for all i

$$B \subset K_i \subset \sqrt{n} B,$$

where B denotes the origin-centered unit ball. The Blaschke selection theorem now guarantees the existence of a body at which p_p attains a maximum. Since this maximizing body is the limit (with respect to the Hausdorff metric) of a subsequence of the K_i, it follows that this maximizing body contains the origin (in fact the interior of the unit ball) in its interior.

We will use a *class reduction* technique to show that all bodies at which the maximum of p_p is attained must be sufficiently smooth. This

reduction will be critical in our proof. The proof is completed by the fundamental fact (Lemma 14) that Steiner symmetrization intertwines with the polar L_p-projection operator.

Finally, to prove that $\mathrm{b}_p < 1$ for all star bodies (except ellipsoids), we will use a class reduction result to show that this follows from the fact that $\mathrm{p}_p < 1$ for a small class of convex bodies.

Although we will not use either the Petty projection inequality nor the Busemann-Petty centroid inequality to prove their L_p analogues, we do not wish to reprove these classical inequalities. Thus, throughout we shall restrict our attention solely to the case of real $p > 1$. Of course, by taking limits (as $p \to 1$) one may recover the classical inequalities (but not necessarily their equality conditions) from their L_p analogues. We note again that we will be tacitly assuming throughout that all bodies contain the origin in their interiors.

3. Mixed and dual mixed volumes and the operators Π_p^* and Γ_p

For quick reference, we recall some basic properties of L_p-mixed and dual mixed volumes. Some recent applications of dual mixed volumes can be found in [7], [36], [37] and [38]. For general reference the reader may wish to consult the books of Gardner [8] and Schneider [32]. We emphasize again that we are assuming throughout that $1 < p < \infty$ and that our convex (and star-shaped) bodies all contain the origin in their interiors.

The radial function, $\rho_K = \rho(K, \cdot) : \mathbb{R}^n \setminus \{0\} \to [0, \infty)$, of a compact, star-shaped (about the origin) $K \subset \mathbb{R}^n$, is defined, for $x \neq 0$, by $\rho(K, x) = \max\{ \lambda \geq 0 : \lambda x \in K \}$. If ρ_K is positive and continuous, call K a star body (about the origin). Two star bodies K and L are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

If K is a convex body, then it follows from the definitions of support and radial functions, and the definition of polar body, that

$$h_{K^*} = 1/\rho_K \qquad \text{and} \qquad \rho_{K^*} = 1/h_K. \tag{5}$$

For star bodies K, L, and $\varepsilon > 0$, the L_p-harmonic radial combination $K \mathbin{\widetilde{+}} \varepsilon \cdot L$ is the star body defined by

$$\rho(K \mathbin{\widetilde{+}} \varepsilon \cdot L, \cdot)^{-p} = \rho(K, \cdot)^{-p} + \varepsilon \rho(L, \cdot)^{-p}.$$

While this addition and scalar multiplication are obviously dependent on p, we have not made this explicit in our notation. The dual mixed volume $\widetilde{V}_{-p}(K,L)$ of the star bodies K, L, can be defined by

$$\frac{n}{-p}\,\widetilde{V}_{-p}(K,L) = \lim_{\varepsilon\to 0^+} \frac{V(K\,\widetilde{+}\,\varepsilon\cdot L) - V(K)}{\varepsilon}.$$

The definition above and the polar coordinate formula for volume give the following integral representation of the dual mixed volume $\widetilde{V}_{-p}(K,L)$ of the star bodies K, L:

$$\widetilde{V}_{-p}(K,L) = \frac{1}{n}\int_{S^{n-1}} \rho_K^{n+p}(v)\rho_L^{-p}(v)\,dS(v), \tag{6}$$

where the integration is with respect to spherical Lebesgue measure S on S^{n-1}.

From the definition of support function, it follows immediately that for a convex body K, an $x \in \mathbb{R}^n$, and a $\phi \in \mathrm{SL}(n)$, we have $h_{\phi K}(x) = h_K(\phi^t x)$, where ϕ^t denotes the transpose of ϕ and $\phi K = \{\phi x : x \in K\}$ is the image of K under ϕ. This and the definition of a Minkowski-Firey L_p-combination show that for a Minkowski-Firey L_p-combination of convex bodies K and L, we have

$$\phi(K \mathbf{+} \varepsilon\cdot L) = \phi K \mathbf{+} \varepsilon\cdot \phi L.$$

This observation together with the definition of the L_p-mixed volume V_p shows that for $\phi \in \mathrm{SL}(n)$ and convex bodies K, L we have $V_p(\phi K, \phi L) = V_p(K,L)$ or equivalently

$$V_p(\phi K, L) = V_p(K, \phi^{-1}L). \tag{7}$$

From the definition of radial function, it follows immediately that for a star body K, an $x \in \mathbb{R}^n$, and a $\phi \in \mathrm{SL}(n)$, we have $\rho_{\phi K}(x) = \rho_K(\phi^{-1}x)$. This and the definition of an L_p-harmonic radial combination shows that for an L_p-harmonic radial combination of star bodies K and L,

$$\phi(K\,\widetilde{+}\,\varepsilon\cdot L) = \phi K\,\widetilde{+}\,\varepsilon\cdot\phi L.$$

This observation together with the definition of the dual mixed volume $\widetilde{V}_{-p}$ shows that for $\phi \in \mathrm{SL}(n)$ and star bodies K, L we have $\widetilde{V}_{-p}(\phi K, \phi L) = \widetilde{V}_{-p}(K,L)$ or equivalently

$$\widetilde{V}_{-p}(\phi K, L) = \widetilde{V}_{-p}(K, \phi^{-1}L). \tag{8}$$

We shall require two basic inequalities regarding the L_p-mixed volumes V_p and the dual mixed volumes $\widetilde{V}_{-p}$. The L_p analogue of the classical Minkowski inequality states that for convex bodies K, L,

$$V_p(K,L) \geq V(K)^{(n-p)/n} V(L)^{p/n}, \tag{9}$$

with equality if and only if K and L are dilates. The L_p-Minkowski inequality was established in [21] by using the Minkowski inequality. The basic inequality for the dual mixed volumes $\widetilde{V}_{-p}$ is that for star bodies K, L,

$$\widetilde{V}_{-p}(K,L) \geq V(K)^{(n+p)/n} V(L)^{-p/n}, \tag{10}$$

with equality if and only if K and L are dilates. This inequality is an immediate consequence of the Hölder inequality and the integral representation (6).

From the definition of the L_p-mixed volume V_p it follows immediately that for each convex body K,

$$V_p(K,K) = V(K). \tag{11}$$

From the definition of the dual mixed volumes $\widetilde{V}_{-p}$ it follows immediately that for each star body K,

$$\widetilde{V}_{-p}(K,K) = V(K). \tag{12}$$

Note that (11) holds only for convex bodies, while identity (12) holds for all star bodies.

An immediate consequence of the dual mixed volume inequality (10) and identity (12) is that if for star bodies K, L we have

$$\widetilde{V}_{-p}(Q,K)/V(Q) = \widetilde{V}_{-p}(Q,L)/V(Q),$$

for all star bodies Q which belong to some class that contains both K and L, then in fact $K = L$.

Lemma 1. *If K is a star body and L is a convex body in $\mathbb{R}^n$, then*

$$V_p(L, \Gamma_p K) = \frac{\omega_n}{V(K)} \widetilde{V}_{-p}(K, \Pi_p^* L).$$

Proof. From the definition of the L_p-centroid body of K,

$$h^p_{\Gamma_p K}(u) = \frac{1}{c_{n,p} V(K)} \int_K |u \cdot x|^p dx,$$

the integral representation (3), Fubini's theorem, (5), and the integral representation (6), it follows that

$$\begin{aligned} V_p(L,\Gamma_p K) &= \frac{1}{n}\int_{S^{n-1}} h^p_{\Gamma_p K}(u)dS_p(L,u) \\ &= \frac{1}{n}\int_{S^{n-1}} \left(\frac{1}{c_{n,p}V(K)}\int_K |u\cdot x|^p dx\right) dS_p(L,u) \\ &= \frac{1}{n(n+p)c_{n,p}V(K)}\int_{S^{n-1}}\int_{S^{n-1}} |u\cdot v|^p \rho_K^{n+p}(v)dS(v)dS_p(L,u) \\ &= \frac{\omega_n}{nV(K)}\int_{S^{n-1}} \rho_K^{n+p}(v)h^p_{\Pi_p L}(v)dS(v) \\ &= \frac{\omega_n}{V(K)}\widetilde{V}_{-p}(K,\Pi_p^* L). \quad \text{q.e.d.} \end{aligned}$$

For $p = 1$ the identity of Lemma 1 was presented in [20].

From (5) and the transformation rules for support and radial functions we see that for a convex body K and $\phi \in \mathrm{SL}(n)$

$$(\phi K)^* = \phi^{-t}K, \tag{13}$$

where ϕ^{-t} denotes the inverse of the transpose of ϕ.

An immediate consequence of the definition of the L_p-centroid body of K is that for $\phi \in \mathrm{SL}(n)$,

$$\Gamma_p \phi K = \phi \Gamma_p K. \tag{14}$$

Lemma 2. *If K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior, $1 < p < \infty$, and $\phi \in \mathrm{SL}(n)$, then*

$$\Pi_p \phi K = \phi^{-t}\Pi_p K.$$

Proof. From Lemma 1, followed by (7), (14), Lemma 1 again, and (8) we have for each star body Q,

$$\begin{aligned} \omega_n \widetilde{V}_{-p}(Q,\Pi_p^*\phi K)/V(Q) &= V_p(\phi K,\Gamma_p Q) \\ &= V_p(K,\phi^{-1}\Gamma_p Q) \\ &= V_p(K,\Gamma_p\phi^{-1}Q) \\ &= \omega_n \widetilde{V}_{-p}(\phi^{-1}Q,\Pi_p^* K)/V(Q) \\ &= \omega_n \widetilde{V}_{-p}(Q,\phi\Pi_p^* K)/V(Q). \end{aligned}$$

But $\widetilde{V}_{-p}(Q,\Pi_p^*\phi K)/V(Q) = \widetilde{V}_{-p}(Q,\phi\Pi_p^* K)/V(Q)$ for all star bodies Q implies that

$$\Pi_p^*\phi K = \phi\Pi_p^* K,$$

and now (13) yields the desired conclusion. q.e.d.

Lemma 2 for $p = 1$ was established by Petty [28] by using a completely different argument.

In Lemma 1, take $L = \Gamma_p K$, use (11) and get:

Lemma 3. *If K is a star body in $\mathbb{R}^n$, then*

$$V(\Gamma_p K) = \omega_n \widetilde{V}_{-p}(K,\Pi_p^*\Gamma_p K)/V(K).$$

In Lemma 1, take $K = \Pi_p^* L$, use (12) and get:

Lemma 4. *If L is a convex body in $\mathbb{R}^n$, then*

$$V_p(L,\Gamma_p\Pi_p^* L) = \omega_n.$$

For $p = 1$ this identity was obtained in [20].

Recall that the L_p-Petty projection product of the convex body K was defined by

$$\mathrm{p}_p(K) = V(K)^{(n-p)/p} V(\Pi_p^* K)\omega_n^{-n/p},$$

while the L_p-Busemann-Petty ratio of the star body K was defined by

$$\mathrm{b}_p(K) = V(K)/V(\Gamma_p K).$$

Our ultimate goal is to show that both p_p and b_p never exceed 1 and that in fact they will attain the value of 1 only on ellipsoids.

From Lemma 3 and the dual mixed volume inequality (10) we immediately obtain:

Lemma 5. *If K is a star body in $\mathbb{R}^n$ and $1 < p < \infty$, then*

$$\mathrm{p}_p(\Gamma_p K) \geq \mathrm{b}_p(K),$$

with equality if and only if K and $\Pi_p^*\Gamma_p K$ are dilates.

From Lemma 4 and the L_p mixed volume inequality (9) we immediately obtain:

Lemma 6. *If K is a convex body in $\mathbb{R}^n$ and $1 < p < \infty$, then*

$$\mathrm{b}_p(\Pi_p^* K) \geq \mathrm{p}_p(K),$$

with equality if and only if K and $\Gamma_p \Pi_p^* K$ are dilates.

Combine Lemmas 5 and 6 to get:

Lemma 7. *If K is a convex body in $\mathbb{R}^n$ then*

$$\mathrm{p}_p(\Gamma_p \Pi_p^* K) \geq \mathrm{p}_p(K),$$

with equality if and only if K and $\Gamma_p \Pi_p^* K$ are dilates.

Throughout, a convex body will be called *smooth* if its boundary is C^2 with everywhere positive curvature. Thus smooth bodies have curvature bounded away from 0 and ∞. In the literature smooth bodies are often called C_+^2.

Petty [27] proved that all centroid bodies are smooth. The fact that this is also the case for the L_p analogues of centroid bodies for $p > 1$ is much easier to see. Lemma 7 shows that any body at which p_p attains a maximum must be smooth and origin-symmetric. Such class reduction methods were presented in [19].

Our aim is to show that given any maximal body K for p_p and any direction $u \in S^{n-1}$, the midpoints of the chords of K in the direction u are coplanar. This together with the classical Bertrand–Brunn theorem (see e.g., Thompson [34]) will allow us to conclude that K is an ellipsoid. To this end a few preliminary lemmas are needed.

4. Some basic facts and lemmas

First we shall need the following trivial elementary inequality:

Lemma 8. *If $a, b \geq 0$ and $c, d > 0$, then for $p > 1$*

$$(a+b)^p (c+d)^{1-p} \leq a^p c^{1-p} + b^p d^{1-p},$$

with equality if and only if $ad = bc$.

Rewriting the inequality as

$$\lambda \frac{a}{c} + (1-\lambda)\frac{b}{d} \leq \left\{ \lambda \left(\frac{a}{c}\right)^p + (1-\lambda)\left(\frac{b}{d}\right)^p \right\}^{1/p},$$

with $\lambda = c/(c+d)$, shows that this is a direct consequence of the convexity of the function $t \mapsto t^p$.

If K is a convex body and ξ is a subspace of codimension 1, then $\mathrm{S}_\xi K$ will denote the Steiner symmetral of K with respect to ξ. Thus if $K \subset \mathbb{R}^{n-1} \times \mathbb{R}$, then

$$\mathrm{S}_{\mathbb{R}^{n-1}} K = \{(x, \tfrac{1}{2}t + \tfrac{1}{2}s) \in \mathbb{R}^{n-1} \times \mathbb{R} : (x,t) \in K,\ (x,-s) \in K\}.$$

If $K \subset \mathbb{R}^{n-1} \times \mathbb{R}$ is a convex body given by

$$K = \{(x,t) \in \mathbb{R}^{n-1} \times \mathbb{R} : -g(x) \le t \le f(x),\ x \in K_o\},$$

where K_o is the image of the orthogonal projection of K onto $\mathbb{R}^{n-1}$ and $f, g : K_o \to \mathbb{R}$, then

$$\mathrm{S}_{\mathbb{R}^{n-1}} K = \{(x,t) \in \mathbb{R}^{n-1} \times \mathbb{R} : -\tfrac{1}{2}(f(x)+g(x)) \le t \le \tfrac{1}{2}(f(x)+g(x)), x \in K_o\},$$

We will need the following often used fact:

Lemma 9. *Suppose $K, L \subset \mathbb{R}^{n-1} \times \mathbb{R}$ are convex bodies. Then*

$$\mathrm{S}_{\mathbb{R}^{n-1}} K^* \subset L^*$$

if and only if

$$h_K(x,t) = 1 = h_K(x,-s),\ \textit{with } t \ne -s \implies h_L(x, \tfrac{1}{2}t + \tfrac{1}{2}s) \le 1.$$

In addition if $\mathrm{S}_{\mathbb{R}^{n-1}} K^* = L^*$, then $h_K(x,t) = 1 = h_K(x,-s)$, with $t \ne -s$ must imply $h_L(x, \frac{1}{2}t + \frac{1}{2}s) = 1$.

Lemma 9 is an immediate consequence of the definition of Steiner symmetrization, identities (5) and the obvious fact that for each body Q, we have $x \in Q \backslash \partial Q$ if and only if $\rho_Q(x) > 1$.

Lemma 10. *Suppose $K \subset \mathbb{R}^{n-1} \times \mathbb{R}$ is a smooth convex body given by*

$$K = \{(x,t) \in \mathbb{R}^{n-1} \times \mathbb{R} : -g(x) \le t \le f(x),\ x \in K_o\},$$

where K_o is the image of the orthogonal projection of K onto $\mathbb{R}^{n-1}$ and $f, g : K_o \to \mathbb{R}$. If $h : S^{n-1} \to \mathbb{R}$ is a continuous function then

$$\int_{S^{n-1}} \frac{h(u)}{\kappa_K(u)}\, dS(u) = \int_{\operatorname{int} K_o} [h(u_x^+)\sqrt{1+|\nabla f(x)|^2} + h(u_x^-)\sqrt{1+|\nabla g(x)|^2}]\, dx,$$

where $\kappa_K(u)$ is the Gauss curvature of ∂K at the point of ∂K whose outer unit normal is u, while u_x^+ is the outer unit normal to K at $(x, f(x))$ and u_x^- is the outer unit normal to K at $(x, -g(x))$.

The Lemma follows from the formula for the surface area of a graph and the fact that the Gauss curvature is equal to the Jacobian of the Gauss map.

Suppose A is the interior of a convex subset of $\mathbb{R}^{n-1}$ and $f : A \to \mathbb{R}$ is a C^1 function, then $\langle f \rangle \colon A \to \mathbb{R}$ is the function defined by

$$\langle f \rangle(x) = f(x) - x \cdot \nabla f(x),$$

for each $x \in A$. Note that $\langle \cdot \rangle$ is a linear operator; i.e., if $f_1, f_2 : A \to \mathbb{R}$ and $\lambda_1, \lambda_2 \in \mathbb{R}$, then

$$\langle \lambda_1 f_1 + \lambda_2 f_2 \rangle = \lambda_1 \langle f_1 \rangle + \lambda_2 \langle f_2 \rangle .$$

We shall need the fact that the kernel of the operator $\langle \cdot \rangle$ consists only of linear functions; i.e.,

$$\langle f \rangle(x) = 0 \quad \text{for all } x \in A \quad \Longrightarrow \quad f \text{ is linear on } A.$$

Finally, we shall require the trivial observation that if A is origin-symmetric, then

$$f_1(-x) = f_2(x) \quad \text{for all } x \in A \quad \Longrightarrow \quad \langle f_1 \rangle (-x) = \langle f_2 \rangle (x) \quad \text{for all } x \in A. \tag{15}$$

Lemma 11. *Suppose $K \subset \mathbb{R}^{n-1} \times \mathbb{R}$ is a smooth convex body given by*

$$K = \{(x, t) \in \mathbb{R}^{n-1} \times \mathbb{R} \,:\, -g(x) \le t \le f(x),\ x \in K_o\},$$

where K_o is the image of the orthogonal projection of K onto $\mathbb{R}^{n-1}$ and $f, g : K_o \to \mathbb{R}$. Then

$$h_K(-\nabla f(x), 1) = \langle f \rangle(x)$$

and

$$h_K(-\nabla g(x), -1) = \langle g \rangle(x),$$

for all $x \in \operatorname{int} K_o$.

To see this note that for $x \in \operatorname{int} K_o$, the outer unit normal to ∂K at the point $(x, f(x))$ is

$$u_x^+ = \frac{(-\nabla f(x), 1)}{\sqrt{1 + |\nabla f(x)|^2}} \tag{16$^+$}$$

and the outer unit normal to ∂K at the point $(x, -g(x))$ is

$$u_x^- = \frac{(-\nabla g(x), -1)}{\sqrt{1 + |\nabla g(x)|^2}}. \tag{16$^-$}$$

Hence

$$h_K\left(\frac{(-\nabla f(x), 1)}{\sqrt{1 + |\nabla f(x)|^2}}\right) = h_K(u_x^+) = (x, f(x))\cdot u_x^+ = \frac{f(x) - x\cdot\nabla f(x)}{\sqrt{1 + |\nabla f(x)|^2}}$$

and

$$h_K\left(\frac{(-\nabla g(x), -1)}{\sqrt{1 + |\nabla g(x)|^2}}\right) = h_K(u_x^-) = (x, -g(x))\cdot u_x^- = \frac{g(x) - x\cdot\nabla g(x)}{\sqrt{1 + |\nabla g(x)|^2}}.$$

The homogeneity (of degree 1) of h_K now gives the identities of the Lemma.

As an aside, we note that since K contains the origin in its interior, it follows that for $x \in \operatorname{int} K_o$

$$\langle f\rangle(x) = h_K(-\nabla f(x), 1) > 0,$$

and

$$\langle g\rangle(x) = h_K(-\nabla g(x), -1) > 0.$$

5. Steiner symmetrization and the operator Π_p^*

Lemma 12. *Suppose $K \subset \mathbb{R}^{n-1} \times \mathbb{R}$ is a smooth convex body given by*

$$K = \{(x, t) \in \mathbb{R}^{n-1} \times \mathbb{R} : -g(x) \le t \le f(x),\ x \in K_o\},$$

where K_o is the image of the orthogonal projection of K onto $\mathbb{R}^{n-1}$ and $f, g : K_o \to \mathbb{R}$. Then the support function of $\Pi_p K$ at $(y, t) \in \mathbb{R}^{n-1} \times \mathbb{R}$ is given by

$$n\omega_n c_{n-2,p}\, h^p_{\Pi_p K}(y, t) = \int_{\operatorname{int} K_o} \left[|t - y\cdot\nabla f(x)|^p \langle f\rangle(x)^{1-p} + |t + y\cdot\nabla g(x)|^p \langle g\rangle(x)^{1-p}\right] dx.$$

Proof. It was shown in [21] that the L_p-surface area measure $S_p(K,\cdot)$ is absolutely continuous with respect to the classical surface area measure S_K and that the Radon-Nikodym derivative

$$\frac{dS_p(K,\ \cdot\ )}{dS_K} = h_K^{1-p}.$$

Since K is smooth, the measure S_K is absolutely continuous with respect to spherical Lebesgue measure S and the Radon-Nikodym derivative

$$\frac{dS_K}{dS} = 1/\kappa_K,$$

where $\kappa_K : S^{n-1} \to (0,\infty)$ is the Gauss curvature of ∂K viewed as a function of the outer normals (i.e., $\kappa_K(u)$, for $u \in S^{n-1}$, is the Gauss curvature at the point of ∂K whose outer unit normal is u).

These observations together with the definition of $\Pi_p K$ show that for $(y,t) \in \mathbb{R}^{n-1} \times \mathbb{R}$

$$h^p_{\Pi_p K}(y,t) = \frac{1}{n\omega_n c_{n-2,p}} \int_{S^{n-1}} |(y,t)\cdot u|^p \frac{h_K^{1-p}(u)}{\kappa_K(u)}\, dS(u).$$

Now if $h : \mathbb{R}^n \to \mathbb{R}$ is any continuous function that is homogeneous of degree 1, then from (16^+), (16^-), and Lemma 10 it follows that

$$\int_{S^{n-1}} \frac{h(u)}{\kappa_K(u)}\, dS(u) = \int_{\operatorname{int} K_o} [h(-\nabla f(x),1) + h(-\nabla g(x),-1)]\, dx.$$

The desired result now follows from Lemma 11.

If in addition, K is also origin-symmetric, then $g(-x) = f(x)$ for all $x \in \operatorname{int} K_o$. Now (15) shows that in this case Lemma 12 becomes:

Lemma 13. *Suppose $K \subset \mathbb{R}^{n-1} \times \mathbb{R}$ is a smooth origin-symmetric convex body given by*

$$K = \{(x,t) \in \mathbb{R}^{n-1} \times \mathbb{R} : -g(x) \le t \le f(x), x \in K_o\},$$

where K_o is the image of the orthogonal projection of K onto $\mathbb{R}^{n-1}$ and $f, g : K_o \to \mathbb{R}$. Then the support function of $\Pi_p K$ at $(y,t) \in \mathbb{R}^{n-1} \times \mathbb{R}$ is given by

$$h^p_{\Pi_p K}(y,t) = \frac{2}{n\omega_n c_{n-2,p}} \int_{\operatorname{int} K_o} |t - y\cdot\nabla f(x)|^p \, \langle f\rangle(x)^{1-p}\, dx,$$

and

$$h^p_{\Pi_p K}(y,t) = \frac{2}{n\omega_n c_{n-2,p}} \int_{\operatorname{int} K_o} |t + y\cdot\nabla g(x)|^p \, \langle g\rangle(x)^{1-p}\, dx.$$

Lemma 14. *Suppose $1 < p < \infty$ and K is a smooth origin-symmetric convex body in $\mathbb{R}^n$. If ξ is a subspace of codimension 1, then*

$$\mathrm{S}_\xi\, \Pi_p^* K \subset \Pi_p^*\, \mathrm{S}_\xi\, K,$$

with equality if and only if the chords of K orthogonal to ξ have midpoints that are coplanar.

Proof. Without loss of generality assume $\xi = \mathbb{R}^{n-1}$ and that $K \subset \mathbb{R}^{n-1} \times \mathbb{R}$ is given by

$$K = \{(x,t) \in \mathbb{R}^{n-1} \times \mathbb{R} : -g(x) \le t \le f(x),\ x \in K_o\},$$

where K_o is the image of the orthogonal projection of K onto $\mathbb{R}^{n-1}$ and $f, g : K_o \to \mathbb{R}$, while

$$\mathrm{S}_{\mathbb{R}^{n-1}} K = \\ \{(x,t) \in \mathbb{R}^{n-1} \times \mathbb{R} : -\tfrac{1}{2}(f(x)+g(x)) \le t \le \tfrac{1}{2}(f(x)+g(x)), x \in K_o\}.$$

Now suppose

$$h_{\Pi_p K}(y,t) = 1 = h_{\Pi_p K}(y,-s), \text{ with } t \ne -s.$$

Since K is smooth and origin-symmetric, obviously so is $\mathrm{S}_{\mathbb{R}^{n-1}} K$. Now Lemma 13, the triangle inequality, Lemma 8, and Lemma 13 again, give

$$\begin{aligned}
&h^p_{\Pi_p \mathrm{S}_\xi K}(y, \tfrac{1}{2}t + \tfrac{1}{2}s) \\
&= \frac{2}{n\omega_n c_{n-2,p}} \int_{\operatorname{int} K_o} \left|(\tfrac{1}{2}t + \tfrac{1}{2}s) - y\cdot\nabla(\tfrac{1}{2}f + \tfrac{1}{2}g)(x)\right|^p \left\langle \tfrac{1}{2}f + \tfrac{1}{2}g \right\rangle (x)^{1-p} dx \\
&\le \frac{1}{n\omega_n c_{n-2,p}} \int_{\operatorname{int} K_o} (|t - y\cdot\nabla f(x)| + |s - y\cdot\nabla g(x)|)^p (\langle f\rangle(x) + \langle g\rangle(x))^{1-p} dx \\
&\le \frac{1}{n\omega_n c_{n-2,p}} \int_{\operatorname{int} K_o} [|t - y\cdot\nabla f(x)|^p \, \langle f\rangle(x)^{1-p} + |s - y\cdot\nabla g(x)|^p \, \langle g\rangle(x)^{1-p}] dx \\
&= \tfrac{1}{2} h^p_{\Pi_p K}(y,t) + \tfrac{1}{2} h^p_{\Pi_p K}(y,-s) \\
&= 1.
\end{aligned}$$

Thus $h_{\Pi_p \mathrm{S}_\xi K}(y, \frac{1}{2}t + \frac{1}{2}s) \leq 1$ which by Lemma 9 yields the desired inclusion.

If $\mathrm{S}_\xi \Pi_p^* K = \Pi_p^* \mathrm{S}_\xi K$ then by Lemma 9, we have

$$h^p_{\Pi_p^* \mathrm{S}_\xi K}(y, \tfrac{1}{2}t + \tfrac{1}{2}s) = 1$$

which would force equality in the inequalities above. The equality conditions of Lemma 8 now forces

$$|t - y\cdot\nabla f(x)|\, \langle g\rangle(x) = |s - y\cdot\nabla g(x)|\, \langle f\rangle(x)$$

for all $x \in \operatorname{int} K_o$. Choose $y = 0$ and (since s, t are such that $(0, t), (0, -s) \in \partial\Pi_p^* K$) we must have $s = t$ and thus $\langle g\rangle(x) = \langle f\rangle(x)$ for all $x \in \operatorname{int} K_o$. But

$$\langle f - g\rangle = 0$$

implies that $f - g$ is linear and hence that the chords of K orthogonal to $\mathbb{R}^{n-1}$ have coplanar midpoints.

The fact that the coplanarity of the midpoints of the chords of K that are orthogonal to ξ forces $\mathrm{S}_\xi \Pi_p^* K = \Pi_p^* \mathrm{S}_\xi K$ is left to the reader (and will not be used in this article). q.e.d.

Note that an obvious limit argument shows the inclusion of Lemma 14 for $p > 1$ and K origin-symmetric and smooth immediately yields exactly the same inclusion for all $p \geq 1$ and all arbitrary origin-symmetric bodies K. However, the critical equality conditions are lost in the limit process.

6. Proofs of the theorems

Since the volume of convex bodies is obviously unaffected by Steiner symmetrization, Lemma 14 and the definition of p_p immediately yield:

Lemma 15. *If K is a smooth origin-symmetric convex body and ξ is a codimension* 1 *subspace, then*

$$\mathrm{p}_p(K) \leq \mathrm{p}_p(\mathrm{S}_\xi K),$$

with equality if and only if the chords of K orthogonal to ξ have coplanar midpoints.

Now a body at which p_p attains a maximum must be (by our class reduction arguments) both origin-symmetric and smooth. But

the body's maximality together with Lemma 15 shows that any parallel set of chords of the body must have coplanar midpoints. The classical Bertrand-Brunn theorem now allows us to conclude that this maximal body can only be an ellipsoid. This proves Theorem 2.

Lemma 5 shows that Theorem 2 immediately gives Theorem 1.

7. Open problems

Question. Suppose $1 \leq p < \infty$ and K is a convex body in $\mathbb{R}^n$. If ξ is a subspace of $\mathbb{R}^n$ of codimension 1, then is it the case that

$$\mathrm{S}_\xi \, \Pi_p^* K \subset \Pi_p^* \, \mathrm{S}_\xi \, K,$$

with equality if and only if the chords of K orthogonal to ξ have midpoints that are coplanar?

A positive answer to this question would immediately and directly yield a proof of Theorem 2 (and thus Theorem 1 as well).

Conjecture. *If K is a convex body such that $\Gamma_p \Pi_p^* K$ is a dilate of K, then K must be an ellipsoid.*

A proof of this together with Lemma 7 would immediately prove Theorem 2 (and thus Theorem 1 as well). Obviously, the conjecture is true for $p = 2$.

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