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HAUSDORFF DIMENSION AND CONFORMAL DYNAMICS I: STRONG CONVERGENCE OF KLEINIAN GROUPS

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Abstract

This paper investigates the behavior of the Hausdorff dimensions of the limit sets Λ_n and Λ of a sequence of Kleinian groups $\Gamma_n \to \Gamma$, where $M = \mathbb{H}^3/\Gamma$ is geometrically finite. We show if $\Gamma_n \to \Gamma$ strongly, then:

(a) $M_n = \mathbb{H}^3/\Gamma_n$ is geometrically finite for all $n \gg 0$,

(b) $\Lambda_n \to \Lambda$ in the Hausdorff topology, and

(c) $\mathrm{H.dim}(\Lambda_n) \to \mathrm{H.dim}(\Lambda)$, if $\mathrm{H.dim}(\Lambda) \geq 1$.

On the other hand, we give examples showing the dimension can vary *discontinuously* under strong limits when $\mathrm{H.dim}(\Lambda) < 1$. Continuity can be recovered by requiring that accidental parabolics converge radially.
Similar results hold for higher-dimensional manifolds. Applications are given to quasifuchsian groups and their limits.

1. Introduction

To any complete hyperbolic manifold M one may associate a *conformal dynamical system*, by considering the action of $\Gamma = \pi_1(M)$ on the sphere at infinity for the universal cover, $S^d_\infty = \partial\mathbb{H}^{d+1}$. For 3-manifolds one obtains in this way the classical Kleinian groups acting on the Riemann sphere $\widehat{\mathbb{C}}$.

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A fundamental invariant of M and Γ is the Hausdorff dimension of the *limit set* $\Lambda \subset S^d_\infty$, the set of accumulation points of any orbit $\Gamma x \subset \mathbb{H}^{d+1}$. When Γ is geometrically finite, $D = \mathrm{H.dim}(\Lambda)$ coincides with several other invariants of Γ, and is related to the bottom of the spectrum of the Laplacian on M by

$$\lambda_0(M) = D(d-D),$$

when $D \geq d/2$.

Moreover the limit set of a geometrically finite group supports a canonical *conformal density* of dimension D. That is, there is unique probability measure μ on Λ transforming by $|\gamma'|^D$ under the action of Γ. In the absence of cusps, μ is simply the normalized D-dimensional Hausdorff measure on Λ.

In this paper we study the behavior of $\mathrm{H.dim}(\Lambda)$ for a sequence of Kleinian groups. Recall that $\Gamma_n \to \Gamma$ *geometrically* if the groups converge in the Hausdorff topology on closed subsets of $\mathrm{Isom}(\mathbb{H}^{d+1})$. We say $\Gamma_n \to \Gamma$ *strongly* if, in addition, there are surjective homomorphisms

$$\chi_n : \Gamma \to \Gamma_n$$

converging pointwise to the identity. Equivalently, χ_n tends to the inclusion $\Gamma \subset \mathrm{Isom}(\mathbb{H}^{d+1})$.

It has been conjectured that the Hausdorff dimension of the limit set varies continuously under strong limits (see e.g. [33, Conj. 5.6]). We will show this conjecture is *false* in general, and at the same time give positive theorems to guarantee continuity.

In the 3-dimensional case we obtain the following results.

Theorem 1.1. *Let $\Gamma_n \to \Gamma$ strongly, where $M = \mathbb{H}^3/\Gamma$ is geometrically finite. Then Γ_n is geometrically finite for all $n \gg 0$, and the limits sets satisfy $\Lambda_n \to \Lambda$ in the Hausdorff topology.*

Theorem 1.2. *If, in addition,* $\mathrm{H.dim}(\Lambda) \geq 1$, *then*

$$\mathrm{H.dim}(\Lambda_n) \to \mathrm{H.dim}(\Lambda);$$

and if $\mathrm{H.dim}(\Lambda) > 1$ *then the canonical densities satisfy $\mu_n \to \mu$ in the weak topology on measures.*

On the other hand we find new phenomena when $\mathrm{H.dim}(\Lambda) < 1$:

Theorem 1.3. *For any ϵ with $0 < \epsilon < 1/2$, there exist geometrically finite groups Γ and Γ_n, such that $\Gamma_n \to \Gamma$ strongly but*

$$\mathrm{H.dim}(\Lambda_n) \to 1 > \mathrm{H.dim}(\Lambda) = 1/2 + \epsilon.$$

To recover continuity of dimension in general, one must sharpen the notion of convergence. Recall that $\Gamma_n \to \Gamma$ *algebraically* if there are *isomorphisms* $\chi_n : \Gamma \to \Gamma_n$ converging to the identity. A parabolic element $g \in \Gamma$ is an *accidental parabolic* if $\chi_n(g)$ is hyperbolic for infinitely many n. Then the complex length $L_n + i\theta_n$ of $\chi_n(g)$ tends to zero; it does so *radially* if $\theta_n = O(L_n)$, and *horocyclically* if $\theta_n^2/L_n \to 0$.

Theorem 1.4. *Let $M = \mathbb{H}^3/\Gamma$ be geometrically finite, and suppose $\Gamma_n \to \Gamma$ algebraically. Then:*

(a) $\Gamma_n \to \Gamma$ *strongly* $\iff$ *all accidental parabolics converge horocyclically; and*

(b) $\mathrm{H.dim}(\Lambda_n) \to \mathrm{H.dim}(\Lambda)$ *and* $\mu_n \to \mu$ *if all accidental parabolics converge radially.*

The examples of Theorem 1.3 reside in the gap between horocyclic and radial convergence.

Higher dimensional manifolds. Theorems 1.1 and 1.2 generalize to Kleinian groups acting on S^d_∞, $d \geq 3$, if we replace $\mathrm{H.dim}(\Lambda) \geq 1$ by the condition

$$D = \mathrm{H.dim}(\Lambda) \geq (d-1)/2.$$

Since $\lambda_0(M)$ is sensitive to the dimension of the limit set only when $D > d/2$, we find:

Theorem 1.5. *If M is a geometrically finite manifold of any dimension and $M_n \to M$ strongly, then $\lambda_0(M_n) \to \lambda_0(M)$.*

Tame 3-manifolds. A hyperbolic manifold is *topologically tame* if it is homeomorphic to the interior of a compact manifold. If $M = \mathbb{H}^3/\Gamma$ is geometrically infinite but topologically tame, then $\mathrm{H.dim}(\Lambda) = 2$, and it is easy to see $\mathrm{H.dim}(\Lambda_n) \to \mathrm{H.dim}(\Lambda)$ whenever $\Gamma_n \to \Gamma$ geometrically, algebraically or strongly (§7).

On the other hand, the limit set of such a manifold generally carries many different conformal densities μ of dimension 2 (see §3), so the discussion of convergence of these measures must be reserved for the geometrically finite case.

Quasifuchsian groups. As applications of the results above, one can study a sequence of quasifuchsian manifolds $M_n = Q(X_n, Y)$ in Bers' model for the Teichmüller space of a surface S. Here are four examples, treated in detail in §9.

1. If $X_n = \tau^n(X_0)$, where τ is Dehn twist, then M_n has distinct algebraic and geometric limits M_A and M_G as $n \to \infty$. We find $M_n \to M_G$ strongly and the limit sets satisfy
$$\mathrm{H.dim}(\Lambda_n) \to \mathrm{H.dim}(\Lambda_G) > \mathrm{H.dim}(\Lambda_A).$$
2. If $X_n = \phi^n(X_0)$, where ϕ is pseudo-Anosov on S or any subsurface of S, then all geometric limits are geometrically infinite and
$$\mathrm{H.dim}(\Lambda_n) \to 2.$$
3. If X_n is obtained by pinching a system of disjoint simple closed curves on X_0 (in Fenchel-Nielsen coordinates), then M_n tends strongly to a geometrically finite manifold M and
$$\mathrm{H.dim}(\Lambda_n) \to \mathrm{H.dim}(\Lambda) < 2.$$
4. Finally consider the manifolds $M_t = Q(\tau^t(X), Y)$, $t \in \mathbb{R}$, obtained by performing a Fenchel-Nielsen twist of length t about a simple geodesic C on X. In this case we show there is a continuous function $\delta(t)$, periodic under $t \mapsto t + \ell_C(X)$, such that
$$\lim_{t\to\infty} |\,\mathrm{H.dim}(\Lambda_t) - \delta(t)| = 0.$$
It seems likely that $\delta(t)$ is nonconstant, and thus $\mathrm{H.dim}(\Lambda_t)$ oscillates along this twist path to infinity in Teichmüller space.

This last example was inspired by a discovery of Douady, Sentenac and Zinsmeister in the dynamics of quadratic polynomials. These authors show the Hausdorff dimension of the Julia set $J(z^2 + c)$ is also asymptotically periodic, and probably oscillatory, as $c \searrow 1/4$ along the real axis [17].

Plan of the paper. In §2 and §3 we consolidate known material on the dimension of the limit set of a Kleinian groups. Some short proofs are included for ease of reference. The main results on continuity of dimension and λ_0 are obtained in §7. Examples of discontinuity, including Theorem 1.3, are given in §8.

The general argument to establish continuity of dimension is to take a weak limit ν of the canonical densities μ_n, and show $\nu = \mu$. It turns out that $\mu \neq \nu$ only if ν is an atomic measure supported on cusps points in the limit set. To rule this out, we explicitly control the concentration of μ_n near incipient cusps. Part of the control comes from convergence of the convex core (§4), and part from estimates for the Poincaré series (§6). The theory of accidental parabolics is developed along the way (§5), and the applications to quasifuchsian groups are given in §9.

Notes and references. The continuity of $\mathrm{H.dim}(\Lambda)$ was also studied independently and contemporaneously by Canary and Taylor. Using spectral methods, Canary and Taylor show that if $\Gamma_n \to \Gamma$ strongly and $M^3 = \mathbb{H}^3/\Gamma$ is *not* a handlebody, then $\mathrm{H.dim}(\Lambda_n) \to \mathrm{H.dim}(\Lambda)$ [15]. The condition that M^3 is not a handlebody guarantees $\mathrm{H.dim}(\Lambda) \geq 1$, so when M^3 is geometrically finite their result is also covered by Theorem 1.2. On the other hand, the theorems of this paper do provide continuity of $\mathrm{H.dim}(\Lambda)$ for geometrically finite handlebodies, so long as a condition such as $\mathrm{H.dim}(\Lambda) > 1$ or radial convergence $\Gamma_n \to \Gamma$ is assumed. Note that our counterexamples to continuity of dimension come from geometrically finite handlebodies with cusps (§8).

A version of Theorem 1.1 (without continuity of Λ) was proved by Taylor [33]. See Anderson and Canary for related results on cores and limits [2], [3].

This paper belongs to a three-part series [25], [24]. Part II gives parallel results in the setting of iterated rational maps. The theory of conformal densities is available for both rational maps and Kleinian groups; it is in anticipation of the applications in Part II that we work with conformal densities here, rather than with eigenfunctions of the Laplacian.

Part III presents explicit dimension calculations for families of conformal dynamical systems.

For background on hyperbolic manifolds, see the texts [34], [37], [6] and [28].

Notation. $A \asymp B$ means $A/C < B < CB$ for some implicit constant C; $n \gg 0$ means for all n sufficiently large.

2. The basic invariants

Let $\mathbb{H}^{d+1}$ be the hyperbolic space of constant curvature -1, and let $S^d_\infty = \mathbb{R}^d_\infty \cup \{\infty\}$ denote its sphere at infinity. Let $M = \mathbb{H}^{d+1}/\Gamma$ be a complete hyperbolic manifold. In this section we recall the relation between:

- $\lambda_0(M)$, the bottom of the spectrum of the Laplacian;
- $\delta(\Gamma)$, the critical exponent of the Poincaré series for Γ;
- $\alpha(\Gamma)$, the minimum dimension of a Γ-invariant density; and
- $\mathrm{H.dim}(\Lambda_{\mathrm{rad}})$, the Hausdorff dimension of the radial limit set.

Groups and limit sets. A *Kleinian group* is a discrete subgroup $\Gamma \subset \mathrm{Isom}(\mathbb{H}^{d+1})$. Every complete manifold of constant curvature -1 can be presented as a quotient $M = \mathbb{H}^{d+1}/\Gamma$ where Γ is a Kleinian group.

For simplicity we will generally assume that Γ is torsion-free and *nonelementary*, i.e. any subgroup of finite index is nonabelian.

The *limit set* Λ of Γ is the subset of S^d_∞ defined for any $x \in \mathbb{H}^{d+1}$ by

$$\Lambda = \overline{\Gamma x} \cap S^d_\infty.$$

It complement $\Omega = S^d_\infty - \Lambda$ is the *domain of discontinuity.*

The *radial limit set* $\Lambda_{\mathrm{rad}} \subset \Lambda$ consists of those $y \in S^d_\infty$ such that there is a sequence $\gamma_n x \to y$ which remains within a bounded distance of a geodesic landing at y. Equivalently, $y \in \Lambda_{\mathrm{rad}}$ iff y corresponds to a recurrent geodesic on M.

Cusps. An element $\gamma \in \Gamma$ is *parabolic* if it has a unique fixed-point $c \in S^d_\infty$.

We say $c \in S^d_\infty$ is a *cusp point*, and its stabilizer $L \subset \Gamma$ is a *parabolic subgroup*, if L contains a parabolic element. Then L contains a subgroup of finite index $L_0 \cong \mathbb{Z}^r$, $r > 0$; and we say c and L belong to a *cusp of rank* r (compare [37, §4]). All cusp points belong to the limit set, and all elements of $L - \{\mathrm{id}\}$ are parabolic.

Invariants. The *bottom of the spectrum of the Laplacian* is defined by

$$\begin{aligned}
(2.1) \quad \lambda_0(M) \;&=\; \inf\left\{ \frac{\int_M |\nabla f|^2}{\int_M |f|^2} \;:\; f \in C_0^\infty(M) \right\} \\
(2.2) \quad &=\; \sup\{\lambda \geq 0 \;:\; \exists\, f > 0 \text{ on } M \text{ with } \Delta f = \lambda f\}.
\end{aligned}$$

Here Δ denotes the positive Laplacian; for example $\lambda_0(\mathbb{H}^{d+1}) = d^2/4$. The equivalence of the two definitions above (on any Riemannian manifold) is shown in [16].

The *Poincaré series* is defined for $x \in \mathbb{H}^{d+1} \cup \Omega$ by

$$P_s(\Gamma, x) = \begin{cases} \sum_{\gamma\in\Gamma} e^{-sd(x,\gamma x)} & \text{if } x \in \mathbb{H}^{d+1}, \\ \sum_{\gamma\in\Gamma} |\gamma'(x)|^s & \text{if } x \in \Omega. \end{cases}$$

Here and below $|\gamma'|$ is measured in the spherical metric. The *critical exponent* is given by

$$\delta(\Gamma) = \inf\{s \geq 0 \ : \ P_s(\Gamma, x) < \infty\};$$

it is independent of the choice of x.

A Γ-*invariant conformal density of dimension* α is a positive measure μ on S^d_∞ such that

$$\mu(\gamma E) = \int_E |\gamma'|^\alpha \, d\mu \tag{2.3}$$

for every Borel set E and $\gamma \in \Gamma$. A density is *normalized* if $\mu(S^d_\infty) = 1$.

From a more functorial point of view, a conformal density of dimension α is a map

$$\mu : (\text{conformal metrics } \rho(z)|dz| \text{ on } S^d_\infty) \to (\text{measures on } S^d_\infty)$$

such that

$$\frac{d\mu(\rho_1)}{d\mu(\rho_2)} = \left(\frac{\rho_1}{\rho_2}\right)^\alpha.$$

Conformal maps act on densities in a natural way, and (2.3) says $\gamma_*(\mu) = \mu$. We implicitly identify μ with the measure $\mu(\sigma)$ where

$$\sigma = 2|dx|/(1 + |x|^2)$$

is the spherical metric.

The *critical dimension* of Γ is given by

$$\alpha(\Gamma) = \inf\{\alpha \geq 0 \ : \ \exists \text{ a } \Gamma\text{-invariant density of dimension } \alpha\}. \tag{2.4}$$

In (2.2) and (2.4) it is easy to see that the inf and sup are achieved: there exist a positive eigenfunction with eigenvalue $\lambda_0(M)$, and a Γ-invariant density of dimension $\alpha(\Gamma)$. In particular we have $\alpha(\Gamma) > 0$, because Λ has no Γ-invariant measure.

The *Hausdorff dimension* of the radial limit set, denoted $\mathrm{H.dim}(\Lambda_{\mathrm{rad}})$, is the infimum of those $\delta > 0$ such that Λ_{rad} admits coverings $\langle B_i \rangle$ with $\sum(\operatorname{diam} B_i)^\delta \to 0$.

Combining the results of Sullivan and Bishop-Jones, namely [29, Cor. 4], [32, Thm. 2.17] and [8, Thm 1.1] (generalized to arbitrary d), we may now state:

Theorem 2.1. *Any nonelementary complete hyperbolic manifold $M = \mathbb{H}^{d+1}/\Gamma$ satisfies*

$$\mathrm{H.dim}(\Lambda_{\mathrm{rad}}) = \delta(\Gamma) = \alpha(\Gamma)$$

and

$$\lambda_0(M) = \begin{cases} d^2/4 & \text{if } \delta(\Gamma) \le d/2 , \\ \delta(\Gamma)(d - \delta(\Gamma)) & \text{if } \delta(\Gamma) \ge d/2 . \end{cases}$$

For later use we record:

Corollary 2.2. *If Γ has a cusp of rank r, then $\delta(\Gamma) > r/2$.*

Proof. Let $c \in S^d_\infty$ be a cusp point whose stabilizer contains a subgroup $L \cong \mathbb{Z}^r$. Change coordinates so $c = \infty$ in $S^d_\infty = \mathbb{R}^d_\infty \cup \{\infty\}$, and so L acts by translations on $\mathbb{R}^r \subset \mathbb{R}^d_\infty$. (In general L can rotate the planes $\mathbb{R}^{d-r}$ orthogonal to $\mathbb{R}^r$.) Then $g \mapsto g(0)$ embeds L as a discrete lattice in $\mathbb{R}^r$.

Choose a ball $B = B(0,s) \subset \mathbb{R}^d_\infty$ with $\mu(B) > 0$, where μ is an invariant density of dimension $\delta = \delta(\Gamma)$. Then we have

$$\sum_{g \in L} \mu(gB) \asymp \sum |g'(0)|^\delta_\sigma < \infty,$$

where $|g'|_\sigma$ is measured in the spherical metric $\sigma = 2|dx|/(1+|x|^2)$. Since

$$\sum_L |g'(0)|^\delta_\sigma = \sum_L (1 + |g(0)|^2)^{-\delta} \asymp \int_{\mathbb{R}^r} (1+|x|^2)^{-\delta}\, dx,$$

the above integral converges, and thus $\delta > r/2$. $\square$

Here is another useful criterion for the critical exponent [29, Cor. 20]:

Theorem 2.3. *If μ is a Γ-invariant density of dimension α, and μ gives positive measure to the radial limit set, then $\alpha = \delta(\Gamma)$.*

We also note:

Theorem 2.4. *If the Poincaré series diverges at the critical exponent, then any invariant density μ of dimension $\delta(\Gamma)$ is supported on the limit set.*

Proof. Let $F \subset \Omega(\Gamma)$ be a fundamental domain for the action of Γ. Then

$$\sum_{\Gamma} \mu(gF) = \sum_{\Gamma} \int_F |g'(x)|_\sigma^\delta \, d\mu(x) = \int_F P_\delta(x)\, d\mu(x) \le \mu(S^d_\infty) < \infty;$$

since $P_\delta(\Gamma, x) = \infty$ for $x \in \Omega(\Gamma)$, we have $\mu(F) = 0$. $\square$

3. Geometrically finite groups

To obtain more precise results, it is useful to impose geometric conditions on the hyperbolic manifold $M = \mathbb{H}^{d+1}/\Gamma$.

The *convex core* $K(M)$ is the quotient by Γ of the smallest convex set in $\mathbb{H}^{d+1}$ containing all geodesics with both endpoints in the limit set. The manifold M and the group Γ are said to be *geometrically finite* if the convex core meets the Margulis thick part of M in a compact set. For such a manifold, $\Lambda - \Lambda_{\mathrm{rad}}$ is equal to the countable set of *cusp points* (fixed-points of parabolic elements of Γ). If Γ is geometrically finite without cusps, then $K(M)$ is compact, $\Lambda = \Lambda_{\mathrm{rad}}$ and we say Γ is *convex cocompact.*

By [31] and Theorem 2.4 we have:

Theorem 3.1. *If $M = \mathbb{H}^{d+1}/\Gamma$ is geometrically finite, then*

$$\mathrm{H.dim}(\Lambda_{\mathrm{rad}}) = \mathrm{H.dim}(\Lambda).$$

Moreover, the sphere carries a unique Γ-invariant density μ of dimension $\delta(\Gamma)$ and total mass one; μ is nonatomic and supported on Λ; and the Poincaré series diverges at the critical exponent.

Corollary 3.2. *If the limit set of a geometrically finite group is not the whole sphere, then* $\mathrm{H.dim}(\Lambda) < d$.

Proof. Otherwise Lebesgue measure on the sphere would be a second invariant density of dimension $\delta(\Gamma) = d$. $\square$

Corollary 3.3. *Any normalized Γ-invariant density supported on the limit set of a geometrically finite group Γ is either:*

- *the canonical density of dimension* $\delta(\Gamma)$, *or*
- *an atomic measure supported on the cusp points in* Λ *and of dimension* $\alpha > \delta(\Gamma)$.

Proof. If the dimension α of μ is more than $\delta(\Gamma)$, then $\mu(\Lambda_{\mathrm{rad}}) = 0$ by Theorem 2.3, so μ is supported on the countable set of cusps. □

If Γ has cusps then these atomic measures actually exist for any $\alpha > \delta(\Gamma)$ [32, Thm 2.19].

Corollary 3.4. *The limit set of a convex cocompact group supports a unique normalized* Γ*-invariant density.*

Recall $M = \mathbb{H}^{d+1}/\Gamma$ is topologically tame if it is homeomorphic to the interior of a compact $(d+1)$-manifold.

Theorem 3.5. *If* $M = \mathbb{H}^3/\Gamma$ *is geometrically infinite but topologically tame, then*

$$\mathrm{H.\,dim}(\Lambda_{\mathrm{rad}}) = \mathrm{H.\,dim}(\Lambda) = 2.$$

Proof. In [13] it is shown that $\lambda_0(M) = 0$ when M is geometrically infinite but topologically tame. Since $\lambda_0 = \delta(2-\delta)$ and $\delta > 0$, the result follows from Theorem 2.1. □

Non-uniqueness of μ for topologically tame manifolds. When $M^3 = \mathbb{H}^3/\Gamma$ is topologically tame but geometrically infinite, there may be more than one normalized invariant density μ in the critical dimension $\delta(\Gamma) = 2$.

To sketch an example, we start with any compact, acylindrical, atoroidal 3-manifold N such that ∂N has 3 components. By Thurston's hyperbolization theorem, N admits convex hyperbolic structures $M(X, Y, Z)$ parameterized by conformal structures (X, Y, Z) on the pieces of ∂N. (See, e.g. [26, §5].)

Let ϕ and ψ be pseudo-Anosov mapping classes, and let M be any algebraic limit of $M_n = M(\phi^n(X), \psi^n(Y), Z)$ (such limits exist by compactness of $AH(N)$ [35]). Then $M = \mathbb{H}^3/\Gamma$ has two geometrically infinite and asymptotically period ends E_1 and E_2, as well as a geometrically finite end corresponding to Z. (Compare [23, §3].)

Let $g_n : M \to (0, \infty]$ be the Green's with a pole at $p_n \to \infty$ in E_1, scaled so $g_n(*) = 1$ at a fixed basepoint. By Harnack's principle there is a convergent subsequence, with limit a positive harmonic function h_1. The function h_1 tends to infinity in the end E_1, is bounded in E_2 and tends to zero at Z (compare [30]). There is a similar positive harmonic function h_2 tending to infinity in E_2 and bounded in E_1. Then h_1 and h_2 are linearly independent, and at infinity they determine mutually singular invariant densities μ_1 and μ_2 of dimension two.

It seems likely that for topologically tame hyperbolic 3-manifolds, the space of invariant densities is finite-dimensional and its dimension is controlled by the number of ends of M.

Notes and references.

1. The unique density guaranteed in Theorem 3.1 can be related to the Hausdorff measure or packing measure of the limit set in dimension $\delta(\Gamma)$ [31].

2. Some papers we cite define geometric finiteness in terms of a convex fundamental polyhedron. This definition agrees with ours when $\dim(M) \leq 3$, and the results we quote remain valid with the present definition. See [10] for a discussion of the definition of geometric finiteness.

3. It is conjectured that $M = \mathbb{H}^3/\Gamma$ is topologically tame whenever Γ is finitely generated, and tameness is known in many cases [9], [14]. So Theorem 3.5 suggests:

Conjecture 3.6. *For any finitely generated Kleinian group* $\Gamma \subset \mathrm{Isom}(\mathbb{H}^3)$, *we have* $\mathrm{H.dim}(\Lambda_{\mathrm{rad}}) = \mathrm{H.dim}(\Lambda)$.

Any counterexample to this conjecture must have a limit set of positive area [8, Thm 1.7].

1. See also [5], [27], and [38] for results treated in this section.

4. Cores and geometric limits

In this section we introduce the algebraic, geometric and strong topologies on the space of all Kleinian groups. The main result is the following criterion for the limit set and the truncated convex core to move continuously.

Theorem 4.1 (Convergence of cores). *Suppose* $\Gamma_n \to \Gamma$ *strongly, where* Γ *is geometrically finite. Then:*

1. *The manifold* $M_n = \mathbb{H}^{d+1}/\Gamma_n$ *is geometrically finite for all* $n \gg 0$,

2. *The limit sets satisfy* $\Lambda_n \to \Lambda$ *in the Hausdorff topology, and*

3. *The truncated convex cores of the quotient manifolds satisfy* $K_\epsilon(M_n) \to K_\epsilon(M)$ *strongly, for all* $\epsilon > 0$.

Here is a criterion to promote algebraic convergence to strong convergence:

Theorem 4.2. *Let* $\Gamma_n \to \Gamma_A$ *algebraically, where* Γ_A *is geometrically finite. Then* $\Gamma_n \to \Gamma_A$ *strongly if and only if* $L_n \to L_A$ *geometrically for each maximal parabolic subgroup* $L_A \subset \Gamma_A$.

In the statement above, $L_n = \chi_n(L_A)$ are the subgroups of Γ_n corresponding algebraically to L_A.

Corollary 4.3. *If* Γ_A *is convex cocompact, then algebraic convergence implies strong convergence.*

Algebraic and geometric limits. Let $\Gamma_n \subset \mathrm{Isom}(\mathbb{H}^{d+1})$ be a sequence of Kleinian groups. There are several possible notions of convergence of the sequence Γ_n. We say $\Gamma_n \to \Gamma_G$ *geometrically* if we have convergence in the Hausdorff topology on closed subsets of $\mathrm{Isom}(\mathbb{H}^{d+1})$. We say $\Gamma_n \to \Gamma_A$ *algebraically* if there exist isomorphisms $\chi_n : \Gamma_A \to \Gamma_n$ such that $\chi_n(g) \to g$ for each $g \in \Gamma_A$.

A sequence Γ_n has at most one geometric limit Γ_G, but it may have many algebraic limits Γ_A (coming from different 'markings' of Γ_n). If the geometric limit exists, then it contains all the algebraic limits.

Here is a description of geometric convergence from the point of view of quotient manifolds. By choosing a standard baseframe at one point in $\mathbb{H}^{d+1}$, we obtain a bijective correspondence between (torsion-free) Kleinian groups and baseframed hyperbolic manifolds (M, ω). Then

$$(M_n, \omega_n) \to (M, \omega)$$

geometrically if and only if, for each compact submanifold $K \subset M$ containing ω, there are smooth embeddings $\phi_n : K \to M_n$ for $n \gg 0$ such that ϕ_n sends ω to ω_n and ϕ_n tends to an isometry in the C^∞ topology. (See [6, Thm. E.1.13].)

Strong convergence. We say $\Gamma_n \to \Gamma_S$ *strongly* if

(a) $\Gamma_n \to \Gamma_S$ geometrically, and

(b) for $n \gg 0$ there exist surjective homomorphisms

$$\chi_n : \Gamma_S \to \Gamma_n$$

such that $\chi_n(g) \to g$ for all $g \in \Gamma_S$.

If Γ_n converges to Γ both geometrically and algebraically, then it converges strongly. However strong convergence is more general, since we require only a surjection (instead of an isomorphism) in (b). This generality accommodates situations like Dehn filling in 3-manifolds.

Note that when Γ_S is finitely generated, any two choices for χ_n in (b) agree for all $n \gg 0$.

Lemma 4.4. *Suppose Γ_n has algebraic and geometric limits $\Gamma_A \subset \Gamma_G$, and $M_G = \mathbb{H}^{d+1}/\Gamma_G$ is topologically tame. Then $\Gamma_n \to \Gamma_G$ strongly.*

Proof. By tameness there is a compact submanifold $K \subset M_G$ such that the inclusion induces an isomorphism on π_1. Then we may identify $\pi_1(K)$ with Γ_G. Geometric convergence provides nearly isometric embeddings $\phi_n : K \to M_n = \mathbb{H}^{d+1}/\Gamma_n$ for $n \gg 0$; on the level of π_1, these give homomorphisms $\chi_n : \Gamma_G \to \Gamma_n$ converging to the identity.

By algebraic convergence, there are also isomorphisms $\chi_n' : \Gamma_A \to \Gamma_n$ converging to the identity. Since $\Gamma_A \subset \Gamma_G$ and $\chi_n | \Gamma_A = \chi_n'$ for all $n \gg 0$, the maps χ_n are surjective. $\square$

Lemma 4.5. *Let $\Gamma_n \to \Gamma_A$ algebraically. Then $\Gamma_n \to \Gamma_A$ strongly iff for all sequences g_n in Γ_A,*

$$(g_n \to \infty \textit{ in } \Gamma_A) \implies (\chi_n(g_n) \to \infty \textit{ in } \operatorname{Isom}(\mathbb{H}^{d+1})) \tag{4.1}$$

where $\chi_n : \Gamma_A \to \Gamma_n$ are isomorphisms converging to the identity.

Remark. We say $x_n \to \infty$ in X if $\{x_n\} \cap K$ is finite for every compact $K \subset X$.

Proof. Assume (4.1), and consider any $h \in \operatorname{Isom}(\mathbb{H}^{d+1})$ on which Γ_n accumulates. Then passing to a subsequence we can write $h = \lim \chi_n(g_n) \in \Gamma_n$. By (4.1) g_n returns infinitely often to a compact (hence finite) subset of Γ_A, so g_n equals some fixed g for infinitely many n. Then $h = \lim \chi_n(g) = g$, and therefore $\Gamma_G = \Gamma_A$.

Conversely, suppose Γ_n converges geometrically to Γ_A. Consider any $g_n \in \Gamma_A$ such that $\chi_n(g_n) \not\to \infty$. Then $\chi_n(g_n) \to h$ along a subsequence, so $h \in \Gamma_A$ and $\chi_n(h^{-1}g_n) \to \mathrm{id}$. Therefore $g_n = h$ for all $n \gg 0$ and $g_n \not\to \infty$, as required by (4.1). $\square$

The truncated convex core. We now turn to an analysis of convergence when the limiting manifold is geometrically finite.

Given a hyperbolic manifold M, let $K(M)$ denote its convex core, and for $\epsilon > 0$ let $M^{<\epsilon}$ denote the ϵ-thin part of M (where the injectivity radius is less than ϵ). The *truncated core* is defined by

$$K_\epsilon(M) = K(M) - M^{<\epsilon}.$$

Note that M is geometrically finite iff $K_\epsilon(M)$ is compact when ϵ is the Margulis constant. If M is geometrically finite, and ϵ is less than both the Margulis constant and the length of the shortest geodesic on M, then there is a retraction

$$\rho : M \to K_\epsilon(M).$$

(First take the nearest point projection to $K(M)$, and then use the product structure in the cusps.)

Proof of Theorem 4.2. Clearly strong convergence of Γ_n to Γ_A implies strong convergence of parabolic subgroups.

Now assume $L_n \to L_A$ for each maximal parabolic subgroup $L_A \subset \Gamma_A$. We will show $\Gamma_n \to \Gamma_A$ geometrically, and hence strongly.

Pass to any subsequence such that the geometric limit Γ_G of Γ_n exists. Then Γ_A is a subgroup of Γ_G, so on the level of quotient manifolds we have the diagram:

$$\begin{tikzcd} M_A \arrow[d, "\pi"'] & \\ M_G \arrow[r, dashed, "\phi_n"] & M_n \end{tikzcd}$$

where π is a covering map and ϕ_n are nearly isometric embeddings defined on larger and larger compact submanifolds of M_G (as suggested by the notation $\phi_n : M_G \dashrightarrow M_n$). Each manifold is equipped with a baseframe, chosen for simplicity in its convex core, and the maps above preserve baseframes.

The convex cocompact case. To give the idea of the proof, first consider the case where Γ_A is convex cocompact. Then $K(M_A)$ is compact and homotopy equivalent to M_A. The composition $\phi_n \circ \pi$ is C^∞ close to a (local) isometry on $K(M_A)$ for all $n \gg 0$, and the isomorphisms

$$\chi_n : \Gamma_A \to \Gamma_n$$

converging to the identity are the same as the maps on fundamental group

$$(\phi_n \circ \pi)_* : \pi_1(K(M_A), *) \to \pi_1(M_n, *)$$

defined for all $n \gg 0$.

Let $g_n \to \infty$ in Γ_A, and let $\gamma_n \subset K(M_A)$ be the geodesic paths beginning and ending at the basepoint and representing $g_n \in \pi_1(M_A, *)$. Then $\gamma_n' = \phi_n \circ \pi(\gamma_n)$ has small geodesic curvature and length comparable to γ_n, so the geodesic representative of γ_n' is also long. Therefore $\chi_n(g_n) \to \infty$ in $\mathrm{Isom}(\mathbb{H}^{d+1})$. By Lemma 4.5 above, $\Gamma_n \to \Gamma_A$ geometrically, and hence strongly.

The geometrically finite case. Now suppose only that Γ_A is geometrically finite. Choose $\epsilon > 0$ less than both the Margulis constant and the length of the shortest geodesic on M_A. As noted above, there is a smooth retraction

$$\rho : M_A \to K_\epsilon(M_A)$$

from the manifold to its truncated core.

Consider again the geodesic representatives $\gamma_n \subset M_A$ of a sequence $g_n \to \infty$ in Γ_A. Focusing attention on a particular $n \gg 0$, write

$$\gamma_n = \delta \cup \xi_1 \cup \cdots \cup \xi_s, \tag{4.2}$$

where $\delta = \gamma \cap K_\epsilon(M_A)$, and the $\{\xi_i\}$ are geodesic segments in $M_A^{\leq\epsilon}$. These segments account for excursions of γ_n into the cusps of M_A. Modify this decomposition slightly by absorbing into δ any short ξ_i (say of length less than 1).

Let $\delta' = \phi_n \circ \pi(\delta)$, and define $\xi_i' \subset M_n$ by first retracting ξ_i to the truncated core $K_\epsilon(M_A)$, then straightening $\phi_n \circ \pi \circ \rho(\xi_i)$ *rel* its endpoints. Then

$$\gamma_n' = \delta' \cup \xi_1' \cup \cdots \cup \xi_s'$$

is a based piecewise geodesic segment representing the homotopy class $\chi_n(g_n)$ in M_n. See Figure 1.

Strong convergence of cusps (the condition $L_n \to L_A$ geometrically) implies the length of ξ_i' tends to infinity as the length of ξ_i tends to infinity, by Lemma 4.5.

We claim γ_n' is nearly a geodesic. More precisely, the geodesic segments making up γ_n' have definite length and meet in small angles, and

M_A γ_n

$K_\epsilon(M_A)$ $\phi_n \circ \pi$

M_n

γ'_n

FIGURE 1. A piecewise-geodesic representative of γ_n in M_n.

these angles tend to zero as $n \to \infty$. Indeed, if one such segment ξ_i is very long, then ξ'_i is also very long, so ξ'_i and δ' are both nearly perpendicular to the boundary of the thin part, hence nearly parallel. On the other hand, if ξ_i has only moderate length, then ξ_i is within a compact neighborhood of $K_\epsilon(M_A)$. Hence $\xi'_i \cup \delta'$ is close to $\phi_n \circ \pi(\xi_i \cup \delta)$, so there is a small angle in this case too.

Now as $n \to \infty$, we have $\ell(\gamma_n) \to \infty$, so in the decomposition (4.2) we either have a large number of segments or a long individual segment. In either case the same is true of γ'_n, so $\ell(\gamma'_n) \to \infty$. Therefore $\chi_n(g_n) \to \infty$ and we have again established strong convergence by Lemma 4.5.

□

Convergence of cores. Next we explain the condition on cores in the statement of Theorem 4.1. Suppose $(M_n, \omega_n) \to (M, \omega)$ geometrically, and $\phi_n : M \dashrightarrow M_n$ are almost-isometries defined on larger and larger compact submanifolds. If $K_n \subset M_n$, $K \subset M$ are compact sets, we say $K_n \to K$ *strongly* if:

(i) K_n is contained in a unit neighborhood of $\phi_n(K)$ for all $n \gg 0$, and

(ii) $\phi_n^{-1}(K_n) \to K$ in the Hausdorff topology on compact subsets of M.

Note that (i) prevents any part of K_n from disappearing in the limit by tending to infinity. An equivalent formulation is that the Hausdorff distance between K_n and $\phi_n(K)$ tends to zero.

For later reference we quote [23, Prop. 2.4]:

Proposition 4.6. *If $\Gamma_n \to \Gamma$ geometrically, and the injectivity radius of the quotient manifold M_n in its convex core $K(M_n)$ is bounded above, independent of n, then*

$$\Lambda(\Gamma_n) \to \Lambda(\Gamma)$$

in the Hausdorff topology.

Proof of Theorem 4.1 (Convergence of cores). We first show $K_\epsilon(M_n) \to K_\epsilon(M)$ strongly. The argument is very similar to the proof of Theorem 4.2.

Consider any $\epsilon > 0$ less than the Margulis constant and the length of the shortest geodesic on M. As before there is a retraction

$$\rho : M \to K_\epsilon(M),$$

and we can identify Γ with $\pi_1(K_\epsilon(M))$. By strong convergence, the nearly isometric embedding

$$\phi_n : K_\epsilon(M) \to M_n$$

determines a surjective homomorphism

$$\chi_n : \Gamma \to \Gamma_n$$

for all $n \gg 0$. Moreover, ϕ_n sends the thin part into the thin part, so geodesics passing through the thick part of M_n are represented by geodesics on M.

Consider any closed geodesic γ_n passing through the ϵ-thick part of M_n. Let γ be the shortest closed geodesic in M such that $\phi_n(\rho(\gamma))$ is homotopic to γ_n. As before, we can write

$$\gamma = \delta \cup \xi_1 \cup \cdots \cup \xi_s,$$

where $\delta = \gamma \cap K_\epsilon(M)$, and the $\{\xi_i\}$ are geodesic segments in $M^{\leq \epsilon}$. Then γ_n is homotopic to the broken geodesic

$$\gamma_n' = \delta' \cup \xi_1' \cup \cdots \cup \xi_s',$$

where $\delta' = \phi_n(\delta)$ and ξ_i' is obtained by straightening $\phi_n \circ \rho(\xi_i)$ while holding its endpoints fixed.

Now we claim ξ_i' is long whenever ξ_i is long. Indeed, for any $R > 0$ there exists an $N(R)$ such that ϕ_n is defined and nearly isometric on an R-neighborhood of $K_\epsilon(M)$ when $n > N(R)$. For such n, the straightened segment ξ_i' is longer than $R/2$ whenever ξ_i is longer than R; otherwise we could replace ξ_i with $\phi_n^{-1}(\xi_i')$ and shorten our representative γ while keeping $\phi_n(\gamma)$ homotopic to γ_n.

Thus one can repeat the proof of Theorem 4.2 to conclude that γ_n' is nearly a geodesic, and therefore:

> *$(*)$ The loops γ_n' and γ_n are C^1-close in the thick part $M_n^{\geq\epsilon}$, with a bound depending only on n and tending to zero as $n \to \infty$.*

Now any $x \in K_\epsilon(M_n)$ lies in the image of an ideal simplex $S \subset \mathbb{H}^{d+1} = \widetilde{M_n}$ with vertices in the limit set. Approximating the edges of S by lifts of long closed geodesics and applying $(*)$, we conclude that x lies close to $\phi_n(K_\epsilon(M))$.

In particular $K_\epsilon(M_n)$ is compact, so M_n is geometrically finite. The same reasoning shows any $x \in K_\epsilon(M)$ is close to $\phi_n^{-1}(K_\epsilon(M_n))$, so the truncated cores converge strongly.

By convergence of cores, the injectivity radius in $K(M_n)$ is uniformly bounded above for all $n \gg 0$. Thus Proposition 4.6 shows $\Lambda_n \to \Lambda$. $\square$

Another proof of Theorem 4.1(a), similar in spirit and with many details, can be found in [33].

5. Accidental parabolics

Cusps play a central role in the theory of deformations of geometrically finite manifolds. In this section we discuss the ways in which a closed geodesic can shrink to form a cusp in the algebraic limit of a sequence of hyperbolic 3-manifolds. This shrinking governs the difference between algebraic and strong convergence, and we will prove:

Theorem 5.1. *Let $\Gamma_n \subset \mathrm{Isom}(\mathbb{H}^3)$ converge algebraically to a geometrically finite group Γ_A. Then $\Gamma_n \to \Gamma_A$ strongly iff all accidental parabolics converge horocyclically.*

On the other hand, we will see in §7 and §8 that the stronger condition of *radial* convergence is required to obtain convergence of the dimension of the limit set.

1 1

FIGURE 2. Radial and horocyclic convergence of multipliers.

Radial and horocyclically convergence. Consider a sequence $t_n = iL_n + \theta_n$ in the upper half-plane $\mathbb{H} \subset \mathbb{C}$ with $t_n \to 0$. We say $t_n \to 0$ *radially* if there exists an M such that

$$|\theta_n|/L_n < M$$

for all n, while $t_n \to 0$ *horocyclically* if

$$\theta_n^2/L_n \to 0.$$

Radial convergence means t_n remains within a bounded hyperbolic distance of a geodesic converging to $t = 0$, while horocyclic convergence means any horoball resting on $t = 0$ contains t_n for all $n \gg 0$.

Now suppose $\lambda_n \to 1$ in $\mathbb{C}^*$ and $|\lambda_n| > 1$. We say $\lambda_n \to 1$ radially or horocyclically if $t_n = i \log \lambda_n \to 0$ radially or horocyclically in $\mathbb{H}$. See Figure 2. Radial convergence is equivalent to the condition

$$|\lambda_n - 1| \leq M||\lambda_n| - 1|$$

for some M.

The *complex length* of a hyperbolic element $g \in \mathrm{Isom}^+(\mathbb{H}^3)$ is given by

$$\mathcal{L}(g) = L + i\theta = \log \lambda,$$

where the *multiplier* $\lambda = g'(x)$ is the derivative of g at its repelling fixed-point. The real part L is the length of the core geodesic of the solid torus $\mathbb{H}^3/\langle g \rangle$, and θ is the torsion of parallel transport around this geodesic. When g is nearly parabolic (λ is close to 1) there is a natural choice of θ close to 0.

Now suppose $\Gamma_n \to \Gamma_A$ algebraically, with isomorphisms $\chi_n : \Gamma_A \to \Gamma_n$ converging to the identity. An *accidental parabolic* $g \in \Gamma_A$ is a parabolic element such that $g_n = \chi_n(g)$ is hyperbolic for infinitely many n. The complex lengths of these g_n satisfy

$$i\mathcal{L}(g_n) \to 0 \text{ in } \mathbb{H}.$$

If the convergence above is radial (or horocyclic), we say *all accidental parabolics converge radially* (or *horocyclically).* (For $\Gamma_A \not\subset \mathrm{Isom}^+(\mathbb{H}^3)$ we apply the condition to its orientation preserving subgroup.)

Proof of Theorem 5.1. Suppose $\Gamma_n \to \Gamma_A$ algebraically. By Lemma 4.5, $\Gamma_n \to \Gamma_A$ strongly iff $L_n \to L_A$ geometrically for each maximal parabolic subgroup $L_A \subset \Gamma_A$, where $L_n = \chi_n(L_A)$.

If L_n is a parabolic subgroup for all $n \gg 0$, then it is conjugate to a group of translations on $\mathbb{C}$, and geometric convergence is easily verified (cf. Theorem 6.2 below).

Otherwise L_n is hyperbolic for infinitely many n, so L_A is of rank-1 and generated by an accidental parabolic. Pass to the subsequence where L_n is hyperbolic, and let $\lambda_n \to 1$ denote the multiplier of a generator of L_n. To analyze the geometric limit in this case, consider the quotient torus

$$X_n \quad = \quad \Omega(L_n)/L_n \cong \mathbb{C}^*/\lambda_n^{\mathbb{Z}} \cong \mathbb{C}/(\mathbb{Z}2\pi i \oplus \mathbb{Z}\log\lambda)$$

and the quotient cylinder

$$X_A \quad = \quad \Omega(L_A)/L_A \cong \mathbb{C}/\mathbb{Z}.$$

Suppose $\lambda_n \to 1$ horocyclically. Then $t_n = i(\log\lambda)/2\pi \to 0$ horocyclically in $\mathbb{H}$. But this means $[X_n] \to \infty$ in the moduli space of complex tori $\mathcal{M}_1 = \mathbb{H}/SL_2(\mathbb{Z})$, since horoballs resting on $z = 0$ form neighborhoods of the cusp of the modular surface. Thus X_n becomes long and thin as $n \to \infty$, so it converges geometrically (as a surface with a complex affine structure) to the cylinder X_A. Therefore $L_n \to L_A$ geometrically as well.

On the other hand, if the convergence is not horocyclic, then after passing to a subsequence the tori $[X_n]$ converge to a torus $X_G \in \mathcal{M}_1$. Thus L_n converges geometrically to a rank-2 parabolic subgroup L_G, with

$$X_G \cong \Omega(L_G)/L_G.$$

Hence $L_G \neq L_A$ and strong convergence fails whenever horocyclic convergence fails. $\square$

6. Cusps and Poincaré series

In this section we continue our study of cusps from a more analytical point of view. We consider a single parabolic group L and a deformation given by a family of representations $\chi_n : L \to L_n$ converging to the identity. The results we develop control the Poincaré series for L_n as $n \to \infty$.

This control will be applied in the next section to establish continuity of the Hausdorff dimension of the limit set.

Deformations of cusps. Let L be an elementary Kleinian group. Then L is a finite extension of a free abelian group $\mathbb{Z}^r$, and we set $\mathrm{rank}(L) = r$. We say L is *hyperbolic* if it contains a hyperbolic element; in this case $\mathrm{rank}(L) = 1$. Otherwise L is *parabolic*, its elements share a common fixed-point c, $|g'(c)| = 1$ for all $g \in L$ and $0 \leq \mathrm{rank}(L) \leq d$.

Now consider a deformation given by a sequence of representations $\chi_n : L \to L_n$, and a sequence of real numbers δ_n, satisfying the following conditions:

1. L_n and L are elementary Kleinian groups fixing a common point $c \in S^d_\infty$;
2. L is parabolic, with $\mathrm{rank}(L) \geq 1$;
3. $\chi_n : L \to L_n$ is a surjective homomorphism, converging pointwise to the identity; and
4. $\delta_n \to \delta > \mathrm{rank}(L)/2$.

Example: Dehn filling. As a typical example, these conditions arise naturally when one performs (p_n, q_n) Dehn-filling on a rank two parabolic subgroup L of a Kleinian group Γ [34, Ch. 4], [6, E.5-E.6]. In the filled group Γ_n, the cusp becomes a short geodesic with stabilizer $L_n = \langle g_n \rangle$, and there is a surjective filling homomorphism $\chi_n : L \to L_n$. This example shows we can have $\mathrm{rank}(L_n) < \mathrm{rank}(L)$ for all n. In addition, L_n need not have *any* algebraic limit; if p_n and q_n tend to infinity, then $g_n \to \infty$ in $\mathrm{Isom}(\mathbb{H}^3)$ (although L_n converges to L geometrically). In our applications the exponents will be given by $\delta_n = \delta(\Gamma_n)$.

Uniform convergence of Poincaré series. Recall that for $\Gamma \subset \mathrm{Isom}(\mathbb{H}^{d+1})$, $x \in S^d_\infty$ and $\delta \geq 0$, the absolute Poincaré series is defined by

$$P_\delta(\Gamma, x) = \sum_{g\in\Gamma} |g'(x)|_\sigma^\delta,$$

where the derivative is measured in the spherical metric σ. To study the rate of convergence we define for any open set U the sub-sum

$$P_\delta(\Gamma, U, x) = \sum_{g(x)\in U} |g'(x)|_\sigma^\delta.$$

Let us say the Poincaré series for (L_n, δ_n) *converge uniformly* if for any compact set $K \subset S^d_\infty - \{c\}$ and $\epsilon > 0$, there is a neighborhood U of c such that

$$P_{\delta_n}(L_n, U, x) < \epsilon$$

for all $n \gg 0$ and all $x \in K$. This means the tail of the series can be made small, independent of n, by choosing U small enough.

We will establish uniform convergence under the following 3 conditions.

Theorem 6.1. *If $L_n \to L$ geometrically and*

$$\delta > \begin{cases} 1 & \text{if } d = 2, \text{ or} \\ (d-1)/2 & \text{if } d \neq 2, \end{cases}$$

then the Poincaré series for (L_n, δ_n) converge uniformly.

Theorem 6.2. *If L_n is parabolic with* $\operatorname{rank}(L_n) \geq d-1$ *for all n, then $L_n \to L$ strongly and the Poincaré series converge uniformly.*

Theorem 6.3. *If $L_n \to L \subset$* $\mathrm{Isom}(\mathbb{H}^3)$ *algebraically, L_n are hyperbolic and all accidental parabolics converge radially, then $L_n \to L$ strongly and the Poincaré series converge uniformly.*

Cusps in high-dimensional manifolds. We remark that the conclusion of Theorem 6.2 holds for *all* cusps in the case of hyperbolic 3-manifolds (since $d = 2$), but it fails for cusps of low rank in high-dimensional hyperbolic manifolds ($d > 2$). A related fact is that the rank of a cusp can increase in the geometric limit.

For a standard example, let R_n be the isometry of $\mathbb{R}^3_\infty$ given by rotating angle $2\pi/n$ about the line $(x = 0, y = n)$; let $T_n(x, y, z) =$

$(x, y, z+1/n)$; and let $g_n = T_n \circ R_n$. Then $L_n = \langle g_n \rangle$ is a rank 1 parabolic group, converging geometrically to the rank 2 cusp $L_G = \langle g, h \rangle$ where

$$\begin{aligned} g(x,y,z) &= \lim g_n(x,y,z) = (x + 2\pi, y, z), \\ h(x,y,z) &= \lim g_n^n(x,y,z) = (x, y, z+1). \end{aligned}$$

The algebraic limit $L = \langle g \rangle$ also exists and with the obvious isomorphism $\chi_n : L \to L_n$ and $\delta_n = \delta = 1/2 + \epsilon$ we have the setup for Theorem 6.2, except $\mathrm{rank}(L_n) = d - 2$. But L_n does not converge strongly to L (because $L \neq L_G$), and the Poincaré series do not converge uniformly (because $P_\delta(L_G, x) = \infty$).

Proof of Theorem 6.1. For simplicity, assume $K = \{p\}$ is a single point; the case of a general compact set is similar.

Normalize coordinates on $S^d_\infty = \mathbb{R}^d_\infty \cup \{\infty\}$ so $c = 0$ and $Lp \subset \mathbb{R}^d_\infty$. Since L is parabolic, it acts freely and properly discontinuously on $S^d_\infty - \{c\}$. Thus we can choose a ball $B = B(p, r)$, $r \ll |p|$, such that its translates $L \cdot B$ are bounded and disjoint. Since $L_n \to L$ geometrically, we can also arrange that the balls $L_n \cdot B$ are bounded and disjoint for all $n \gg 0$.

To give the main idea of the proof, we first treat the case where we have $\delta > d/2$ and L_n is parabolic for all n. Then all L_n act isometrically with respect to the metric

$$\rho = \frac{|dx|}{|x|^2}$$

obtained by pulling back the Euclidean metric $|dx|$ under an inversion sending the cusp point c to ∞. Consequently $\mathrm{diam}(gB) \asymp d(0, gB)^2$. Thus

$$\begin{aligned} P_\delta(L,p) &= \sum_L |g'(p)|_\sigma^\delta \asymp \sum \mathrm{diam}(gB)^\delta = \sum \mathrm{diam}(gB)^d \, \mathrm{diam}(gB)^{\delta - d} \\ &\asymp \sum \int_{gB} |x|^{2(\delta-d)} \, |dx|^d = \int_{\bigcup gB} |x|^{2(\delta-d)} \, |dx|^d < \infty, \end{aligned}$$

because

$$2(\delta - d) > 2(d/2 - d) = -d,$$

and $-d$ is the critical exponent for integrability of the singularity $|x|^\alpha$ on $\mathbb{R}^d$.

To obtain uniform convergence, choose α such that

$$2(\delta_n - d) > \alpha > -d$$

for all $n \gg 0$. Consider a small neighborhood $U = B(0, s)$ of the cusp point c. Then for all $n \gg 0$ we have similarly

$$\begin{aligned} P_{\delta_n}(L_n, U, p) &\asymp \int_{\bigcup\{gB \,:\, gp \in U\}} |x|^{2(\delta_n - d)} \, |dx|^d \\ &= O\left(\int_U |x|^\alpha \, |dx|^d\right) = O(s^{d+\alpha}). \end{aligned}$$

Since $d + \alpha > 0$, this bound tends to zero as $s \to 0$ and we have established uniform convergence of the Poincaré series for (L_n, δ_n).

Next we treat the case where L_n is hyperbolic. Then the geodesic stabilized by L_n has endpoints $\{a_n, c\}$ on the sphere at infinity. In this case the invariant metric becomes

$$\rho_n = \frac{|dx|}{|x - a_n||x|},$$

and we have the estimate

$$\operatorname{diam}(B) \asymp d(0, B) d(a_n, B).$$

The calculation above becomes

$$\begin{aligned} P_{\delta_n}(L_n, U, p) &= O\left(\int_U |x - a_n|^{\alpha/2} |x|^{\alpha/2} \, |dx|^d\right) \\ &= O\left(\int_U |x - a_n|^\alpha + |x|^\alpha \, |dx|^d\right) = O(s^{d+\alpha}), \end{aligned}$$

and as before this bound gives uniform convergence.

Since $d/2 = 1$ when $d = 2$, the proof is now complete in the case of classical Kleinian groups. Also when $d = 1$ we have

$$\delta > \operatorname{rank}(L)/2 = 1/2 = d/2,$$

so we have covered this case as well.

Finally we explain how $d/2$ can be improved to $(d-1)/2$ for $d \geq 3$. Replacing L with a suitable abelian subgroup of finite index, we can assume that for each n there is a maximal connected abelian Lie group G_n with

$$L_n \subset G_n \subset \operatorname{Isom}^+(\mathbb{H}^{d+1}).$$

Fixing attention on a particular n, let $D = \dim(G_n)$. If L_n is hyperbolic, then G_n is conjugate into $\mathbb{R}^* \times \mathrm{SO}(d)$ and we have

$$D = 1 + [d/2],$$

since $[d/2]$ is the dimension of a maximal torus in $\mathrm{SO}(d)$. If L_n is parabolic of rank r, then G_n is conjugate into $\mathbb{R}^r \times \mathrm{SO}(d-r)$, so we have

$$D = r + [(d-r)/2] = [(d+r)/2].$$

Now for $D < d$ the orbit $L_n p$ is rather sparse, since it is confined to the D-dimensional submanifold $G_n p$. Due to this sparseness, the critical dimension for integrability of $|x|^\alpha$ becomes $-D$ instead of $-d$. Thus we need only guarantee $\delta > D/2$ to achieve uniform convergence of the Poincaré series. (For a detailed proof of this statement, it is useful to change coordinates so $c = \infty$ and $p = 0$. Moving p distance ≤ 1 if necessary, we can assume the stabilizer of p in G_n is trivial, and that the injectivity radius of the submanifold $G_n p \subset \mathbb{R}^d_\infty$ at p is bounded below. Then for L_n parabolic the orbit of p in $\mathbb{R}^d_\infty$ is a cylinder:

$$G_n p \cong (S^1)^{D-r} \times \mathbb{R}^r,$$

while for L_n hyperbolic it is a cone with vertex a_n:

$$G_n p \cong (S^1)^{D-1} \times \mathbb{R}_+.$$

The bound on the Poincaré series becomes

$$\int_{G_n p} \frac{|dx|^D}{(1+|x|^2)^\delta},$$

and this integral is uniformly bounded, independent of the shape of the cylinder or cone, if $\delta > D/2$. To see this uniformity, compare the integral above to one where $G_n p$ is replaced by a D-plane through p.)

It remains only to check that $\delta > (d-1)/2$ implies $\delta > D/2$. If L_n is hyperbolic then $d \geq 3$ implies

$$D \leq 1 + [d/2] \leq d - 1,$$

while if L_n is parabolic of rank $r \leq d-1$ we have

$$D \leq [(2d-1)/2] = d - 1;$$

in either case, we have $\delta > (d-1)/2 \geq D/2$ as desired. Finally if $\mathrm{rank}(L_n) = d$ then $D = d$ and we have

$$\delta > \mathrm{rank}(L)/2 = d/2 = D/2,$$

so we are done. $\square$

Proof of Theorem 6.2. As before we assume $K = \{p\}$ is a single point. Normalize coordinates so $c = \infty$ and $p = 0$, and pass to a subgroup of index two if necessary, so L and L_n preserve orientation for all $n \gg 0$. Then the rank restriction implies L_n and L act by pure translations on $\mathbb{R}^d_\infty$. Thus for $g \in L$ we can write $g(z) = z + \ell(g)$ and $(\chi_n g)(z) = z + \ell_n(g)$.

Since χ_n converges to the identity, the length ratio satisfies

$$\sup_{L-\{0\}} \frac{|\ell_n(g)|}{|\ell(g)|} \to 1 \tag{6.1}$$

as $n \to \infty$. In particular, χ_n is an isomorphism for all $n \gg 0$, so $L_n \to L$ algebraically. Moreover large elements of L map to large elements of L_n, so $L_n \to L$ strongly (by Lemma 4.5).

Now recall the spherical metric is given by $\sigma = 2|dx|/(1+|x|^2)$. Let $U = \{z \,:\, |z| > R\}$ be a neighborhood of ∞. Then by (6.1), for $n \gg 0$, the Poincaré series satisfies

$$\begin{aligned} P_{\delta_n}(L_n, U, p) &= \sum_{g(p)\in U} |g'(p)|_\sigma^{\delta_n} = \sum_{\{g\in L\,:\,|\ell_n(g)|>R\}} \left(\frac{1}{1+|\ell_n(g)|^2}\right)^{\delta_n} \\ &\leq \sum_{|\ell_n(g)|>R} |\ell_n(g)|^{-2\delta_n} = O\left(\sum_{|\ell(g)|>R/2} |\ell(g)|^{-2\alpha}\right), \end{aligned}$$

where α is chosen so $\delta_n > \alpha > \operatorname{rank}(L)/2$ for all $n \gg 0$. Since the last sum converges, it can be made arbitrarily small by a suitable choice of R. Thus the Poincaré series converges uniformly. $\square$

Proof of Theorem 6.3. Once again we assume $K = \{p\}$. By algebraic convergence, we can write $L = \langle g \rangle$ and $\chi_n(g) = g_n$. Normalize coordinates so $c = \infty$, $p = 0$, and $g(p) = 1$, where we have identified $\mathbb{R}^2_\infty$ with $\mathbb{C}$. Changing L_n by a conjugacy tending to the identity, we can assume

$$\begin{aligned} g(z) &= z + 1, \\ g_n(z) &= \lambda_n z + 1, \end{aligned}$$

and $|\lambda_n| > 1$.

By hypothesis, $|\lambda_n - 1| \le M(|\lambda_n| - 1)$ for some M. So for $k > 0$, we have

$$\begin{aligned} |g_n^k(0)| &= |1 + \lambda_n + \lambda_n^2 + \cdots + \lambda_n^{k-1}| = \frac{|\lambda_n^k - 1|}{|\lambda_n - 1|} \\ &\ge \frac{|\lambda_n|^k - 1}{M(|\lambda_n| - 1)} = \frac{1}{M}(1 + |\lambda_n| + |\lambda_n|^2 + \cdots + |\lambda_n|^{k-1}) \\ &\ge \frac{k|\lambda_n|^{k/2}}{2M}. \end{aligned}$$

Since the last bound is independent of n, we see g_n^k is large whenever k is large, and thus $L_n \to L$ strongly (by Lemma 4.5). Moreover

$$\begin{aligned} \sum_{k>K} |(g_n^k)'(0)|_\sigma^{\delta_n} &= \sum_{k>K} \left(\frac{|\lambda_n|^k}{1 + |g_n^k(0)|^2} \right)^{\delta_n} \\ &\le \sum_{k>K} \left(\frac{(2M)^2|\lambda_n|^k}{k^2|\lambda_n|^k} \right)^{\delta_n} \\ &= O\left(\sum_{k>K} k^{-2\alpha} \right), \end{aligned}$$

where α is chosen so $\delta_n > \alpha > \operatorname{rank}(L)/2 = 1/2$ for all $n \gg 0$. Since the last sum converges, it can be made arbitrarily small by taking K sufficiently large.

Carrying out the same argument with the other fixed-point a_n of g_n normalized to be at $c = \infty$, we conclude that for any $\epsilon > 0$ there is a K such that

$$\sum_{|k|>K} |(g_n^k)'(p)|_\sigma^{\delta_n} < \epsilon.$$

Choose a neighborhood U of $c = \infty$ such that $g_n^k(p) \notin U$ for all n and $|k| \le K$. Then $P_{\delta_n}(L_n, U, p) < \epsilon$, and we have shown that the Poincaré series converge uniformly. $\square$

7. Continuity of Hausdorff dimension

In this section we establish conditions for continuity of the Hausdorff dimension of the limit set.

The continuity of dimension will generally come along with a package of additional properties. For economy of language, we say $\Gamma_n \to \Gamma$ *dynamically* if:

D1. $\Gamma_n \to \Gamma$ strongly;

D2. The limit sets satisfy $\Lambda_n \to \Lambda$ in the Hausdorff topology on closed subsets of S^d_∞;

D3. $\mathrm{H.dim}(\Lambda_n) \to \mathrm{H.dim}(\Lambda)$;

D4. The critical exponents satisfy $\delta(\Gamma_n) \to \delta(\Gamma)$;

D5. The groups Γ_n and Γ are geometrically finite for all $n \gg 0$; and

D6. The normalized canonical densities satisfy $\mu_n \to \mu$ in the weak topology on measures.

Actually when Γ is geometrically finite, conditions D1 and D6 imply the rest (by Theorems 3.1 and 4.1). The terminology is meant to suggest that the dynamical and statistical features of Γ_n (as reflected in its limit set and invariant density) converge to those of Γ.

Here is the prototypical example:

Theorem 7.1. *If $\Gamma_n \to \Gamma$ algebraically and Γ is convex cocompact, then $\Gamma_n \to \Gamma$ dynamically.*

Proof. Since Γ has no parabolic subgroups, $\Gamma_n \to \Gamma$ strongly (Corollary 4.3). By Theorem 4.1, Γ_n is geometrically finite for all $n \gg 0$ and $\Lambda_n \to \Lambda$. Now pass to any subsequence such that $\delta(\Gamma_n) \to \alpha$ and $\mu_n \to \nu$ weakly; then ν is a Γ-invariant density supported on the limit set. Such a density is unique for a convex compact group (Corollary 3.4), so $\nu = \mu$. We have thus verified D1 and D6, and D2-D5 follow. $\square$

Our goal in this section is to obtain dynamic convergence in the presence of cusps. We will establish the following results.

Theorem 7.2 (Dynamic convergence). *Let $\Gamma_n \subset \mathrm{Isom}(\mathbb{H}^{d+1})$ converge strongly to a geometrically finite group Γ. If*

$$\liminf \delta(\Gamma_n) > \begin{cases} 1 & \text{if } d = 2, \text{ or} \\ (d-1)/2 & \text{if } d \neq 2, \end{cases}$$

then $\Gamma_n \to \Gamma$ dynamically.

For hyperbolic 3-manifolds, there is a simple condition on parabolics that promotes algebraic convergence to dynamic convergence.

Theorem 7.3 (Radial limits). *Let* $\Gamma_n \to \Gamma$ *algebraically, where* $\Gamma \subset \mathrm{Isom}(\mathbb{H}^3)$ *is geometrically finite. If all accidental parabolics converge radially, then* $\Gamma_n \to \Gamma$ *dynamically.*

Corollary 7.4. *If* $\Gamma_n \to \Gamma$ *is an algebraically convergent sequence of finitely-generated Fuchsian groups, then* $\mathrm{H.dim}(\Lambda_n) \to \mathrm{H.dim}(\Lambda)$.

The bottom of the spectrum λ_0 is insensitive to subtleties of limit sets of dimension $d/2$ or less, so from Theorem 7.2 we obtain:

Corollary 7.5. *If* Γ_n *converges strongly to a geometrically finite group* Γ*, then* $\lambda_0(M_n) \to \lambda_0(M)$ *for the corresponding quotient manifolds.*

Corollary 7.6 (Strong limits). *Suppose* $M = \mathbb{H}^3/\Gamma$ *is topologically tame, and* $\Gamma_n \to \Gamma$ *strongly. Then the quotient manifolds* $M_n = \mathbb{H}^3/\Gamma_n$ *satisfy*

$$\lambda_0(M_n) \to \lambda_0(M);$$

and if $\mathrm{H.dim}(\Lambda) \geq 1$*, then we also have*

$$\mathrm{H.dim}(\Lambda_n) \to \mathrm{H.dim}(\Lambda).$$

In the next section we will give several examples of the *discontinuous* behavior of dimension. In particular we will show strong convergence alone is *insufficient* to guarantee $\mathrm{H.dim}(\Lambda_n) \to \mathrm{H.dim}(\Lambda)$, even for geometrically finite 3-manifolds.

Semicontinuity of dimension. One inequality for the critical dimension and the bottom of the spectrum for a geometric limit is general and immediate:

Theorem 7.7. *If* $\Gamma_n \to \Gamma_G$ *geometrically, then*

$$\begin{aligned} \delta(\Gamma_G) &\leq \liminf \delta(\Gamma_n) \qquad \textit{and} \\ \lambda_0(M_G) &\geq \limsup \lambda_0(M_n) \end{aligned}$$

for the corresponding quotient manifolds.

Proof. By Theorem 2.1, the critical dimension δ is the same as the minimal dimension of an invariant conformal density. Thus for each n there exists a normalized Γ_n-invariant density μ_n of dimension

$\delta(\Gamma_n)$. Taking the weak limit of a subsequence, we obtain a Γ_G-invariant density μ of dimension $\liminf \delta(\Gamma_n)$, so $\delta(\Gamma_G) \leq \liminf \delta(\Gamma_n)$, again by Theorem 2.1.

Similarly, we can choose an $f \in C_0^\infty(M_G)$ such that the Ritz-Rayleigh quotient (2.1) is less than $\lambda_0(M_G)+\epsilon$. This f is supported on a compact submanifold $K \subset M_G$ that can be nearly isometrically embedded in M_n for all $n \gg 0$. It follows that $\limsup \lambda_0(M_n) \leq \lambda_0(M_G) + \epsilon$; now let $\epsilon \to 0$. □

Corollary 7.8. *Let $M = \mathbb{H}^3/\Gamma$ be a geometrically infinite, topologically tame hyperbolic 3-manifold. Then*

$$\begin{aligned} \delta(\Gamma_n) &\to \delta(\Gamma) = 2 \qquad \textit{and} \\ \lambda_0(M_n) &\to \lambda_0(M) = 0 \end{aligned}$$

whenever $\Gamma_n \to \Gamma$ algebraically or geometrically.

Proof. We have $\delta(\Gamma) = 2$ by Theorem 3.5. By the preceding result we have

$$2 \geq \limsup \delta(\Gamma_n) \geq \liminf \delta(\Gamma_n) \geq \delta(\Gamma) = 2$$

if the convergence is geometric. For the case of algebraic convergence, use the fact that any geometric limit Γ' contains the algebraic limit and thus $\liminf \delta(\Gamma_n) \geq \delta(\Gamma') = \delta(\Gamma) = 2$. The relation $\lambda_0 = \delta(2-\delta)$ completes the proof. □

Proof of Theorem 7.2 (Dynamic convergence). By strong convergence, for $n \gg 0$ there are surjective homomorphisms

$$\chi_n : \Gamma \to \Gamma_n$$

converging to the identity. By Theorem 4.1 we also know Γ_n is geometrically finite, the limit sets satisfy $\Lambda_n \to \Lambda$ and the truncated cores satisfy $K_\epsilon(M_n) \to K_\epsilon(M)$ strongly.

To complete the proof, we must show (D6): that the canonical measures μ_n for Γ_n converge weakly to the canonical measure μ for Γ. Pass to any subsequence such that μ_n has a weak limit ν, and $\delta_n = \delta(\Gamma_n)$ converges to some limit δ. Clearly ν is a Γ-invariant density of dimension δ.

To prove $\mu = \nu$, recall that μ is the unique Γ-invariant density supported on Λ with no atoms at the cusps (Corollary 3.3). Since $\Lambda_n \to$

Λ, ν is supported on Λ, so we need only check that $\nu(c) = 0$ for each cusp point $c \in \Lambda$.

To this end, fix $\epsilon > 0$ and a cusp point $c \in \Lambda$. We will construct a neighborhood U of c such that $\mu_n(U) < \epsilon$ for all $n \gg 0$.

Let $L \subset \Gamma$ be the stabilizer of c and let $L_n = \chi_n(L) \subset \Gamma_n$. Adjusting L_n by a conjugacy converging to the identity, we can assume c is also fixed by L_n. By Corollary 2.2, we have

$$\delta = \lim \delta(\Gamma_n) \geq \delta(\Gamma) > \operatorname{rank}(L)/2,$$

so the sequence (L_n, δ_n) fits into the setup discussed in §6.

We claim there exists a compact set $F \subset \mathbb{R}^d$ such that

$$\Lambda \subset L \cdot F \cup \{\infty\} \tag{7.1}$$

and, for all $n \gg 0$,

$$\Lambda_n \subset L_n \cdot F \cup \{\infty\}. \tag{7.2}$$

Indeed, by Theorem 4.1 there is an $\epsilon > 0$ less than the Margulis constant such that the truncated core $K_\epsilon(M)$ is the strong limit of $K_\epsilon(M_n)$. Let $H \subset \mathbb{H}^{d+1}$ be the horosphere such that $H/L \subset M$ is the component of $\partial M^{<\epsilon}$ corresponding to c. Projection along geodesics rays from c gives a natural bijection

$$\mathbb{R}^d_\infty / L \cong H/L \subset M.$$

Since every geodesic from c to Λ lies in the convex hull, this bijection sends Λ/L into $K_\epsilon(M)$. By compactness of $K_\epsilon(M)$, there is a compact set $F \subset \mathbb{R}^d$ such that

$$\Lambda/L \subset F/L \subset \mathbb{R}^d_\infty / L.$$

Thus we have (7.1), and strong convergence of $K_\epsilon(M_n)$ to $K_\epsilon(M)$ implies that a slight enlargement of F satisfies (7.2).

By hypothesis, we have $\delta > 1$ (if $d = 2$) or $\delta > (d-1)/2$ (if $d \neq 2$). So by Theorem 6.1, the Poincaré series for (L_n, δ_n) converge uniformly. Since F is compact, this means we can choose a neighborhood U of c such that

$$P_{\delta_n}(L_n, U, x) < \epsilon$$

for all $x \in F$ and $n \gg 0$.

Since $\mu_n(c) = 0$ and $L_n \cdot F$ covers the rest of the limit set Λ_n, we have

$$\begin{aligned}
\mu_n(U) &= \mu_n(U \cap (L_n \cdot F)) \leq \sum_{L_n} \mu_n(U \cap gF) \\
&= \sum_{L_n} \int_{\{x \in F \,:\, g(x) \in U\}} |g'(x)|_\sigma^{\delta_n} \, d\mu_n \\
&= \int_F P_{\delta_n}(L_n, U, x) \, d\mu_n \quad \leq \quad \epsilon \mu_n(F) \quad \leq \quad \epsilon.
\end{aligned}$$

Since ϵ was arbitrary, the weak limit ν has no atom at c. Thus $\mu = \nu$ and we have established the dynamic convergence. □

Proof of Theorem 7.3 (Radial limits). We first show $\Gamma_n \to \Gamma$ strongly. Let $\chi_n : \Gamma \to \Gamma_n$ be isomorphisms converging to the identity, and consider any maximal parabolic subgroup $L \subset \Gamma$. Set $L_n = \chi_n(L) \subset \Gamma_n$. The cusps of a hyperbolic 3-manifold have rank 1 or 2 ($= d$ or $d-1$), so if L_n is parabolic it converges strongly to L by Theorem 6.2. Otherwise L is generated by an accidental parabolic; but since we are assuming accidental parabolics converge radially, $L_n \to L$ strongly by Theorem 6.3. Thus all parabolic subgroups converge strongly, so $\Gamma_n \to \Gamma$ strongly by Theorem 4.2.

We therefore have the setup for Theorem 7.2. As before it suffices to show a weak limit ν of the canonical measure ν_n has no mass at a cusp point c. But Theorems 6.2 and 6.3 imply that the Poincaré series converges uniformly, so $\nu(c) = 0$ by the same reasoning as above. □

Proof of Corollary 7.4. Finitely generated Fuchsian groups are always geometrically finite, and all accidental parabolics converge radially (since the multipliers are in $\mathbb{R}^*$). Thus Theorem 7.3 applies. □

Proof of Corollary 7.5. Consider any subsequence such that $\lambda_0(M_n)$ converges to a limiting value λ. If $\lambda = d^2/4$, then

$$\frac{d^2}{4} \geq \lambda_0(M_G) \geq \lim \lambda_0(M_n) = \frac{d^2}{4}$$

by Theorem 7.7, so we have convergence of λ_0. Otherwise $\lambda < d^2/4$ and the relation $\lambda_0 = \delta(d - \delta)$ (Theorem 2.1) shows

$$\lim \delta(\Gamma_n) = \lambda(d - \lambda) > d/2.$$

By Theorem 7.2, $\delta(\Gamma_n) \to \delta(\Gamma_G)$ and thus $\lambda_0(M_n) \to \lambda(M_G)$ in this case as well. □

Proof of Corollary 7.6 (Strong limits). First suppose M is geometrically finite. Then the convergence of λ_0 is contained in Corollary 7.5. Since

$$1 \leq \mathrm{H.dim}(\Lambda) \leq \liminf \mathrm{H.dim}(\Lambda_n),$$

and M_n is geometrically finite for all $n \gg 0$, $\mathrm{H.dim}(\Lambda_n)$ is determined by $\lambda_0(M_n)$ (via the relation $\lambda_0 = \delta(2-\delta)$), so the dimensions of the limit sets also converge.

For M geometrically infinite, the convergence of dimension and λ_0 follows from Corollary 7.8. □

8. Examples of discontinuity

In this section we sketch three examples of the *discontinuous* behavior of the $\delta(\Gamma)$ as Γ varies. We only consider geometrically finite groups; thus we have $\delta(\Gamma) = \mathrm{H.dim}(\Lambda(\Gamma))$ throughout, and so these examples also demonstrate discontinuity of the dimension of the limit set.

I. $\delta(\Gamma)$ is not continuous in the geometric topology. Let $\Gamma_0 \subset \mathrm{Isom}(\mathbb{H}^{d+1})$ be a cocompact Kleinian group. By a result of Mal'cev, any finitely generated linear group is residually finite [21, Thm. VII]. Thus there is a descending sequence of subgroups of finite index,

$$\Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \dots,$$

such that $\bigcap \Gamma_n = \Gamma = \{1\}$.

The limit sets of these groups satisfy $\Lambda(\Gamma_n) = S^d_\infty$ for all n, while $\Lambda(\Gamma) = \emptyset$. So although we have $\Gamma_n \to \Gamma$ geometrically, the critical dimension $\delta(\Gamma_n) = d$ does not converge to $\delta(\Gamma) = 0$.

Note that the strong convergence fails dramatically in this example, since there is no surjection $\Gamma \to \Gamma_n$.

II. Convergence of limit sets vs. convergence of $\delta(\Gamma)$. Here is an example where δ is discontinuous even though Λ varies continuously.

Consider a sequence of open hyperbolic Riemann surfaces M_n of genus two with one end of infinite area. Let γ_n be a simple geodesic separating M_n into two subsurfaces X_n, Y_n of genus one, with $\mathrm{area}(X_n) = \infty$ and $\mathrm{area}(Y_n) = \pi$. Suppose that as $n \to \infty$ the length of γ_n tends to zero (see Figure 3). Then

$$\lambda_0(M_n) \leq \int |\nabla \phi_n|^2 \Big/ \int |\phi_n|^2 \to 0,$$

X_n Y_n

γ_n

FIGURE 3. Pinching an open Riemann surface.

where ϕ_n is supported on Y_n, $|\nabla\phi_n| = 1$ on small area neighborhood of γ_n, and $\phi_n = 1$ on the rest of Y_n. Thus the limit sets satisfy $\mathrm{H.dim}(\Lambda_n) \to 1$.

On the other hand, choosing base-frames in X_n, we can arrange the example so (M_n, ω_n) converges geometrically to a surface (M, ω) of genus one with one cusp and one infinite volume end. The area of $K(M_n)$ is constant by Gauss-Bonnet, so the injectivity radius in $K(M_n)$ is bounded above. Letting Λ denote the limit set of (M, ω), we then have

$$\Lambda_n \to \Lambda$$

in the Hausdorff topology by Proposition 4.6, but

$$\mathrm{H.dim}(\Lambda_n) \to 1 > \mathrm{H.dim}(\Lambda),$$

so the dimension is discontinuous.

In terms of the Laplacian, the positive ground state $\phi_n \in L^2(M_n)$ of norm 1 becomes more and more concentrated in Y_n as $n \to \infty$, and Y_n disappears in the limit. In terms of invariant densities, the canonical density μ_n for Γ_n becomes concentrated at the cusps of Γ, and any limit $\nu = \lim \mu_n$ is purely atomic.

III: $\delta(\Gamma)$ is not continuous in the strong topology. Our last example establishes Theorem 1.3 of the introduction, by showing $\mathrm{H.dim}(\Lambda)$ can jump even for a strongly convergent sequence of Kleinian groups.

Let Γ_t be the elementary Kleinian group generated by

$$\gamma_t(z) = e^{2\pi i t} z + 1,$$

where $t = 0$ or $\mathrm{Im}(t) > 0$. Then $M_t = \mathbb{H}^3/\Gamma_t$ is homeomorphic to a solid torus $S^1 \times D^2$. For $t \in \mathbb{H}$ the fundamental group of M_t is generated by a geodesic, while M_0 has a rank-one cusp.

As we saw in the proof of Theorem 5.1, we have:

1. *$\Gamma_t \to \Gamma_0$ strongly iff $t \to 0$ horocyclically.*

2. *If $t \to 0$ along a horocycle in $\mathbb{H}$, then a subsequence of Γ_t converges geometrically to a rank-two parabolic group $\Gamma' \supset \Gamma_0$.*

Indeed, the strong convergence occurs exactly when the complex torus $X_t = \Omega(\Gamma_t)/\Gamma_t$ converges geometrically to an infinite cylinder, and this means $t \to 0$ horocyclically. On the other hand, if $t \to 0$ along a horocycle, then $[X_t]$ remains in a compact subset of the moduli space $\mathbb{H}/SL_2(\mathbb{Z})$, and a subsequence converges to a torus $X_0 \cong \mathbb{C}/\Gamma'$.

Now the idea is to use the fact that $\delta(\Gamma) = r/2$ if $\Gamma \cong \mathbb{Z}^r$ is an elementary parabolic group of rank r. The discrepancy between the ranks of Γ_0 and Γ' will lead to a discontinuity in δ.

We would also like our example groups to be nonelementary. To this end, for $R > 4$ let

$$G_0 = \left\langle z \mapsto \frac{z}{Rz+1},\ z \mapsto z+1 \right\rangle.$$

Then $G_0 \cong \mathbb{Z} * \mathbb{Z}$ is a Fuchsian group; $\mathbb{H}/G_0$ is a pair of pants with two cusps and one infinite volume end. (When $R = 4$ the quotient is the triply-punctured sphere.) Since G_0 is geometrically finite and its limit set is a proper subset of $S^1_\infty = \mathbb{R} \cup \{\infty\}$, we have

$$\delta(G_0) < 1.$$

In fact, $\delta(G_0)$ tends to $1/2$ continuously as $R \to \infty$, so for any $0 < \epsilon < 1/2$ we can choose $R > 4$ such that

$$\delta(G_0) = 1/2 + \epsilon.$$

(See the discussion of Hecke groups in [24]).

We can also think of G_0 in terms of the Klein-Maskit combination theory. A fundamental domain for $z \mapsto z/(Rz+1)$ is the region exterior to the two tangent disks

$$D = \{z\ :\ |z \pm 1/R| < 1/R\}.$$

Since we assume $R > 4$, D is properly contained in a fundamental domain

$$F = \{z\ :\ |\operatorname{Re} z| \leq 1/2\}$$

$F - D$

$-1/2$ $-2/R$ 0 $2/R$ $1/2$

FIGURE 4. A fundamental domain for G_0.

for the other generator $\gamma_0(z) = z+1$ of G_0. Thus $F-D$ is a fundamental domain for G_0 (Figure 4). From this picture one sees G_0 is discrete and free on its generators, and the manifold $N_0 = \mathbb{H}^3/G_0$ is isomorphic to a connect sum of two copies of M_0.

Now let

$$G_t = \left\langle z \mapsto \frac{z}{Rz+1},\ z \mapsto e^{2\pi i t} z + 1 \right\rangle .$$

Note that $G_t \supset \Gamma_t$. For all t inside a small horoball resting on $t = 0$, we also have $D \subset F_t$ for a suitable fundamental domain for Γ_t. (This is because the torus $\Omega(\Gamma_t)/\Gamma_t$ approximates the infinite cylinder $\Omega(\Gamma_0)/\Gamma_0$.) Thus G_t is also free and discrete, and $N_t = \mathbb{H}^3/G_t$ is the connect sum of M_t and M_0. As in the discussion of Γ_t, we have $G_t \to G_0$ strongly iff $t \to 0$ horocyclically (by Theorem 5.1).

> *3. Any sufficiently small horocycle in $\mathbb{H}$ resting on $t = 0$ contains parameters t such that $\delta(G_t) > 1$.*

Indeed, by Theorem 7.7 any geometric limit G' of G_t satisfies $\delta(G') \leq \liminf \delta(G_t)$, while for t moving along a horocycle (2) above says G' contains a rank two parabolic subgroup Γ', and thus $\delta(G') > 1$ (Corollary 2.2).

> *4. There is a sequence $t_n \in \mathbb{H}$ such that $G_{t_n} \to G_0$ strongly, but*
>
> $$\lim \delta(G_{t_n}) = 1 > \delta(G_0) = \frac{1}{2} + \epsilon.$$

To construct t_n, simply choose horocycles H_n converging to 0 and $t_n \in H_n$ such that $\delta(G_{t_n}) > 1$; then $t_n \to 0$ horocyclically, so $G_{t_n} \to G_0$

strongly, but $\delta(G_0) = 1/2 + \epsilon < 1$. It follows that $\delta(G_{t_n}) \to 1$, since otherwise δ would be continuous by Theorem 7.2.

By (4), δ is *discontinuous* in the strong topology.

Sharpness. This jump in dimension from 1 down to $1/2+\epsilon$ is essentially sharp. Indeed, suppose $G_0 \subset \mathrm{Isom}(\mathbb{H}^3)$ is geometrically finite and $G_n \to G_0$ strongly. If $\lim \delta(G_n) > 1$, then $\delta(G_n) \to \delta(G_0)$ by Theorem 7.2; and if $\delta(G_0) \leq 1/2$, then G_0 is convex cocompact (Corollary 2.2), so we have continuity by Theorem 7.1.

9. Quasifuchsian groups

The Teichmüller space of a surface S leads, via Bers embedding, to many examples of Kleinian groups with interesting algebraic and geometric limits. In this section we study the dimension of the limit set $\Lambda(X, Y)$ of the quasifuchsian group obtained by gluing together the universals covers of two surfaces X and Y in the Teichmüller space of S.

Dimension as a function on Teichmüller space. We begin by recalling some facts from Teichmüller theory [7], [23, §3]. Let S be a connected compact oriented surface with $\chi(S) < 0$, and let $\mathrm{Teich}(S)$ denote the Teichmüller space of Riemann surfaces X marked by S. The Teichmüller metric is defined by

$$d(X, X') \quad = \quad \frac{1}{2} \inf \, \log K(\phi),$$

where $\phi : X \to X'$ ranges over all quasiconformal mappings compatible with markings, and $K(\phi) \geq 1$ denotes the quasiconformal dilatation of ϕ.

Let $AH(S)$ denote the discrete faithful representations of $\pi_1(S)$ into $\mathrm{Isom}^+(\mathbb{H}^3)$, modulo conjugacy, equipped with the topology of algebraic convergence. One can also think of $AH(S)$ as the space of complete hyperbolic 3-manifolds M homotopy equivalent to S.

Let $\overline{S}$ denote S with its orientation reversed. For any pair of Riemann surfaces

$$(X, Y) \in \mathrm{Teich}(S) \times \mathrm{Teich}(\overline{S})$$

we can construct a *quasifuchsian manifold*

$$Q(X, Y) = \mathbb{H}^3 / \Gamma(X, Y)$$

marked by $S\times[0,1]$ and hence residing in $AH(S)$. The limit set $\Lambda(X,Y)$ is a quasicircle, and the domain of discontinuity satisfies

$$\Omega(X,Y)/\Gamma(X,Y)\cong X\sqcup Y$$

with markings respected.

Proposition 9.1. *The function* $\mathrm{H.dim}\,\Lambda(X,Y)$ *is uniformly Lipschitz on* $\mathrm{Teich}(S)\times\mathrm{Teich}(\overline{S})$.

Proof. Let $K=e^{2t}$ where $t=\max(d(X_1,X_2),d(Y_1,Y_2))$. Then there is a K-quasiconformal conjugacy between $\Gamma(X_1,Y_1)$ and $\Gamma(X_2,Y_2)$, and hence a K-quasiconformal map between their limit sets Λ_1 and Λ_2, where $K=e^{2t}$. Since K-quasiconformal maps are $1/K$-Hölder continuous [1, III.C], by general properties of Hausdorff dimension we have

$$\frac{1}{K}\,\mathrm{H.dim}(\Lambda_2)\le\mathrm{H.dim}(\Lambda_1)\le K\,\mathrm{H.dim}(\Lambda_2),$$

and therefore

$$|\,\mathrm{H.dim}(\Lambda_1)-\mathrm{H.dim}(\Lambda_2)|\le 2(K-1)\le 4t+O(t^2),$$

using the fact that both dimensions are ≤ 2. Thus $\mathrm{H.dim}\,\Lambda(X,Y)$ is Lipschitz with constant 4. $\square$

Sharper estimates can be obtained using [4].

Iteration on a Bers slice. The subspace

$$B_Y=\{Q(X,Y)\ :\ X\in\mathrm{Teich}(S)\}\subset QF(S)$$

is a *Bers slice* of quasifuchsian space; it gives a complex-analytic model for $\mathrm{Teich}(S)$ as a space of Kleinian groups.

Each element ϕ in the mapping class group $\mathrm{Mod}(S)$ determines an automorphism of B_Y by sending $Q(X,Y)$ to $Q(\phi(X),Y)$. It is a fundamental fact that a Bers slice B_Y has compact closure in $AH(S)$ [7], so it is interesting to investigate the behavior of $Q(\phi^n(X),Y)$ as $n\to\infty$.

We begin with the case of Dehn twists. A *partition* $\mathcal{P}=\{C_1,\dots,C_r\}$ on S is a collection of isotopy classes of essential disjoint simple closed curves, with no two parallel and none parallel to ∂S. Let $M_{\mathcal{P}}(X,Y)$ denote the unique geometrically finite hyperbolic 3-manifold with

$$M_{\mathcal{P}}(X,Y)\cong\mathrm{int}\left(S\times[0,1]-\bigcup_{\mathcal{P}}C_i\times\{1/2\}\right)$$

as a topological space, with cusps of rank 1 along $\partial S \times [0,1]$ and of rank 2 along $\bigcup C_i \times \{1/2\}$, and with conformal boundary $X \sqcup Y$ corresponding to $S \times \{0,1\}$.

By [20] and [11], [12] we have:

Theorem 9.2. *Let $\tau \in \mathrm{Mod}(S)$ be a product of Dehn twists*

$$\tau = \tau_{C_1}^{p_1} \cdots \tau_{C_r}^{p_r}$$

about the curves in a partition $\mathcal{P}$, with each $p_i \neq 0$. Then as $n \to \infty$,

$$M_n = Q(\tau^n(X), Y)$$

converges to algebraic and geometric limits M_A and M_G, with

$$M_G \cong M_{\mathcal{P}}(X, Y)$$

and with M_A the covering space of M_G corresponding to $\pi_1(Y)$.

More precisely, there exists a choice of baseframes $\omega_n \in M_n$ such that algebraic and geometric convergence as above is obtained.

We can now apply the results of §7 to arrive at:

Theorem 9.3. *For any partition $\mathcal{P} \neq \emptyset$ and product of Dehn twists τ as above, the limit sets satisfy*

$$\mathrm{H.dim}(\Lambda_A) < \mathrm{H.dim}(\Lambda_G) = \lim \mathrm{H.dim}(\Lambda(\tau^n(X), Y)) < 2.$$

Proof. The algebraic and geometric limits M_A and M_G both exist, and M_G is geometrically finite, so $M_n \to M_G$ strongly by Lemma 4.4. Thus $\mathrm{H.dim}(\Lambda_n) \to \mathrm{H.dim}(\Lambda_G)$ by Theorem 7.2, and $\mathrm{H.dim}(\Lambda_G) < 2$ since the limit set is not the whole sphere.

To show the dimension of Λ_A is strictly less than that of Λ_G, first note that M_A is also geometrically finite. If the dimensions of the limit sets were to agree, then the canonical measures would satisfy $\mu_A = \mu_G$ by Theorem 3.1, since there is a unique normalized Γ_A-invariant density in the critical dimension $\delta(\Gamma_A)$. But then the supports of the canonical measures would coincide, which is impossible since $\Lambda_A \neq \Lambda_G$. $\square$

Example. Figure 5 shows the limit sets Λ_A and Λ_G for τ a single Dehn twist on a surface of genus two. The parameters for this example were computed by Jeff Brock [11], [12].

FIGURE 5. Algebraic and geometric limits.

Corollary 9.4. *For any mapping class* $\phi \in \operatorname{Mod}(S)$*, there is an* $i > 0$ *such that*

$$\delta(X,Y) = \lim_{n\to\infty} \operatorname{H.dim}(\Lambda(\phi^{ni}(X),Y))$$

exists for every (X,Y)*, and either*

1. ϕ^i *is a product of disjoint Dehn twists, and* $\delta(X,Y) < 2$*, or*
2. ϕ^i *is pseudo-Anosov on a subsurface, and* $\delta(X,Y) = 2$*.*

Proof. The fact that an iterate ϕ^i is either a (possibly trivial) product of Dehn twists, or pseudo-Anosov on a subsurface, follows from Thurston's classification of surface diffeomorphisms [18], [36]. The first case is handled by the preceding result. For the second (pseudo-Anosov) case, by [11], [12], i can be chosen so the manifolds $Q(\phi^{ni}(X),Y)$ converge geometrically to a topologically tame, geometrically infinite manifold M_G, and hence $\delta = 2$ by Corollary 7.8. $\square$

Pinching a geodesic. Next we investigate approach to infinity by pinching the curves in a partition $\mathcal{P}$.

Theorem 9.5. *Suppose* $X_n \to \infty$ *in* $\operatorname{Teich}(S)$ *and for all simple geodesics* C *we have*

$$\ell_C(X_n) \to L(C) \in [0,\infty].$$

Let $\mathcal{P}$ be the partition consisting of all C with $L(C) = 0$, and suppose $L(C) < \infty$ if C is disjoint from $\bigcup \mathcal{P}$. Then there is a geometrically finite manifold M_A such that

1. $Q(X_n, Y) \to M_A$ *strongly;*
2. $\mathcal{P}$ *is the set of all accidental parabolics on* M_A*;*
3. *All accidental parabolics converge radially; and*
4. $\mathrm{H.}\dim \Lambda(X_n, Y) \to \mathrm{H.}\dim \Lambda_A$.

Proof. By the conditions on $\ell_C(X_n)$, the sequence X_n tends to a limit Z in the Deligne-Mumford compactification of the moduli space of S. Here Z is a 'surface with nodes' naturally marked by

$$T = S - \bigcup \mathcal{P};$$

it has cusps along the components of $\mathcal{P}$, while its geometry outside $\mathcal{P}$ is fixed by the limiting lengths of curves $C \notin \mathcal{P}$. The convergence is compatible with marking outside $\mathcal{P}$. (Compare [19, Appendix A].)

Pass to any subsequence such that the algebraic limit $M_A \in AH(S)$ exists. We claim M_A is the unique geometrically finite manifold with rank-1 cusps along $\mathcal{P}$ and with marked conformal boundary

$$\partial M_A = \Omega(\Gamma_A)/\Gamma_A = Z \sqcup Y.$$

To see this, first note that M_A has a cusp at each $C \in \mathcal{P}$ because

$$\ell_C(M_A) = \lim \ell_C(Q(X_n, Y)) \le 2 \lim \ell_C(X_n) = 0$$

(by Bers' inequality, cf. [7], [22, §6.3]). Next we show $\partial M_A = Z \sqcup Y$. Consider on T any pair of simple curves C and D with positive geometric intersection number. The geodesic representative $\gamma(C)$ intersects the ruled cylinder $D \times I$ between the representatives of D on the two faces of the convex core of $Q(X_n, Y)$. Since $\ell_D(X_n)$ and $\ell_D(Y)$ are bounded, $\gamma(C)$ meets an essential loop $\delta(D) \subset D \times I$ of bounded length such that $\langle \gamma(C), \delta(D) \rangle \subset \pi_1(Q(X_n, Y))$ is a nonelementary group. By the Margulis lemma, the length of $\gamma(C)$ is bounded below, and hence the two faces of the convex hull of $Q(X_n, Y)$ are a bounded distance apart (independent of n) along C. It follows that T persists as a subsurface of the conformal boundary of the algebraic limit, and thus $Z \sqcup Y \subset$

∂M_A. But the conformal boundary can be no larger than this by area considerations.

We conclude that M_A is the geometrically finite manifold described above. Now for each $C \in \mathcal{P}$, the quotient torus for $\pi_1(C) \subset \Gamma(X_n, Y)$ becomes long and thin as $n \to \infty$, since it contains the $\pi_1(C)$-covering space of X_n, itself an annulus of large modulus. Thus each accidental parabolic converges horocyclically, so $Q(X_n, Y) \to M_A$ strongly by Theorem 5.1 (or by inspection). Since the limit set is connected, we have $\mathrm{H.dim}(\Lambda_A) \geq 1$; therefore $\mathrm{H.dim}(\Lambda(X_n, Y) \to \mathrm{H.dim}(\Lambda_A)$ by Corollary 7.6. $\square$

Fenchel-Nielsen coordinates. To construct a sequence X_n as above, let $\mathcal{P}$ be a maximal partition of S. Then one obtains an isomorphism

$$\mathrm{Teich}(S) \cong \mathbb{R}_+^{\mathcal{P}} \times \mathbb{R}^{\mathcal{P}}$$

by the map

$$X \mapsto (\ell_C(X), \tau_C(X))$$

sending a surface to its Fenchel-Nielsen length and twist parameters (see, e.g. [19]). Consider any sequence of these coordinates $(L_n(C), T_n(C))$, $C \in \mathcal{P}$, tending to limiting values $(L(C), T(C))$ where some $L(C) = 0$. Then it is easy to see the corresponding Riemann surfaces X_n satisfy the hypotheses of the Theorem above. The manifolds $Q(X_n, Y)$ tend to M_A strongly, where $\partial M_A = Z \sqcup Y$ and Z is a surface with nodes obtained by gluing together (possibly degenerate) pairs of pants with the limiting length and twist parameters $(L(C), T(C))$.

In fact one need only require that $T_n(C)$ converge for those $C \in \mathcal{P}$ with $L(C) > 0$, since twists along the accidental parabolics have no effect on Z. With this proviso, any sequence X_n satisfying the Theorem arises via the construction above.

The Fenchel-Nielsen twist. Finally we describe some interesting periodic behavior that occurs for the continuous version of a Dehn twist. Fix a basepoint $X_0 \in \mathrm{Teich}(S)$ and a simple closed geodesic C on X_0 of length L. Define X_t by cutting along C, twisting distance tL to the right, and regluing. (In terms of Fenchel-Nielsen coordinates, this means only the twist parameter $\tau_C(X)$ is varied.) The resulting continuous path in Teichmüller space represents a Fenchel-Nielsen twist deformation of X_0.

By construction

$$X_{t+1} = \tau(X_t),$$

where $\tau \in \mathrm{Mod}(S)$ is a right Dehn twist about C. Thus the geometric limit

$$M_G(t) \;=\; \lim_{n\to\infty} Q(X_{t+n}, Y) \;=\; \lim Q(\tau^n(X_t), Y) \;=\; M_{\mathcal{P}}(X_t, Y)$$

satisfies $M_G(t+1) = M_G(t)$; here $\mathcal{P} = \{C\}$. Let $\delta(t)$ denote the Hausdorff dimension of the limit set of $M_G(t)$.

Theorem 9.6. *The function $\delta(t)$ is continuous, $\delta(t+1) = \delta(t)$ and*

$$\lim_{t\to\infty} |\operatorname{H.dim}(\Lambda(X_t, Y)) - \delta(t)| = 0.$$

Proof. By definition, $\delta(t+1) = \delta(t)$. To see δ is continuous, first note that by Theorem 9.3 we have

$$\delta(t) = \lim_{n\to\infty} f(t+n), \tag{9.1}$$

where $f(t) = \operatorname{H.dim}\Lambda(X_t, Y)$. The map $t \mapsto X_t$ is uniformly continuous in the Teichmüller metric, because $X_{t+1} = \tau(X_t)$ and τ is an isometry; since $\operatorname{H.dim}\Lambda(X, Y)$ is Lipschitz, $f(t)$ is also uniformly continuous. Thus (9.1) converges uniformly for $t \in [0, 1]$; therefore $\delta(t)$ is continuous and the Theorem follows. $\square$

Now recall we have $Q(X_t, Y) \to M_A$ algebraically as $t \to \infty$. If $\delta(t)$ is nonconstant (as seems likely), then the Hausdorff dimension of the limit set of $Q(X, Y)$ oscillates like $\sin(1/x)$ as $Q(X, Y)$ converges to M_A along a Fenchel-Nielsen horocycle.

See [17] for similar results on the dimension of the Julia set of $z^2 + c$ as $c \to 1/4$.

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