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# EQUIVARIANT HOLOMORPHIC MORSE INEQUALITIES II: TORUS AND NON-ABELIAN GROUP ACTIONS

SIYE WU

**Abstract**

We extend the equivariant holomorphic Morse inequalities of circle actions to cases with torus and non-Abelian group actions on holomorphic vector bundles over Kähler manifolds and show the necessity of the Kähler condition. For torus actions, there is a set of inequalities for each choice of action chambers specifying directions in the Lie algebra of the torus. We apply the results to invariant line bundles over toric manifolds. If the group is non-Abelian, there is in addition an action of the Weyl group on the fixed-point set of its maximal torus. The sum over the fixed points can be rearranged into sums over the Weyl group (having incorporated the character of the isotropy representation on the fiber) and over the orbits of the Weyl group in the fixed-point set.

## 1. Introduction

Index theorems express analytical indices of elliptic complexes in terms of topological invariants; information on the individual cohomology groups are usually obtained with the aid of vanishing theorems. Taking the de Rham complex for example, the Euler number is not enough to determine the Betti numbers. However, if we consider a Morse function, then the Morse inequalities bound each Betti number by the data of the critical points. In this paper, we consider a holomorphic setting in which a compact group acts holomorphically on a holomorphic vector bundle over a Kähler manifold. The index theorem is the Atiyah-Bott fixed-point formula [2] (when the fixed points are isolated), which expresses the equivariant index, the alternating sum of the characters of the Dolbeault cohomology groups, in terms of the

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fixed-point data. The corresponding equivariant holomorphic Morse inequalities when the group is the circle group was obtained by Witten [24] and was first proved analytically using the heat kernel method by Mathai and the present author [19] when the fixed points are isolated. In this paper, we extend the result to cases with torus and non-Abelian group actions. We also show that the Kähler condition is essential for such Morse-type inequalities although, in contrast, not necessary for the equivariant index formula of Atiyah and Bott.

Let $M$ be a compact Kähler manifold of complex dimension $n$ and $E$, a holomorphic vector bundle over $M$. The Dolbeault cohomology groups $H^*(M,\mathcal{O}(E))$ with coefficients in $E$ are cohomologies of the twisted Dolbeault complex $(\Omega^{0,*}(M,E),\bar\partial_E)$ and are independent of the choice of holomorphic connections. Let $G$ be a compact, connected Lie group whose Lie algebra is denoted by $\mathrm{Lie}(G)$. Let $\mathfrak{g}=\sqrt{-1}\,\mathrm{Lie}(G)$. We assume that $G$ acts holomorphically and effectively on $M$ preserving the Kähler form $\omega$. If the fixed-point set of a maximal torus in $G$ is non-empty, then the $G$-action is Hamiltonian [8], i.e., there is a moment map $\mu\colon M\to\mathfrak{g}^*$ such that for any $x\in\mathfrak{g}$, the corresponding vector field $V_x$ on $M$ satisfies $i_{V_x}\omega=d\langle\mu,x\rangle$. If the action of $g\in G$ on $M$ (still denoted by $g$) can be lifted holomorphically to $\tilde g$ on $E$, then $G$ acts on the space of sections $\Gamma(M,E)$ by $g\colon s\mapsto\tilde g\circ s\circ g^{-1}$ $(g\in G)$ and similarly on $\Omega^*(M,E)$. In this case, $G$ commutes with the twisted Dolbeault operator $\bar\partial_E$ and hence acts on the cohomology groups $H^k=H^k(M,\mathcal{O}(E))$ $(0\le k\le n)$. The purpose of this paper is to study the decomposition of each $H^k$ in terms of the irreducible representations of $G$.

In sections 2 and 3, the group acting on $M$ is a torus $T$. Section 2 shows that there is a set of Morse-type inequalities for each choice of action chambers specifying the directions in the Lie algebra $\mathfrak{t}$. This is obtained by applying the result of [24], [19] to various circle subgroups of $T$. Section 3 applies the result of the previous section to $T$-invariant line bundles over toric manifolds (including projective spaces and Hirzebruch surfaces). In section 4, we demonstrate that for non-Kähler manifolds, the strong equivariant holomorphic Morse inequalities need not hold. Violation of strong inequalities is also used to show the non-existence of invariant Kähler structures. In section 5, the group $G$ is a general compact non-Abelian group. The main result is obtained by applying that of section 2 to a maximal torus $T$ of $G$, assuming that the $T$-fixed-point set $F$ is discrete. The novelties are that the cohomology groups $H^k$ $(0\le k\le n)$ are representations of the non-Abelian group $G$ (hence the structure of $H^k$ is more rigid) and that there exists an action of the

Weyl group $W$ on $F$ (hence the set $F$ can be organized into $W$-orbits).

Throughout this paper, $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}$ will denote the sets of complex numbers, real numbers, rational numbers, integers, non-negative integers, respectively.

## 2. Equivariant holomorphic Morse inequalities with torus actions

In this section, we assume that the compact Lie group that acts on the Kähler manifold is a torus group $T$. Let $\mathfrak{t} = \sqrt{-1}\,\mathrm{Lie}(T)$ be the Lie algebra of $T$ and $\mathcal{L}$, the integral lattice in $\mathfrak{t}$; the dual lattice $\mathcal{L}^*$ in $\mathfrak{t}^*$ is the weight lattice.

**Definition 2.1.** Let $\mathbb{Z}[\mathcal{L}^*]$ be the formal character ring of $T$ consisting of elements $q = \sum_{\xi\in\mathcal{L}^*} q_\xi e^\xi$ ($q_\xi \in \mathbb{Z}$). We say $q \geq 0$ if $q_\xi \geq 0$ for all $\xi \in \mathcal{L}^*$. The support of $q$ is the set $\mathrm{supp}(q) = \{\xi \in \mathcal{L}^* \mid q_\xi \neq 0\}$. Let $Q(t) = \sum_{k=0}^{n} q_k t^k \in \mathbb{Z}[\mathcal{L}^*][t]$ be a polynomial of degree $n$ with coefficients in $\mathbb{Z}[\mathcal{L}^*]$. We say $Q(t) \geq 0$ if $q_k \geq 0$ in $\mathbb{Z}[\mathcal{L}^*]$ for all $k$. For two such polynomials $P(t)$ and $Q(t)$, we say $P(t) \leq Q(t)$ if $Q(t) - P(t) \geq 0$.

For example, if $V$ is a finite dimensional representation of $T$, then its character $\mathrm{char}(V) \geq 0$ in $\mathbb{Z}[\mathcal{L}^*]$. Let the *support* of $V$, denoted by $\mathrm{supp}(V)$, be the set $\mathrm{supp}(\mathrm{char}(V))$ of weights whose multiplicity in $V$ is non-zero. For any $\theta \in \mathfrak{t}$, there is a homomorphism $\mathbb{Z}[\mathcal{L}^*] \to \mathbb{C}$ given by $e^\xi \mapsto e^{\sqrt{-1}\langle\xi,\theta\rangle}$. For instance, $\mathrm{char}(V) \mapsto \mathrm{tr}_V e^{\sqrt{-1}\theta}$ under this homomorphism. Another important type of elements in $\mathbb{Z}[\mathcal{L}^*]$ is given by the series

$$\frac{e^\eta}{1-e^\xi} \overset{\text{def.}}{=\!=} \sum_{k=0}^{\infty} e^{k\xi+\eta}, \quad \xi, \eta \in \mathcal{L}^*. \tag{2.1}$$

We emphasize here that in (2.1) the left-hand side is a notation for the formal series on the right-hand side.

Recall we assumed that $T$ acts holomorphically and effectively on a compact Kähler manifold $M$ with a non-empty and discrete fixed-point set $F$. We also assume that the $T$-action preserves the Kähler form $\omega$ and hence is Hamiltonian [8]; let $\mu\colon M \to \mathfrak{t}^*$ be the moment map. For any $p \in F$, let $\lambda_1^p, \cdots, \lambda_n^p \in \mathcal{L}^*\backslash\{0\}$ be the weights of the isotropy representation of $T$ on $T_pM$ (Figure 2.1(a)). The hyperplanes $(\lambda_k^p)^\perp \subset \mathfrak{t}$ cut $\mathfrak{t}$ into open polyhedral cones called *action chambers* [12], [13], [21]. (When $M$ is a coadjoint orbit, the action chambers are precisely the Weyl chambers.) We fix a *positive* action chamber $C$ (Figure 2.1(b)). Let $\lambda_k^{p,C} = \pm\lambda_k^p$ be the *polarized weights*, with the sign chosen so that $\lambda^{p,C} \in C^*$. (Here $C^*$ is the dual cone in $\mathfrak{t}^*$ defined by $C^* = \{\xi \in \mathfrak{t}^* \,|\, \langle \xi, C\rangle > 0\}$.) We define the *polarizing index* $n^{p,C}$ of $p$ with respect to $C$ as the number of weights $\lambda_k^p \in -C^*$. We also assume that there is a holomorphic vector bundle $E$ over $M$ on which the $T$-action lifts holomorphically. The torus $T$ acts on the fiber $E_p$ over $p \in F$ and on the cohomology groups $H^k = H^k(M, \mathcal{O}(E))$ $(0 \le k \le n)$. Following [24], [19], we denote $E_p(\theta) = \mathrm{tr}_{E_p} e^{\sqrt{-1}\theta}$ $(p \in F)$ and $H^k(\theta) = \mathrm{tr}_{H^k} e^{\sqrt{-1}\theta}$.

**Theorem 2.2.** *For each choice of the positive action chamber $C$, we have the strong equivariant holomorphic Morse inequalities*

$$\begin{aligned}\sum_{p\in F} t^{n^{p,C}} \operatorname{char}(E_p) \prod_{\lambda_k^p \in C^*} \frac{1}{1-e^{-\lambda_k^p}} \prod_{\lambda_k^p \in -C^*} \frac{e^{-\lambda_k^{p,C}}}{1-e^{-\lambda_k^{p,C}}} \\ = \sum_k t^k \operatorname{char}(H^k) + (1+t) Q^C(t)\end{aligned} \tag{2.2}$$

*for some $Q^C(t) \ge 0$ in $\mathbb{Z}[\mathcal{L}^*]$.*

**Remark 2.3.** The strong inequalities (2.2) imply the following weak equivariant holomorphic Morse inequalities:

$$\operatorname{char}(H^k) \le \sum_{n^{p,C}=k} \operatorname{char}(E_p) \prod_{\lambda_j^p \in C^*} \frac{1}{1-e^{-\lambda_j^p}} \prod_{\lambda_j^p \in -C^*} \frac{e^{-\lambda_j^{p,C}}}{1-e^{-\lambda_j^{p,C}}} \quad (0 \le k \le n). \tag{2.3}$$

Setting $t = -1$ in (2.2), we recover the Atiyah-Bott fixed-point theorem

$$\sum_{p\in F} \frac{\operatorname{char}(E_p)}{\prod_{k=1}^n (1-e^{-\lambda_k^p})} = \sum_{k=0}^n (-1)^k \operatorname{char}(H^k) = \mathrm{Ind}. \tag{2.4}$$

**Remark 2.4.** If $T$ is the circle group $S^1$, then

$$\mathfrak{t} = \sqrt{-1}\,\mathrm{Lie}(S^1) \cong \mathbb{R}.$$

There are only two action chambers $C^\pm = \{\theta \in \mathbb{R} \,|\, \pm\theta > 0\}$. The polarizing index $n^{p,C^+}$ of $p$ with respect to $C^+$ is the number of weights

$\lambda^p_k < 0$, denoted by $n_p$. Theorem 2.2 reduces to the results in [24], [19]. In particular, for $C = C^+$, (2.2) becomes

$$\begin{aligned}\sum_{p\in F} t^{n_p} E_p(\theta) \prod_{\lambda^p_k>0} \frac{1}{1-e^{-\sqrt{-1}\lambda^p_k\theta}} \prod_{\lambda^p_k<0} \frac{e^{-\sqrt{-1}|\lambda^p_k|\theta}}{1-e^{-\sqrt{-1}|\lambda^p_k|\theta}} \\ = \sum_{k=0}^{n} t^k H^k(\theta) + (1+t) Q^+(\theta,t),\end{aligned} \tag{2.5}$$

where $Q^+(\theta,t) \ge 0$ in $\mathbb{R}((e^{\sqrt{-1}\theta}))$, the ring of formal characters of $S^1$ [19].

*Proof of Theorem 2.2.* We notice that (2.2) is equivalent to

$$\begin{aligned}\sum_{p\in F} t^{n^{p,C}} E_p(\theta) \prod_{\lambda^p_k\in C^*} \frac{1}{1-e^{-\sqrt{-1}\langle\lambda^p_k,\theta\rangle}} \prod_{\lambda^p_k\in -C^*} \frac{e^{-\sqrt{-1}\langle\lambda^{p,C}_k,\theta\rangle}}{1-e^{-\sqrt{-1}\langle\lambda^{p,C}_k,\theta\rangle}} \\ = \sum_k t^k H^k(\theta) + (1+t) Q^C(t)(\sqrt{-1}\theta)\end{aligned} \tag{2.6}$$

regarded as an equality of analytic functions in $\theta \in \mathfrak{t}^{\mathbb{C}}$ for $\operatorname{Im}\theta \in -C$. (2.2) implies (2.6) because for any $\delta \in C$ the Taylor expansions on the left-hand side of (2.6) are uniformly convergent in $\theta \in \mathfrak{t}^{\mathbb{C}}$ if $\operatorname{Im}\theta \in -(\delta+\overline{C})$. Conversely, if (2.6) is true for complex $\theta \in \mathfrak{t}^{\mathbb{C}}$ with $\operatorname{Im}\theta \in -C$, we take $\operatorname{Im}\theta \to 0$ within the cone $-C$. The equality (2.2) of formal series follows from the uniqueness of the Fourier expansion of tempered distributions [16]. For $S^1$-actions [24], [19], (2.5) is a true equality both in $\mathbb{R}((e^{\sqrt{-1}\theta}))$ and as analytic functions for $\operatorname{Im}\theta < 0$. The rest of the proof, which is similar to that of [21], Theorem 2.2, shows (2.6) using (2.5). By analyticity, it suffices to prove (2.6) for purely imaginary $\theta$. Pick an integral point $\theta_1 \in C \cap \mathcal{L}$. Since $\langle\lambda^p_k,\theta_1\rangle \ne 0$ for all $p \in F$ and $1 \le k \le n$, $h = \langle\mu,\theta_1\rangle$ generates a Hamiltonian $S^1$-action on $M$ with the same fixed-point set $F$. Moreover the weights of $S^1$ at $p \in F$ are $\langle\lambda^p_k,\theta_1\rangle$ $(1 \le k \le n)$ and $|\langle\lambda^p_k,\theta_1\rangle| = \langle\lambda^{p,C}_k,\theta_1\rangle > 0$ for all $p \in F$ and $1 \le k \le n$. (2.5) implies (2.6) for $\theta = -\sqrt{-1}s\theta_1$ $(s > 0)$. The result follows from continuity since such $\theta$'s form a dense subset of $-\sqrt{-1}C$.
q.e.d.

The inequalities (2.3) or (2.2) show that the multiplicities of weights in the cohomology groups $H^k$ $(0 \le k \le n)$ are constrained by the fixed-point data. Given an action chamber $C$, the support of $H^k$ is contained in a suitably shifted cone $-C^*$ in $\mathcal{L}^*$. By choosing different chambers, it is possible to bound supp$H^*$ $(0 \le k \le n)$ in various directions in $\mathfrak{t}^*$. We need the following definition. (See Figure 2.1(c),(d).)

**Definition 2.5.** For a given choice of positive action chamber $C$ and $p \in F$, let

$$\Gamma^{p,C} = \{\xi - \textstyle\sum_{k=1}^n r_k \lambda_k^{p,C} | \xi \in \operatorname{supp}(E_p), r_k \ge 0 \quad \textit{and } r_k > 0 \textit{ if } \lambda_k^p \in -C^*\} \tag{2.7}$$

and

$$\Gamma^{k,C} = \bigcup_{p\in F, n^{p,C}=k} \Gamma^{p,C}. \tag{2.8}$$

We set $\Gamma^k = \bigcap_C \Gamma^{k,C}$.

**Proposition 2.6.** *For any* $0 \le k \le n$,

$$\operatorname{supp}(H^k) \subset \mathcal{L}^* \cap \bigcap_C \Gamma^{k,C} = \mathcal{L}^* \cap \Gamma^k, \tag{2.9}$$

$$\operatorname{supp}(H^k) \supset \mathcal{L}^* \cap \bigcup_C \Gamma^{k,C} \backslash (\Gamma^{k-1,C} \cup \Gamma^{k+1,C}). \tag{2.10}$$

*Proof.* (2.9) follows from the weak inequalities with all choices of $C$. (2.10) follows from the strong inequalities: If $\xi \in \mathcal{L}^* \cap \Gamma^{k,C}$ but $\xi \notin \Gamma^{k\pm1,C}$, then the polynomial $Q^C(t) = \sum_{k=0}^n \sum_{\xi\in\mathcal{L}^*} q_\xi^k e^\xi t^k$ in (2.2) has coefficients $q_\xi^{k-1} = q_\xi^k = 0$. This means that $(1+t)Q^C(t)$ does not contain the term $e^\xi t^k$. Hence $\xi \in \operatorname{supp}(H^k)$. q.e.d.

Recall that a Kähler manifold $(M,\omega)$ is *quantizable* if $\frac{\omega}{2\pi}$ represents an integral de Rham class. In this case, a *pre-quantum line bundle* over $M$ is a line bundle whose curvature is $\frac{\omega}{\sqrt{-1}}$.

**Corollary 2.7.** *If $L$ is a $T$-invariant pre-quantum line bundle of a quantizable Kähler manifold $(M,\omega)$, then* supp$H^k(M,\mathcal{O}(L))$ *is contained in the moment polytope $\Delta$ for any $k = 0,\cdots,n$.*

*Proof.* Let $\mu\colon M \to \mathfrak{t}^*$ be the moment map. The image $\Delta = \mu(M)$ is a convex polytope [1], [14]. For any facet of $\Delta$, there is $x \in \mathfrak{t}$ such that the hyperplane $x^\perp$ contains the facet and $\langle \Delta, x\rangle \le 0$. Choose an action chamber $C$ with an edge containing $x$. For any fixed point $p \in F$, the weight of the torus action on the fiber of the pre-quantum line bundle over $p$ is $\mu(p) \in \Delta$, hence $\langle \mu(p), x\rangle \leq 0$. Since all $\langle \lambda_k^{p,C}, x\rangle \geq 0$ we conclude that $\langle \Gamma^{p,C}, x\rangle \leq 0$ for all $p \in F$. By (2.9) of Proposition 2.6, $\langle \operatorname{supp}(H^k), x\rangle \leq 0$ for any facet of $\Delta$. Since $\Delta$ is convex, we get $\operatorname{supp}(H^k) \subset \Delta$ for any $k$. q.e.d.

In sections 3 and 5, we will apply Theorem 2.2 and Proposition 2.6 to toric manifolds and to cases with general non-Abelian group actions.

## 3. Applications to toric manifolds

The toric manifolds which we consider here are smooth complex manifolds $M$, each equipped with an effective action of $T = T^n$ ($n = \dim_{\mathbb{C}} M$). Such a manifold can be characterized combinatorially by a fan $\Sigma$, a collection of cones in $\mathfrak{t} = \sqrt{-1}\,\mathrm{Lie}(T)$ that satisfy certain compatibility and integrality conditions. (For reviews, see for example [5], [20], [3], [9].) $M$ is compact if and only if the corresponding fan $\Sigma$ is complete, i.e., the union of all the cones $|\Sigma| = \mathfrak{t}$. Top dimensional cones $\sigma$ in the $n$-skeleton $\Sigma^{(n)}$ are in one-to-one correspondent with the $T$-fixed points $p_\sigma \in F$. The weights of isotropic representations on $T_{p_\sigma}M$ span the cone $-\sigma^* \subset \mathfrak{t}^*$. The hyperplanes containing $(n-1)$-dimensional cones cut $\mathfrak{t}$ into action chambers. The $T$-equivariant holomorphic line bundles over $M$ are characterized by (continuous) piecewise linear functions $\varphi$ on $\Sigma$ modulo $\mathfrak{t}^*$ (the space of globally linear functionals). For each $\sigma \in \Sigma^{(n)}$, $\varphi|_\sigma \in \mathfrak{t}^*$ is the weight of the $T$-action on the fiber over $p_\sigma$. We denote this line bundle by $L_\varphi$. The cohomology groups $H^k = H^k(M, \mathcal{O}(L_\varphi))$ ($k = 0, \cdots, n$) are representations of $T$. For each weight $\xi \in \mathcal{L}^*$, the multiplicity of $\xi$ in $H^k$ can be computed using $\xi$, $\varphi$ and $\Sigma$ (see Remark 4.3 below). The purpose of this section is to find information on such multiplicities using the results of the previous section. This method provides much geometric insight, especially on how the multiplicity varies according to the position of the weight relative to the image of the (generalized) moment map. The multiplicity in the equivariant index was studied in [18], [10] with similar considerations by using the Atiyah-Bott fixed-point formula (2.4).

We consider a toric manifold $M$ with a $T$-invariant Kähler structure. This is equivalent to the existence of a $T$-invariant ample line bundle over $M$; the Kähler structure is then induced by the projective embedding. Such a line bundle corresponds to a strictly convex piecewise linear function. From the symplectic point of view, if $M$ admits a (non-degenerate) symplectic form, then the image of the moment map is a convex polytope which does define a strictly convex piecewise linear function on its fan (with a possible perturbation of the Kähler form). Therefore a symplectic toric manifold is Kähler. An explicit construction of a Kähler structure using the Delzant construction [6] was given by [11].

**Proposition 3.1.** *Let $M$ be a compact smooth toric manifold of complex dimension $n$ with a $T$-invariant Kähler structure and let $H^k = H^k(M, \mathcal{O}(L_\varphi))$ $(k = 0, \cdots, n)$, where the line bundle is determined by a piecewise linear function $\varphi$ on the fan. Then the following hold:*

1. *If $0 < k < n$, the sets $\mathrm{supp}(H^0)$, $\mathrm{supp}(H^k)$ and $\mathrm{supp}(H^n)$ are mutually disjoint;*
2. $\mathrm{supp}(H^0) = \mathcal{L}^* \cap \Gamma^0$, $\mathrm{supp}(H^n) = \mathcal{L}^* \cap \Gamma^n$, *where each weight is of multiplicity* 1.

*Proof.* Choose any action chamber $C$. Let $\sigma_C \in \Sigma^{(n)}$ be the unique cone containing $-C$. It is clear that the polarizing indices $n^{\sigma,C}$ of $p_\sigma \in F$ are given by $n^{\sigma_C,C} = 0$ and $n^{\sigma,C} > 0$ for all $\sigma \neq \sigma_C$. Therefore in the weak inequality (2.3) for $k = 0$, there is only one term that bounds $\mathrm{char}(H^0)$. Since $\dim_{\mathbb{C}} M = \dim \mathfrak{t}$, the number of isotropy weights at $p_\sigma$ is equal to $n$, hence the multiplicity of any weight in $H^0$ is not greater than 1. Let $C$ run through all action chambers. Since $\Gamma^{\sigma_C,C} = \varphi|_{\sigma_C} + \sigma_C^*$, we obtain $\Gamma^0 = \bigcap_{\sigma\in\Sigma^{(n)}}(\varphi|_\sigma + \sigma^*)$. Using again $\dim_{\mathbb{C}} M = \dim \mathfrak{t} = n$, for any $\sigma \in \Sigma^{(n)}$ and any action chamber $C$ such that $\sigma_C \neq \sigma$, we have $(\varphi|_\sigma + \sigma^*) \cap \Gamma^{\sigma,C} = \emptyset$. Therefore $\Gamma^0 \cap \Gamma^{\sigma,C} = \emptyset$. Taking the union of $\sigma \in \Sigma^{(n)}$ with $n^{\sigma,C} = k > 0$, we get $\Gamma^0 \cap \Gamma^{k,C} = \emptyset$, hence $\Gamma^0 \cap \Gamma^k = \emptyset$ for $k > 0$. By (2.9) of Proposition 2.6, $\mathrm{supp}(H^0) \cap \mathrm{supp}(H^k) = \emptyset$ for $k > 0$. Also, since $\Gamma^0 \cap \Gamma^{k,C} = \emptyset$ for any $k > 0$ and any $C$, by the Atiyah-Bott fixed-point theorem (2.4) or by (2.10) of Proposition 2.6, we have $\mathrm{supp}(H^0) \supset \mathcal{L}^* \cap \Gamma^0$. Hence $\mathrm{supp}(H^0) = \mathcal{L}^* \cap \Gamma^0$, each weight with multiplicity 1. The results on $H^n$ can be proved similarly. q.e.d.

**Corollary 3.2.** *If $\varphi$ is a strictly convex piecewise linear function, then $H^k = 0$ for $k > 0$ and* $\mathrm{Ind} = \mathrm{char}(H^0)$*; if $-\varphi$ is convex, then $H^k = 0$ for $k < n$ and* $\mathrm{Ind} = (-1)^n \mathrm{char}(H^n)$*.*

*Proof.* In the first case, there is a $T$-invariant Kähler form such that $L_\varphi$ is the pre-quantum line bundle and $\mu(p) = \varphi|_{p_\sigma}$ for all $\sigma \in \Sigma^{(n)}$. The moment polytope $\Delta = \bigcap_{\sigma\in\Sigma^{(n)}}(\mu(p_\sigma) + \sigma^*) = \Gamma^0$. From Corollary

2.7 and Proposition 3.1, we obtain $\Gamma^k \subset \Delta$ for all $k$, and $\Gamma^0 \cap \Gamma^k = \emptyset$ for $k > 0$. Therefore $H^k = 0$ for $k > 0$, and $\mathrm{Ind} = \mathrm{char}(H^0)$, with $\mathrm{supp}(H^0) = \mathcal{L}^* \cap \Delta$. The second case can be proved similarly. q.e.d.

**Example 3.3.** Consider the projective space $\mathbb{C}P^n = (\mathbb{C}^{n+1}\backslash\{0\})/\mathbb{C}^*$ with the standard $T^n$-action given by $(u_1, \cdots, u_n)\colon [z_1, \cdots, z_n.z_{n+1}] \mapsto [u_1z_1, \cdots .u_nz_n, z_{n+1}]$ in homogeneous coordinates. $\mathfrak{t} = \sqrt{-1}\,\mathrm{Lie}(T^n) \cong \mathbb{R}^n$ is spanned by $e_1, \cdots, e_n$.

1. The case $n = 1$ has been considered in [24] using the inequalities in (2.5) and their counterparts for $C = C^-$. If $L$ is a line bundle over $\mathbb{C}P^1$ with $c_1(L) = r$, then $\dim H^0(\mathbb{C}P^1, \mathcal{O}(L)) = r + 1$, $H^1 = 0$ if $r \geq 0$, and $H^0 = 0$, $\dim H^1 = |r| - 1$ if $r < 0$.

2. It is instructive to work out the case $n = 2$ directly using the results of the previous section. There are three fixed points $p_1 = [1, 0, 0]$, $p_2 = [0, 1, 0]$, $p_3 = [0, 0, 1]$ with isotropy weights $\{e_1^*, e_1^* - e_2^*\}$, $\{e_2^*, e_2^* - e_1^*\}$, $\{-e_1^*, -e_2^*\}$, respectively. There are six action chambers. Let $C$ be a chamber spanned by $\{e_1, e_1 + e_2\}$ (Figure 3.1(a)). The line bundle $L = (\mathbb{C}^3\backslash\{0\} \times \mathbb{C})/\mathbb{C}^*$, where the quotient is $(z_1, z_2, z_3, w) \sim (uz_1, uz_2, uz_3, u^r w)$, $u \in \mathbb{C}^*$, has $c_1(L) = r \in \mathbb{Z}$. The $T^2$-action lifts to $L$ by

$$(u_1, u_2)\colon [z_1, z_2, z_3, w] \mapsto [u_1z_1, u_2z_2, z_3, w].$$

The weights on $L_{p_1}$, $L_{p_2}$, $L_{p_3}$ are $\xi_1 = re_1^*$, $\xi_2 = re_2^*$, $\xi_3 = 0$, respectively (Figure 3.1(b)). It is easy to see that $n^{p,C} = 0, 1, 2$ for $p = p_1, p_2, p_3$, respectively and that

$$\begin{aligned}
\Gamma^{p_1,C} &= \{x_1e_1^* + x_2e_2^* \,|\, x_2 \geq 0, x_1 + x_2 \leq r\},\\
\Gamma^{p_2,C} &= \{x_1 < 0, x_1 + x_2 \leq r\},\\
\Gamma^{p_3,C} &= \{x_1, x_2 < 0\}.
\end{aligned}$$

On the other hand, $n^{p,-C} = 2, 1, 0$ for $p = p_1, p_2, p_3$, respectively and

$$\begin{aligned}
\Gamma^{p_1,-C} &= \{x_1 + x_2 > r, x_2 < 0\},\\
\Gamma^{p_2,-C} &= \{x_1 \geq 0, x_1 + x_2 > r\},\\
\Gamma^{p_3,-C} &= \{x_1, x_2 \geq 0\}.
\end{aligned}$$

If $r \geq 0$, by (2.9) of Proposition 2.6,

$$\Gamma^0 \subset \Gamma^{p_1,C} \cap \Gamma^{p_3,-C} = \{x_1, x_2 \geq 0, x_1 + x_2 \leq r\}$$

(Figure 3.1(c)),

$$\Gamma^1 \subset \Gamma^{p_2,C} \cap \Gamma^{p_2,-C} = \emptyset$$

(Figure 3.1(d)), and

$$\Gamma^2 \subset \Gamma^{p_1,C} \cap \Gamma^{p_3,-C} = \emptyset$$

(Figure 3.1(e)). By (2.10) of Proposition 2.6,

$$\mathrm{supp}(H^0) \supset \mathbb{Z}^2 \cap (\Gamma^{p_1,C} \backslash \Gamma^{p_2,C}).$$

Hence $\mathrm{supp}(H^0) = \mathbb{Z}^2 \cap \Gamma^0$, where $\Gamma^0 = \{x_1, x_2 \geq 0, x_1 + x_2 \leq r\}$, and $H^1 = H^2 = 0$. Similarly, if $r < 0$, then $H^0 = H^1 = 0$, and $H^2 = \mathbb{Z}^2 \cap \Gamma^2$, where $\Gamma^2 = \{x_1, x_2 < 0, x_1 + x_2 > r\}$.

3. For general $n$, $\mathbb{C}P^n$ as a toric manifold can be described by a fan $\Sigma$: any $k$ vectors in $\{v_0 = -\sum_{i=1}^n e_i, v_i = e_i \,(1 \leq i \leq n)\}$ span a $k$-dimensional cone in the fan $\Sigma$. For any piecewise linear function $\varphi$ on $\Sigma$, either $\varphi$ or $-\varphi$ is convex. In fact by shifting a linear functional, $\varphi$ can be brought to the standard form $\varphi(v_0) = -r$, $\varphi(v_i) = 0$ $(1 \leq i \leq n)$; $r$ is the first Chern number of the line bundle. $\varphi$ is convex if and only if $r \geq 0$. In this case, Corollary 3.2 implies that $H^k = 0$ for $k > 0$ and that $\mathrm{supp}(H^0)$ contains the integer points in the simplex $\{\sum_{i=1}^n x_i e_i^* \,|\, x_i \geq 0 \,(1 \leq i \leq n), \sum_{i=1}^n x_i \leq r\}$ in $\mathfrak{t}^* \cong \mathbb{R}^n$, each with multiplicity 1. If $r < 0$, then $H^k = 0$ for $k < n$, and $\mathrm{supp}(H^n) = \mathbb{Z}^2 \cap \{\sum_{i=1}^n x_i e_i^* \,|\, x_i < 0 \,(1 \leq i \leq n), \sum_{i=1}^n x_i > r\}$, with multiplicity 1.

**Corollary 3.4.** *If* $\dim_{\mathbb{C}} M = 2$, *then*

$$\mathrm{supp}(\mathrm{Ind}) = \mathrm{supp}(H^0) \cup \mathrm{supp}(H^1) \cup \mathrm{supp}(H^2) \tag{3.1}$$

*is a disjoint union.*

*Proof.* This follows from part 1 of Proposition 3.1 since the only choice of $k$ is $k = 1$. q.e.d.

**Example 3.5.** We consider the Hirzebruch surface, given by a fan $\Sigma$ in $\mathbb{R}^2$ whose 1-skeleton $\Sigma^{(1)}$ is spanned by vectors $v_1 = e_1$, $v_2 = e_2$, $v_3 = -e_1$, $v_4 = -ae_1 - e_2$ $(a \in \mathbb{N})$. We choose an action chamber $C$ spanned by $\{-e_1, -ae_1 - e_2\}$ (Figure 3.2(a)). Consider a piecewise linear function given by $\varphi(v_1) = \varphi(v_2) = 0$, $\varphi(v_3) = -r$ and $\varphi(v_4) = -s$. We assume that $r, s > 0$. It is easy to see that if $s \geq ar$, then $\varphi$ is convex, hence $H^1 = H^2 = 0$, $\mathrm{supp}(H^0) = \mathbb{Z}^2 \cap \Gamma^0$, with multiplicity 1, where $\Gamma^0 = \{0 \leq x_1 \leq s, 0 \leq x_2 \leq r - ax_1\}$ is a 4-gon. If $s < ar$, since isotropy weights of the four fixed points $p_i$ $(1 \leq i \leq 4)$ are the negatives of $\{e_1^*, e_2^*\}$, $\{-e_1^*, e_2^*\}$, $\{-e_2^*, -e_1^* + ae_2^*\}$, $\{-e_2^*, e_1^* - ae_2^*\}$, respectively (Figure 3.2(b)), we have $n^{p,C} = 0, 1, 2, 1$ for $p = p_1, p_2, p_3, p_4$, and $\Gamma^{p_1,C} = \{x_1, x_2 \geq 0\}$, $\Gamma^{p_2,C} = \{x_1 > r, x_2 \geq 0\}$, $\Gamma^{p_3,C} = \{x_1 > r, ax_1 + x_2 > s\}$, $\Gamma^{p_4,C} = \{x_1 \geq 0, ax_1 + x_2 > s\}$. On the other hand, $n^{p,-C} = 2, 1, 0, 1$ for $p = p_1, p_2, p_3, p_4$, respectively, and

$$\begin{aligned}
\Gamma^{p_1,-C} &= \{x_1, x_2 < 0\}, \\
\Gamma^{p_2,-C} &= \{x_1 \leq r, x_2 < 0\}, \\
\Gamma^{p_3,-C} &= \{x_1 \leq r, ax_1 + x_2 \leq s\}, \\
\Gamma^{p_4,-C} &= \{x_1 < 0, ax_1 + x_2 \leq s\}.
\end{aligned}$$

Using Proposition 2.6, we obtain $\mathrm{supp}(H^0) = \mathbb{Z}^2 \cap \Gamma^0$, $\mathrm{supp}(H^1) = \mathbb{Z}^2 \cap \Gamma^1$ (the equalities follow from part 2), where

$$\Gamma^0 = \Gamma^{p_1,C} \cap \Gamma^{p_3,-C} = \{x_1, x_2 \geq 0, ax_1 + x_2 \leq r\}$$

(Figure 3.2(c)),

$$\Gamma^1 = (\Gamma^{p_2,C} \cup \Gamma^{p_4,C}) \cap (\Gamma^{p_2,-C} \cup \Gamma^{p_4,-C}) = \{x_1 \leq r, x_2 < 0, ax_1 + x_2 > s\}$$

(Figure 3.2(d)), and $H^2 = 0$ since

$$\Gamma^2 \subset \Gamma^{p_3,C} \cap \Gamma^{p_1,-C} = \emptyset$$

(Figure 3.2(e)). The results are in accord with Corollary 3.4. An alternative way of obtaining the same results is to calculate the equivariant index (2.4) and use Corollary 3.4.

**Remark 3.6.** We make several observations on the multiplicity of weights in $H^k$ when $0 < k < n$.

1. First, the multiplicity of a weight in $H^k$ $(0 < k < n)$ is not necessarily 1, even when $\dim_{\mathbb{C}} M = 2$. For example, it was shown in [18, Example 5.6], that for a certain line bundle over the blow-up of $\mathbb{C}P^2$ at three points, a weight can appear in the index with multiplicity $-2$. Using Corollary 3.4, one concludes that the multiplicity of that weight in $H^1$ is 2.

2. When $\dim_{\mathbb{C}} M \geq 3$, a weight can appear simultaneously in cohomology groups of different degrees. To see this, we construct a fan $\Sigma$ in $\mathbb{R}^3$. Set $e_+ = e_1 + e_2 + e_3$. Let the top-dimensional cones be spanned by

$\{e_1, e_2, e_+\}$, $\{e_1, e_2, -e_3\}$, $\{-e_1, -e_2, e_3\}$, $\{-e_1, -e_2, -e_+\}$, and those obtained by cyclic permutations of the basis. Define a piecewise linear function $\varphi$ on $\Sigma$ by $\varphi(e_+) = \varphi(-e_i) = 1$, $\varphi(-e_+) = \varphi(e_i) = -1$ $(i = 1, 2, 3)$. Then $\varphi$ determines a line bundle such that 0 is a weight of both $H^1$ and $H^2$.

3. It would be interesting to investigate the general conditions on the fan under which any weight is of multiplicity 1 or can appear in only one of the cohomology groups. One knows that if $M_1$ and $M_2$ are toric manifolds of one of these types, then so is any $M_1$-fibered toric manifolds over $M_2$. An interesting class of examples is the Bott-Samelson manifolds studied from the symplectic point of view in [10].

## 4. Necessity of the Kähler condition

It is well-known that the Hirzebruch-Riemann-Roch theorem or its equivariant counterpart, the Atiyah-Bott fixed-point theorem, are valid for holomorphic vector bundles over arbitrary compact complex manifolds. Therefore it comes as a surprise that the strong equivariant holomorphic Morse inequalities holds for Kähler manifolds only. In this section, we show that the strong inequalities (2.2) are violated on a suitable non-Kähler toric manifold. We also use the violation of the strong inequalities to show the non-existence of invariant Kähler structures on certain symplectic manifolds.

Recall that a toric manifold of complex dimension $n$ is Kähler if and only if its fan $\Sigma$ admits strictly convex piecewise linear function. When the complex dimension $n = 2$, such a convex function always exists. Therefore every toric 2-manifold is Kähler. Notice that in this dimension, the strong inequalities (2.2) are equivalent to the weak ones (2.3). When $n = 3$, there are fans in $\mathfrak{t} \cong \mathbb{R}^3$ which does not admit any convex piecewise linear functions. An example is the fan with 8 top-dimensional cones $\sigma_i$ $(0 \le i \le 6)$ and $\sigma'$, whose stereographic projection is given by Figure 4.1(a) (see for example [5], [9]). Suppose that there were a strictly convex piecewise linear function $\varphi$ on $\Sigma$. For the choice of coordinates in Figure 4.1(a), convexity on the adjacent cones $\sigma_1$ and $\sigma_4$ means that $2\varphi(a) + \varphi(f) > \frac{1}{2}\varphi(c) + \frac{1}{2}\varphi(d)$. Summing over this and the other two inequalities obtained by the cyclic permutations of $(a, d)$, $(b, e)$, $(c, f)$, we get $3(\varphi(a)+\varphi(b)+\varphi(c))+\varphi(d)+\varphi(e)+\varphi(f) > 0$. On the other hand, convexity on $\sigma_4$ and $\sigma_2$ implies that $-2\varphi(a) > \varphi(b) + \varphi(f)$, and hence $3(\varphi(a)+\varphi(b)+\varphi(c))+\varphi(d)+\varphi(e)+\varphi(f) < 0$, a contradiction.

The corresponding toric variety has an orbifold singularity because the cone $\sigma'$ is not spanned by a $\mathbb{Z}$-basis. (In general, a toric variety has at most orbifold-type singularities if all the top-dimensional cones in the fan are simplicial. It is smooth if these cones are spanned by a $\mathbb{Z}$-basis.) The singularity can be avoided by a further triangulation of $\sigma'$ into 15 cones $\sigma_i$ ($7 \leq i \leq 21$) by Jurkiewicz [17], as shown in Figure 4.1(b). This results a smooth toric 3-manifold with 22 fixed points.

**Theorem 4.1.** *There exists a $T^3$-invariant line bundle over the smooth toric 3-manifold corresponding to the fan of Figure 4.1 such that the strong equivariant holomorphic Morse inequalities are not satisfied.*

*Proof.* Let $\{e_1, e_2, e_3\}$ be the standard basis in $\mathfrak{t} \cong \mathbb{R}^3$, and $\{e_1^*, e_2^*, e_3^*\}$, its dual basis in $\mathfrak{t}^*$. We choose a piecewise linear function $\varphi$ on the fan in Figure 4.1 such that $\varphi = -(e_1^*+e_2^*+e_3^*)$ in the cones $\sigma_i$, $7 \leq i \leq 21$ (those inside $\sigma'$), $\varphi = 0$ in $\sigma_0$, and $\varphi$ is given by linear interpolations between the cones $\sigma'$ and $\sigma_0$, for example, $\varphi|_{\sigma_1} = -3e_2^* + 3e_3^*$, $\varphi|_{\sigma_4} = -3e_3^*$. Choose the action chamber $C = \sigma_7$ (shown in Figure 4.1(b)). It is straightforward to check that

1) $0 \in \Gamma^{k,C}$ for $k = 0, 7$ only and $n^{p_0,C} = 0$, $n^{p_7,C} = 3$,

2) $0 \notin \Gamma^{k,-C}$ for any $k$.

If the strong inequalities (2.2) were to hold, then according to Proposition 2.6, the above claims imply the following:

1) 0 is a weight in $H^0$ and $H^3$ with multiplicity 1,

2) 0 is not a weight in $H^0$ or in $H^3$,

respectively. This is a contradiction. q.e.d.

We actually showed that the strong inequalities (2.5) are violated for the action of any circle subgroup of $T^3$ whose generator is in $C = \sigma_7$. The example is consistent with the Atiyah-Bott fixed-point formula (2.4) since possible contributions to $H^0$ and $H^3$ cancel in the equivariant index.

**Remark 4.2.** In [26], the author shows that the strong holomorphic Morse inequalities hold for any meromorphic group action on a complex manifold with a filterable Białynicki-Birula decomposition, which is a filtration of the manifold by closed subvarieties compactible with the group action. This condition is weaker than the Kähler assumption made in the current paper. The toric manifold in Theorem 4.1 was initially constructed to show that Białynicki-Birula decompositions need not be filterable [17]. Therefore the work of [26] provides a deeper understanding this counterexample of strong inequalities.

We now make some remarks on the weak inequalities.

**Remark 4.3.** Proposition 3.1 on toric manifolds was derived from the weak inequalities (2.3), together with the Atiyah-Bott fixed-point formula (2.4). For a general (possibly non-Kähler) toric variety $M$ with a $T$-invariant holomorphic line bundle $L_\varphi$, Demazure's result [7] (see also [5], [20], [9]) states that the multiplicity of $\xi \in \mathcal{L}^*$ in $H^k$ is equal to the dimension of the local cohomology group $H^k_{Z(\xi)}(\mathfrak{t})$, where $Z(\xi) = \{x \in \mathfrak{t} \,|\, \xi(x) \geq \varphi(x)\}$. In fact, Demazure's result implies Proposition 3.1. ($\xi \in \mathcal{L}^* \cap \Gamma^0$ if and only if $Z(\xi) = \mathfrak{t}$, in which case $H^0 = \mathbb{C}$ and $H^k = 0$ for $k > 0$. That the multiplicity in $H^0$ is not greater than 1 also follows from the existence of a dense open $T^{\mathbb{C}}$-orbit in $M$.) This suggests that (at least in the completely integrable cases) the weak holomorphic Morse inequalities could be valid for a larger class of complex manifolds. For example, the weak inequalities are not violated for the line bundle in Theorem 4.1. Notice also that the lowest complex dimensions for a toric manifold to be non-Kähler is 3; this is when the weak Morse inequalities become *weaker* than the strong ones. This leaves open the possibility that weak inequalities might hold when the manifold is complex but not Kähler.

**Remark 4.4.** More generally, consider a complex manifold $M$ with a holomorphic torus $T$-action. Assume that the fixed-point set $F$ is discrete. For any $p \in F$, let $\lambda_1^p, \cdots, \lambda_n^p \in \mathcal{L}^* \backslash \{0\}$ be the weights of $T$ on $T_pM$. Let $E$ be a holomorphic vector bundle over $M$ with a lifted $T$-action. Let $e_1^p, \cdots, e_r^p$ form a basis of $E_p$, with weights $\mu_1^p, \cdots, \mu_r^p \in \mathcal{L}^*$, respectively. Near a fixed point $p \in F$, the $T$-action on $M$ has the local model $\mathbb{C}^n$ equipped with the action

$$e^{\sqrt{-1}\theta} \colon (z_1, \cdots, z_n) \mapsto (e^{\sqrt{-1}\langle \lambda_1^p, \theta\rangle} z_1, \cdots, e^{\sqrt{-1}\langle \lambda_n^p, \theta\rangle} z_n),$$

$\theta \in \mathfrak{t}$. Holomorphic sections of $E$ in a neighborhood of $p$ correspond to linear combinations of $z_1^{m_1} \cdots z_n^{m_n} \otimes e_i^p$ ($m_1, \cdots, m_n \in \mathbb{N}$, $1 \leq i \leq r$), whose weight under the $T$-action is $\mu_i^p - \sum_{k=0}^n m_k \lambda_k^p$. (The minus sign is explained in [19] in $S^1$-cases.) Not all such local sections extend to $M$. So

$$\operatorname{supp}(H^0) \subset \Big\{ \mu_i^p - \sum_{k=0}^n m_k \lambda_k^p | m_1, \cdots, m_n \in \mathbb{N}, 1 \leq i \leq r \Big\},$$

counting multiplicities. This is part of the weak Morse inequalities. For similar reasons, other weak inequalities in (2.3) are likely to be true when the manifold $M$ admits a $T$-invariant complex structure.

Violation of strong holomorphic Morse inequalities can be used as an obstruction to the existence of $T$-invariant Kähler structures. Recently, Tolman [22] constructed a simply-connected six dimensional symplectic manifold $(M,\omega)$ that has a Hamiltonian $T^2$-action with isolated fixed points but does not admit any $T^2$-invariant Kähler structure. (See [25] for another perspective.) Let $\{e_1, e_2\}$ be the standard basis in $\mathfrak{t} \cong \mathbb{R}^2$ and $\{e_1^*, e_2^*\}$, the dual basis in $\mathfrak{t}^*$. The six fixed points $p_i$ $(1 \le i \le 6)$ in $M$ correspond to the six vertices $\mu_i = \mu(p_i)$ of the moment polytope in Figure 4.2(a) (reproduced from [22], Figure 2). Their isotropy weights are the negatives of $\{e_1^*, e_2^*, e_1^*+e_2^*\}$, $\{e_1^*, e_2^*, -e_1^*-e_2^*\}$, $\{e_1^*, -e_2^*, e_1^*-e_2^*\}$, $\{-e_1^*, -e_2^*, e_1^*-e_2^*\}$, $\{-e_1^*, -e_1^*+e_2^*, -2e_1^*+e_2^*\}$, $\{-e_1^*, -e_1^*+e_2^*, 2e_1^*-e_2^*\}$, respectively. It can be shown that the strong inequalities (2.2) do not hold in this example, therefore giving an alternative proof of the non-existence of $T^2$-invariant Kähler structures. (Most of this comes up in a discussion with S. Tolman and C. Woodward.)

**Proposition 4.5** (Tolman [22]). *There is no $T^2$-invariant Kähler structure on Tolman's manifold M.*

*Proof.* There is a $T^2$-invariant line bundle $L$ such that the weights on $L_{p_i}$ $(1 \le i \le 6)$ are $\xi_1 = 0$, $\xi_2 = 3e_1^* + 3e_2^*$, $\xi_3 = 2e_2^*$, $\xi_4 = 3e_1^* + 2e_2^*$, $\xi_5 = 5e_1^*$, $\xi_6 = -e_1^* + 3e_2^*$ (Figure 4.2(b), from [22], Figure 3, Case $0 < t < s$). If there is a $T^2$-invariant Kähler form $\omega'$, which may be different from the original symplectic form $\omega$, then Figure 4.2(b) is a deformation of the moment polytope associated to $\omega'$. Since all the symplectic quotients of $M$ by the $T^2$-action are $\mathbb{C}P^1$, where the complex structure is unique, the curvature of $L$ is still a $(1,1)$-form on $M$. Therefore $L$ can be equipped with a holomorphic structure, and it is invariant under the $T^2$-action. Let $C$ be the cone spanned by $\{e_1 - e_2, -e_2\}$. Then $e_1^* + 2e_2^* \in \Gamma^{p,C}$ for $p = p_1, p_5$ only and $n^{p_1,C} = 1$, $n^{p_5,C} = 0$ (Figure 4.2(c)). If the strong inequalities (2.2) were to hold, by (2.9) of Proposition 2.6, $e_1^* + 2e_2^* \notin \operatorname{supp}(H^2)$. On the other hand, $e_1^* + 2e_2^* \in \Gamma^{p,-C}$ only for $p = p_3, p_6$ and $n^{p_3,-C} = 0$, $n^{p_6,-C} = 2$ (Figure 4.2(d)). By (2.10) of Proposition 2.6, $e_1^* + 2e_2^* \in \operatorname{supp}(H^2)$, a contradiction. q.e.d.

In fact, this argument shows further that there is no Kähler structure invariant under any $S^1$-subgroup of $T^2$ whose generator lies in the cone

$C$.

**Remark 4.6.** It is not known whether there is a $T^2$-invariant complex structure on Tolman's manifold. If it exists, then the strong inequalities (2.2) need not be true even when there is a moment map. On the other hand since the toric 3-manifold in Theorem 4.1 is not symplectic, though generalized moment maps in the sense of [18] do exist, there is a possibility that the strong inequalities (2.2) could be true for Hamiltonian torus actions on symplectic manifolds with invariant complex structures not necessarily *calibrated* [3], §II.1.5 by the symplectic forms. If so, then Tolman's example does not admit any $T^2$-invariant complex structure.

## 5. Non-Abelian equivariant holomorphic Morse inequalities

In this section we consider a compact Kähler manifold $M$ of complex dimension $n$ with a holomorphic and Hamiltonian action of a compact connected real Lie group $G$, assuming that the fixed-point set of a maximal torus $T \subset G$ is discrete. Let $E$ be a holomorphic vector bundle over $M$ with a lifted holomorphic $G$-action. Let $G^{\mathbb{C}}$ be the complexification of $G$ that contains $G$ as a maximal compact subgroup. Then $G^{\mathbb{C}}$ acts holomorphically on $M$ and $E$ [15]. Choose an (isolated) $T$-fixed-point $p \in F$ and denote the isotropy group of $p$ in $G$ and $G^{\mathbb{C}}$ by $H = G_p$ and $U = (G^{\mathbb{C}})_p$, respectively. (In this way, we have for simplicity dropped the subscript $p$ unless ambiguity occurs.) Clearly $H \supset T$ and $U$ is a complex subgroup of $G^{\mathbb{C}}$ containing $H^{\mathbb{C}}$ such that $U \cap G = H$. We need a few group-theoretic results on $G^{\mathbb{C}}$, $U$, $H^{\mathbb{C}}$ satisfying the above constraints. These results will determine the behavior of the fixed-point set, the isotropy weights and the fiber of the vector bundle over the fixed points. For this purpose, it suffices that $M$ is a complex manifold with a holomorphic $G$-action; the Kähler condition will be used only to establish the equivariant holomorphic Morse inequalities.

Let $H_0$, $U_0$ be the identity components of $H$ and $U$, and let $W = N_G(T)/T = N_{G^{\mathbb{C}}}(T^{\mathbb{C}})/T^{\mathbb{C}}$, $W_H$, $W_{H_0}$, $W_U$, $W_{U_0}$ be the Weyl groups of $G$, $H$, $H_0$, $U$, $U_0$, respectively.

**Lemma 5.1.** *$W$ acts transitively on the $T^{\mathbb{C}}$-fixed-point set in $G^{\mathbb{C}}/U$ with isotropy group $W_U$.*

*Proof.* The fixed-point set is $\{gU \mid g \in G^{\mathbb{C}}, g^{-1}T^{\mathbb{C}}g \subset U\}$. It has a well-defined action of $W$ because $h \in N_{G^{\mathbb{C}}}(T^{\mathbb{C}})$ acts on the fixed points by $gU \mapsto hgU$ and for any $t \in T^{\mathbb{C}}$, $htgU = hg(g^{-1}tg)U = hgU$. To show that the action is transitive, choose any fixed point $gU$. Since $g^{-1}T^{\mathbb{C}}g$ and $T^{\mathbb{C}}$ are two maximal tori in $U_0$, there is an element $u \in U_0$ such that $u^{-1}g^{-1}T^{\mathbb{C}}gu = T^{\mathbb{C}}$. So $gu \in N_{G^{\mathbb{C}}}(T^{\mathbb{C}})$ and $gU = (gu)U$ can be obtained from $U$ by an element of $W$. Finally, $h \in N_{G^{\mathbb{C}}}(T^{\mathbb{C}})$ fixes $U$ (i.e., $hU = U$) if and only if $h \in U \cap N_{G^{\mathbb{C}}}(T^{\mathbb{C}}) = N_U(T^{\mathbb{C}})$. So the isotropy group in $W$ is $N_U(T^{\mathbb{C}})/T^{\mathbb{C}} = W_U$. q.e.d.

**Proposition 5.2.** *There is an action of $W$ on the $T$-fixed-point set $F$ in $M$. Each $W$-orbit in $F$ is the intersection of $F$ with a $G$- or $G^{\mathbb{C}}$-orbit in $M$, which do not depend on the choice of $T$.*

*Proof.* Lemma 5.1 implies that for $p \in F$, the $T$-fixed points in $G^{\mathbb{C}} \cdot p \cong G^{\mathbb{C}}/U$ form a single $W$-orbit. For a different maximal torus $gTg^{-1}$ ($g \in G$), the new fixed-point set $gF$ is contained in the same $G$-orbits. q.e.d.

**Example 5.3.** We consider the diagonal action of $G = SO(3)$ on $M = S^2 \times S^2$, where $G$ acts on $S^2$ by standard rotations. The maximal torus in $G$ is $T = S^1$. Let $n$, $s$ be the poles in $S^2$ which are fixed by $T$. Then the $T$-fixed-point set in $M$ is $F = \{(n,n),(s,s),(n,s),(s,n)\}$. Though the isotropy groups $G_p = T$ for all $p \in F$, the situation is different if we consider the action of the complexification $G^{\mathbb{C}} = SL(2,\mathbb{C})/\mathbb{Z}_2$. It is easy to see that the stabilizer $(G^{\mathbb{C}})_{(n,n)} = B$, a Borel subgroup in $G^{\mathbb{C}}$, whereas the stabilizer $(G^{\mathbb{C}})_{(n,s)} = T^{\mathbb{C}}$. Consequently, the orbit $G^{\mathbb{C}} \cdot (n,n) = \{(x,x) \mid x \in S^2\} \cong G^{\mathbb{C}}/B$ is compact and contains another fixed point $(s,s)$ related to $(n,n)$ by $W = \mathbb{Z}_2$, whereas $G^{\mathbb{C}} \cdot (n,s) = \{(x,y) \mid x \neq y \in S^2\} \cong G^{\mathbb{C}}/T^{\mathbb{C}}$ is non-compact and contains $(s,n)$, related to $(n,s)$ also by $\mathbb{Z}_2$. In this case, $M$ is the union of two $G^{\mathbb{C}}$-orbits.

Let $\Delta$, $\Delta_{H_0}$, $\Delta_{U_0}$ be the set of roots of the pairs $(G^{\mathbb{C}},T^{\mathbb{C}})$, $(H_0^{\mathbb{C}},T^{\mathbb{C}})$, $(U_0,T^{\mathbb{C}})$, respectively. Choose a set of positive roots $\Delta^+ \subset \Delta$ and let $\Delta^- = -\Delta^+$, $\Delta^{\pm}_{H_0} = \Delta^{\pm} \cap \Delta_{H_0}$. The length of $w \in W$ is $l(w) = |w\Delta^+ \cap \Delta^-|$. Let $l_{H_0}(w) = |w(\Delta^+ \backslash \Delta^+_{H_0}) \cap \Delta^-|$ be the length of $w$ *relative* to the subgroup $H_0$. Notice that $l_T(w) = l(w)$ ($w \in W$).

**Lemma 5.4.** *There is a $T^{\mathbb{C}}$-fixed-point in $G^{\mathbb{C}}/U$ at which the set of weights of the isotropy representation contains $\Delta^+ \backslash \Delta^+_{H_0}$.*

*Proof.* Since the sets of isotropy weights at two fixed-points $p$ and $wp$ ($w \in W$) are related by the action of $w$ and since any two choices of $\Delta^+$ are conjugate under a $w \in W$, it suffices to show that the lemma holds for any given $p$ under a particular choice of $\Delta^+$ depending on $p$. Because $U$ is a complex subgroup of $G$, $\Delta_{U_0}$ is a *closed* subset of $\Delta$, i.e., if $\alpha, \beta \in \Delta_{U_0}$ and $\alpha+\beta \in \Delta$, then $\alpha+\beta \in \Delta_{U_0}$. Furthermore, since $U \cap G = H$, $\Delta_{U_0} \cap (-\Delta_{U_0}) = \Delta_{H_0}$. It is easy to see that $\Delta' = \Delta_{U_0} \backslash \Delta_{H_0}$ is also closed (if $\alpha, \beta \in \Delta'$ but $\alpha+\beta \in \Delta_{H_0}$, then $-\beta = (-\alpha-\beta)+\alpha \in \Delta_{U_0}$, a contradiction) and satisfies $\Delta' \cap (-\Delta') = \emptyset$. So $\Delta'$ is contained in some choice of $\Delta^-$ [4]. Consequently, there is a parabolic subgroup $P \supset U_0$ and $P \cap G = H_0$. Therefore the set of weights of isotropy representation at $p$ is $\Delta \backslash \Delta_{U_0} \supset \Delta \backslash \Delta_P = \Delta^+ \backslash \Delta^+_{H_0}$. q.e.d.

For each $W$-orbit $S \in F/W$, choose $p \in S$ and let $H = G_p$ be the isotropy group in $G$. Denote $\Delta_S = \Delta_{H_0}$, $l_S(w) = l_{H_0}(w)$ and $\det_S w = (-1)^{l_S(w)} \det w$ for $w \in W$; they do not depend on the choice of $p$ in $S$.

**Proposition 5.5.** *In each orbit $S \in F/W$ we can choose a representative, denoted by $p_S$, such that the weights of the isotropy representation at $p_S$ contains $\Delta^+ \backslash \Delta^+_S$.*

*Proof.* This is a direct consequence of Lemma 5.4. q.e.d.

**Remark 5.6.** In [13] (see also [21]), under a regularity assumption that implies $H = T$, it was shown that the set of isotropy weights contains $\Delta^+$ up to signs that depend on the roots. Proposition 5.5 is a refinement of this result when $M$ is complex and the $G$-action preserves the complex structure. Let $\lambda^S_k$ ($k = 1, \cdots, n - |\Delta^+ \backslash \Delta^+_S|$) be the other weights at $p_S$. Since $T_{p_S}M$ is also a representation of $U$, the set of weights $(\Delta^+ \backslash \Delta^+_S) \cup \{\lambda^S_k\}$ is $W_U$-invariant. At another fixed point $wp_S \in S$, the corresponding set of isotropy weights is $w((\Delta^+ \backslash \Delta^+_S) \cup \{\lambda^S_k\})$, which depends only on the coset $\bar{w} \in W/W_U$. By $W$-invariance, any action chamber can be transformed into one that intersects the positive Weyl chamber $\mathfrak{t}^+ = \{x \in \mathfrak{t} \,|\, \langle \Delta^+, x \rangle > 0\}$ in an open cone; let $C$ be one of such. At $p_S$, the polarizing index $n^C_{p_S}$ is equal to $n^C(\{\lambda^S_k\})$, that of the set $\{\lambda^S_k\}$. At $wp_S$, $n^C_{wp_S} = l_S(w) + n^C(\{w\lambda^S_k\})$.

The following lemma and example are communicated to the author by D. Vogan.

**Lemma 5.7.** $N(\Delta_{U_0}) = \{w \in W \,|\, w\Delta_{U_0} = \Delta_{U_0}\}$ *contains* $W_{H_0}$ *as a normal subgroup. Moreover,* $U/U_0 \cong W_U/W_{H_0}$ *is a subgroup of* $N(\Delta_{U_0})/W_{H_0}$.

*Proof.* Since $H_0^{\mathbb{C}}$ is a subgroup of $U$, $W_{H_0}$ and $W_U$ preserve the set $\Delta_{U_0}$, hence $W_{H_0} < W_U < N(\Delta_{U_0})$. For any $\alpha \in \Delta_{H_0}$, $w \in N(\Delta_{U_0})$, since $wr_\alpha w^{-1} = r_{w\alpha}$, and $w\alpha \in w\Delta_{U_0} \cap (-w\Delta_{U_0}) = \Delta_{H_0}$, $W_{H_0}$ is a normal subgroup of $N(\Delta_{U_0})$. Finally, we prove $U/U_0 \cong W_U/W_{H_0}$. Define a homomorphism $N_U(T^{\mathbb{C}}) \to U/U_0$ by $u \mapsto uU_0$. Its kernel is $N_{U_0}(T^{\mathbb{C}})$; we show that it is onto. For any $u \in U$, $u^{-1}T^{\mathbb{C}}u$ is a maximal torus in $U_0$. So there is $u_0 \in U_0$ such that $u_0^{-1}u^{-1}T^{\mathbb{C}}uu_0 = T^{\mathbb{C}}$. This implies that $uu_0 \in N_U(T^{\mathbb{C}})$ and $uu_0 \mapsto uU_0$. By homomorphism theorems, $U/U_0 \cong N_U(T^{\mathbb{C}})/N_{U_0}(T^{\mathbb{C}}) \cong W_U/W_{U_0}$. Finally, let $P$ be a parabolic subgroup that contains $U_0$ and such that $P \cap G = H_0$. Then $W_{H_0} < W_{U_0} < W_P$. Since $W_P = W_{H_0}$, we conclude that $W_{U_0} = W_{H_0}$.

q.e.d.

**Example 5.8.** The isotropy group $U$ can be disconnected. Let the action of $G = SO(3)$ on $M = \mathbb{C}P^2$ be the projectivization of the adjoint representation on $\mathfrak{so}\,(3)^{\mathbb{C}} \cong \mathbb{C}^3$. If the root space decomposition is $\mathfrak{so}\,(3)^{\mathbb{C}} = \mathbb{C}e_0 \oplus \mathbb{C}e_+ \oplus \mathbb{C}e_-$, where $\mathbb{C}e_0 = \mathrm{Lie}(T^{\mathbb{C}})$ and $[e_0, e_\pm] = \pm e_\pm$, then the $T$-fixed-point set $F = \{[e_0], [e_+], [e_-]\}$. The isotropy groups of $[e_\pm]$ in $G^{\mathbb{C}}$ are Borel subgroups while that of $[e_0]$ is the full normalizer $N_{G^{\mathbb{C}}}(T^{\mathbb{C}})$ of $T^{\mathbb{C}}$, which is disconnected.

Since $G$ is compact, $\mathfrak{t}$ is equipped with an invariant inner product; this induces one on $\mathfrak{t}^*$, which will be denoted by $(\cdot,\cdot)$. Let $\mathcal{D} = \{\lambda \in \mathcal{L}^* \mid (\lambda, \Delta^+) \geq 0\}$ and $\mathcal{D}_{H_0} = \{\lambda \in \mathcal{L}^* \mid (\lambda, \Delta^+_{H_0}) \geq 0\}$ be the sets of dominant weights with respect to $G$ and $H_0$, respectively. We denote the irreducible representation of $G$ ($H_0$, respectively) of the highest weight $\lambda \in \mathcal{D}$ ($\lambda \in \mathcal{D}_{H_0}$, respectively) by $R^G_\lambda$ ($R^{H_0}_\lambda$, respectively). Since $W_U$ preserves $\Delta_{U_0}$, hence $\Delta_{U_0} \cap (-\Delta_{U_0}) = \Delta_{H_0}$, the action of $W_U$ permutes the Weyl chambers of the pair $(H_0^{\mathbb{C}}, T^{\mathbb{C}})$. Let $\mathcal{D}_U \subset \mathcal{D}_{H_0}$ be a fundamental region of the group $W_U$ in $\mathcal{L}^*$.

**Lemma 5.9.** *Let $V$ be a finite dimensional representation of $U$. Then*

$$\mathrm{char}(V) = \sum_{\Lambda \in \mathcal{D}_U} m_\Lambda \sum_{w \in W_U} w \frac{e^\Lambda}{\prod_{\alpha \in \Delta^+_{H_0}} (1 - e^{-\alpha})}. \tag{5.1}$$

*Here $m_\Lambda \in \mathbb{Q}$; if $U$ is connected, then $m_\Lambda \in \mathbb{N}$ is the multiplicity of $R^H_\Lambda$ in $V$.*

*Proof.* As a representation of $H_0$, $V = \bigoplus_\lambda R^{H_0}_\lambda$; the direct sum is over some $\lambda \in \mathcal{D}_{H_0}$ with possible multiplicities. Since $V$ is a representation of $U$, $W_U/W_{H_0}$ acts on the set $\{R^{H_0}_\lambda\}$. For a given $W_U/W_{H_0}$-orbit through $R^{H_0}_\Lambda$, let $W_\Lambda$ be the isotropy group. The contribution of this orbit to the character is

$$
\begin{aligned}
(5.2)\qquad & \frac{1}{|W_\Lambda|} \sum_{\bar{w}\in W_U/W_{H_0}} \bar{w}\,\mathrm{char}(R^{H_0}_\Lambda) \\
&= \frac{1}{|W_\Lambda|} \sum_{w\in W_U} w\, \frac{e^\Lambda}{\prod_{\alpha\in\Delta^+_{H_0}}(1-e^{-\alpha})}.
\end{aligned}
$$

Equation (5.1) follows readily. If $U$ is connected, then $W_U/W_{H_0}=\{1\}$, hence all $m_\Lambda\in\mathbb{N}$. q.e.d.

At $p_S$ ($S\in F/W$) whose isotropy group in $G^{\mathbb{C}}$ is $U$, set $\mathcal{D}_S=\mathcal{D}_U$. According to the above lemma,

$$
(5.3)\qquad \mathrm{char}(E_{p_S}) = \sum_{\Lambda\in\mathcal{D}_S} m^S_\Lambda \sum_{w\in W_U} w \frac{e^\Lambda}{\prod_{\alpha\in\Delta^+_S}(1-e^{-\alpha})},
$$

where $m^S_\Lambda\in\mathbb{Q}$; if the isotropy group $U$ is connected, then $m^S_\Lambda\in\mathbb{N}$ is the multiplicity of $R^H_\Lambda$ in $E_{p_S}$.

**Theorem 5.10.** *Let $G$ be a compact connected Lie group acting Hamiltonianly on a compact Kähler manifold $M$ preserving the complex structure. Assume that the fixed-point set $F$ of the maximal torus $T$ is discrete. If $E$ is a holomorphic vector bundle over $M$ where the $G$-action lifts holomorphically, then $G$ acts on the cohomology groups $H^k(M,\mathcal{O}(E))$ $(0\le k\le n)$. Let $H^k(M,\mathcal{O}(E))=\bigoplus_{\Lambda\in\mathcal{D}} m^k_\Lambda R^G_\Lambda$ $(m^k_\Lambda\in\mathbb{N})$ be the decompositions into irreducible representations $R^G_\Lambda$ of $G$. Choose an action chamber $C$ that intersects $\mathfrak{t}^+$ in an open cone. Then under the above notation, we have the following non-Abelian equivariant holomorphic Morse inequalities*

$$
\begin{aligned}
(5.4)\qquad & \sum_{S\in F/W}\sum_{w\in W} \det{}_S w\; t^{l_S(w)+n^C(\{w\lambda^S_k\})} \\
&\quad \cdot \sum_{\Lambda\in\mathcal{D}_S} m^S_\Lambda\, e^{w(\Lambda+\rho)-\rho} \prod_{w\lambda^S_k\in C^*} \frac{1}{1-e^{-w\lambda^S_k}} \prod_{w\lambda^S_k\in -C^*} \frac{e^{w\lambda^S_k}}{1-e^{w\lambda^S_k}} \\
&= \sum_{k=0}^n t^k \sum_{w\in W} \det w \sum_{\Lambda\in\mathcal{D}} m^k_\Lambda\, e^{w(\Lambda+\rho)-\rho} \\
&\quad + (1+t)Q^C(t) \prod_{\alpha\in\Delta^+}(1-e^{-\alpha})
\end{aligned}
$$

*for some $Q^C(t)\ge 0$.*

*Proof.* We apply (2.2) to the maximal torus $T$ of $G$. The contribution of one orbit $S \in W/F$ is a sum over cosets $\bar{w} \in W/W_U$. This sum can be combined with that over $W_U$ in (5.3). Since the set $w((\Delta^+ \backslash \Delta_S^+) \cup \{\lambda_k^S\})$ depends only on the coset $\bar{w} \in W/W_U$ and $\mathrm{char}(E_{\bar{w}p_S}) = \bar{w}\,\mathrm{char}(E_{p_S})$, the result is

$$\begin{aligned}
&\sum_{w \in W} t^{n^C_{wp_S}} \prod_{w\alpha \in w(\Delta^+ \backslash \Delta_S^+) \cap \Delta^+} \frac{1}{1-e^{-w\alpha}} \prod_{w\alpha \in w(\Delta^+ \backslash \Delta_S^+) \cap \Delta^-} \frac{e^{w\alpha}}{1-e^{w\alpha}} \\
&\qquad \cdot \sum_{\Lambda \in \mathcal{D}_S} m_\Lambda^S \, w \left( \frac{e^\Lambda}{\prod_{\alpha \in \Delta_S^+}(1-e^{-\alpha})} \right) L^C_{w,S} \\
(5.5) \quad &= \sum_{w \in W} (-1)^{l(w)-l_S(w)} t^{l_S(w)+n^C(\{w\lambda_k^S\})}\, w \left( \prod_{\alpha \in \Delta^+ \backslash \Delta_S^+} \frac{1}{1-e^{-\alpha}} \right) \\
&\qquad \cdot \sum_{\Lambda \in \mathcal{D}_S} m_\Lambda^S w \left( \frac{e^\Lambda}{\prod_{\alpha \in \Delta_S^+}(1-e^{-\alpha})} \right) L^C_{w,S} \\
&= \sum_{w \in W} \mathrm{det}_S w \; t^{l_S(w)+n^C(\{w\lambda_k^S\})} \frac{\det w \sum_{\Lambda \in \mathcal{D}_p} m_\Lambda^S \, e^{w(\Lambda+\rho)-\rho}}{\prod_{\alpha \in \Delta^+}(1-e^{-\alpha})} L^C_{w,S},
\end{aligned}$$

where

$$(5.6) \qquad L^C_{w,S} = \prod_{w\lambda_k^S \in C^*} \frac{1}{1-e^{-w\lambda_k^S}} \prod_{w\lambda_k^S \in -C^*} \frac{e^{w\lambda_k^S}}{1-e^{w\lambda_k^S}}$$

is a factor from weights not in $\Delta^+ \backslash \Delta_S^+$. Using Weyl's character formula, the contribution to (2.2) from the cohomology group is

$$(5.7) \qquad \sum_{k=0}^n t^k \,\mathrm{char}(H^k) = \sum_{k=0}^n t^k \sum_{\Lambda \in \mathcal{D}} m_\Lambda^k \frac{\sum_{w \in W} \det w \; e^{w(\Lambda+\rho)-\rho}}{\prod_{\alpha \in \Delta^+}(1-e^{-\alpha})}.$$

The result follows after multiplying both sides of (2.2) by $\prod_{\alpha \in \Delta^+}(1-e^{-\alpha})$. q.e.d.

**Remark 5.11.** Setting $t = -1$ in (5.4), we obtain a fixed-point formula

$$\sum_{S\in F/W}\sum_{w\in W}\det w\sum_{\Lambda\in\mathcal{D}_S} m_\Lambda^S\, e^{w(\Lambda+\rho)-\rho}\prod_{k=1}^{n-|\Delta^+\setminus\Delta_S^+|}\frac{1}{1-e^{-w\lambda_k^S}} = \sum_{k=0}^{n}(-1)^k\sum_{w\in W}\det w\sum_{\Lambda\in\mathcal{D}} m_\Lambda^k\, e^{w(\Lambda+\rho)-\rho} \tag{5.8}$$

for non-Abelian group actions.

## Acknowledgement

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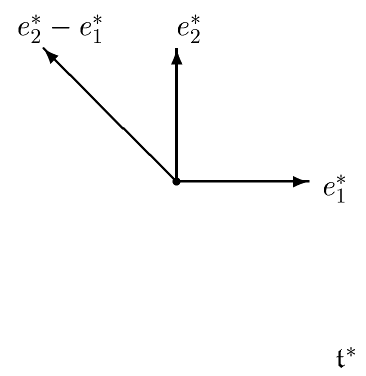
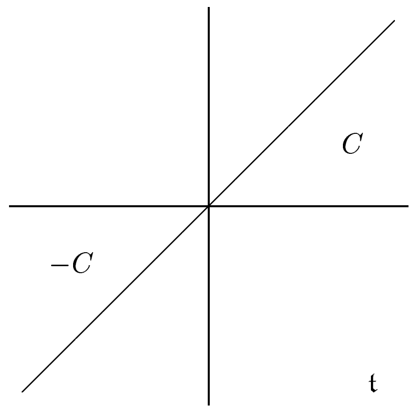
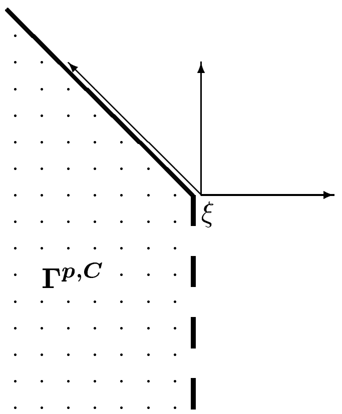


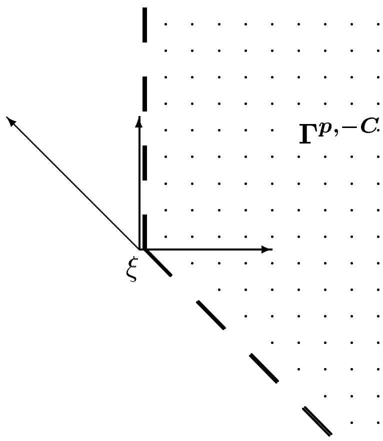

(a) An example of isotropy weights at a fixed point $p \in F$ when $\dim T = 2$, $\dim_{\mathbb{C}} M = 3$.




(b) The hyperplanes cut $\mathfrak{t}$ into 6 chambers. (Weights at other fixed points may cut them further.) A chamber $C$ is chosen.

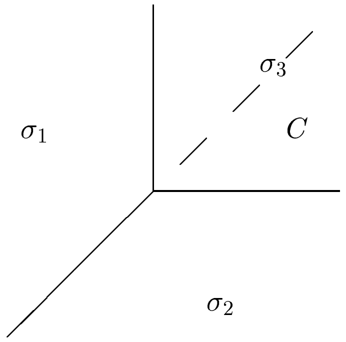



(c) The cone $\Gamma^{p,C}$ when $\text{supp}E_p = \{\xi\}$. Dotted line stands for boundaries not included. The polarizing index $n^{p,C} = 1$.

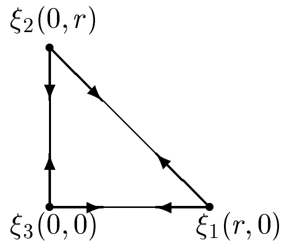



(d) The cone $\Gamma^{p,-C}$ when $\text{supp}E_p = \{\xi\}$. Dotted line stands for boundaries not included. The polarizing index $n^{p,-C} = 2$.

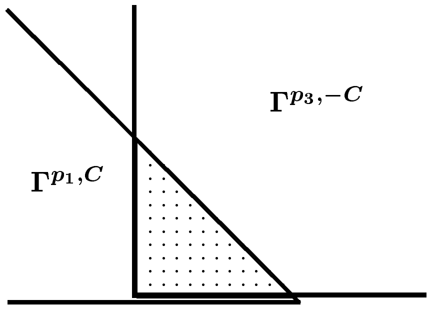
Figure 1: Illustration of isotropy weights, action chambers and the regions $\Gamma^{p,C}$.



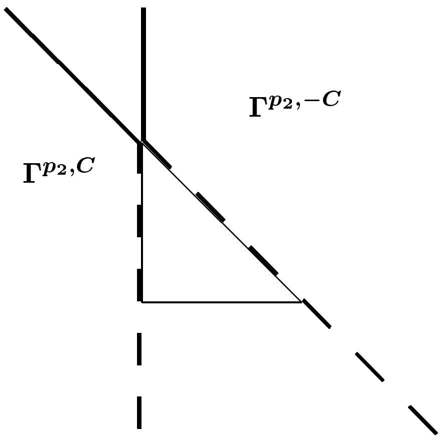

(a) The fan of $\mathbb{C}P^2$ with three top dimensional cones $\sigma_i$ $(1 \le i \le 3)$. An action chamber $C = \sigma_3 \cap (-\sigma_1)$ is chosen.



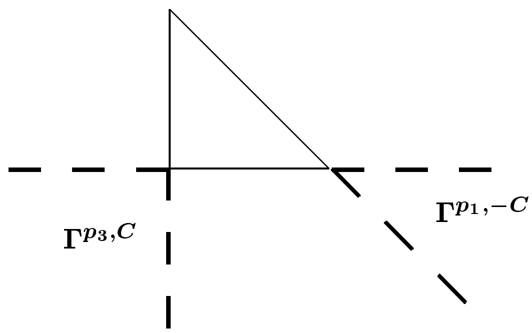

(b) Position of weights $\xi_i$ in the fiber over $p_i \in F$. The two arrows at $\xi_i$ are the negative of isotropy weights at $p_i$.




(c) The two cones $\Gamma^{p_1,C}$ and $\Gamma^{p_3,-C}$, with $n^{p_1,C} = n^{p_3,-C} = 0$, intersect at the dotted region.




(d) The two cones $\Gamma^{p_2,C}$ and $\Gamma^{p_2,-C}$, with $n^{p_2,C} = n^{p_2,-C} = 1$, have empty intersection.




(e) The two cones $\Gamma^{p_3,C}$ and $\Gamma^{p_1,-C}$, with $n^{p_2,C} = n^{p_2,-C} = 2$, have empty intersection.

Figure 1: The projective space $\mathbb{C}P^2$ with an invariant line bundle of first Chern number $r \geq 0$

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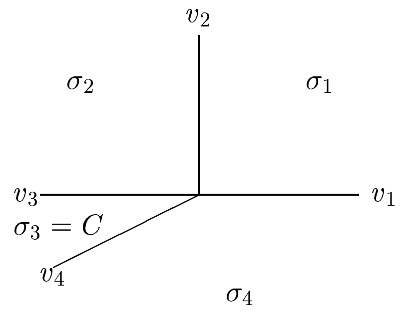
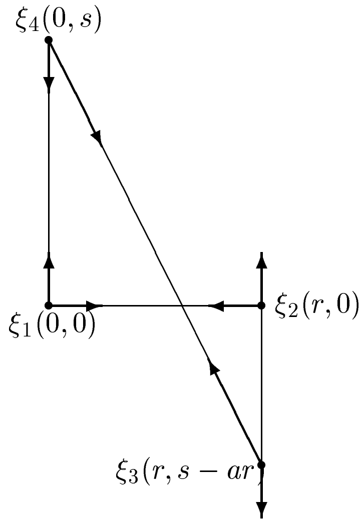
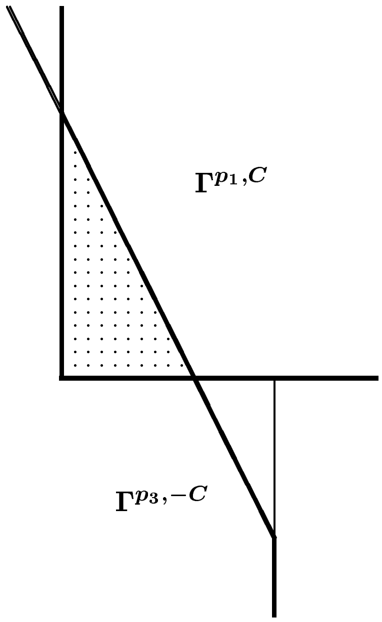
International Centre for Theoretical Physics, Italy
Mathematical Sciences Research Institute, Berkeley



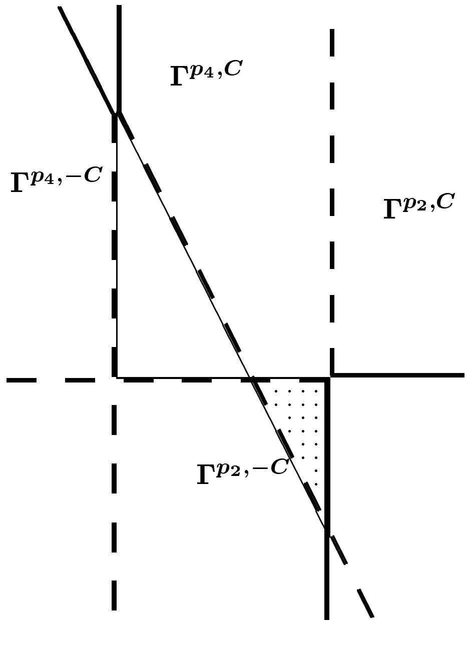

(a) The fan of the Hirzebruch surface with four top dimensional cones $\sigma_i$ ($1 \leq i \leq 4$). A chamber $C = \sigma_3$ is chosen.



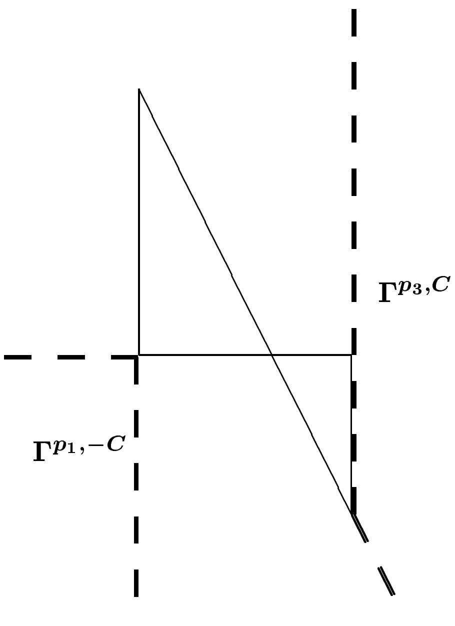

(b) Position of weights $\xi_i$ in the fiber over $p_i \in F$. The two arrows at $\xi_i$ are the negative of isotropy weights at $p_i$.




(c) The two cones $\Gamma^{p_1,C}$ and $\Gamma^{p_3,-C}$, with $n^{p_1,C} = n^{p_3,-C} = 0$, intersect at the dotted region.

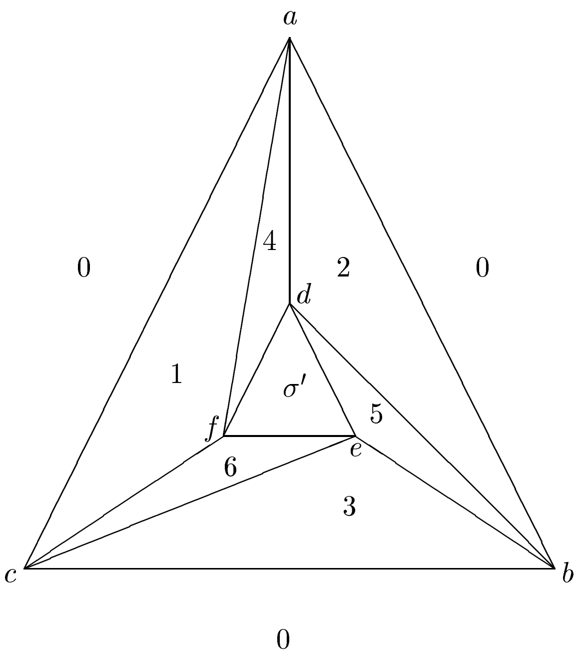


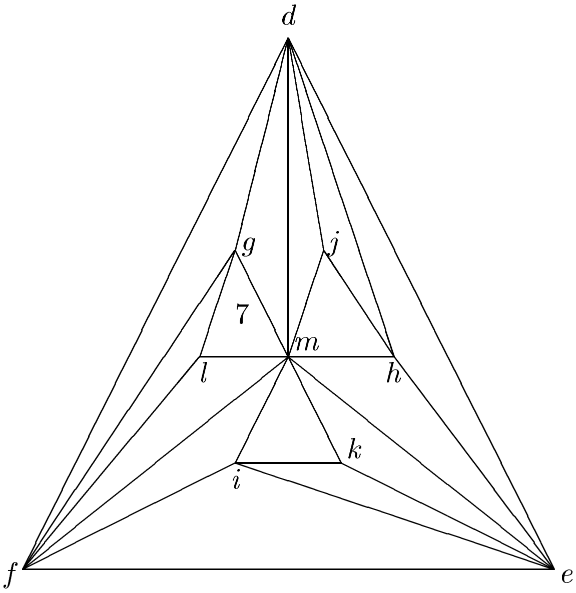

(d) The four cones with polarizing indices 1. $\Gamma^{p_4,C}$ and $\Gamma^{p_2,-C}$ intersect at the dotted region.




(e) The two cones $\Gamma^{p_3,C}$ and $\Gamma^{p_1,-C}$, with $n^{p_3,C} = n^{p_1,-C} = 2$, have empty intersection.

Figure 2: Hirzebruch surface with an invariant line bundle, case $s < ar$

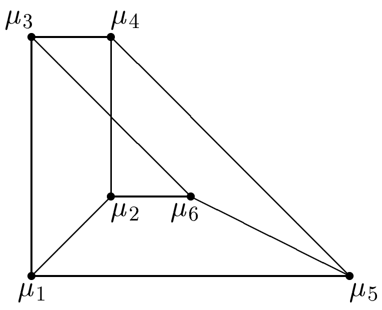


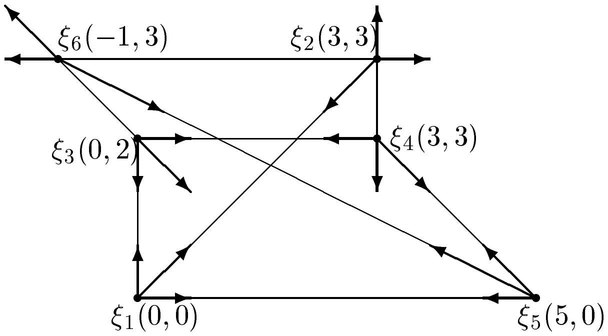

(a) A fan that does not admit a strictly convex piecewise linear function. The edges are spanned by $a(1,0,0)$, $b(0,1,0)$, $c(0,0,1)$, $d(0,-2,-1)$, $e(-1,0,-2)$, $f(-2,-1,0)$.

(b) The triangulation of $\sigma'$ by Jurkiewicz. New edges are generated by $g(-1,-2,-1)$, $h(-1,-1,-2)$, $i(-2,-1,-1)$, $j(-1,-2,-2)$, $k(-2,-1,-2)$, $l(-2,-2,-1)$, $m(-1,-1,-1)$.

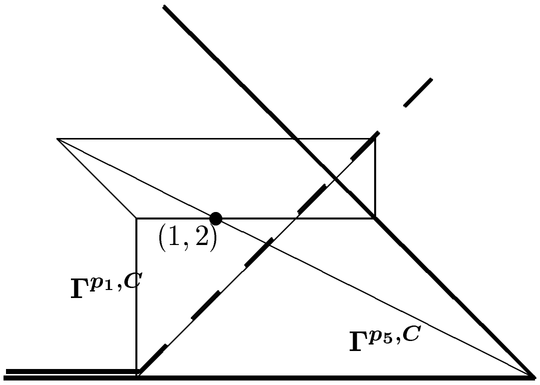
Figure 1: Stereographic projection of the fan $\Sigma$ corresponding to a non-Kähler toric 3-manifold. There are 22 top-dimensional cones $\sigma_i$ $(0 \leq i \leq 21)$ in $\Sigma$. The cones $\sigma_i$ $(0 \leq i \leq 6)$ are shown in (a). The cones $\sigma_i$ $(7 \leq i \leq 21)$ are contained in $\sigma'$. Only $\sigma_7$ is labeled in (b).



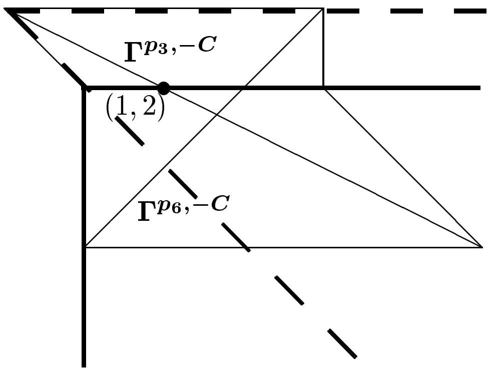

(a) The moment polytope of Tolman's symplectic manifold. The vertices $\mu_i$ are the values of the moment map $\mu$ at $p_i \in F$.




(b) Position of the weights $\xi_i$ in the fiber of a line bundle over $p_i \in F$. The negative of isotropy weights are also shown.




(c) Two cones $\Gamma^{p_1,C}$, $\Gamma^{p_5,C}$ containing $e_1^* + 2e_2^*$. The polarizing indices $n^{p_1,C} = 1$, $n^{p_5,C} = 0$.




(d) Two cones $\Gamma^{p_3,-C}$, $\Gamma^{p_6,-C}$ containing $e_1^* + 2e_2^*$. The polarizing indices $n^{p_3,-C} = 0$, $n^{p_6,-C} = 2$.

Figure 2: Moment polytope and a $T^2$-invariant line bundle over Tolman's manifold