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AN ANALYTIC COMPACTIFICATION OF SYMPLECTIC GROUP

HONGYU L. HE

Introduction

In this paper, we propose an analytic compactification from the symplectic group $Sp_{2n}(\mathbb{R})$ to the symmetric space $U(2n)/O_{2n}(\mathbb{R})$. We obtain this compactification through the study of Bargmann-Segal model of the oscillator (metaplectic) representation.

In the theory of symmetric spaces, a Hermitian symmetric space of noncompact type can be realized as a bounded domain in a complex vector space. For example, $SL(2,\mathbb{R})/SO(2)$ can be realized as the Poincaré disc. Harish-Chandra studied harmonic analysis on such a domain. The studies along this line had been quite fruitful for holomorphic discrete representations. For a noncompact reductive group G, we also wish to do analysis on an appropriate compactification $\overline{G}$ of G. If the "push forward" of matrix coefficients of unitary representations of G behaves reasonably well, we may gain a better understanding of unitary representations of G through the study of functions on $\overline{G}$ (see [5]).

To begin with, let X be an analytic manifold. We say $(i,\overline{X})$ is an analytic compactification of X, if $\overline{X}$ is a compact analytic manifold and

$$i : X \to \overline{X}$$

is an analytic embedding such that $i(X)$ is open dense in $\overline{X}$. Let G be the standard symplectic group. Then G has a KAK decomposition, where K is $U(n) = Sp_{2n}(\mathbb{R}) \cap SO_{2n}(\mathbb{R})$ and $A \cong \mathbb{R}^n$. Let K^o be the opposite group. Then G has a $K \times K^o$ action. For the symmetric

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space $Y = U(2n)/O_{2n}(\mathbb{R})$, one can also define a $K \times K$ action on Y, where $K \times K$ is embedded diagonally into $U(2n)$. We define a group isomorphism $\tau : K \times K^o \to K \times K$ by

$$\tau(k_1, k_2) = (\overline{k_1}, k_2^{-1}) \qquad (k_1, k_2 \in K).$$

Thus $K \times K^o$ can be identified with $K \times K$ through τ. In this paper, we prove the following theorem.

Theorem 0.1. *There exists a $U(n) \times U(n)^o$-equivariant analytic embedding:*

$$\mathcal{H} : Sp_{2n}(\mathbb{R}) \to U(2n)/O_{2n}(\mathbb{R})$$

such that $(\mathcal{H}, U(2n)/O_{2n}(\mathbb{R}))$ is an analytic compactification.

If f is a $U(n)$-finite matrix coefficient of an irreducible nontrival unitary representation of $Sp_{2n}(\mathbb{R})$, then f vanishes at infinity (see Theorem 5.4 [3]). Thus f can automatically be extended into a continuous function f^0 on $U(2n)/O_{2n}(\mathbb{R})$ by zero. It is well-known that f is analytic on $Sp_{2n}(\mathbb{R})$. Therefore f^0 is analytical on $\mathcal{H}(Sp_{2n}(\mathbb{R}))$. The intrigue question is whether f^0 is analytic over the boundary. If it is the case, is the matrix coefficients' being analytic a consequence of unitarity of the underlying representation? In other words, can there be a nonunitary irreducible representation whose $U(n)$-finite matrix coefficient can be extended to an analytic function on $U(2n)/O(2n)$? Of course, these questions are far beyond the scope of this paper, we wish to address these questions in the future.

In fact, $U(2n)/O_{2n}$ can be realized as a space of matrices. Let $\mathcal{S}_{2n}$ be the space of symmetric unitary matrices of the following form

$$\{X^t X \mid X \in U(2n)\}.$$

If $2n$ is fixed, we will write $\mathcal{S}$. Now $g \in U(2n)$ acts on $\mathcal{S}$ by

$$\tau(g) : s \to gsg^t \qquad (s \in \mathcal{S}).$$

We compute the isotropic subgroup at the identity,

$$U(2n)_I = \{U^t U = I \mid U \in U(2n)\} = O_{2n}.$$

Therefore $\mathcal{S}$ can be identified with $U(2n)/O_{2n}$. In this paper, we establish the compactification on the model $\mathcal{S}$ through the study of the Bargmann-Segal model of the oscillator representation.

Roughly speaking, the Bargmann-Segal model is the "minimal" unitary representation of the double covering of $Sp_{2n}(\mathbb{R})$. The underlying Hilbert space is the space of L^2-analytic functions with respect to the Gaussian measure. Then the group action of $\widetilde{Sp_{2n}}(\mathbb{R})$ can be expressed as integration operators. We observe some nice structure in the integration kernel which leads to the compactification $(\mathcal{H}, \mathcal{S})$.

Recall that $Sp_2(\mathbb{R}) = SL(2, \mathbb{R})$ and $SL(2, \mathbb{R}) \cong SU(1,1)$. The latter is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \frac{1}{2} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}.$$

The compactification of these groups are given by the following theorem:

Theorem 0.2. *We have for* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$

$$\mathcal{H}(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} \frac{(a-d)+(b+c)i}{(a+d)-(c-b)i} & \frac{-2i}{(a+d)-(c-b)i} \\ \frac{-2i}{(a+d)-(c-b)i} & \frac{(a-d)-(b+c)i}{(a+d)-(c-b)i} \end{pmatrix} \in \mathcal{S}_2,$$

and for $\begin{pmatrix} p & q \\ \overline{q} & \overline{p} \end{pmatrix} \in SU(1,1)$,

$$\mathcal{H}(\begin{pmatrix} p & q \\ \overline{q} & \overline{p} \end{pmatrix}) = \begin{pmatrix} \frac{iq}{\overline{p}} & \frac{-i}{\overline{p}} \\ \frac{-i}{\overline{p}} & \frac{-i\overline{q}}{\overline{p}} \end{pmatrix}.$$

Finally, let $\mathcal{D} = \mathcal{S} - Im(\mathcal{H})$ be the boundary at infinity. In this paper, we prove

Theorem 0.3. *$\mathcal{D}$ is an irreducible subvariety of codimension* 1 *in* $\mathcal{S}$. *The generic $U(n) \times U(n)$-orbit in $\mathcal{D}$ is given by*

$$U(n) \times U(n) / \{(U, V) \mid U = diag(\pm 1, \pm 1, \ldots \pm 1) = V\}.$$

Therefore $\mathcal{D}$ can be regarded as the divisor at infinity of the compactification $(\mathcal{H}, \mathcal{S})$.

The following is what is covered here in this paper. In Chapter 1, we introduce the Bargmann-Segal model of $Sp_{2n}(\mathbb{R})$. In Chapter 2, we introduce the structure theory of $Sp_{2n}(\mathbb{R})$ and define a continuous and one-to-one mapping $\mathcal{H}$ from $Sp_{2n}(\mathbb{R})$ to $\mathcal{S}$. In Chapter 3, we prove that $d\mathcal{H}$ is nondegenerate. Thus $\mathcal{H}$ is a local diffeomorphism. We show that $\mathcal{H}$ is in fact analytic. Therefore, $\mathcal{H}$ is an analytic embedding. In Chapter 4, we use generalized Cartan decomposition to show that the image of $\mathcal{H}$ is in fact dense in $U(2n)/O_{2n}(\mathbb{R})$. We also provide the proofs of Theorems 0.1 and 0.2. In Chapter 5, we prove Theorem 0.3.

I should make a final remark here. It can be shown that $SO_{p,q}(\mathbb{R})$ also possesses an analytic compactification, namely, $SO_{p+q}(\mathbb{R})$. This compactification is established in [4] using reductive dual pair and Corallary 3.5 in this paper. The author would like to thank Professors Helgason, Schlichtkrull and Vogan for their advices and the referee for pointing out the question regarding divisor at infinity.

1. Bargmann-Segal model

Let V be an $n-$dimensional complex Hilbert space with the standard inner product $(*,*)$. Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis of V. We write

$$(u,v) = Re(u,v) + iIm(u,v) \qquad (u,v \in V).$$

Then $\Omega(u,v) = Im(u,v)$ is a real symplectic form on V. Notice that

$$i\Omega(ie_j, e_k) = iIm(ie_j, e_k) = i\delta_j^k.$$

We fix a real basis once for all

$$\{ie_1, ie_2, \ldots, ie_n, e_1, \ldots, e_n\} = \{\xi_1, \xi_2, \ldots, \xi_n, e_1, \ldots, e_n\}.$$

If we regard V as a real vector space under such a basis, then Ω is the standard symplectic form, and $Re(,)$ is the standard (real) inner product. From now on, whenever we regard V as a complex space, we will add a subscript $\mathbb{C}$. For a linear endomorphism g of V, without the subscript $\mathbb{C}$, g will be a real linear transform. However $g_{\mathbb{C}}$ will be a complex linear transform of V.

Let $O_{2n}(\mathbb{R})$ be the subgroup of $GL(V)$ fixing $Re(,)$, and $Sp_{2n}(\mathbb{R})$ be the subgroup of $GL(V)$ fixing $\Omega(,)$. Let $U(n)$ be the subgroup of $GL(V)$ fixing the (complex) inner product $(,)$. Then

$$U(n) = O_{2n}(\mathbb{R}) \cap Sp_{2n}(\mathbb{R}).$$

In terms of real basis, the complex multiplication by imaginary i can be identified with left multiplication by

$$J = \begin{pmatrix} O & I \\ -I & 0 \end{pmatrix}.$$

For arbitrary $g \in Sp_{2n}(\mathbb{R})$, g can be decomposed into

$$g = C_g + A_g,$$

where C_g commutes with J, and A_g anticommutes with J. Thus $C_g \in End_{\mathbb{C}}(V)$, and A_g is complex-conjugate linear. Explicitly,

$$C_g = \frac{1}{2}(g - JgJ), \qquad A_g = \frac{1}{2}(g + JgJ).$$

It is known that $C_g \in GL_{\mathbb{C}}(V)$ (see [8]). Let $\Pi_{\mathbb{C}}(V)$ be the set of $T \in GL_{\mathbb{C}}(V)$ for which $Re(Tv, v)$ is strictly positive for all nonzero $v \in V$. According to [2], $\Pi_{\mathbb{C}}(V)$ is a contractible open domain of the identity in $GL_{\mathbb{C}}(V)$. Consequently, there is a unique continuous function

$$\det{}_{\mathbb{C}}^{\frac{1}{2}} : \Pi_{\mathbb{C}}(V) \to \mathbb{C}$$

such that

$$\det{}_{\mathbb{C}}^{\frac{1}{2}}(I) = 1, \qquad (\det{}_{\mathbb{C}}^{\frac{1}{2}}(T))^2 = \det{}_{\mathbb{C}} T, \quad (T \in \Pi_{\mathbb{C}}(V)).$$

Notice here $\det_{\mathbb{C}} T$ is the determinant of T as a complex matrix. Now we define

$$Z_g = C_g^{-1} A_g, \qquad (g \in Sp(V, \Omega)).$$

According to [8], we have $I - Z_{g_1} Z_{g_2} \in \Pi_{\mathbb{C}}(V)$ for $g_1, g_2 \in Sp(V, \Omega)$.

Let $Mp(V, \Omega)$ be the double cover of $Sp_{2n}(\mathbb{R})$. This group is often called the metaplectic group. Sometimes, we denote it by $\widetilde{Sp_{2n}}(\mathbb{R})$. There is in fact a nice way to represent this group (see [8]).

Theorem 1.1.

$$Mp(V, \Omega) = \{(\lambda, g) \mid g \in Sp(V, \Omega), \lambda \in \mathbb{C}, \lambda^2 \det{}_{\mathbb{C}}(C_g) = 1\}.$$

In addition, the multiplicative structure is given by

$$(\lambda_1, g_1)(\lambda_2, g_2) = (\lambda_1 \lambda_2 (\det{}_{\mathbb{C}}^{\frac{1}{2}}(I - Z_{g_1} Z_{g_2^{-1}}))^{-1}, g_1 g_2).$$

Now we will construct the Bargmann-Segal model. Let dx be the Euclidean measure on V. Let

$$d\mu(x) = \exp(-\frac{1}{2}(x, x)) dx$$

be the Gaussian measure. Let $\mathcal{P}_n$ or simply $\mathcal{P}$ be the polynomial ring on $V_{\mathbb{C}}$. We define an inner product on $\mathcal{P}$ by

$$(f, g) = \int_V f(x)\overline{g(x)} d\mu(x), \qquad (f, g \in \mathcal{P}).$$

Let $\|f\|^2 = (f, f)$. Let $\mathcal{F}$ be the completion of $\mathcal{P}$ under $\| * \|$. Then $\mathcal{F}$ is exactly the space of square Gaussian integrable analytic functions. In particular, $\| * \|$-covergence implies pointwise convergence. I should refer the reader to Bargmann's original paper [1] for details.

Theorem 1.2 (Bargmann-Segal model). *Let $(\lambda, g) \in Mp(V, \Omega)$. For every $f \in \mathcal{F}$, we define*

$$\omega(\lambda, g)f(z) = \int_V \lambda \exp \frac{1}{4}(2(C_g^{-1}z, w) - (z, Z_{g^{-1}}z) - (Z_g w, w))f(w)d\mu(w).$$

Then ω is a faithful unitary representation of $Mp(V, \Omega)$. Let

$$\mathcal{H}(g, z, w) = 2(C_g^{-1}z, w) - (z, Z_{g^{-1}}z) - (Z_g w, w).$$

If $g \neq g'$, then as functions of complex variables z and w

$$\mathcal{H}(g, z, w) \neq \mathcal{H}(g', z, w).$$

Proof. A proof of the first part of the theorem can be found in [8]. Suppose $g \neq g'$, but

$$\mathcal{H}(g, z, w) = \mathcal{H}(g', z, w).$$

Then $C_g = C_{g'}$. Let $\lambda \in \mathbb{C}$, such that

$$\lambda^2 \det_{\mathbb{C}}(C_g) = 1.$$

Then $(\lambda, g), (\lambda, g') \in Mp(V, \Omega)$, and

$$\omega(\lambda, g) = \omega(\lambda, g').$$

This implies that $g = g'$, a contradiction. q.e.d.

2. Some structure theory

Since $K = U(n)$ is a maximal compact subgroup of $Sp_{2n}(\mathbb{R})$, we can choose

$$A = \{diag(\lambda_1, \ldots, \lambda_n, \lambda_1^{-1}, \ldots, \lambda_n^{-1}) \mid \lambda_i \in \mathbb{R}^+\}$$

to be the maximal split Abelian subgroup. Then $Sp_{2n}(\mathbb{R})$ possesses a KAK decomposition.

Theorem 2.1. *For $g \in Sp_{2n}(\mathbb{R})$, let $g = k_1 a k_2$ be a KAK decomposition. Let $a = \exp(H)$, $H \in \mathfrak{a}$. Then we have*

$$C_g = k_1 \cosh(H) k_2, \qquad A_g = k_1 \sinh(H) k_2,$$

$$Z_g = k_2^{-1} \tanh(H) k_2, \qquad Z_{g^{-1}} = -k_1 \tanh(H) k_1^{-1},$$

where

$$\cosh(H) = \frac{1}{2}[\exp(H)+\exp(-H)]; \qquad \sinh(H) = \frac{1}{2}[\exp(H)-\exp(-H)]$$

$$\tanh(H) = (\cosh(H))^{-1}\sinh(H).$$

Proof. In $Sp_{2n}(\mathbb{R})$, the action of K commutes with J. Thus

$$\begin{aligned} C_g &= \frac{1}{2}(g - JgJ) \\ &= \frac{1}{2}(k_1 a k_2 - J k_1 a k_2 J) \\ &= \frac{1}{2}(k_1 a k_2 - k_1 J a J k_2) \\ &= k_1 C_a k_2. \end{aligned} \tag{1}$$

Similarly, we have

$$A_g = k_1 A_a k_2.$$

Thus

$$Z_g = C_g^{-1} A_g = (k_1 C_a k_2)^{-1}(k_1 A_a k_2) = k_2^{-1}(C_a^{-1} A_a) k_2.$$

Since $g^{-1} = k_2^{-1} a^{-1} k_1^{-1}$, we have

$$Z_{g^{-1}} = k_1 (C_{a^{-1}})^{-1} A_{a^{-1}} k_1^{-1}.$$

Now a simple computation shows that

$$JaJ = -a^{-1}, \qquad (a \in A).$$

Thus

$$C_a = \frac{1}{2}[\exp(H) + \exp(-H)] = \cosh(H),$$

$$A_a = \frac{1}{2}[\exp(H) - \exp(-H)] = \sinh(H),$$

$$Z_a = C_a^{-1} A_a = \tanh(H).$$

$$Z_{a^{-1}} = \tanh(-H) = -\tanh(H).$$

Therefore

$$C_g = k_1 \cosh(H) k_2, \qquad A_g = k_1 \sinh(H) k_2,$$

$$Z_g = k_2^{-1} \tanh(H) k_2, \qquad Z_{g^{-1}} = -k_1 \tanh(H) k_1^{-1}.$$

q.e.d.

We define

$$\operatorname{sech}(H) = (\cosh(H))^{-1}, \qquad \coth(H) = (\tanh(H))^{-1}.$$

Combined with Theorem 1.2, we have

Theorem 2.2. *Let (λ, g) be an element in $Mp(V,\Omega)$, and $g = k_1 \exp(H) k_2$ be a KAK decomposition. Then*

$$\begin{aligned}\mathcal{H}(g,z,w) =& 2(\operatorname{sech}(H) k_1^{-1} z, k_2 w) \\ &+ (k_1^{-1} z, \tanh(H) k_1^{-1} z) \\ &- (\tanh(H) k_2 w, k_2 w).\end{aligned}$$

In particular, the right-hand side does not depend on the KAK decomposition.

Recall that C_a is always complex linear and A_a is always complex-conjugate linear. Suppose

$$H = diag(H_1, \ldots, H_n, -H_1, \ldots, -H_n), \qquad (H_i \in \mathbb{R}).$$

We write

$$H_{\mathbb{C}} = diag(H_1, H_2, \ldots, H_n).$$

Then

$$(\operatorname{sech}(H) z, w) = (\operatorname{sech}(H_{\mathbb{C}}) z, w).$$

Now we want to compute $(\tanh(H)z, w)$. Let $z = iy + x$ with $x, y \in \mathbb{R}^n$. Then

$$\begin{aligned}\tanh(H) z =& \tanh(H)(x + iy) \\ =& i \tanh(H_{\mathbb{C}}) y - \tanh(H_{\mathbb{C}}) x \\ =& -\tanh(H_{\mathbb{C}}) \overline{z}.\end{aligned}$$

Therefore

$$(\tanh(H)z, w) = -(\tanh(H_{\mathbb{C}})\overline{z}, w),$$
$$(z, \tanh(H)w) = -(z, \tanh(H_{\mathbb{C}})\overline{w}).$$

Now

$$\begin{aligned}
\mathcal{H}(g,z,w) =& 2(\operatorname{sech}(H_{\mathbb{C}})k_1^{-1}z, k_2 w) - (k_1^{-1}z, \tanh(H_{\mathbb{C}})\overline{k_1^{-1}z}) \\
& + (\tanh(H_{\mathbb{C}})\overline{k_2 w}, k_2 w) \\
=& 2\overline{w^t k_2^t}\operatorname{sech}(H_{\mathbb{C}})k_1^{-1}z - z^t\overline{k_1}\tanh(H_{\mathbb{C}})k_1^{-1}z \\
& + \overline{w^t k_2^t}\tanh(H_{\mathbb{C}})\overline{k_2 w} \\
=& (iz^t, \overline{w^t}) \begin{pmatrix} \overline{k_1} & 0 \\ 0 & \overline{k_2^t} \end{pmatrix} \begin{pmatrix} \tanh(H_{\mathbb{C}}) & -i\operatorname{sech}(H_{\mathbb{C}}) \\ -i\operatorname{sech}(H_{\mathbb{C}}) & \tanh(H_{\mathbb{C}}) \end{pmatrix} \\
& \cdot \begin{pmatrix} k_1^{-1} & 0 \\ 0 & \overline{k_2} \end{pmatrix} \begin{pmatrix} iz \\ \overline{w} \end{pmatrix}.
\end{aligned} \tag{2}$$

Definition 2.1. We define

$$\begin{aligned}
&\mathcal{H}(k_1 \exp(H) k_2) \\
&= \begin{pmatrix} \overline{k_1} & 0 \\ 0 & \overline{k_2^t} \end{pmatrix} \begin{pmatrix} \tanh(H_{\mathbb{C}}) & -i\operatorname{sech}(H_{\mathbb{C}}) \\ -i\operatorname{sech}(H_{\mathbb{C}}) & \tanh(H_{\mathbb{C}}) \end{pmatrix} \begin{pmatrix} k_1^{-1} & 0 \\ 0 & \overline{k_2} \end{pmatrix}.
\end{aligned}$$

Notice that k_1, k_2 are unitary. One critical observation is that the images of $\mathcal{H}$ are symmetric unitary matrices. Therefore this definition of $\mathcal{H}$ is uniquely determined by the following equation

$$\mathcal{H}(g,z,w) = (iz^t, \overline{w^t})\mathcal{H}(g) \begin{pmatrix} iz \\ \overline{w} \end{pmatrix}. \tag{3}$$

Theorem 2.3. *The map $\mathcal{H}$ is a continuous injection from $Sp_{2n}(\mathbb{R})$ into $U(2n)$.*

Proof. First of all, if $\mathcal{H}(g) = \mathcal{H}(g')$, then

$$\mathcal{H}(g,z,w) = \mathcal{H}(g',z,w), \qquad (\forall\ \ z, w \in V).$$

According to Theorem 1.2, we have $g = g'$. Therefore $\mathcal{H}$ is an injection. Since the maps $g \to C_g^{-1}$, $g \to Z_g$, and $g \to Z_{g^{-1}}$ are all continuous, for every $z, w \in V$, the map $g \to \mathcal{H}(g,z,w)$ is continuous. From Equation 3 and by linearity, every entry of the matrix $\mathcal{H}(g)$ is a continuous function of $Sp_{2n}(\mathbb{R})$. Therefore, $\mathcal{H}(g)$ is continuous as well. q.e.d.

3. Analytic properties of $\mathcal{H}$

We define $\mathbb{T}_n$ in $U(2n)$ to be the space of matrices of the following form

$$T(\theta) = \begin{pmatrix} diag(\cos\theta_1, \dots, \cos\theta_n) & diag(-i\sin\theta_1, \dots, -i\sin\theta_n) \\ diag(-i\sin\theta_1, \dots, -i\sin\theta_n) & diag(\cos\theta_1, \dots, \cos\theta_n) \end{pmatrix}$$

$$(\theta = (\theta_1, \dots, \theta_n)).$$

We want to analyze the map $\kappa : A \to \mathbb{T}^n$, defined to be the restriction of $\mathcal{H}$ on A. Without loss of generality, let $n = 1$. Then

$$\kappa(\exp H) = \begin{pmatrix} \tanh H & -i\mathrm{sech} H \\ -i\mathrm{sech} H & \tanh H \end{pmatrix} \in \mathbb{T}.$$

κ can be regarded as a homeomorphism from $\mathbb{R}$ to $(0, \pi)$. Therefore, θ can be regarded as a continuous function of H, and H can also be regarded as a continuous function of θ. Notice that from $\tanh H = \cos\theta$, we obtain

$$(\mathrm{sech}(H))^2 dH = -(\sin\theta) d\theta.$$

Therefore

$$\frac{d\theta}{dH} = -\mathrm{sech}(H) \neq 0, \qquad \frac{dH}{d\theta} = -\csc\theta \neq 0.$$

Since all these functions are (real) analytic, κ is an analytic embedding from A to $\mathbb{T}^n$. From the $K \times K^o$ action on $Sp_{2n}(\mathbb{R})$ one may guess that $\mathcal{H}$ is in fact an embedding; however, in order to prove this, knowing the fact that κ is an embedding is not sufficient.

Let $\mathcal{S} = \{U^t U \mid U \in U(2n)\}$ be a subset of $U(2n)$. Then $\kappa(A)$ is contained in $\mathcal{S}$. Thus $\mathcal{H}(Sp_{2n}(\mathbb{R}))$ is in fact contained in $\mathcal{S}$. We obtain

Lemma 3.1.

- *Let $U(2n)$ act on $\mathcal{S}$ by*

  $$g \to U^t g U, \qquad (g \in \mathcal{S}, U \in U(2n)).$$

  Then $\mathcal{S} \cong U(2n)/O_{2n}(\mathbb{R})$.

- *The map $\mathcal{H}$ is a continuous map from $Sp_{2n}(\mathbb{R})$ into $\mathcal{S}$.*

- *Let $U(n)^o$ be the opposite group of $U(n)$. Let $U(n) \times U(n)^o$ act on $Sp_{2n}(\mathbb{R})$ by left and right multiplications respectively. Let*

$$\tau : U(n) \times U(n)^o \to U(n) \times U(n)$$

be a group isomorphism defined as follows:

$$\tau(k_1, k_2) = (\overline{k_1}, {k_2}^{-1}).$$

If we identify these two groups through τ, then $\mathcal{H}$ is equivariant with respect to these two group actions.

Now we want to compute the differential of $\mathcal{H}$,

$$d\mathcal{H} : TSp_{2n}(\mathbb{R}) \to T\mathcal{S}.$$

Let $g(t)$ be a germ of a smooth curve in a neigborhood of $g \in Sp_{2n}(\mathbb{R})$. Let dg be the tangent vector represented by this germ $g(t)$. Since $Sp_{2n}(\mathbb{R})$ is contained in the space of $2n \times 2n$ matrices, we can engage all our discussion in the space of $2n \times 2n$ matrices. Thus the tangent vector dg in $Sp_{2n}(\mathbb{R})$ can be identified with a $2n \times 2n$ matrix. This is going to be the perspective we take in interpreting all the equations we will have. From $gg^{-1} = 1$, we obtain

$$(dg)g^{-1} + g(dg^{-1}) = 0.$$

Therefore we have

$$dg^{-1} = -g^{-1}(dg)g^{-1}.$$

By standard calculus, we can prove the following lemma.

Lemma 3.2.

1. $dg^{-1} = -g^{-1}(dg)g^{-1}$;
2. $dC_g^{-1} = -C_g^{-1}(dC_g)C_g^{-1}$;
3. $dC_{g^{-1}} = -\frac{1}{2}(g^{-1}(dg)g^{-1} - Jg^{-1}(dg)g^{-1}J)$;
4. $dA_{g^{-1}} = -\frac{1}{2}(g^{-1}(dg)g^{-1} + Jg^{-1}(dg)g^{-1}J)$;
5. $dZ_g = -C_g^{-1}(dC_g)Z_g + C_g^{-1}(dA_g)$; *where*

$$dC_g = \frac{1}{2}(dg - J(dg)J), \qquad dA_g = \frac{1}{2}(dg + J(dg)J);$$

6. $dZ_{g^{-1}} = -C_{g^{-1}}^{-1}(dC_{g^{-1}})Z_{g^{-1}} + C_{g^{-1}}^{-1}(dA_{g^{-1}})$;
7. $dZ_g = -C_g^{-1}(dC_g)Z_g + C_g^{-1}(dA_g)$; *where*

$$dC_g = \frac{1}{2}(dg - J(dg)J), \qquad dA_g = \frac{1}{2}(dg + J(dg)J);$$

8. $dZ_{g^{-1}} = -C_{g^{-1}}^{-1}(dC_{g^{-1}})Z_{g^{-1}} + C_{g^{-1}}^{-1}(dA_{g^{-1}})$.

Now we can compute $d\mathcal{H}$. Let $\mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{sp}_{2n}(\mathbb{R})$ with $\mathfrak{k} = \mathfrak{u}(n)$. In fact, it can be shown that:

Lemma 3.3. *The space $\mathfrak{k}$ is complex linear and $\mathfrak{p}$ is complex-conjugate linear in $End(V)$.*

Now we have the following theorem.

Theorem 3.1. *The map $(d\mathcal{H})_g : T_g(Sp_{2n}(\mathbb{R})) \to T_{\mathcal{H}(g)}(\mathcal{S})$ is bijective.*

Proof. For an arbitrary $g \in Sp_{2n}(\mathbb{R})$, let $g = k_1 \exp H k_2$ be a KAK decomposition. Because of the action of $U(n) \times U(n)^o$, without loss of generality, we assume that $g = \exp H$, $H \in \mathfrak{a}$.

1. First notice that

   $$\dim(\mathcal{S}) = \dim(U(2n)) - \dim(O_{2n}(\mathbb{R})) = n(2n+1) = \dim(Sp_{2n}(\mathbb{R})).$$

   It suffices to show that the kernel of $(d\mathcal{H})_g$ is trivial.

2. Let dg be the equivalence class of the germ $g \exp tk$ with $k \in \mathfrak{k}$. Then we may write $dg = gk$. We have

   $$\begin{aligned} dC_g =& \frac{1}{2}(dg - J(dg)J) \\ =& \frac{1}{2}(gk - JgkJ) \\ =& \frac{1}{2}(g - JgJ)k \\ =& C_g k. \end{aligned} \tag{4}$$

   Similarly, we have

   $$dC_{g^{-1}} = -kC_{g^{-1}},$$
   $$dA_g = A_g k, \qquad dA_{g^{-1}} = -kA_{g^{-1}}.$$

Thus we obtain

$$\begin{aligned} dC_g^{-1} &= -\,C_g^{-1}(dC_g)C_g^{-1} = -kC_g^{-1}, \\ dZ_g &= -\,C_g^{-1}(dC_g)Z_g + C_g^{-1}(dA_g) = -kZ_g + Z_g k, \\ dZ_{g^{-1}} &= -\,C_{g^{-1}}^{-1}(dC_{g^{-1}})C_{g^{-1}}^{-1}A_{g^{-1}} + C_{g^{-1}}^{-1}dA_{g^{-1}} \\ &= C_{g^{-1}}^{-1}kA_{g^{-1}} - C_{g^{-1}}^{-1}kA_{g^{-1}} = 0. \end{aligned}$$

Now we have proved

$$d(z, Z_{g^{-1}}z) = 0, \tag{5}$$

$$d\mathcal{H}(g,z,w) = 2(-kC_g^{-1}z, w) + ((kZ_g - Z_g k)w, w). \tag{6}$$

Since $C_g \in GL_{\mathbb{C}}(V)$, we can see that:

$$\begin{aligned} & d\mathcal{H}(g,z,w) = 0 \ (\forall\ z, w \in V) \\ & \Longrightarrow (-kC_g^{-1}z, w) = 0 (\forall\ z, w \in V) \\ & \Longrightarrow k = 0. \end{aligned} \tag{7}$$

Conversely, $k = 0$ implies that $d\mathcal{H}(g,z,w) = 0(\forall\ z, w \in V)$.

3. On the other hand, let dg be the equivalence class of the germ $g \exp tp$ with $p \in \mathfrak{p}$. Then we may write $dg = gp$. Since $pJ = -Jp$, we have

$$dC_g = \frac{1}{2}(dg - JdgJ) = \frac{1}{2}(gp - JgpJ) = \frac{1}{2}(gp + JgJp) = A_g p,$$

$$dA_g = C_g p, \qquad dA_{g^{-1}} = -pC_{g^{-1}}, \qquad dC_{g^{-1}} = -pA_{g^{-1}}.$$

Then

$$dC_g^{-1} = -C_g^{-1}(dC_g)C_g^{-1} = -Z_g p C_g^{-1},$$

$$dZ_g = -C_g^{-1}(dC_g)Z_g + C_g^{-1}(dA_g) = -Z_g p Z_g + p,$$

$$\begin{aligned} dZ_{g^{-1}} &= -C_{g^{-1}}^{-1}(dC_{g^{-1}})Z_{g^{-1}} + C_{g^{-1}}^{-1}dA_{g^{-1}} \\ &= C_{g^{-1}}^{-1}pA_{g^{-1}}Z_{g^{-1}} - C_{g^{-1}}^{-1}pC_{g^{-1}} \\ &= C_{g^{-1}}^{-1}p(A_{g^{-1}}Z_{g^{-1}} - C_{g^{-1}}). \end{aligned} \tag{8}$$

Since $g = \exp H$, and $H \in \mathfrak{a}$, we have

$$\begin{aligned} A_{g^{-1}}Z_{g^{-1}} - C_{g^{-1}} =& \sinh(H)\tanh(H) - \cosh(H) \\ =&(\cosh(H))^{-1}(\sinh(H)^2 - \cosh(H)^2) \\ =& -(\cosh(H))^{-1}. \end{aligned} \tag{9}$$

Notice that $\mathfrak{a}$ is commutative. Thus our computation above is valid. The above equation implies

$$\begin{aligned} d(z, Z_{g^{-1}}z) =&(z, -\mathrm{sech}(H)(p)\mathrm{sech}(H)z) \\ =& -(z, \mathrm{sech}(H_{\mathbb{C}})(p)\mathrm{sech}(H_{\mathbb{C}})z). \end{aligned}$$

Suppose under the real basis $\{ie_j, e_j\}_1^n$,

$$p = \begin{pmatrix} A & B \\ B & -A \end{pmatrix}, \qquad (A^t = A, B^t = B). \tag{10}$$

Therefore

$$\begin{aligned} p(yi + x) =& i(Ay + Bx) + (By - Ax) = (Bi - A)(x - iy) \\ =&(Bi - A)(\overline{x + iy}). \end{aligned}$$

We see that

$$\begin{aligned} d(z, Z_{g^{-1}}z) =& -(z, \mathrm{sech}(H_{\mathbb{C}})(p)\mathrm{sech}(H_{\mathbb{C}})z) \\ =& -(z, \mathrm{sech}(H_{\mathbb{C}})(Bi - A)\mathrm{sech}(H_{\mathbb{C}})\overline{z}) \\ =& -z^t\mathrm{sech}(H_{\mathbb{C}})(-Bi - A)\mathrm{sech}(H_{\mathbb{C}})z \\ =& z^t\mathrm{sech}(H_{\mathbb{C}})(A + Bi)\mathrm{sech}(H_{\mathbb{C}})z. \end{aligned} \tag{11}$$

Since $A + Bi$ is a symmetric matrix and $\mathrm{sech}(H_{\mathbb{C}})$ is invertible, we have

$$d(z, Z_{g^{-1}}z) = 0 \ \ (\forall\ z \in V) \Longleftrightarrow A + Bi = 0 \Longleftrightarrow p = 0. \tag{12}$$

4. For an arbitrary $X = k + p \in \mathfrak{g}$, $g = \exp H$, we fix a germ $g(t) = g\exp(tX)$. Let p be defined as in Equation 10. Suppose that $d\mathcal{H}(g, z, w) = 0$. Then combined with Equation 5 and Equation 11, we see that

$$d(z, Z_{g^{-1}}z) = z^t\mathrm{sech}(H_{\mathbb{C}})(A + Bi)\mathrm{sech}(H_{\mathbb{C}})z = 0.$$

Thus we have

$$d(z, Z_{g^{-1}}z) = 0(\forall\ z \in V) \Longrightarrow p = 0$$

Now $X = k$. From Equation 7, we see that $k = 0$. Therefore, $X = 0$. Thus we have proved that

$$d\mathcal{H}(g, z, w) = 0\ (\forall\ z, w \in V) \Longrightarrow X = 0.$$

By Equation 3 we conclude that

$$d\mathcal{H}(g) = 0 \Longrightarrow X = 0.$$

5. Since $Sp_{2n}(\mathbb{R})$ is a Lie group, the tangent space $T_g(Sp_{2n}(\mathbb{R}))$ can be identified with those germs

$$g\exp(tX) \qquad (X \in \mathfrak{g}).$$

Thus

$$d\mathcal{H}|_g : T_g(Sp_{2n}(\mathbb{R})) \to T_{\mathcal{H}(g)}(\mathcal{S})$$

is injective. Because of the left and right K-action, this is true for all $g \in Sp_{2n}(\mathbb{R})$. q.e.d.

This theorem shows that $\mathcal{H}$ is an immersion, locally homeomorphism. It is also one-to-one. Thus $\mathcal{H}$ is a homeomorphism from $Sp_{2n}(\mathbb{R})$ onto an open submanifold of $\mathcal{S}$. In fact $\mathcal{H}$ is analytic.

Theorem 3.2. *The map $\mathcal{H} : Sp_{2n}(\mathbb{R}) \to \mathcal{S}$ is analytic.*

Proof. In this proof V will be regarded as a real vector space. Then $\mathcal{S}$ is an analytic submanifold of $B(V \oplus V, \mathbb{C})$, the space of symmetric complex-valued bilinear forms on $V \oplus V$. It suffices to show that

$$\mathcal{H} : Sp_{2n}(\mathbb{R}) \to B(V \oplus V, \mathbb{C})$$

is analytic. Recall that under the real basis $\{ie_j, (j = 1, \dots, n), e_j, (j = 1, \dots, n)\}$, multiplication by i can be regarded as left multiplication by J, and taking conjugation can be regarded as left multiplication by

$$B = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}.$$

Therefore

$$2(C_g^{-1}z, w) = 2\bar{w}^t C_g^{-1} z = 2\bar{w}^t C_g^{-1}(-J)iz,$$

$$(z, Z_{g^{-1}}z) = \overline{z^t \overline{Z_{g^{-1}}}^t} z = -(iz^t)B\overline{Z_{g^{-1}}}^t(iz),$$
$$(Z_g w, w) = \overline{w^t} Z_g w = \overline{w^t} Z_g B \overline{w}.$$

Since the maps $g \to g^{-1}$, $g \to C_g^{-1}$, $g \to Z_g$ are all real analytic, we conclude that

$$\mathcal{H} : Sp_{2n}(\mathbb{R}) \to B(V \oplus V, \mathbb{C})$$

is analytic. q.e.d.

Notice that the maps $g \to g^{-1}$, $g \to C_g^{-1}$, $g \to Z_g$ are all rational functions. Therefore by the same argument, we have

Theorem 3.3. *$\mathcal{H} : Sp_{2n}(\mathbb{R}) \to \mathcal{S}$ is a rational function.*

Now we have shown that $d\mathcal{H}_g$ is bijective and $\mathcal{H} : Sp_{2n}(\mathbb{R}) \to \mathcal{S}$ is analytic and one-to-one. From the classical theorem on inverse functions (see page 21 [9]), we obtain the following theorem.

Theorem 3.4. *The map $\mathcal{H} : Sp_{2n}(\mathbb{R}) \to \mathcal{S}$ is an analytic embedding.*

In a more general setting, we have

Theorem 3.5. *Let G be an arbitrary Lie group with a faithful representation into $Sp_{2n}(\mathbb{R})$. Suppose the closure of $\mathcal{H}(G)$, denoted by $\overline{G}$, is a compact smooth submanifold of $\mathcal{S}$. Then $(\mathcal{H}|_G, \overline{G})$ is an analytic compactification of G.*

4. Generalized Cartan decomposition and some remarks

Let G be a connected compact Lie group. For a subgroup H of G, let $N_G(H), Z_G(H)$ be the normalizer and centralizer of H in G. For a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, let $N_G(\mathfrak{h})$ and $Z_G(\mathfrak{h})$ be the normalizer and centralizer of $\mathfrak{h}$ in G. Suppose G is a compact connected Lie group. Let σ, τ be a pair of commuting involutions of G. Let K and H be the fixed point sets of σ and τ respectively. Let $\mathfrak{p}$ be the -1 eigenspace of σ, and $\mathfrak{q}$ the -1 eigenspace of τ. Let $\mathfrak{t}_{\mathfrak{pq}}$ be the maximal Abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$. Let $T_{\mathfrak{pq}}$ be the analytic group of $\mathfrak{t}_{\mathfrak{pq}}$. We define the Weyl group

$$W_{\mathfrak{pq}} = N_K(\mathfrak{t}_{\mathfrak{pq}})/Z_K(\mathfrak{t}_{\mathfrak{pq}}) \cong N_H(\mathfrak{t}_{\mathfrak{pq}})/Z_H(\mathfrak{t}_{\mathfrak{pq}}).$$

Theorem 4.1 (Generalized Cartan Decomposition). *The group G possesses a $KT_{\mathfrak{pq}}H$ decomposition. In other words,*

$$m : K \times T_{\mathfrak{pq}} \to G/H$$

is surjective. In addition, for $g = kth$, t is unique up to the action of $W_{\mathfrak{pq}}$ and a multiplication of $T_{\mathfrak{pq}} \cap Z_K(\mathfrak{t}_{\mathfrak{pq}})Z_H(\mathfrak{t}_{\mathfrak{pq}})$.

This theorem is essentially due to Hoogenboom (see p. 194 in [7]). Now for $G = U(2n)$, let

$$\sigma(x) = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix} x \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix} \qquad (x \in U(2n)),$$

$$\tau(x) = \overline{x}, \qquad (x \in U(2n)).$$

It is obvious that

$$\tau\sigma = \sigma\tau$$

and

$$K = U(n) \times U(n) \qquad H = O_{2n}(\mathbb{R}),$$

$$\mathfrak{p} = \{\begin{pmatrix} 0 & A \\ -\overline{A}^t & 0 \end{pmatrix} \mid A \in gl(n, \mathbb{C})\},$$

$$\mathfrak{q} = \{iB \mid B^t = B, B \in gl(2n, \mathbb{R})\}.$$

Thus

$$\mathfrak{p} \cap \mathfrak{q} = \{\begin{pmatrix} 0 & iA \\ iA^t & 0 \end{pmatrix} \mid A \in gl(n, \mathbb{R})\}$$

We may choose $T_{\mathfrak{pq}} = \mathbb{T}^n \subseteq U(2n)$. Then

$$\mathfrak{t}_{\mathfrak{pq}} = \Big\{ \mathfrak{t}_\theta = \begin{pmatrix} 0 & -diag(i\theta_1, \dots, i\theta_n) \\ -diag(i\theta_1, \dots, i\theta_n) & 0 \end{pmatrix} \cdot \mid \theta_i \in \mathbb{R}, i \in [1, n] \Big\}.$$

Hence $W_{\mathfrak{pq}}$ is simply the Weyl group of type B_n Lie algebra. More precisely, $W_{\mathfrak{pq}}$ acts on $\mathfrak{t}_\theta$ by permuting θ_i's and changing the signs of θ_i's. We identify $\mathbb{T}^n$ with $(\mathbb{T})^n$. According to the generalized Cartan decomposition, we have

Theorem 4.2. *The group $U(2n)$ possesses a $K\mathbb{T}^n H$ decomposition, where K is $U(n) \times U(n)$ embedded diagonally, and H is $O_{2n}(\mathbb{R})$. In addition, for $g = kth$, $t = \exp \mathfrak{t}_\theta$ is unique up to a reordering of $(\mathbb{T})^n$ and conjugations on any factor $\mathbb{T}$ in $\mathbb{T}^n$. If we define $\psi : K \times \mathbb{T}^n \to \mathcal{S}$ by*

$$\psi(k, t) = ktk^t \in \mathcal{S} \cong U(2n)/H \qquad (k \in K, t \in \mathbb{T}^n),$$

then ψ is surjective.

In particular, due to the action of Weyl group, we may assume that $\sin\theta_i \geq 0$ for every $i \in [1,n]$, i.e.,

$$\theta_i \in [0,\pi] \qquad (i \in [1,n]).$$

We observe that the set

$$Im(\mathcal{H}) = \psi(K \times (\mathcal{H}(A)) = \psi(K \times \{T_\theta \mid \theta_i \in (0,\pi)\})$$

is dense in $\mathcal{S}$. Combined with Theorem 3.4 we have shown

Theorem 4.3 (Compactification of $Sp_{2n}(\mathbb{R})$). *$(\mathcal{H},\mathcal{S})$ is an analytic compactification of $Sp_{2n}(\mathbb{R})$.*

For any function $f \in C(Sp_{2n}(\mathbb{R})$, let f^0 be the push-forward of f, defined to be

$$f^0(s) = f(\mathcal{H}^{-1}(s)), \qquad (s \in Im(\mathcal{H})),$$

and zero otherwise. Let (π, H) be a nontrivial irreducible unitary representation of $Sp_{2n}(\mathbb{R})$. Suppose now f is a $U(n)$-finite matrix coefficient of π. In other words, $f(g) = (\pi(g)u,v)$ with u, v $U(n)-$finite. It is well known that $f(g)$ vanishes at infinity (see Theorem 5.4 [3]). Let $\mathcal{D} = \mathcal{S} - Im(\mathcal{H})$ be the boundary. Since the push forward f^0 vanishes on the boundary $\mathcal{D}$, f^0 is continuous on $\mathcal{D}$. f^0 is also analytic on $Im(\mathcal{H})$. Therefore f^0 is continuous on $\mathcal{S}$. For the trivial representation, the matrix coefficients are constant functions. We can simply extend the constant function to $\mathcal{S}$.

We will compute the exact formula for the compactification of $Sp_2(\mathbb{R}) = SL(2,\mathbb{R})$.

Theorem 4.4.

$$\mathcal{H}(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} \frac{(a-d)+(b+c)i}{(a+d)-(c-b)i} & \frac{-2i}{(a+d)-(c-b)i} \\ \frac{-2i}{(a+d)-(c-b)i} & \frac{(a-d)-(b+c)i}{(a+d)-(c-b)i} \end{pmatrix} \in \mathcal{S}_2.$$

Proof. Let

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad g^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Then

$$C_g = \frac{1}{2}\begin{pmatrix} a+d & b-c \\ c-b & a+d \end{pmatrix}, \qquad A_g = \frac{1}{2}\begin{pmatrix} a-d & b+c \\ b+c & d-a \end{pmatrix}.$$

Thus we obtain

$$\det C_g = \frac{1}{4}(a^2+b^2+c^2+d^2+2ad-2bc) = \frac{1}{4}(a^2+b^2+c^2+d^2+2).$$

Let $\xi = a^2+b^2+c^2+d^2+2$. Then

$$C_g^{-1} = \frac{2}{\xi}\begin{pmatrix} a+d & c-b \\ b-c & a+d \end{pmatrix},$$

$$Z_g = C_g^{-1}A_g = \frac{1}{\xi}\begin{pmatrix} a^2-d^2+c^2-b^2 & 2(ab+cd) \\ 2(ab+cd) & b^2-c^2+d^2-a^2 \end{pmatrix}.$$

Recalling that the real basis of $V=\mathbb{C}$ is $i, 1$, we have

$$2(C_g^{-1}z,w) = \frac{4}{\xi}((a+d)+(c-b)i)z\overline{w} = \frac{-4i}{(a+d)+(b-c)i}(iz)\overline{w},$$

$$\begin{aligned}(Z_g w,w) =& \frac{1}{\xi}((b^2-c^2+d^2-a^2)+2(ab+cd)i)\overline{w}\overline{w} \\ =& \frac{(b+ai)^2+(d+ci)^2}{\xi}\overline{w}\overline{w} \\ =& \frac{(d-a)+(c+b)i}{(d+a)-(c-b)i}\overline{w}\overline{w}.\end{aligned}$$

Interchanging $a \leftrightarrow d, b \leftrightarrow -b, c \leftrightarrow -c$ gives

$$(Z_{g^{-1}}z,z) = \frac{(a-d)-(c+b)i}{(a+d)-(b-c)i}\overline{z}\overline{z}.$$

Thus

$$(z, Z_{g^{-1}}z) = \overline{(Z_{g^{-1}}z,z)} = \frac{(a-d)+(b+c)i}{(a+d)+(b-c)i}zz.$$

From

$$\mathcal{H}(g,z,w) = (iz,\overline{w})\mathcal{H}(g)\begin{pmatrix} iz, \overline{w} \end{pmatrix}$$

it follows that

$$\mathcal{H}(g) = \begin{pmatrix} \frac{(a-d)+(b+c)i}{(a+d)+(b-c)i} & \frac{-2i}{(a+d)+(b-c)i} \\ \frac{-2i}{(a+d)+(b-c)i} & \frac{(a-d)-(c+b)i}{(d+a)+(b-c)i} \end{pmatrix}.$$

It is easy to check that $\mathcal{H}(g) \in \mathcal{S}$. q.e.d.

5. Divisor at infinity

Now let us look at the boundary $\mathcal{D}$. The first question one may ask is whether $\mathcal{D}$ is a subvariety of codimension 1 in $\mathcal{S}$. If this is the case, is $\mathcal{D}$ irreducible? What is the defining function for this divisor? Let us first recall that

$$Im(\mathcal{H}) = \psi(K \times (\mathcal{H}(A)) = \psi(K \times \{T_\theta \mid \theta_i \in (0, \pi) \ \forall\ i \in [1, n]\}).$$

Then

$$\mathcal{D} = \psi(K \times \{T(\theta) \mid \exists\ i \in [1, n], \theta_i \in \{0, \pi\}\})$$

Observe that

$$\mathcal{D} \cap \mathbb{T}^n = \cup_{i=1}^n \mathbb{T}^n_{\pm,i},$$

where $\mathbb{T}^n_{\pm,i} = \{T(\theta) \mid \cos(\theta_i) = \pm 1\}$. In fact, each $\mathbb{T}^n_{\pm,i}$ can be identified with $\mathbb{T}^{n-1}$, hence is irreducible. We start with the following lemma.

Lemma 5.1. *The geometric dimension of $\mathcal{D}$ is $2n^2 + n - 1$.*

Proof. To compute the dimension of $\mathcal{D}$, we compute the isotropic algebra of the action of $\mathfrak{u}(n) \times \mathfrak{u}(n)$ on $T(\theta)$ (previously denoted by ψ). Suppose $(U, V) \in \mathfrak{u}(n) \times \mathfrak{u}(n)$ such that

$$\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \begin{pmatrix} \cos\theta & -i\sin\theta \\ -i\sin\theta & \cos\theta \end{pmatrix} + \begin{pmatrix} \cos\theta & -i\sin\theta \\ -i\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} U^t & 0 \\ 0 & V^t \end{pmatrix} = 0,$$

which implies that

$$\begin{pmatrix} U\cos\theta & -iU\sin\theta \\ -iV\sin\theta & V\cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta\overline{U} & -i\sin\theta\overline{V} \\ -i\sin\theta\overline{U} & \cos\theta\overline{V} \end{pmatrix}.$$

Thus

$$U\cos\theta = \cos\theta\overline{U}, \qquad V\cos\theta = \cos\theta\overline{V},$$

and therefore it follows that

$$\forall\ (i, j) \in [1, n], \qquad U_{i,j}\cos\theta_j = \cos\theta_i\overline{U_{i,j}} \qquad V_{i,j}\cos\theta_j = \cos\theta_i\overline{V_{i,j}},$$

so that

$$\forall\ (i, j) \in [1, n], \qquad \overline{U_{i,j}} = \pm U_{i,j} \qquad \overline{V_{i,j}} = \pm V_{i,j}.$$

Combining with the former equation yields

$$\forall\ (i,j) \in [1,n], \qquad U_{i,j}(\cos\theta_j \pm \cos\theta_i) = 0, \qquad V_{i,j}(\cos\theta_j \pm \cos\theta_i) = 0.$$

We assume that

$$\forall\ (i \neq j) \in [1,n], \qquad \cos\theta_i \pm \cos\theta_j \neq 0.$$

Then

$$\forall\ (i \neq j) \in [1,n], \qquad U_{ij} = 0, \qquad V_{ij} = 0.$$

For $i = j$, since $\overline{U_{i,i}} = -U_{i,i}$, we assume that

$$\cos\theta_i \neq 0.$$

Under these two assumptions on $T(\theta)$, we must have $U = 0$ and $V = 0$. Therefore, the isotropy algebra of $T(\theta)$ is trivial. This shows that if

$$\cos\theta_i \pm \cos\theta_j \neq 0, \qquad (\forall\ i \neq j \in [1,n]),$$

and $\cos(\theta_i) \neq 0$ for all i, then

$$\dim \psi((U(n) \times U(n)) \times T(\theta)) = 2n^2.$$

Since

$$\dim(\mathbb{T}^n_{\pm,i} \cap \{T(\theta) \mid \forall\, i \neq j \in [1,n], \cos\theta_i \pm \cos\theta_j \neq 0; \cos\theta_i \neq 0\}) = n-1,$$

we have

$$\dim(\mathcal{D}) = 2n^2 + n - 1.$$

q.e.d.

One may now ask the question, what does the (generic) $U(n) \times U(n)$-orbit look like in the boundary $\mathcal{D}$? Without loss of generality, we assume that $\cos\theta_1 = 1$ and

$$\forall\ (i \neq j) \in [1,n], \qquad \cos\theta_i \pm \cos\theta_j \neq 0; \cos\theta_i \neq 0.$$

Suppose that $X = (U,V) \in U(n) \times U(n)$ such that $XT(\theta)X^t = T(\theta)$. Then $XT(\theta) = T(\theta)\overline{X}$, and we obtain the following equations:

$$U\cos\theta = \cos\theta\overline{U}, \qquad V\cos\theta = \cos\theta\overline{V},$$

$$U\sin\theta = \sin\theta\overline{V}, \qquad V\sin\theta = \sin\theta\overline{U}.$$

For $i \neq j$, applying the argument in the proof of the lemma, we obtain

$$U_{ij} = 0, \qquad V_{ij} = 0.$$

Since U and V are diagonal, we see that

$$U = diag(\pm 1, \pm 1, \dots, \pm 1) = V.$$

Now we have proved the following theorem:

Theorem 5.1. *The generic $U(n) \times U(n)$-orbit on $\mathcal{D}$ is given by*

$$U(n) \times U(n)/\{(U, V) \mid U = diag(\pm 1, \pm 1, \dots, \pm 1) = V\}.$$

From a purely algebraic point of view, all these $U(n) \times U(n)$-orbits are closed and algebraic. Therefore they can all be defined by algebraic equations. The proof of the next theorem should give us some flavor about how we can construct the defining functions.

Theorem 5.2. *There exists an algebraic function $f \in \mathcal{O}_{\mathcal{S}}$ such that $zero(f) = \mathcal{D}$. Therefore, $\mathcal{D}$ is a subvariety of $\mathcal{S}$. In addition, $\mathcal{D}$ is an irreducible divisor.*

Before I go ahead proving this theorem, I should say that one can give a much easier proof for the existence of f. But such a proof cannot be generalized to produce an algorithm to compute the defining functions for any closed $U(n) \times U(n)$-subvariety of $\mathcal{S}$. Therefore I choose a more general construction here.

Proof.

1. We will construct a $U(n) \times U(n)$-invariant function $f \in \mathcal{O}_{\mathcal{S}}$ such that

   $$f(T(\theta)) = \prod_{i=1}^{n} \sin^2 \theta_i.$$

   Therefore

   $$zero(f) = \psi(K \times \{T(\theta) \mid \exists\, i \in [1, n], \theta_i \in \{0, \pi\}\}) = \mathcal{D}.$$

   Thus $\mathcal{D}$ is a closed subvareity.

2. We observe that

   - $\prod_{i=1}^{n} \sin^2 \theta_i = \prod_{i=1}^{n} -(\frac{1}{2}(\exp(i\theta_i) - \exp(-i\theta_i)))^2$
     $= \prod_{i=1}^{n} \frac{1}{2}(1 - \cos(2\theta_i))$.

- The following two linear independent bases

$$\{1, \exp(2i\alpha) + \exp(-2i\alpha), \ldots, \exp(2ki\alpha) + \exp(-2ki\alpha)\},$$

$$\{1, \exp(2i\alpha) + \exp(-2i\alpha), \ldots, (\exp(2i\alpha) + \exp(-2i\alpha))^k\}$$

span the same vector space. In other words,

$$\{1, \cos(2\alpha), \ldots, \cos(2k\alpha)\},$$

$$\{1, \cos(2\alpha), \ldots, (\cos(2\alpha))^k\}$$

span the same vector space.

- By the theory of symmetric functions, the function

$$\prod_{i=1}^{n}(1 - \cos(2\theta_i))$$

can be written as a function of

$$\{s_k = \sum_{i=1}^{n}(\cos(2\theta_i))^k \mid k \in \mathbb{N}\}.$$

- Therefore $\prod_{i=1}^{n} \sin^2 \theta_i$ can be expressed as a function of

$$\Big\{\sum_{i=1}^{n} \cos(2k\theta_i) \mid k \in \mathbb{N}\Big\}.$$

3. Now let $B = diag(-I_n, I_n)$. We look at the function $\phi_1(s) = Tr(B\overline{s}Bs)$. Since for every $g \in U(n) \times U(n)$, we have

$$\phi_1(gsg^t) = Tr(B\overline{gsg^t}Bgsg^t) = Tr((g^tB\overline{g})\overline{s}(\overline{g^t}Bg)s) = Tr(B\overline{s}Bs),$$

which shows that the function ϕ_1 is $U(n) \times U(n)$-invariant. On the other hand we have

$$B\overline{T(\theta)}BT(\theta) = BT(-\theta)BT(\theta) = T(2\theta),$$

$$\phi_1(T(\theta)) = \sum_{i=1}^{n} 2\cos(2\theta_i).$$

Similarly, we may define

$$\phi_k(s) = Tr((B\overline{s}Bs)^k).$$

Then $\phi_k(s)$ is $U(n) \times U(n)$-invariant, and

$$\phi_k(T(\theta)) = \sum_{i=1}^{n} 2\cos(2k\theta_i).$$

Now we can conclude that f can be expressed as a function of

$$\{\phi_k(s) \mid k \in \mathbb{N}\}.$$

It is not difficult to see that each ϕ_k is $U(2n)$-finite. Therefore $\phi_k \in \mathcal{O}_{\mathcal{S}}$. Thus f is algebraic and $\mathcal{D}$ is a closed subvareity.

4. From the lemma, we have

$$\dim(\mathcal{D}) = 2n^2 + n - 1.$$

Hence the subvariety $\mathcal{D}$ is of codimension 1.

5. Now suppose that $D = \sum \mathcal{D}_j$ with $\mathcal{D}_j$ irreducible. Since the group $U(n) \times U(n)$ acts on $\mathcal{S}$ algebraically, $U(n) \times U(n)$ acts on the irreducible components of $\mathcal{D}$. Since $U(n) \times U(n)$ is connected, it acts on the set of irreducible components trivially. Therefore, $U(n) \times U(n)$ acts on each $\mathcal{D}_j$. Now the intersection of $\mathcal{D}_j$ with $\mathbb{T}^n$ must be a $W_{\mathfrak{p}q}$-stable subvariety. However $\mathcal{D} \cap \mathbb{T}^n = \cup_{i=1}^{n} \mathbb{T}^n_{\pm,i}$ is the only $W_{\mathfrak{p}q}$-stable subvariety of codimension 1 of $\mathbb{T}^n$; that is contained in $\mathcal{D} \cap \mathbb{T}^n$. Therefore there is only one irreducible component of $\mathcal{D}$. This implies that $\mathcal{D}$ is irreducible. q.e.d.

We should make a final remark. In the setting of Cartan decomposition for a compact group G, we have the map

$$\psi : K \times A \to G/K.$$

Even though the singular points in the maximal torus A is of codimension 1, the singular points in G/K is always of codimension less or equal to two (see Ch VII.3 in [6]). The reason that in our case $\mathcal{D}$ is of codimension 1 is that $\mathcal{D}$ is only related to the sign changes in the Weyl group $W_{\mathfrak{p}q}$.

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