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COLLAPSED MANIFOLDS WITH PINCHED POSITIVE SECTIONAL CURVATURE

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Abstract

Let M^n be a manifold of sectional curvature, $0 < \delta \le K_{M^n} \le 1$, let X be an Alexandrov space of curvature ≥ -1. Suppose the Gromov-Hausdorff distance of M^n and X is less than $\epsilon(n, \delta) > 0$. Our main results are: (A) If X has the lowest possible dimension, $\frac{n-1}{2}$, then a covering space of M^n of order $\le \frac{n+1}{2}$ is diffeomorphic to a lens space, $S^n/\mathbb{Z}_q$, such that $0 < c(n,\delta)[vol(M^n)]^{-1} \le q \le vol(S^n_\delta)[vol(M^n)]^{-1}$, where S^n_δ is the sphere of constant curvature δ. (B) If X has nonempty boundary, then a covering space of M^n of order $\le \frac{n+1}{2}$ is diffeomorphic to a lens space, provided ϵ depends also on the Hausdorff measure of X.

0. Introduction

Let d_{GH} denote the Gromov-Hausdorff distance between two metric spaces, cf. [20]. Gromov's theory of almost flat manifolds asserts that a compact manifold, M^n, whose finite normal covering of order $\le i(n)$ (the Margulis constant) is diffeomorphic to a compact nilpotent manifold, N/Γ, if and only if M^n admits a metric with sectional curvature $|K_{M^n}| \le 1$ and $d_{GH}(M^n, pt) < \epsilon(n)$, a small constant depending only on n, see [6], [19] and [36].

In this paper, one of the problems we shall be concerned with is to characterize a compact manifold, M^n, which admits a metric with $0 < \delta \le K_{M^n} \le 1$ such that $d_{GH}(M^n, X)$ is sufficiently small depending only on n and δ, where X is an *Alexandrov space* of the lowest dimension with n and δ fixed, cf. [4] (see Theorem 0.4). Since the diameter of M^n

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is bounded above by $\pi/\sqrt{\delta}$ and since if n is even, the volume of M^n is bounded below by a half of the volume of the round sphere of radius one ([25]), we can assume that n is *odd*, cf. [7], [20]. The classification of space form in [38] implies $\dim(X) < n$. Consequently, M^n has small volume, i.e., M^n is *collapsed.*

In general, let X be an Alexandrov space of dimension $< n$. Suppose that $d_{GH}(M^n, X)$ is sufficiently small depending only on n and δ. We shall study problems concerning interactions between the geometry and topology of M^n and that of X. Let $S(X)$ denote the set of *singular points* of X; see Section 1, cf. [4]. A specific problem is determining

How the geometry and topology of M^n are reflected by $dim(X)$ or $codim(S(X))$?

Note that in order for the above problem to make sense, it must be assumed that X is not collapsed, i.e., the *s-Hausdorff measure*, $m_H(X)$, of X has a definite lower bound, where s is equal to the dimension of X; cf. [30], [37].

We now begin to state the main results of this paper.

Recall that the singular set of an Alexandrov space is of at most codimension 1, and is of codimension 1 if and only if the *boundary* is not empty. A boundary point is one at which the *space of directions* has non-empty boundary (note that a 1-dimensional compact Alexandrov space with boundary is a closed interval); cf. [4].

Theorem 0.1. *Let M^n be a compact manifold of $\delta \le K_{M^n} \le 1$, and let X be an Alexandrov space of $cur(X) \ge -1$ and $m_H(X) \ge m_0 > 0$. Suppose $d_{GH}(M^n, X) < \epsilon(n, \delta, m_0)$. If X has non-empty boundary, then a covering space of M^n with order $\le \frac{n+1}{2}$ is diffeomorphic to a lens space, $S^n/\mathbb{Z}_q$.*

For examples of lens spaces and non-lens spaces in Theorem 0.1, see Examples 5.1 and 6.3 respectively.

Let $\mathcal{M}_\delta^n$ be the set of compact n-manifolds with $\delta \le K \le 1$. Then, $\mathcal{M}_\delta^n$ has a compact closure with respect to d_{GH} ([20]), and each limit is an Alexandrov space of curvature $\ge \delta$ ([4]).

Theorem 0.1 has the following interesting consequence.

Corollary 0.2. *Let $M_i^n \xrightarrow{d_{GH}} X$ be a sequence in $\mathcal{M}_\delta^n$ of simply connected and diffeomorphically distinct. Then X has empty boundary.*

Examples of Corollary 0.2 have been known in dimensions 7 and 13 with $\delta < \frac{1}{37}$ ([1], [2], [13], [33]). Corollary 0.2 provides a constraint for any possible similar examples of higher dimensions.

The point of departure is the equivariant and parameterized fibration theorem adapting to collapsed manifolds of pinched sectional curvature. It asserts that there is a small constant, $v(n,\delta)>0$, such that if $M^n \in \mathcal{M}^n_\delta$ has volume $< v(n,\delta)$, then the universal covering space, $\tilde{M}^n$, admits an *almost* isometric π_1-*invariant* T^k-*action* without fixed points ([9], [15], [16]), and a nearby *invariant metric of positive curvature* ([34]); see Theorem 3.1. In terms of the terminology of [12], this π_1-invariant T^k-action is called *collapsible.*

A π_1-invariant T^k-action is a usual T^k-action on $\tilde{M}^n$ and a homomorphism, $\rho : \pi_1(M^n) \to Aut(T^k)$, such that the T^k-action extends to a $\pi_1(M^n) \ltimes_\rho T^k$-action. In particular, the T^k-action is the lift of a T^k-action on M if ρ is the identity map. In this case, we also say that the π_1-invariant T^k-action *descends* to M. Note that the notion of a π_1-invariant T^k-action is an alternative formulation for a *pure F-structure* on a manifold of finite fundamental group (see [10], [11]).

Let $\bar{M}^n = \tilde{M}^n/[\pi_1(M^n) \ltimes_\rho T^k]$. We will call $\bar{M}^n$ the *orbit space* of the π_1-invariant T^k-action.

By employing the above fibration theorem, together with Cheeger-Gromov's compactness theorem ([7], [20]), Perel'man's stability theorem ([30]) and a theorem of Grove-Searle ([22]), we get

Theorem 0.3. *Let M^n be a compact manifold of $\delta \le K_{M^n} \le 1$, and let X be an Alexandrov space of $cur(X) \ge -1$. Suppose $d_{GH}(M^n,X) < \epsilon(n,\delta)$. Then, $dim(X) \ge \frac{n-1}{2}$. Suppose, in addition, $d_{GH}(M^n,X) < \epsilon(n,\delta,m_0)$ with $m_H(X) \ge m_0 > 0$. If $dim(X) < n$, then M^n admits a collapsible π_1-invariant torus action such that $\bar{M}^n$ is homeomorphic to X, and otherwise M^n is homeomorphic to X.*

In view of Theorem 0.3, the following theorem will characterize a compact n-manifold which admits a δ-pinched metric such that it is close to an Alexandrov space of the lowest possible dimension.

Theorem 0.4. *Let M^n and X be as in Theorem 0.3 such that $d_{GH}(M^n,X) < \epsilon(n,\delta)$. Suppose $dim(X) = \frac{n-1}{2}$. Then a covering space of M^n with order $\le \frac{n+1}{2}$ is diffeomorphic to a lens space, $S^n/\mathbb{Z}_q$, such that*

$$0 < \frac{c(n,\delta)}{vol(M^n)} \le q \le \frac{vol(S^n_\delta)}{vol(M^n)},$$

where the constant $c(n,\delta)$ bounds from below the volume of the pullback metric on S^n, and S^n_δ denotes the sphere of constant curvature δ.

Note that from the proof, X in Theorem 0.4 has nonempty boundary (see Theorem 0.1). Also, $c(n,\delta)\to 0$ as $\delta\to 0$.

Corollary 0.5. *Let $M_i^n \xrightarrow{d_{GH}} X$ be a sequence in $\mathcal{M}_\delta^n$ of simply connected manifolds. Then $dim(X)\geq \frac{n+1}{2}$.*

A conjecture of Fukaya ([18]) asserts that X in Corollary 0.5 has dimension $\geq n-1$. This conjecture is false by a recent example in [32] of a sequence of Eschenburg 7-manifolds of positively pinched curvature that converges to a bi-quotient space, $T^2\setminus SU(3)/T^2$. This example shows that the lower bound in Corollary 0.5 is *sharp* for $n=7$.

For the completeness, we shall give the following result which will yield examples of lens spaces in Theorems 0.1 and 0.4 (see Example 5.1), compare [38].

Theorem 0.6. *Let M be a compact manifold which admits an isometric T^k-action. Suppose, in addition, that the fixed point set is empty. Then, there is a sequence of cyclic subgroups, $\{\Gamma_j\}$, of T^k freely acting on M^n such that, equipped with the quotient metrics, $M^n/\Gamma_j \xrightarrow{d_{GH}} M^n/T^k$.*

Before proceeding further, we would like to make some comments.

Remark 0.7. For a given M^n, it is possible that there are X, X' of different dimensions in Theorem 0.1 (see Example 5.1). Roughly, the choice of X depends an observer's scale of collapsing. At the end of the proof, we will show the existence of a constant, $m(n,\delta)$, such that for (M^n, X) in Theorem 0.1 with $m_H(X)\geq m_0$, if $m_0 << m(n,\delta)$, then there exists another X' satisfying Theorem 0.1 with $m_H(X')\geq m(n,\delta)$, see Lemma 3.4 and the discussion following it.

Remark 0.8. In the proof of Theorem 0.1 (resp. Theorem 0.4), we will show that X admits a metric with curvature $\geq\delta$ in the Alexandrov sense (resp. and X has nonempty boundary). According to [30], X is contractible (compare [22]). This should explain why the diffeomorphism type of M^n does not rely on a particular X, compare Remark 0.7.

Remark 0.9. Theorem 0.1 is quite *optimal* in certain sense. First, the order estimate cannot be improved, see Example 6.3. If one removes either of the (normalized) condition, $\mathrm{cur}(X)\geq -1$, or the dependence of $\epsilon(n,\delta,m_0)$ on m_0, without imposing further restriction, then counterexamples will occur, see Example 6.1. Note that by scaling, the pinching condition becomes $1 \leq K_{M^n} \leq \Lambda$, where $\Lambda = \delta^{-1}$. Then, Theorem 0.1 will be false if one removes the dependence on Λ without imposing any further restriction. For instance, following the construction of Example 2.4 in [21], a complex projective space admits a sequence of metrics of curvature ≥ 1 converging to a closed interval.

Remark 0.10. The Klingenberg-Sakai conjecture asserts that on a manifold which supports a δ-pinched metric, the infimum of the volumes of all possible δ-pinched metrics is positive (cf. [26]). This conjecture implies that the estimate as in Theorem 0.4 should also hold for Theorem 0.1.

Note that the conjecture is true in dimension three (see [5]) and a closed $2n$-dimensional manifold of $0 < K \leq 1$ has volume $\geq vol(S_1^{2n})/2$ (see [25]).

Remark 0.11. In view of Theorem 0.1, the following problem naturally arises: In each dimension, is there a *universal pinching constant* $\delta(n) > 0$. Namely, a positively curved metric on compact M^n implies a metric of $\delta(n) \leq K \leq 1$; compare [34]. Note that Hamilton's work provides an affirmative answer for $n = 3$ ([23]).

We now give an indication for the proofs of Theorems 0.1, 0.4.

Recall that $\bar{M}^n$ is, equipped with the quotient metric, an Alexandrov space and the property that $\bar{M}^n$ has non-empty boundary is topological (see p.16, p.54, [4]; compare Section 1, **b.**).

Given Theorem 0.3, the following result will imply Theorem 0.1.

Theorem 0.12. *Let M be a compact manifold of positive sectional curvature. If M admits a π_1-invariant isometric T^k-action such that the orbit space, $\bar{M}$, has non-empty boundary, then a covering space of M with order $\leq k$ is diffeomorphic to a lens space or a complex projective space.*

Note that the order estimate in Theorem 0.12 cannot be improved; see Example 6.3.

Theorem 0.12 generalizes a theorem of Grove-Searle ([22]) which asserts that a compact manifold, M^n, is diffeomorphic to a lens space or a complex projective space if and only if it admits a metric of $K_{M^n} > 0$ and an isometric circle action with fixed point set codimension 2 (see Theorem 2.1). A homeomorphic classification for compact 4-manifolds of $K > 0$ which admit isometric circle actions was obtained by [24].

We will first show that a π_1-invariant torus-action has a circle subgroup with fixed point set codimension 2 if and only if the orbit space has non-empty boundary (Corollary 1.5). Let S^1 denote any circle subgroup with fixed point set, $\tilde{F}_0$, of codimension 2. The *difference* here is that the S^1-action is not necessarily descendible when n is *odd* (see Example 6.3). If not, M^n is not necessarily diffeomorphic to a lens space.

Let H be the subgroup of $\pi_1(M^n)$ consisting of γ such that $\rho(\gamma) = id$ when restricting to S^1. A priori, H could be trivial (see Example 6.2). Nevertheless, since S^1-action descends to $\tilde{M}^n/H$, $\tilde{M}^n/H$ is diffeomorphic to a lens space by [22].

The problem is to estimate the index, $[\pi_1(M^n), H]$. It turns out that the index is bounded above by the number of circle subgroups of T^k with fixed point set codimension 2 which is less or equal to k. Since this is completely false without either of the assumptions on codimension 2 or positive curvature, these conditions have to be essential, and that is where theorems of Synge type on compact manifolds of positive sectional curvature are being used (see Lemmas 2.5, 2.6).

In Theorem 0.4, a half of the inequality is from the standard volume comparison in Riemannian geometry. The other half inequality is a special case of the Klingenberg-Sakai conjecture (see Remark 0.10). If not true, as seen earlier, then $\tilde{M}^n$ admits a *second* collapsible T^s-action. If there is a T^s-*invariant* totally geodesic submanifold of $\tilde{M}^n$ which is diffeomorphic to a three sphere, then the three sphere necessarily has small volume, a contradiction to the solution of Klingenberg-Sakai conjecture in dimension 3 by Burago-Toponogov (Theorem 3.1).

The maximal symmetry implies a $T^{\frac{n+1}{2}}$-*invariant* totally geodesic submanifold of $\tilde{M}^n$ which is diffeomorphic to a three-sphere (note that the $T^{\frac{n+1}{2}}$-action on $\tilde{M}^n$ is not *collapsible*). We then get a contradiction by observing, from the construction of [9], that the T^s-action can be viewed as a subtorus-action of the $T^{\frac{n+1}{2}}$-action and therefore the three-sphere is also T^s-invariant.

The rest of the paper is divided as follows:

In Section 1, we will provide the necessary material required in this paper and establish some preliminary results. In Section 2, we will prove Theorems 0.12. In Section 3, we will prove Theorem 0.3, thereby completing a proof of Theorem 0.1. In Section 4, we will prove Theorem 0.4. In Section 5, we will prove Theorem 0.6. In Section 6, we will supply examples mentioned in the introduction.

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1. Preliminaries

In this section, we will provide a necessary background and establish some preliminary results required in the sequel. Note that Lemmas 1.4, 1.8 will be used in the proof of Theorem 0.12 in the next section. For the convenience of readers, we will also recall some basic facts about a π_1-invariant T^k-action which can be found in [35] in terms of a pure F-structure.

a. π_1-invariant torus actions

Let M be a manifold, and let $\pi : \tilde{M} \to M$ denote the universal covering space. A π_1-invariant T^k-action on M is a T^k- action on $\tilde{M}$ and a homomorphism, $\rho : \pi_1(M) \to Aut(T^k)$, the group of automorphisms of T^k, such that

$$\gamma(t\tilde{x}) = \rho(\gamma)(t)(\gamma(\tilde{x})), \tag{1.1}$$

for all $\tilde{x} \in \tilde{M}, \gamma \in \pi_1(M)$ and $t \in T^k$. Equivalently, the T^k-action extends to the action of the semi-direct product, $\pi_1(M) \ltimes_\rho T^k$. We will call ρ the *holonomy representation of* $\pi_1(M)$. A metric on M is called *invariant* if the $\pi_1(M) \ltimes_\rho T^k$-action on $\tilde{M}$ preserves the pullback metric.

It is clear that a π_1-invariant T^k-action descends to a T^k-action on M if and only if ρ is the identity map. In particular, on a simply connected manifold, a π_1-invariant T^k-action is the usual T^k-action.

Note that the π_1-invariance implies that the orbit structure on $\tilde{M}$ descends to an orbit structure on M. We call the quotient space, $\bar{M} = \tilde{M}/[\pi_1(M) \ltimes_\rho T^k]$, the *orbit space of* M (by the π_1-invariant torus action).

It is easy to see that the notion of π_1-invariant T^k-action is equivalent to that of *pure F-structure* on a manifold of finite fundamental group; cf. [10], [11]. We refer to [10], [11] for examples.

b. Singular sets on $\tilde{M}, M$ and $\bar{M}$

Given an effective T^k-action on $\tilde{M}$, recall that an orbit is *principal* if the isotropy group is trivial. A non-principal orbit is either *exceptional* or *singular* if its dimension is equal to or less than that of a principal orbit. The set of non-principal orbits has a stratification by submanifolds; cf. [34], [35].

Let $\tilde S$ denote the union of all singular orbits. When restricting the stratified structure of non-principal orbits to $\tilde S$, one obtains a stratification of $\tilde S$, $\tilde S = \bigcup_i \tilde S_i$, which satisfies the following conditions:
(1.2.1) The closure, $cl(\tilde S_i) = \bigcup_{j<i} \tilde S_j$ and each component, $\tilde S_{ij}$, of $\tilde S_i$ is a submanifold. In particular, $\max_l\{dim(\tilde S_{jl})\} < \max_k\{dim(\tilde S_{ik})\}$.
(1.2.2) On each stratum, $\tilde S_{ij}$, the orbit projection is a fibration map (and the orbit projection is *not* a fibration map on any two adjacent strata)
(1.2.3) Each point in $\tilde S_{ij}$ has the same isotropy group, and $cl(\tilde S_{ij})$ is a component of the fixed point set of the isotropy group.

The dimension of $\tilde S$ is defined to be $\max_k\{dim(\tilde S_{1k})\}$. Since each $cl(\tilde S_{ij})$ is the fixed point set of the isotropy group of $\tilde S_{ij}$, $\tilde S$ has at most codimension 2; cf. [3].

Now let $\pi : \tilde M \to M$ be the universal covering space, and let the T^k-action on $\tilde M$ be π_1-invariant. An orbit on M is called *regular* if it has an invariant neighborhood in which the orbits form a fibration. Similarly, we call a non-regular orbit *exceptional* or *singular* if its dimension is equal to or less than that of a regular orbit. Let S be the union of all singular orbits on M. It is clear that $\pi(\tilde S) = S$ and $\pi^{-1}(S) = \tilde S$. We define $\dim(\tilde S) = \dim(S)$.

Note that unlike the singular set, the projection of non-principal orbits in $\tilde M$ may be a proper subset of the non-regular set of M; see Examples 4.9, 4.10 in [35]. This is a part of the reason why we choose to work with a singular set.

Let $p : M \to \bar M$ be the orbit projection. Put $\bar S = p(S)$. We will call $\bar x \in \bar S$ a *singular point.* This will be justified as follows.

Recall that the quotient space equipped with the quotient metric of an Alexandrov space by compact any group of isometries is an Alexandrov space with the same curvature lower bound; cf. [4]. In particular, given any invariant metric on M, $\bar M$ is an Alexandrov space. For each $\bar x \in \bar M$, the *space of directions* at $\bar x$ is given by $S^\perp_{\tilde x}/(\Gamma_{\tilde x} \ltimes_\rho T^k_{\tilde x})$, where $\tilde x \in \tilde M$ such that $\bar x = p(\pi(\tilde x))$, $\Gamma_{\tilde x}$ is the subgroup of deck transformations which preserve the T^k-orbit at $\tilde x$, $T^k_{\tilde x}$ is the isotropy group at $\tilde x$, and $S^\perp_{\tilde x}$ the normal sphere to the orbit at $\tilde x$.

In view of the notion of a singular point in an Alexandrov space ([4]), we immediately see:

Lemma 1.3. *Let M admit a π_1-invariant T^k-action. The orbit at $x \in M$ is non-regular if and only if $\bar x \in \bar M$ is singular in the Alexandrov sense, with respect to the quotient metric of any invariant metric on*

M.

Recall that at each point, x, in an Alexandrov space, X, the space of directions, S_x, is a compact Alexandrov space of dimension $= dim(X) - 1$. If a 1-dimensional Alexandrov space is a closed interval, we call each end point a *boundary point.* In general, a point in X is called a *boundary point* if the space of directions has non-empty boundary.

We call $\bar{x} \in \bar{M}$ a *boundary point* if $\bar{x}$ is a boundary point in the Alexandrov sense. Clearly, whether or not $\bar{x} \in M$ is a boundary point is independent of a particular invariant metric involved. Note that if $\bar{x}$ is a boundary point, then the orbit at x with $p(x) = \bar{x}$ is singular since the quotient space of a sphere by a finite group has empty boundary.

We now give equivalent conditions for a singular set with the maximal possible dimension.

Lemma 1.4. *Let $\tilde{M}^n$ be a T^k-manifold. The following conditions are equivalent:*

(1.4.1) There is a circle subgroup with fixed point set codimension 2.
(1.4.2) The singular set $\tilde{S}$ has codimension 2.
(1.4.3) The orbit space, $\tilde{M}/T^k$, has singular set of codimension 1.
(1.4.4) The orbit space, $\tilde{M}/T^k$, has non-empty boundary.

Proof. (1.4.1) $\iff$ (1.4.2). One direction is obvious. Assume $\dim(\tilde{S}) = n-2$. By definition, there is a $\tilde{S}_{1j}$ such that $\dim(\tilde{S}_{1j}) = n-2$. Let T^k_{1j} be the isotropy group of $\tilde{S}_{1j}$ (see (1.2.3)). Then, T^k_{1j} has rank ≥ 1. By (1.2.3), $\mathrm{cl}(\tilde{S}_{1j})$ is a component of the fixed point set $F(\tilde{M}, T^k_{1j})$. Now given any invariant metric on $\tilde{M}$, $\mathrm{cl}(\tilde{S}_{1j})$ is a totally geodesic submanifold and T^k_{1j} acts effectively on the normal space to $\mathrm{cl}(\tilde{S}_{1j})$. Since $\dim(cl(\tilde{S}_{1j})) = n-2$, T^k_{1j} is a circle.

(1.4.1) $\Rightarrow$ (1.4.3). Let $\tilde{F}_0$ denote a component of $F(\tilde{M}, S^1)$ with codimension 2. First, $\tilde{M}/S^1$ has non-empty boundary F_0. Consider the effective T^k/S^1-action on $\tilde{M}/S^1$. Note that the restriction of T^k/S^1 on $\tilde{F}_0$ is also effective. Let $\bar{x} \in F_0$ such that the T^k/S^1-orbit at $\bar{x}$ is principal. Since principal orbits form an open dense subset, there is an invariant neighborhood, U, of $\bar{x}$ in which all T^k/S^1-orbits are principal. It then follows that $(\tilde{F}_0 \cap U)/(T^k/S^1)$ is a boundary of $U/(T^k/S^1)$. Consequently $\tilde{M}/T^k$ has non-empty boundary since $\tilde{M}/T^k = (\tilde{M}/S^1)/(T^k/S^1)$.

(1.4.3) $\Rightarrow$ (1.4.1). Put $p : \tilde{M} \to \tilde{M}/T^k$ and $\partial\tilde{M} = p^{-1}(\partial(\tilde{M}/T^k))$. By the early discussion each point in $\partial\tilde{M}$ is singular. From the stratified structure for $\tilde{S}$ (see (1.2.1)—(1.2.3)), there is a singular stratum, $\tilde{S}_{1j} \subseteq \partial\tilde{M}$, such that $\dim(\tilde{S}_{1j}) = dim(\partial\tilde{M})$ and the isotropy group of $\tilde{S}_{1j}$ acts freely on the normal space of $\tilde{S}_{1j}$. The isotropy group has to be a circle since a torus of rank ≥ 2 cannot freely act on a sphere; see Theorem 8.5, [3]. Consequently, $\tilde{S}_{1j}$ has codimension 2, since $\tilde{S}_{1j}$ is in the boundary of $\tilde{M}/T^k$. Now (1.4.1) follows from the equivalence of (1.4.1) and (1.4.2).

We leave the proof of (4.1.1) $\iff$ (4.1.4) to the reader. q.e.d.

Corollary 1.5. *Let M admit a π_1-invariant T^k- action. Then, the following are equivalent conditions:*
(1.5.1) There is a circle subgroup with fixed point set codimension 2.
(1.5.2) M has singular set codimension 2.
(1.5.3) $\bar{M}$ has singular set codimension 1.
(1.5.4) $\bar{M}$ has non-empty boundary.

Proof. Note that $\bar{M}$ has singular set codimension 1 or non-empty boundary if and only if $\tilde{M}/T^k$ does. q.e.d.

2. Proof of Theorem 0.12.

As seen in the introduction, the following theorem of Grove-Searle will play a crucial role in the proof.

Theorem 2.1 ([22]). *Let M^n be a compact manifold of $K_{M^n} > 0$. Suppose M^n admits an isometric T^k-action. Then $k \leq \frac{n}{2}$ (resp. $\leq \frac{n+1}{2}$) if n is even (resp. odd). Moreover, we have:*
(2.1.1) If $k = 1$ and the circle action has fixed point set codimension 2, then M^n is diffeomorphic to S^n, $S^n/\mathbb{Z}_q$ or $\mathbb{C}P^m$.
(2.1.2) If $k = \frac{n}{2}$ (resp. $k = \frac{n+1}{2}$) when n is even (resp. n is odd), then T^k has a circle subgroup of fixed point set codimension 2.

Let M^n be as in Theorem 0.12. Note that one can assume n is *odd*. Otherwise Theorem 0.12 will follow from Theorem 2.1 since the fundamental group is, if not trivial, $\mathbb{Z}_2$.

First, by Corollary 1.5 the π_1-invariant isometric T^k-action has a circle subgroup, S^1, with fixed point set, $\tilde{F}_0$, of codimension 2. Let $\rho : \pi_1(M^n) \to Aut(T^k)$ be the holonomy representation, and let $\rho(\gamma)|_{S^1}$ denote the restriction on S^1.

Lemma 2.2. *Let the assumptions be as in Theorem 0.12. Let H be the subgroup of γ such that $\rho(\gamma)|_{S^1} = id$. Then, $\tilde{M}^n/H$ is diffeomorphic to a lens space, $S^n/\mathbb{Z}_q$.*

Proof. Observe that the isometric S^1-action descends to $\tilde{M}^n/H$ with fixed point set of codimension 2. Then, Lemma 2.2 follows from (2.1.1). q.e.d.

Next, we will give two standard results on compact manifolds of positive curvature that will be used in an estimate for $[\pi_1(M^n), H]$; cf. [27].

Lemma 2.3. *Let M^n be a compact manifold with $K_{M^n} > 0$.*
(2.3.1) If n is even and M^n is orientable, then any orientation preserving isometry has a fixed point. In particular, M^n is simply connected (Synge).
(2.3.2) If n is odd, then M^n is orientable and any orientation reversing isometry has a fixed point.

Lemma 2.4. *Let M^n be a manifold with $K_{M^n} > 0$, and let N_1 and N_2 be two closed totally geodesic submanifolds. If dim $(N_1)+$ dim $(N_2) \geq n$, then N_1 and N_2 have nonempty intersection.*

By definition, for $\gamma_1, \gamma_2 \in \pi_1(M^n)$, $\gamma_1\gamma_2^{-1} \in H$ if and only if $\rho(\gamma_1)|_{S^1} = \rho(\gamma_2)|_{S^1}$.

Lemma 2.5. *Let the assumptions be as in Lemma 2.2. Then, $\gamma_1\gamma_2^{-1} \in H$ if and only if $\rho(\gamma_1)(S^1) = \rho(\gamma_2)(S^1)$ (as subgroups).*

Proof. We only need to check that if $\rho(\gamma)(S^1) = S^1$ (i.e., $\rho(\gamma)|_{S^1} = \pm id$), then $\rho(\gamma)|_{S^1} = id$. We shall show that if $\rho(\gamma)|_{S^1} = -id$, then γ is an orientation reversing isometry of $\tilde{M}^n$, a contraction to (2.3.2).

We shall use two properties (see Lemma 1.1, [22]): 1) $\tilde{F}_0$ is connected (actually diffeomorphic to S^{n-2}). 2) The S^1-action is *free* when restricting to $T_\epsilon(\tilde{F}_0) \setminus \tilde{F}_0$, where $T_\epsilon(\tilde{F}_0)$ is the ϵ-neighborhood of $\tilde{F}_0$.

Since $\rho(\gamma)(S^1) = S^1$, from (1.1) we have $\gamma(\tilde{F}_0) = \tilde{F}_0$. We claim that $\tilde{F}_0$ is orientable and when restricting to $\tilde{F}_0$, γ is orientation preserving. First, $\tilde{F}_0$, being the fixed point set of an isometric S^1-action, is a closed totally geodesic submanifold. Since the induced metric satisfies $K_{\tilde{F}_0} > 0$ and $dim(\tilde{F}_0)$ is odd, by (2.3.2) $\tilde{F}_0$ is orientable. Since γ is a free isometry on $\tilde{F}_0$, by the same reasons γ preserves the orientation of $\tilde{F}_0$.

For any $\tilde{x} \in \tilde{F}_0$, consider the differential $d\gamma : T_{\tilde{x}}(\tilde{M}^n) \to T_{\gamma(\tilde{x})}(\tilde{M}^n)$ of γ. To see that $\rho(\gamma) = -id$ implies that $d\gamma$ is orientation reversing, we take a coordinate as follows: identify $T_\epsilon(\tilde{F}_0)$ with the normal ϵ-disk bundle of $\tilde{F}_0$, and let U denote the (product) normal ϵ-disk bundle over V, where $V \subset \tilde{F}_0$ is a small tubular neighborhood of a simple curve from $\tilde{x}$ to $\gamma(\tilde{x})$ (by 1) $\tilde{F}_0$ is connected). The coordinates on U consists of a polar coordinate in the normal disk, and a coordinate, $(y_1, .., y_{n-2})$, on V. Let $\partial r, \partial\theta, \partial_1, ..., \partial_{n-2}$ be the oriented frame. Note that by 2) the induced S^1-action on U can be viewed as the rotation on the disk and thus $\partial\theta$ is the invariant field.

Since $\rho(\gamma) = -id$, from (1.1) it is clear that $d\gamma(\partial r) = \partial r$ and $d\gamma(\partial\theta) = -\partial\theta$. Since $d\gamma$ preserves the oriented subspace spanned by $\partial_1, ..., \partial_{n-2}$, $d\gamma$ reverses the orientation of $\tilde{M}^n$. q.e.d.

By Lemma 2.5, $[\pi_1(M^n), H] = \#\{\rho(\gamma)(S^1) | \gamma \in \pi_1(M^n)\}$. From (1.1) it follows that $\rho(\gamma)(S^1)$ has fixed point set, $\gamma(\tilde{F}_0)$, of codimension 2.

Lemma 2.6. *Let the assumptions be as in Lemma 2.2. Then,*

$$[\pi_1(M^n), H] = \#\{\rho(\gamma)(S^1) \mid \gamma \in \pi_1(M^n)\} \leq k.$$

Proof. Let $S_1^1, ..., S_r^1$ denote all circle subgroups of T^k such that each S_i^1 has a fixed point component, $\tilde{F}_i$, of dimension $= n-2$. By Lemma 2.5, $[\pi_1(M^n), H] \leq r$. It suffices to prove that $S_1^1, ..., S_r^1$ generate a subtorus, T, of dimension r $(\leq k)$. We verify this by induction on (odd) n starting with a trivial case $n = 3$. For a T^2-action on S^3, the orbit space is a closed interval and thus two circle orbits at endpoints are all nonprinciple orbits. The two S^1-isotropy groups generate T^2 and any other circle subgroup has empty fixed point set.

We claim $\dim(\tilde{F}_i \cap \tilde{F}_1) = n - 4$ $(2 \leq i)$ and $S_i^1 \cap S_1^1 = 1$. First, $\tilde{F}_i \cap \tilde{F}_1 \neq \emptyset$ since $\dim(\tilde{F}_i) + \dim(\tilde{F}_1) = n + (n-4) \geq n+1$ (Lemma 2.4) and thus $\dim(\tilde{F}_i \cap \tilde{F}_1) \geq n-4$. Since the torus generated by S_i^1 and S_1^1 acts effectively on the normal space of $\tilde{F}_i \cap \tilde{F}_j$, $\dim(\tilde{F}_i \cap \tilde{F}_1) < n-2$. By now the first part of claim follows since $\tilde{F}_i \cap \tilde{F}_1$ is fixed by $S_i^1|\tilde{F}_1$ (the restriction of S_i^1 on $\tilde{F}_1$). If $S_i^1 \cap S_1^1 = \mathbb{Z}_q \neq 1$, then $\mathbb{Z}_q$ has a fixed point component containing $\tilde{F}_i \cup \tilde{F}_1$, a contradiction to Property 2) in the proof of Lemma 2.5.

Consider $(\tilde{F}_1 \simeq S^{n-2}, T^k/S_1^1)$. The above implies that each $S_i^1|\tilde{F}_1$ acts (effectively) isometrically on $\tilde{F}_1$ (a totally geodesic submanifold) with fixed point set codimension 2 $(2 \leq i)$. If $\dim(T) < r$, then $\tilde{S}_2^1|\tilde{F}_1$, ...,$S_r^1|\tilde{F}_1$ generate a subtorus of $T^k|\tilde{F}_1$ of dimension $< r-1$. Note that this contradicts to the inductive assumption if $S_2^1|\tilde{F}_1, ..., S_r^1|\tilde{F}_1$ are pairwisely distinct.

Note that $S_i^1|\tilde{F}_1 \neq S_j^1|\tilde{F}_1$ follows if (i) $\tilde{F}_i \cap \tilde{F}_1 \neq \tilde{F}_j \cap \tilde{F}_1$ $(2 \leq i \neq j)$. If (i) is an equality, then $\tilde{F}_i \cap \tilde{F}_j \cap \tilde{F}_1 = \tilde{F}_i \cap \tilde{F}_1$ is fixed by the torus, T^s, generated by S_i^1, S_j^1 and S_1^1 and since $\dim(\tilde F_i \cap \tilde F_j \cap \tilde F_1) = n-4$, $s = 2$. Fix a point $x \in \tilde F_i \cap \tilde F_j \cap \tilde F_1$. The T^2-action in a neighborhood of x is equivalent (via the isotropy representation) to a linear T^2-action on a normal small disk bundle of $\tilde F_i \cap \tilde F_j \cap \tilde F_1$ around x. Since an orthogonal T^2-action on the normal unit 4-ball (or unit 3-sphere) has only two circle subgroups of fixed point set of dimension $= 2$ (or dimension $= 1$), the T^2-action in a neighborhood of x has only two circle subgroups of fixed point set dimension $= n-2$, a contradiction to our situation. q.e.d.

Proof of Theorem 0.12. By Theorem 2.1, $k \le \frac{n+1}{2}$. Then Theorem 0.12 follows from Lemmas 2.2 and 2.6. q.e.d.

3. Proof of Theorem 0.3

As seen in the introduction, the equivariant and parameterized fibration theorem in [9] will play a crucial role in the proof of Theorem 0.3. The following strong version in [34] is required for our purpose.

Theorem 3.1. *For each $\epsilon > 0$, there is a constant, $v(n,\delta,\epsilon) > 0$, such that if $M^n \in \mathcal{M}_\delta^n$ has volume $< v(n,\delta,\epsilon)$, then M^n admits a π_1-invariant T^k-action without fixed points and an invariant metric g_ϵ satisfying*

$$e^{-\epsilon} g \le g_\epsilon \le e^{\epsilon} g, |\nabla^g - \nabla^{g_\epsilon}| < \epsilon, |(\nabla^{g_\epsilon})^i R_{g_\epsilon}| < c(n,i,\epsilon),$$
$$0 < \delta - c(n)\epsilon \le K_\epsilon \le 1.$$

Moreover, each orbit on M^n has diameter less than ϵ.

Remark 3.2. Note that if $\epsilon < \frac{\delta}{2c(n)}$, then the invariant metric has sectional curvature $\ge \delta/2$.

Proof of Theorem 0.3. We shall first show $\dim(X) \ge \frac{n-1}{2}$ by a contradiction argument. Assume a sequence of pairs, M_i^n and X_i, such that $d_{GH}(M_i^n, X_i) < i^{-1}$, which are counterexamples. By passing to a subsequence we can assume that $M_i^n \to Y$ and $X_i \to Y$ ([20]). Note that Y is an Alexandrov space (of curvature $\ge \delta$). Since all X_i have curvature ≥ -1, $\dim(Y) \le \dim(X_i) < \frac{n-1}{2}$ (see p.32, [4]).

According to the equivariant and parameterized fibration theorem in [9] (also [15], [16]), there is a map, $p_i : M_i^n \to Y$ and an $O(n)$-invariant fibration map, $\tilde p_i : F(M_i^n) \to \tilde Y$ such that the following diagram commutes,

$$\begin{array}{ccc} F(\tilde{M}_i^n) & \xrightarrow{\tilde{p}_i} & \tilde{Y} \\ \Big\downarrow \tilde{f} & & \Big\downarrow f \\ M_i^n & \xrightarrow{p_i} & Y \end{array}$$

where $F(M_i^n)$ is the frame bundle equipped with a canonical metric via the Riemannian connection, and $\tilde{Y}$ is a Riemannian manifold such that $\tilde{Y}/O(n) = Y$. Note that in general, a fiber of $\tilde{p}_i$ is a nilpotent manifold. Since $\pi_1(F(M_i^n))$ is finite, the fiber is actually a torus (see Lemma 1.4, [34]). This implies that p_i actually coincides with the projection map to the orbit space of a collapsible π_1-invariant T^k-action (Theorem 3.1), i.e., $Y = \bar{M}_i^n$ and $\tilde{Y} = \overline{F(M_i^n)}$; see Section 1, **c**. According to Theorem 2.1, $k \leq \frac{n+1}{2}$, and thus $\dim(Y) = n - k \geq n - \frac{n+1}{2} = \frac{n-1}{2}$, a contradiction.

We will now prove the second part of Theorem 0.3 by contradiction. Assume there exists a sequence of pairs, M_i^n and X_i, such that $d_{GH}(M^n, X_i) < i^{-1}$, which are counterexamples. As in the above, we assume $M_i^n \to Y$ and $X_i \to Y$.

As in the above, $\dim(Y) \leq \dim(X_i)$. Since $m_H(X_i) \geq m_0 > 0$, we can assume that X_i and Y have the same dimension. Then, by the stability theorem of Perel'man ([30]) X_i is homeomorphic to Y when i is large.

Case 1. Assume $\dim(Y) = n$. This implies that $\mathrm{vol}(M_i^n)$ has a uniform positive lower bound. By Cheeger-Gromov's compactness theorem ([7], [20]), we can assume that M_i^n is diffeomorphic to Y for large i, a contradiction.

Case 2. Assume $\dim(Y) < n$. From the above we see that X_i is homeomorphic to $\bar{M}_i^n$ for i large, a contradiction. q.e.d.

Remark 3.3. Using the equivariant and parameterized fibration theorem in [9], in a straightforward manner one can check that the above argument will yield an extension of the second part of Theorem 0.3 to the class of n-manifolds with $|K| \leq 1$ and diam $\leq d$.

Proof of Theorem 0.1. Since X has a nonempty boundary, by Theorem 0.3 one concludes that $\bar{M}^n$ has non-empty boundary (see p.54, [4]). Now Theorem 0.1 follows from Theorem 0.12. q.e.d.

We will explain the comments in Remark 0.7. A glance of Example 5.1 may help for a motivation.

Let M^n satisfy Theorem 3.1 with respect to $\epsilon = \frac{\delta}{2c(n)}$, and let $v(n,\delta) = v(n,\delta,\epsilon)$ denote the corresponding small constant. In the rest of the discussion, we shall use $\bar{M}^n$ to denote the orbit space of *this* π_1-invariant isometric torus action on M^n.

Let $F(M^n)$ denote the frame bundle equipped with a canonical metric via the Riemannian connection. As seen in the above proof, $F(M^n)$ admits a free isometric π_1-invariant torus action which coincides with the nilpotent Killing structure. According to [9], the quotient metric on the orbit space, $\overline{F(M^n)}$, has injectivity radius bounded below by a constant depending only on n and δ. This property has the following consequence.

Lemma 3.4. $m(n,\delta) = \inf \{m_H(\bar{M}^n);$
$M^n \in \mathcal{M}^n_\delta,\ vol(M^n) < v(n,\delta)\} > 0.$

Let M^n be as in Theorem 0.1 for some X_1. By Theorem 0.3, M^n admits a π_1-invariant isometric T^k-action such that the orbit space has nonempty boundary. Without loss of generality, we can assume that $\mathrm{vol}(M^n) < v(n,\delta)$. We claim that $\bar{M}^n$ also has nonempty boundary. A consequence is that (M^n, X_2) will satisfy Theorem 0.1 for $m_0 = \frac{m(n,\delta)}{100}$ and $X_2 = \bar{M}^n$.

First, without loss of generality, we can assume that the π_1-invariant isometric T^s-action on M^n as given in Theorem 0.3 is a sub-π_1-invariant isometric T^k-action in the above. This is because sufficiently small m_0 implies that the (collapsing) scale used in the construction for the π_1-invariant isometric T^s-action is smaller than the fixed scale, $\epsilon = \frac{\delta}{2c(n)}$, cf. [9]. In view of Lemma 1.4, our claim then follows (if a subtorus has a circle subgroup with fixed point set of codimension 2, so does the torus.).

4. Proof of Theorem 0.4

Let M^n be as in Theorem 0.4. We can assume, without loss of generality, that M^n admits a collapsible π_1-invariant $T^{\frac{n+1}{2}}$-action, i.e., a π_1-invariant isometric $T^{\frac{n+1}{2}}$-action (Theorem 3.1). By (2.1.2), Lemma 1.4 and Theorem 0.12, a covering space of M^n with order $\le \frac{n+1}{2}$ is diffeomorphic to a lens space, $S^n/\mathbb{Z}_q$. Let $\tilde{g}$ denote the pullback metric on S^n.

Proposition 4.1. *Let the assumptions be as in Theorem 0.4. Then,* $vol(S^n, \tilde{g}) \ge c(n,\delta) > 0.$

We will first prove Theorem 0.4 by assuming Proposition 4.1.

Proof of Theorem 0.4. First, $q = \frac{vol(S^n,\tilde{g})}{vol(M^n)}$. By the standard volume comparison in Riemannian geometry ([8]), $\mathrm{vol}(S^n,\tilde{g}) \leq vol(S^n_\delta)$. By Proposition 4.1, $\mathrm{vol}(S^n,\tilde{g}) \geq c(n,\delta)$. q.e.d.

One of the main ingredients in Proposition 4.1 is

Theorem 4.2 ([5]). *Let M^3 be a compact simply connected 3-manifold of $\delta \leq K_{M^3} \leq 1$. Then, vol $(M^3) \geq v(\delta) > 0$, where $v(\delta)$ is a constant depending on δ.*

Note that by [23], M^3 is diffeomorphic to a 3-sphere.
We first observe a simple fact.

Lemma 4.3. *Let M^n be a compact simply connected manifold of $K_{M^n} > 0$. Suppose that M^n admits an isometric $T^{\frac{n+1}{2}}$-action (n odd and ≥ 5). Then, there is an invariant totally geodesic submanifold diffeomorphic to a 3-sphere.*

Proof. By (2.1.2) and (2.1.3), M^n is diffeomorphic to S^n, and there is a circle subgroup, S^1, with fixed point set, F, of codimension 2. Note that F is $T^{\frac{n+1}{2}}$-invariant. From the proof of Theorem 2.1 in [22] one also sees that F is connected and diffeomorphic to S^{n-2}. Note that F is totally geodesic, and $T^{\frac{n-1}{2}} \simeq T^{\frac{n+1}{2}}/S^1$ acts on F by isometries. By an obvious inductive argument, one then completes the proof. q.e.d.

Proof of Proposition 4.1. We argue by contradiction. Assume that $\mathrm{vol}(S^n,\tilde{g})$ can be arbitrarily small. By Theorem 3.1, we can assume that S^n admits a collapsible T^s-action.

We first suppose that the T^s-action is a subtorus action of the T^k-action on S^n. Let S^3 denote a T^k-invariant totally geodesic 3-sphere as in Lemma 4.3. Using Theorem 4.2, we shall derive a contradiction by showing that the volume of S^3 can be arbitrarily small. By Theorem 3.1, the diameters of the orbits of any collapsible π_1-invariant torus action are uniformly small. Thus, it suffices to show that S^3 is preserved by the T^s-action. This follows from S^3 is $T^{\frac{n+1}{2}}$-invariant (Lemma 4.3) and T^s is a subtorus-action of the $T^{\frac{n+1}{2}}$-action.

We now explain why the additional assumption can be satisfied. Recall that the equivariant and parameterized fibration theorem in [9] asserts that if a manifold M of $|K| \leq 1$ and diam$\leq d$ has volume $< \epsilon(n,d)$, then M admits a *pure nilpotent Killing structure* whose orbit at each point contains all short geodesic loops; cf. [9].

As already seen in the proof of Theorem 0.3, in the case that $\pi_1(M)$ is finite, the nilpotent Killing structure is equivalent to a π_1-invariant T^k-action. Therefore, an orbit of any collapsible π_1-invariant torus action contains all short geodesic loops at a point.

Let γ be any short geodesic loop at $\tilde{x} \in S^n$. Then, the projection of γ on M^n is also a geodesic loop with length $\leq$ the length of γ. This means that the projection of each T^s-orbit on M^n is contained in a T^k-orbit, and therefore the T^s-action can be realized (possibly by small deformation, see [9]) as a subtorus-action of the T^k-action. Finally, any T^k-invariant metric is also T^s-invariant. q.e.d.

5. Proof of Theorem 0.6

First, we will prove Theorem 0.6. Then, we will use Theorem 0.6 to construct an example concerning Theorem 0.1 (see the introduction, the discussion following Theorem 0.1)

Recall that, for a compact G-manifold (G a compact Lie group), there are only finitely many conjugacy classes of isotropy groups ([3]). In particular, if $G = T^k$, then there are only finitely many isotropy groups.

Suppose, in addition, that the fixed point set of T^k is empty. Then, it follows from the finiteness of isotropy groups that there is a circle subgroup without any fixed point. To find one such circle subgroup, let H denote any dense one-parameter subgroup of T^k (i.e., $\bar{H} = T^k$) and take a circle subgroup *approximating* to H. Precisely, given an invariant metric on T^k, one can assume that the angle between V and W is sufficiently small, where V and W are invariant vectors tangent to H and S^1 respectively.

Since H is dense in T^k, and since the fixed point set of T^k is empty, V is transversal to any subspace tangent to an isotropy group. Since there are only finitely many such subspaces, it is clear that one can choose W with the same property. Consequently, the circle subgroup has no fixed point.

Proof of Theorem 0.6. First, fix a dense one-parameter subgroup H of T^k. Then, take a sequence of circle subgroups, S^1_i, of T^k which approximates H, so that $d_{GH}(T^k/S^1_i, pt) \to 0$. Consequently, $d_{GH}(M/S^1_i, M/T^k) \to 0$.

Since T^k has no fixed points, by the above discussion we can assume that S^1_i has an empty set of fixed points. For each i, since S^1_i has only finitely many (finite) isotropy groups, we can assume there is a subgroup, $\mathbb{Z}_{q_i}$, of S^1_i that acts freely on M. Moreover, we can choose q_i sufficiently large so that $d_{GH}(M/S^1, M/\mathbb{Z}_{q_i})$ can be arbitrarily small. Finally,

$$\begin{aligned} d_{GH}(M/\mathbb{Z}_{q_i}, M/T^k, M) \leq & 2[d_{GH}(M/S^1_i, M/T^k) \\ & + d_{GH}(M/\mathbb{Z}_{q_i}, M/S^1_i)] \to 0. \end{aligned}$$

q.e.d.

Example 5.1. We will construct a lens space, $M^n = S^n/\mathbb{Z}_q$, of constant curvature one, and two Alexandrov spaces, X_1, X_2, of cur ≥ 1 and nonempty boundaries, and $dim(X_2) > dim(X_1)$. Moreover, M^n can be chosen so that $d_{GH}(M^n, X_1)$ and $d_{GH}(M^n, X_2)$ can be made arbitrarily small. In particular, both (M^n, X_1) and (M^n, X_2) satisfy Theorem 0.1.

Given two torus subgroups, $T^s \subset T^k$ $(s < k)$, of $O(n+1)$ (n odd). Suppose that the T^s-action on S^n has empty fixed point set and the quotient space, S^n/T^s, has nonempty boundary. Put $X_1 = S^n/T^k$.

Note that the T^k-action also has empty fixed point set. Applying Theorem 0.6 to (S^n, T^k), one obtains a sequence of cyclic subgroup, $\{\Gamma_i\}$, of T^k acting freely on S^n such that $d_{GH}(S^n/\Gamma_i, X_1) \to 0$ as $i \to \infty$, where $X_1 = S^n/T^k$, equipped with the limit metric, is an Alexandrov space of $cur(X_1) \geq 1$ (see [4]) and has nonempty boundary. Put $m_1 = m_H(X_1)$ and choose i_0 sufficiently large so that

$$d_{GH}(S^n/\Gamma_{i_0}, X_1) < \epsilon(n, 1, m_1)$$

as in Theorem 0.1. Put $M^n_1 = S^n/\Gamma_{i_0}$.

Note that the torus, T^s/Γ_{i_0}, acts isometrically on M^n_1 with empty fixed point set. Again by Theorem 0.6, there is a sequence of cyclic subgroups, $\{\Gamma'_j\}$, of T^s/Γ_{i_0} acting freely on M^n_1 such that

$$d_{GH}(M^n_1/\Gamma'_j, X_2) \to 0$$

as $j \to \infty$, where $X_2 = M^n_1/T^s$, equipped with the limit metric, is an Alexandrov space with $\mathrm{cur}(X_2) \geq 1$ and has nonempty boundary.

Put $m_2 = m_H(X_2)$. Fix a j_0 sufficiently large so that

$$d_{GH}(M^n_1/\Gamma'_{j_0}, X_2) < \epsilon(n, 1, m_2)$$

as in Theorem 0.1. Put $M^n = M^n_1/\Gamma$, where Γ denotes the subgroup generated by Γ_{i_0} and Γ'_{j_0}.

We claim the following properties:

(i) Γ is a cyclic.

(ii) $d_{GH}(M^n, X_1) \leq d_{GH}(M_1^n, X_1) < \epsilon(n, 1, m_1)$.

Property (i) is a consequence of the following lemma.

Lemma 5.2. *Let M be a compact manifold with positive sectional curvature. Suppose that M admits an isometric T^k-action. If a subgroup acts freely on M, then it is either a circle or a finite cyclic group.*

Proof. If M is even-dimensional, then $\mathbb{Z}_2$ is the only group which could freely and isometrically act on M. Thus, we can assume M is odd-dimensional.

Suppose that there is a subgroup of T^k acting freely on M. First, the subgroup also freely acts on each T^k-orbit. According to Theorem 4.1 in [34], the T^k-action always has a circle orbit. By now the desired result follows. q.e.d.

Property (ii) can be seen as follows. First, from the construction it is clear that $d_{GH}(M_1^n, X_1)$ (resp. $d_{GH}(M^n, X_1)$) can be realized by the largest diameter of T^k-orbits (resp. of T^s-orbits). Note that the quotient of any T^k-orbit by Γ'_{j_0} is a union of T^s-orbits.

6. Examples

In this section, we will construct examples mentioned in the introduction.

Example 6.1. We shall construct two collapsing sequences. These will show that Theorem 0.1 will be false if either of the normalized condition, *cur* ≥ -1 or the dependence of $\epsilon(n, \delta, m_0)$ on m_0 is removed with imposing further restriction; see Remark 0.9.

For $0 < \delta < \frac{1}{37}$, take a sequence, $\{M_i^7\}$, in [1] of simply connected and $\delta \leq K_{M_i^7} \leq 1$ such that $d_{GH}(M_i^7, SU(3)/T^2) \to 0$ as $i \to \infty$ ([33]); compare [2]. Note that none of M_i^7 is homeomorphic to S^7.

Given any $\{\epsilon_i\} \to 0$, let X_i denote $SU(3)/T^2$ with an open ball of small radius (say $\epsilon_i/10$) deleted such that $d_{GH}(SU(3)/T^2, X_i) < \epsilon_i/4$. One can slightly modify the metric near the boundary so that $cur(X_i) \geq -\Lambda_i$ in the Alexandrov sense for some number $\Lambda_i > 0$. For i sufficiently large,

$$d_{GH}(M_i^7, X_i) \leq 2[d_{GH}(M_i^7, SU(3)/T^2) + d_{GH}(SU(3)/T^2, X_i)] < \epsilon_i.$$

Clearly, (M_i^7, X_i) satisfies Theorem 0.1 except $\Lambda_i \to \infty$ as $i \to \infty$.

For all i, put $Y_i = (SU(3)/T^2) \times D^2(\frac{1}{i})$, where $D^2(r)$ denotes a ball of radius r in a plane. Then, $d_{GH}(M_i^7, Y_i) \to 0$ as $i \to \infty$. Clearly, (M_i^7, Y_i) satisfies Theorem 0.1 except $m_H(Y_i) \to 0$ as $i \to \infty$.

Example 6.2. We will construct spherical space form, S^n/Γ (odd $n \geq 3$), such that the action by the maximal torus, $T^{\frac{n+1}{2}}$, of $O(n+1)$ on S^n is Γ-invariant and there is no circle subgroup of fixed point set codimension 2 which descends to S^n/Γ.

Let Γ denote the subgroup generated by γ,

$$\gamma = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ & & & \cdots & \\ 0 & 0 & 0 & \cdots & I \\ -I & 0 & 0 & \cdots & 0 \end{bmatrix} \in O(n+1), \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since γ is a permutation on the complex coordinates,

$$S^n = \{(z_1, ..., z_{\frac{n+1}{2}}) \mid |z_1|^2 + \cdots + |z_{\frac{n+1}{2}}|^2 = 1\},$$

Γ acts freely on S^n. Let $\rho : \Gamma \to Aut(T^{\frac{n+1}{2}})$, which maps γ to the conjugation by γ. Then, the $T^{\frac{n+1}{2}}$-action on S^n is Γ-invariant. Since no circle subgroup of $T^{\frac{n+1}{2}}$ commutes with γ, no circle subgroup of $T^{\frac{n+1}{2}}$ can descend to the circle action on S^n/Γ.

Note that if one can find a finite cyclic subgroup, Γ', of $T^{\frac{n+1}{2}}$ such that Γ and Γ' generate a finite subgroup acting freely on S^n, then one obtains examples of non-lens spaces in Theorem 0.12, as is done in the next example.

Example 6.3. We will construct spherical space forms in Theorem 0.4 which are not lens space. For simplicity, we will present the construction only in dimensions $3, 5$. Note that these examples will also confirm that the order estimate in Theorems 0.1, 0.4 cannot be improved.

1) For each prime number p, choose $\gamma_1, \gamma_2 \in O(4)$,

$$\gamma_1 = \begin{bmatrix} R(\frac{2p+1}{p(p+1)}) & 0 \\ 0 & R(\frac{1}{p(p+1)}) \end{bmatrix}, \gamma_2 = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix},$$

$$R(\theta) = \begin{bmatrix} \cos 2\pi\theta & \sin 2\pi\theta \\ -\sin 2\pi\theta & \cos 2\pi\theta \end{bmatrix}.$$

From [38, p.224], one sees that the finite group, Γ_p, generated by γ_1, γ_2, acts freely on S^3. Since Γ_p converges to a maximal torus of $O(4)$ as $p \to \infty$, S^3/Γ_p converges to S^3/T^2 which is a closed interval.

Clearly, S^3/Γ_p is not a lens space since Γ_p is not cyclic. However, $S^3/(T^2 \cap \Gamma_p)$ is a lens space which is a *double* covering of S^3/Γ_p.

2) For each prime number $p > 9$, choose $\gamma_1, \gamma_2 \in O(6)$,

$$\gamma_1 = \begin{bmatrix} R(\frac{1}{p^3}) & 0 & 0 \\ 0 & R(\frac{p-2}{p^3}) & 0 \\ 0 & 0 & R(\frac{(p-2)^2}{p^3}) \end{bmatrix}, \qquad \gamma_2 = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ -I & 0 & 0 \end{bmatrix}.$$

From [38, p.224], one verifies that the group, Γ_p, generated by γ_1, γ_2 acts freely on S^5. Since Γ_p converges to a maximal torus, T^3, of $O(6)$ as $p \to \infty$, S^5/Γ_p converges to S^5/T^3 which is a closed disk.

Similar as in 1), S^5/Γ_p is not a lens space and $S^5/(T^3 \cap \Gamma_p)$ is a lens space which is a *triple* covering space of S^5/Γ_p. Note that S^5/Γ_p cannot have a double covering lens space since Γ_p has no normal cyclic subgroup of index 2.

Example 6.4. Let M^n be a compact manifold of positive sectional curvature. Suppose that M^n admits an isometric T^k- action. We shall construct a sequence of invariant metrics g_ϵ, on M^n of positive sectional curvature, such that

$$\lim_{\epsilon \to 0} d_{GH}((M^n, g_\epsilon), (M^n/T^k)) = 0,$$

where M^n/T^k is equipped with the quotient metric. Moreover, from the construction for g_ϵ, one will see that $\lim\limits_{\epsilon \to 0} \min K_{g_\epsilon} = 0$.

Let g_{M^n} denote the metric on M^n. Take any invariant metric g on T^k and put $\tilde{g}_\epsilon = \epsilon^2 g$, $0 < \epsilon \le 1$. Let $M^n \times T^k$ be equipped with the product metric, $g_{M^n} \otimes \tilde{g}_\epsilon$. Consider the diagonal action of T^k on $M^n \times T^k$. First, the orbit space of $M^n \times T^k$ by T^k is diffeomorphic to M^n. Since for all ϵ, the diagonal action is free and isometric, the quotient metric, g_ϵ, of $g_{M^n} \otimes \tilde{g}_\epsilon$ is well-defined, and the projection, $M^n \times T^k \to (M^n \times T^k)/T^k \simeq M^n$ is a Riemann submersion. Since $K_{g_{M^n} \otimes \tilde{g}_\epsilon} \ge 0$, by O'Neill's formula, we get $K_{g_\epsilon} \ge 0$. Since $K_{g_{M^n}} > 0$, and since a two-plane tangent to $M^n \times T^k$ has curvature zero if and only if the plane contains a factor, we conclude that $K_{g_\epsilon} > 0$.

To see that $\min K_{g_\epsilon} \to 0$ as $\epsilon \to 0$, we examine a neighborhood of a principal orbit. In this case, one can think of the sequence as formed by rescaling the metric along orbits by ϵ while the metric on the orthogonal direction remains unchanged. Thus, locally, the metric looks more and more like a product and therefore, $\min K_{g_\epsilon} \to 0$ as $\epsilon \to 0$.

Note that in the above, if the T^k-action has non-empty fixed point set, then $\max K_{g_\epsilon}$ is not bounded above. This may suggests that our construction of collapsing sequences is not the same as that constructed using a pure polarization by [10] for which $\max K_{g_\epsilon}$ is bounded above. The later is essentially the same as the well-known Berger's sphere.

Addendum. The recent progress in [31] and [14] (independently in [32]) shows that a δ-pinched simply connected n-manifold of finite second homotopy group has injectivity radius $\geq i(n, \delta) > 0$; compare Remark 0.8.

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