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INSTANTON FLOER HOMOLOGY FOR KNOTS VIA 3-ORBIFOLDS

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Abstract

In this article we construct an instanton Floer Homology for knots in S^3. Rather than working on knot complements, the invariant is constructed using 3-orbifolds singular along a knot, (S^3, K, n). The resulting knot invariant consists of four graded abelian groups $HF_i^{(k)}(S^3, K, n)(0 \leq i \leq 3)$. We give some properties of the invariant and also provide some examples.

1. Introduction

For more than 15 years now, Gauge Theory has provided Low-dimensional Topology with many interesting invariants, sometimes leading to solutions of important problems, but up to now, relatively little has been done to apply Gauge Theory to Knot Theory. The primary goal of this article is to contribute to the development of a proper gauge theoretic setting for the study of knots.

Floer Homology, as initiated by Floer in his seminal paper [22], is by now a well-known gauge theoretic 3-manifold invariant which has been quite useful when related to Donaldson's constructions in 4-dimensional Gauge Theory. This invariant non-trivially generalises Casson's invariant, and therefore is of interest for 3-manifold Topology. One may think of applying Floer's ideas to develop a Floer Theory for knot complements. In the 4-dimensional setting, Kronheimer and Mrowka have

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partially opened the way in [32] and [33], where they used Gauge Theory to study smoothly embedded surfaces in 4-manifolds. This somehow provides a model for the objects to be considered in developing a Floer Theory for knots.

When trying to transpose Floer's ideas from 3-manifolds to knots, there are many advantages in working with 3-dimensional orbifolds singular along a knot rather than with knot complements. The most substantial gains are on the analytical side. As knot complements are either incomplete manifolds (S^3-K) or have a boundary (removing the interior of a tubular neighbourhood of K in S^3) the analytical setting for Gauge Theory is much more complicated than in the case of closed manifolds (see for example work in [26] and comments in [32]). By contrast, if one works with an orbifold, one may encode information about the knot and at the same time work with a compact object, hence simplifying greatly the Analysis required for a Floer Theory.

Here is an outline of the article. In Section 2, we are concerned with introducing 3-orbifolds and set-up the gauge theoretic framework which will be needed to do a Floer Theory. Section 3 gives a detailed construction of the Floer Homology for 3-orbifolds, and is the central part of the article. In Section 4, various properties of the Floer Homology for 3-orbifolds are studied while Section 5 gives some examples. Section 6 is a conclusion, hinting at further developments of the theory and its applications.

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2. 3-orbifolds and gauge theory

2.1 3-orbifolds

Suppose K is a knot in S^3, that is, a smoothly embedded circle in S^3. Denote by N_K a tubular neighbourhood of K in S^3. The fundamental group of the knot complement, $\pi_1(S^3 - N_K)$ is called the *knot group*, and in it are two distinguished elements, the *meridian*, μ, well-defined up to conjugation, and the (trivially framed) *longitude*, λ. Let n be an integer.

In this article, we shall use 3-orbifolds introduced by Thurston (see [46]) and commonly used in 3-dimensional Topology (see [5] and [6]). We shall be very brief here and refer the reader unfamiliar with such objects to [46] or [6].

Definition 2.1. A *3-dimensional orbifold* is given by a triple (S^3, K, n), where S^3 is the underlying topological space, the knot $K \hookrightarrow S^3$ is the singular locus, and the isotropy group around K is $\mathbb{Z}_n$.

There is an alternative approach to (S^3, K, n) which will be useful. Let $V_n(K)$ be the n-fold cyclic branched covering of S^3 along K. It is easy to prove that (S^3, K, n) is the quotient of $V_n(K)$ by the $\mathbb{Z}_n$-action coming from meridional rotations on N_K and covering transformations on the n-fold unbranched covering of $S^3 - N_K$. See, for example, [6] for details.

For 3-orbifolds (S^3, K, n), there is a practical definition of the orbifold fundamental group, via loops, as in the manifold case (see [24] in the case of 2-orbifolds). It is clear from the above construction that the orbifold fundamental group $\pi_1^V(S^3, K, n)$ is

$$\pi_1^V(S^3, K, n) = \pi_1(S^3 - N_K)/ < \mu^n >,$$

where μ is a meridian in $\pi_1(S^3 - N_K)$. The abelianization of the orbifold fundamental group, $H_1^V(S^3, K, n)$ is always isomorphic to $\mathbb{Z}_n$. It follows from the construction of $V_n(K)$ and (S^3, K, n) that there is a short exact sequence

$$1 \longrightarrow \pi_1(V_n(K)) \longrightarrow \pi_1^V(S^3, K, n) \longrightarrow \mathbb{Z}_n \longrightarrow 1,$$

called the *orbifold exact sequence*.

2.2 Gauge theory for orbifolds

We now proceed to introduce the basic gauge theoretical objects that will be needed and developed in the orbifold setting. After introducing rapidly orbifold bundles and connections (for more information on standard material the reader is referred to [43], [28] or [24]), we define a Chern-Simons functional for orbifolds via the Chern-Weil theory of Kronheimer and Mrowka.

The definition of an orbifold vector bundle over orbifolds is very similar to that of a genuine vector bundle, the only difference being that one needs to specify some data near the singular locus. A vector bundle $\pi\colon E \to (S^3, K, n)$ is given locally by trivializations

$$\theta_i\colon E_{|\mathcal{U}_i} \to \tilde{U}_i \times \mathbb{C}^r/G_i,$$

where G_i acts on $\mathbb{C}^r$ by some representation $\tau\colon \mathbb{Z}_n \to Aut(\mathbb{A}^r)$. The standard compatibility (patching) must hold. We will only be concerned with $SU(2)$-bundles. Over the knot complement (a 3-manifold with boundary), an $SU(2)$-bundle is trivial, while over the tubular neighbourhood, an orbifold bundle is determined by its isotropy $\tau\colon \mathbb{Z}_n \to SU(2)$. When we glue the two together to have a bundle over the orbifold, the class will depend on the isotropy and a gluing map

$$\partial(S^3 - N_K) \times \mathbb{C}^2 \to \partial(S^3 - N_K) \times \mathbb{C}^2$$

i.e., a map $T^2 \to SU(2)$. Since $\pi_i(SU(2))$ is trivial for $i = 1, 2$, we have the following:

Proposition 2.2. *The $SU(2)$-bundles over (S^3, K, n) are classified by their isotropy $\tau\colon \mathbb{Z}_n \to SU(2)$.*

We will denote by E_k the orbifold $SU(2)$-vector bundle over (S^3, K, n) whose isotropy is given by

$$\tau(1) = \begin{pmatrix} e^{i2k\pi/n} & 0 \\ 0 & e^{-i2k\pi/n} \end{pmatrix}.$$

Let $\Omega^i(\mathfrak{g}_{E_k})$ denote the space of differential i-forms on (S^3, K, n) with values in the associated bundle $\mathfrak{g}_{E_k}$. On E_k, one can define connections as in the usual way, and form the *space of connections* on E_k, $\mathcal{A}(E_k)$. The *gauge group* $\mathcal{G}(E_k)$ is the group of bundle isomorphisms g of E_k respecting the structure of the fibres and covering the identity. The group $\mathcal{G}(E_k)$ acts on $\mathcal{A}(E_k)$ by

$$g \cdot (d + A) = gdg^{-1} + gAg^{-1},$$

and one can consider the quotient $\mathcal{B}(E_k) = \mathcal{A}(E_k)/\mathcal{G}(E_k)$ the *moduli space of connections*. This space is infinite dimensional and is not necessarily a manifold as it has some singular points, where the action of $\mathcal{G}(E_k)$ on $\mathcal{A}(E_k)$ is not free. A reducible flat connection A on E_k has a non-central stabilizer in the gauge group, $I_A = \{g \in \mathcal{G}(E_k) \mid g \cdot A = A\}$. If E_k is E_0 or $E_{n/2}$ then for a reducible connection, $I_A \simeq SU(2)$. Otherwise, $I_A \simeq S^1$. Denote by $\mathcal{A}^*(E_k)$ the subset in $\mathcal{A}(E_k)$ where the action is free, and let $\mathcal{B}^*(E_k) = \mathcal{A}^*(E_k)/\mathcal{G}(E_k)$. Inside the space $\mathcal{B}(E_k)$, we are particularly interested in the moduli space of flat connections $\mathcal{M}_{flat}(E_k)$, in which the set of irreducible flat connections is denoted $\mathcal{M}^*_{flat}(E_k)$.

Remark 2.3. As (S^3, K, n) is expressible as a global quotient $V_n(K)/\mathbb{Z}_n$, the orbifold setting corresponds to a $\mathbb{Z}_n$-equivariant one over $V_n(K)$. An orbifold bundle E_k over (S^3, K, n) is equivalent to a $\mathbb{Z}_n$-equivariant bundle E'_k over $V_n(K)$, and orbifold gauge transformations are then seen as $\mathbb{Z}_n$-equivariant gauge transformations on E'_k. On the other hand, the $\mathbb{Z}_n$-action on E'_k induces one on forms and hence on connections. Orbifold connections on E_k correspond to invariant connections on E'_k under this induced $\mathbb{Z}_n$-action. For more details on this, see [20], for example.

Let $\chi(S^3, K, n)$ be the $SU(2)$-character variety of the group $\pi_1^V(S^3, K, n)$. Parallel transport yields a homeomorphism between gauge equivalence classes of flat connections on an $SU(2)$-bundle and their corresponding conjugacy classes of holonomy $SU(2)$-representations. See [44, Proposition 4.5] for more details. Contrary to the 3-manifold situation, the orbifold bundles $E_k \to (S^3, K, n)$ are not necessarily trivial, but depend on the isotropy representation $\tau\colon \mathbb{Z}_n \to SU(2)$. Given a representation $\rho\colon \pi_1^V(S^3, K, n) \to SU(2)$, which bundle(s) carries the corresponding flat connection? The reader will have no difficulty in proving the following:

Proposition 2.4. *On a bundle E_k over (S^3, K, n) $(1 \le k \le n/2)$, $\mathcal{M}_{flat}(E_k)$ is homeomorphic to $\chi^k(S^3, K, n)$, where*

$$\chi^k(S^3, K, n) = \{\rho \in \chi(S^3, K, n) \mid tr\ \rho(\mu) = 2cos\ (2\frac{k}{n}\pi)\}.$$

Corollary 2.5. *Each $SU(2)$-bundle E_k over (S^3, K, n) carries one and only one reducible flat connection.*

We now need to outline rapidly relevant aspects of Chern-Weil theory for singular connections in order to define the Chern-Simons functional. Consider $\Sigma \hookrightarrow X^4$ a closed embedded surface in a smooth 4-manifold. In [32] and [33], Kronheimer and Mrowka study singular connections over the non-compact manifold $X^4 - \Sigma$. These depend on a holonomy parameter $\alpha \in (0, 1/2)$, corresponding to the holonomy of the connection along a linking circle of Σ in X^4. There are, moreover, two topological invariants in this situation. Consider an $SU(2)$-bundle

E over X^4. Such a bundle is determined by the integer

$$\frac{1}{4\pi^2}\int_X c_2(E).$$

Let N_Σ be a tubular neighbourhood of Σ. Then over N_Σ, E is trivial and there is a decomposition $E = L \oplus L^*$, where L is a line bundle, and the holonomy is α on L, and $-\alpha$ on L^*. There is no reason why L should be trivial, and this fact provides us with a second topological invariant: the integer

$$\frac{1}{2\pi}\int_\Sigma c_1(L).$$

The Chern-Weil formula of Kronheimer and Mrowka for singular connections may now be given.

Theorem 2.6 ([32, Proposition 5.7]). *Let $\mathbb{A}$ be a singular connection with holonomy parameter $\alpha \in (0,1/2)$. Then the following holds*

$$\frac{1}{8\pi^2}\int_{X-\Sigma} tr\ F_{\mathbb{A}} \wedge F_{\mathbb{A}} = \int_X c_2(E) + 2\alpha \int_\Sigma c_1(L) - \alpha^2 \Sigma\cdot\Sigma,$$

where $\Sigma\cdot\Sigma$ is the self-intersection number of $\Sigma \hookrightarrow X^4$.

Kronheimer and Mrowka explain in some detail how their work can be expressed in terms of orbifolds in the case where $\alpha \in \mathbb{Q}$. Consider an orbifold (X^4, Σ, n), and an orbifold $SU(2)$-bundle $\mathbb{E}_k$ over it. Such a bundle is determined by a bundle E over the underlying manifold X and the isotropy data near the singular locus Σ, say $\xi \in \mathbb{Z}_n$ acting on L by ξ^k. Then for a connection $\mathbb{A}$ on $\mathbb{E}_k$, by the Chern-Weil formula,

$$\frac{1}{8\pi^2}\int_{(X^4,\Sigma,n)} tr\ F_{\mathbb{A}} \wedge F_{\mathbb{A}} = \int_{X^4} c_2(E) + 2\frac{k}{n}\int_\Sigma c_1(L) - \frac{k^2}{n^2}\Sigma\cdot\Sigma.$$

Let us denote $a = \int_{X^4} c_2(E)$ and $b = \int_\Sigma c_1(L)$. The number a is called the *instanton number* while b is the *monopole number*.

We will apply this in our situation to define the Chern-Simons functional. Consider E_k over (S^3, K, n), and $A \in \mathcal{B}(E_k)$. We know that E_k carries one reducible flat connection, θ_k. One can consider the 4-dimensional orbifold $(S^3 \times I, K \times I, n)$ and the pullback bundle of E_k. Let $\mathbb{A}$ be a connection over $(S^3 \times I, K \times I, n)$ such that $\mathbb{A}_{|0} = \theta_k$ and $\mathbb{A}_{|1} = A$.

Definition 2.7. The *Chern-Simons functional* is a function

$$CS\colon \mathcal{B}(E_k) \to S^1$$

defined by

$$CS(A) = \frac{1}{8\pi^2} \int\limits_{(S^3\times I, K\times I, n)} tr\ F_{\mathbb{A}} \wedge F_{\mathbb{A}}.$$

Proposition 2.8. *The Chern-Simons functional is well-defined on $\mathcal{B}(E_k)$ as a function to $\mathbb{R}/\frac{1}{n}\mathbb{Z}$.*

Proof. The definition of CS depends on the connection $\mathbb{A}$ chosen to extend the connections θ_k, $A \in \mathcal{B}(E_k)$, so CS is really well-defined as a functional to a quotient of $\mathbb{R}$. Recall that K has trivial normal bundle in S^3. Identify together the ends $(S^3 \times \{0\}, K \times \{0\}, n)$ and $(S^3 \times \{1\}, K \times \{1\}, n)$ to yield $(S^3 \times S^1, K \times S^1, n)$. Consider an orbifold bundle over this orbifold, and apply the Chern-Weil formula of Theorem 2.12. The formula reads

$$\frac{1}{8\pi^2} \int\limits_{(S^3\times S^1, K\times S^1, n)} tr\ F_{\mathbb{A}} \wedge F_{\mathbb{A}} = \int\limits_{S^3\times S^1} c_2(E) + 2\frac{k}{n} \int\limits_{K\times S^1} c_1(L)$$

which is 0 modulo $\frac{1}{n}\mathbb{Z}$ and therefore the definition does not depend on the instanton and monopole numbers involved in choosing an extension of a connection in $\mathcal{B}(E_k)$. q.e.d.

While all $SU(2)$-bundles over $V_n(K)$ are trivializable, and the trivializations are equivalent up to gauge transformation, there are various non-equivalent ways of equivariantly trivializing $SU(2)$-bundles over $V_n(K)$. Each way is uniquely determined up to equivariant gauge transformation by an isotropy representation $\rho_k\colon \mathbb{Z}_n \to SU(2)$, so as an equivariant bundle, $E_k \cong V_n(K) \times \mathbb{C}^2/\rho_k$. It will be useful to consider an expression of the Chern-Simons functional in terms of a trivialization of E_k:

$$CS(A) = \frac{1}{8\pi^2 n} \int\limits_{V_n(K)} tr\ A \wedge dA + \frac{2}{3} A \wedge A \wedge A.$$

This is of course, up to a scalar factor, the usual Chern-Simons functional for the cyclic branched cover $V_n(K)$, restricted to $\mathbb{Z}_n$-invariant forms.

The critical points of CS are easily computed. These are the flat connections on E_k. At a critical point, the Hessian of CS is

$$*d_A\colon Ker\ d_A^* \to Ker\ d_A^*.$$

For our purposes, it will be more useful to consider the following operator:

$$L_A\colon \Omega^0(\mathfrak{g}_{E_k}) \oplus \Omega^1(\mathfrak{g}_{E_k}) \to \Omega^0(\mathfrak{g}_{E_k}) \oplus \Omega^1(\mathfrak{g}_{E_k}),$$

expressed in matrix notation as

$$L_A = \begin{pmatrix} 0 & d_A^* \\ d_A & *d_A \end{pmatrix}.$$

It may be shown that L_A is an elliptic operator and is self-adjoint. Such an operator has real and discrete spectrum. It will be important to identify the kernels of the operators involved. For $*d_A$ the kernel is H_A^1, the first twisted cohomology group of the complex $(\Omega^i(\mathfrak{g}_{E_k}), d_A)$ of E_k, while $d_A\colon \Omega^0(\mathfrak{g}_{E_k}) \to \Omega^1(\mathfrak{g}_{E_k})$ has kernel the space of covariant sections of $\mathfrak{g}_{E_k}$, which is precisely H_A^0. One has $H_A^0 \neq 0$ if and only if A is a reducible flat connection; on the other hand, if $H_A^1 = 0$, then A is an isolated flat connection. Notice that if $H_A^1 \neq 0$, then A may or may not be isolated in $\mathcal{M}_{flat}$. By Corollary 2.5, on a bundle E_k, there is only one flat connection for which $H_A^0 \neq 0$, so this group will not play an important role. On the other hand, the condition $H_A^1 = 0$ will be important to avoid manifolds of critical points for the functional CS.

Let (X^4, Σ, n) be a 4-dimensional orbifold and $\mathbb{E}_k$ a bundle over it. For a connection $A \in \mathcal{A}(\mathbb{E}_k)$ over a 4-dimensional orbifold, the curvature is split into $F_A = F_A^+ + F_A^-$, and A is called an *anti-self-dual connection* or *instanton* if it satisfies the ASD equation $F_A^+ = 0$. Near A, the linearization of the ASD equation is given by the operator

$$d_A^* \oplus d_A^+\colon \Omega^1(\mathfrak{g}_{\mathbb{E}_k}) \to \Omega^0(\mathfrak{g}_{\mathbb{E}_k}) \oplus \Omega^+(\mathfrak{g}_{\mathbb{E}_k}).$$

The case that will interest us is that of an orbifold of the form $(S^3, K, n) \times \mathbb{R}$. In this situation we will see relations between the 3-dimensional and 4-dimensional settings. Consider E_k over (S^3, K, n). Let $A(t)$ be a path of connections in $\mathcal{B}(E_k)$, and denote by $*_3$ the Hodge *-operator in dimension three. The proofs of the next three propositions are not difficult and entirely analogous those in 3-manifold Floer Homology.

Proposition 2.9 *The gradient vector field of CS is given by the equation*

$$\frac{dA}{dt}(t) = \frac{1}{4\pi^2} *_3 F_{A(t)}.$$

Now extend the bundle E_k over (S^3, K, n) to a bundle $\mathbb{E}_k$ over $(S^3, K, n) \times \mathbb{R}$ and consider a connection $\mathbb{A}$. Then, $\mathbb{A} = A_t + B_t \wedge dt$, for $A_t \in \Omega^1(\mathfrak{g}_{E_k})$ and $B_t \in \Omega^0(\mathfrak{g}_{E_k})$. This is done in $\mathcal{A}(\mathbb{E}_k)$. Say that $\mathbb{A}$ is in *temporal gauge* if it has no dt component in this decomposition.

Proposition 2.10. *Any connection $\mathbb{A}$ on $\mathbb{E}_k$ is gauge-equivalent to a connection in temporal gauge.*

In particular, when working on $\mathcal{B}(\mathbb{E}_k)$ it may be assumed that an instanton over $(S^3, K, n) \times \mathbb{R}$ has no dt component. Then the crucial observation is the following:

Proposition 2.11. *A connection*

$$\mathbb{A} = A_t \in \mathcal{B}(\mathbb{E}_k)$$

over

$$(S^3, K, n) \times \mathbb{R}$$

is an instanton if and only if it satisfies

$$\frac{dA_t}{dt} = *_3 F_{A_t}.$$

Since the orbifold $(S^3, K, n) \times \mathbb{R}$ is not compact, one has to impose an extra condition to the ASD connections over the orbifold cylinder, and restrict to "finite energy" instantons. An instanton $\mathbb{A}$ over $M \times \mathbb{R}$ is said to be of finite energy if its Yang-Mills action

$$YM(\mathbb{A}) = \int_{(S^3,K,n)\times\mathbb{R}} |F_A|^2 dvol$$

is finite. This condition is necessary to have asymptotically flat instantons, that is, instantons that "connect" flat connections over (S^3, K, n).

3. Floer homology for 3-orbifolds

We now give the construction of a Floer Homology for 3-orbifolds (S^3, K, n). This was announced in [10] and developed in [11]. First we define the Floer index of critical points of CS, in order to define the chain groups for the Floer Homology. Then we define a boundary operator between chain groups in adjacent dimensions and obtain the desired Floer homology. Initially we shall impose non-degeneracy conditions in order to define the Floer Homology, and then we treat the problem of perturbations of the Chern-Simons functional so as to define the Homology in degenerate situations.

As we have seen in Section 2, the 3-orbifolds (S^3, K, n) are very much related to the cyclic branched covers $V_n(K)$ of S^3 along the knot K. Before trying to set up a Floer homology for 3-orbifolds, it is therefore legitimate to wonder about the need for such an invariant. Couldn't one study knots using the Floer Homology of cyclic branched coverings? There are some obvious reasons for the preference of 3-orbifolds over cyclic branched coverings.

One good reason is that a Floer Homology for cyclic branched coverings is not always well-defined, as in general the 3-manifolds considered are not homology 3-spheres and therefore Floer's original construction does not apply. In some instances, one could use generalizations of Floer Homology, but this would simply lead one away from Floer's original theory and, at present state of knowledge, still leave some cases untreated; which is not advisable if one wishes to have an effective knot invariant.

Also, a Floer Homology for 3-orbifolds ties in well with other gauge theoretic constructions. For example, in the context of invariants developed in [33], it could play a similar role to the one played by Floer Homology with respect to Donaldson Theory of 4-manifolds. In another direction, various people have recently developed Casson-type and symplectic Floer-type invariants for knots ([3], [27] and [36]) and Floer Homology for 3-orbifolds is related in various ways to these constructions.

3.1 The Floer index

Here we develop the necessary analytical tools to define a relative index between critical points of the Chern-Simons functional. Floer's original construction for homology 3-spheres makes use of the notion of spectral flow, but for the orbifold setting, the Analysis appeared clearer to us if we used the notion of adapted bundles as in [17]. For the most part, what follows is an adaptation to the orbifold case of the work done in [17].

Definition 3.1. An $SU(2)$-bundle $\mathbb{E}$ over $(S^3, K, n) \times \mathbb{R}$ with a fixed flat connection over each end is called an *adapted bundle*. Adapted bundles are said to be equivalent if there is a bundle isomorphism preserving the flat structures over the ends.

When working over a cylinder $(S^3, K, n) \times \mathbb{R}$, there is an identification of the operator which linearises the ASD equation, $D_{\mathbb{A}} = d^*_{\mathbb{A}} + d^+_{\mathbb{A}}$, as

$$D_{\mathbb{A}} = \frac{d}{dt} + L_A.$$

First consider the pullback case. Given a bundle E_k over (S^3, K, n) with a flat, possibly reducible, connection A, which is possibly reducible, pullback E_k to a bundle $\mathbb{E}_k$ over $(S^3, K, n) \times \mathbb{R}$, with an induced connection $\mathbb{A}$. If A is acyclic then

$$D_{\mathbb{A}} \colon L^2_1(\Omega^1(\mathfrak{g}_{\mathbb{E}_k})) \to L^2(\Omega^0(\mathfrak{g}_{\mathbb{E}_k}) \oplus \Omega^+(\mathfrak{g}_{\mathbb{E}_k}))$$

is invertible. However, in the case at hand, $\mathcal{M}_{flat}(E_k)$ contains one connection θ_k which is reducible and is therefore not acyclic, hence some difficulties arise. Indeed, $0 \in Spec\ L_A$ is an eigenvalue and over the cylinder, $D_{\mathbb{A}}$ is not invertible or even Fredholm. To handle this problem we use weighted Sobolev spaces $L^{2,\alpha}_1$, $L^{2,\alpha}$ which are completions of the smooth compactly supported sections over $(S^3, K, n) \times \mathbb{R}$ with respect to norms

$$\| f \|_{L^{2,\alpha}} = \| e^{\alpha t} f \|_{L^2}$$

and

$$\| f \|_{L^{2,\alpha}_1} = \| e^{\alpha t} f \|_{L^2_1} .$$

These definitions extend to an orbifolds with cylindrical ends and a weight α associated to each end.

Proposition 3.2.

$$D_{\mathbb{A}} \colon L^{2,\alpha}_1(\Omega^1(\mathfrak{g}_{\mathbb{E}_k})) \to L^{2,\alpha}(\Omega^0(\mathfrak{g}_{\mathbb{E}_k}) \oplus \Omega^+(\mathfrak{g}_{\mathbb{E}_k}))$$

is an invertible operator over $(S^3, K, n) \times \mathbb{R}$ *for any weight* $\alpha \notin Spec\ L_A$.

Proof. First notice that the problem may be reduced to considering another operator between the corresponding unweighted Sobolev spaces: using the isometries $L^{2,\alpha}_1 \to L^2_1$ and $L^{2,\alpha} \to L^2$ given by multiplying by $e^{\alpha t}$, one can define

$$e^{\alpha t} D_{\mathbb{A}} e^{-\alpha t} \colon L^2_1(\Omega^1(\mathfrak{g}_{\mathbb{E}_k})) \to L^2(\Omega^0(\mathfrak{g}_{\mathbb{E}_k}) \oplus \Omega^+(\mathfrak{g}_{\mathbb{E}_k})),$$

which is equivalent to $D_{\mathbb{A}}$ on weighted spaces. Since

$$e^{\alpha t} D_{\mathbb{A}} e^{-\alpha t}(f) = e^{\alpha t}(\frac{d}{dt} + L_A) e^{-\alpha t}(f) = \frac{d}{dt} f + (L_A - \alpha) f,$$

the problem is reduced to show that for $0 \notin Spect\ L_A$, $\frac{d}{dt} + L_A$ is invertible. The proof of this follows from the next elementary analytical lemma which may be proved using the spectral decomposition of the operator L_A along with elliptic regularity.

Lemma 3.3. *Let $\delta \in \mathbb{R}$ be such that $|\lambda| \geq \delta > 0$ for any $\lambda \in Spec\ L_A$, and ρ a compactly supported section of $\Omega^0(\mathfrak{g}_{\mathbb{E}_k}) \oplus \Omega^+(\mathfrak{g}_{\mathbb{E}_k})$ over $(S^3, K, n) \times \mathbb{R}$. Then there is a section f of $\Omega^1(\mathfrak{g}_{\mathbb{E}_k})$ such that $D_{\mathbb{A}} f = \rho$ and $\| f \|_{L_1^{2,\alpha}} \leq \frac{1}{\delta} \| \rho \|_{L^{2,\alpha}}$.*

To set up an index and establish some of its elementary properties, it is essential to consider not only cylinders, but more generally orbifolds with cylindrical ends. For such orbifolds the invertibility of the operator $D_{\mathbb{A}}$ in appropriate weighted spaces is lost, but one can still prove a Fredholm property. We shall need only particular orbifolds with cylindrical ends: they will be orbifolds (X^4, Σ, n) with underlying space X^4, singular locus Σ with locally a $\mathbb{Z}_n$ isotropy around Σ, such that the ends of the orbifold are modeled on some $(S^3, K, n) \times \mathbb{R}$. Although these orbifolds with cylindrical ends are not necessarily globally quotients of some cyclic branched covering by a $\mathbb{Z}_n$-action, they are at least global quotients on the cylinder part.

Let (X^4, Σ, n) be such an orbifold with m cylindrical ends, and consider weighted Sobolev norms with weight α_i on the i^{th} end of the orbifold. Let $\alpha = (\alpha_1, ..., \alpha_m)$ and let $L_1^{2,\alpha}$, $L^{2,\alpha}$ be the completions with respect to the weighted norms with weights $\alpha_1, ..., \alpha_m$ over the ends. Consider the operator $D_{\mathbb{A}}$ for an adapted bundle over (X^4, Σ, n) with limiting data $(A_1, ..., A_m)$, as before. Over the ends of the orbifold, $D_{\mathbb{A}}$ splits. The following is very important for the construction:

Theorem 3.4. *If none of the α_i lies in the spectrum of L_{A_i}, then*

$$D_{\mathbb{A}} \colon L_1^{2,\alpha}(\Omega^1(\mathfrak{g}_{\mathbb{E}_k})) \to L^{2,\alpha}(\Omega^0(\mathfrak{g}_{\mathbb{E}_k}) \oplus \Omega^+(\mathfrak{g}_{\mathbb{E}_k}))$$

is a Fredholm operator.

Proof. By an analogous reasoning to that of Proposition 3.2, using the isometries $e^{\alpha_i t}$ one may reduce the problem to an operator which is given on each end by $\frac{d}{dt} + L_A - \alpha_i$. For simplicity, we will restrict ourselves to the case where the orbifold has only one cylindrical end. The essential ingredients can be found in Section 2 of [39], but here we follow [17].

First we wish to show that $Ker\ D_{\mathbb{A}}$ is finite dimensional. Take a sequence $\{f_i\} \in Ker\ D_{\mathbb{A}}$ such that $\| f_i \| \leq 1$. Consider a decomposition of (X^4, Σ, n) as

$$(X_0, \Sigma_0, n) \cup (S^3, K, n) \times \mathbb{R}^+$$

into a compact set and a cylinder. Over (X_0, Σ_0, n) the solutions $\{f_i\}$ such that $D_{\mathbb{A}}(f_i) = 0$ converge in L^2 to a limit f_∞, as $D_{\mathbb{A}}$ is an elliptic operator on a compact orbifold. $\| f_\infty \| \leq 1$ and $D_{\mathbb{A}}(f_\infty) = 0$. The problem is thus reduced to showing that $\{f_i\}$ also converges in L^2 over the cylinder $(S^3, K, n) \times (0, \infty)$. For this, the following analytical lemma whose proof, again, uses the spectral decomposition of L_A and elementary ODE results, is needed.

Lemma 3.5. *Any solution to $D_{\mathbb{A}}(f) = 0$ over $(S^3, K, n) \times (0, \infty)$ satisfies the following inequality*

$$\int_{(S^3,K,n)\times(0,\infty)} |f|^2 \leq C_\delta \int_{(S^3,K,n)\times(0.1)} |f|^2$$

for some constant C_δ depending on δ, mentioned in Lemma 3.3.

With this lemma, one gets

$$\int_{(S^3,K,n)\times(0,\infty)} |f_i - f_\infty| \leq C_\delta \int_{(S^3,K,n)\times(0,1)} |f_1 - f_\infty|,$$

where the last integral is taken over a finite volume domain, and therefore this gives L^2-convergence of the $\{f_i\}$ over the cylinder also. This implies that $Ker\ D_{\mathbb{A}}$ is finite dimensional as otherwise one could construct a sequence of solutions without any convergent subsequence.

Next one establishes that $Im\ D_{\mathbb{A}}$ has finite codimension. Decompose again (X^4, Σ, n) into a compact part and a cylindrical part. Over the cylinder $D_{\mathbb{A}}$ has an inverse by Proposition 3.2. For any

$$\rho \in L^2(\Omega^0(\mathfrak{g}_{\mathbb{E}_k}) \oplus \Omega^+(\mathfrak{g}_{\mathbb{E}_k}))$$

let $\tilde{Q}_0(\rho)$ be given by $\tilde{Q}_0(\beta_0\rho)$ where β_0 is a cut-off function which is identically zero on (X_0, Σ_0, n) and takes the value 1 on $\mathcal{U}_0$, the cylinder. Then on the cylinder

$$D_{\mathbb{A}}\tilde{Q}_0(\rho) = \rho.$$

Now cover (X_0, Σ_0, n) with coordinate patches such that $D_{\mathbb{A}}$ has a right inverse over each of the patches. This can be done as $D_{\mathbb{A}}$ is elliptic and (X_0, Σ_0, n) is compact. Thus there is an open cover

$$(X^4, \Sigma, n) = \bigcup_{i=0}^{n} \mathcal{U}_i$$

such that $\mathcal{U}_i \cap \mathcal{U}_j$ is pre-compact and $D_{\mathbb{A}}\tilde{Q}_i(\rho) = \rho$ on a neighbourhood of the closure of $\mathcal{U}_i$. This is the local part of the argument.

Now for a partition of unity subordinate to this cover, $\{\beta_i\}$, define the operator $P\colon L^2(\Omega^0(\mathfrak{g}_{\mathbb{E}_k}) \oplus \Omega^+(\mathfrak{g}_{\mathbb{E}_k})) \to L^2_1(\Omega^1(\mathfrak{g}_{\mathbb{E}_k}))$ by

$$P(\rho) = \sum_{i=0}^{n} \beta_i \tilde{Q}_i(\rho).$$

By the construction of P, the operator $D_{\mathbb{A}}P - I$ is supported on the overlaps $\mathcal{U}_i \cap \mathcal{U}_j$ for $i \neq j$, which is fixed. The Arzelà-Ascoli Theorem implies that $D_{\mathbb{A}}P - I$ is a compact operator. As $D_{\mathbb{A}}P - I$ is compact an elementary result from Functional Analysis (see (11.3.2) p. 315 in [14]) implies that the image of $D_{\mathbb{A}}P$ is closed and has finite codimension in $L^2(\Omega^0(\mathfrak{g}_{\mathbb{E}_k}) \oplus \Omega^+(\mathfrak{g}_{\mathbb{E}_k}))$. Then $D_{\mathbb{A}}$ also has finite codimension in that space as $Im\ D_{\mathbb{A}}P \subset Im\ D_{\mathbb{A}}$. Like $Im\ D_{\mathbb{A}}P$, $Im\ D_{\mathbb{A}}P - I$ is also closed and therefore $Im\ D_{\mathbb{A}}$ is closed. $D_{\mathbb{A}}$ is Fredholm as required. q.e.d.

Therefore we can define an index

$$ind^{(\alpha)}\ D_{\mathbb{A}} = dimKer\ D_{\mathbb{A}} - dimCoker\ D_{\mathbb{A}}.$$

As the index is a deformation invariant, it will be independent of the particular $\mathbb{A}$ chosen on the adapted bundle over (X^4, Σ, n). Also, though $D_{\mathbb{A}}$ was constructed using a metric on (X^4, Σ, n), the index is easily seen to be independent of that metric.

One of the fundamental properties of the index for an adapted bundle is the additivity under the process of gluing. Let (X_1, Σ_1, n) and (X_2, Σ_2, n) be two orbifolds with cylindrical ends such that one of the ends is $(S^3, K, n) \times \mathbb{R}^+$ and $(\bar{S}^3, \bar{K}, n) \times \mathbb{R}^+$ respectively. For simplicity assume that the two orbifolds have no other ends. Take two adapted bundles over these, each with the same flat limits over the ends, and use weighted Sobolev spaces of weight α and $-\alpha$ respectively (to have a compatibility condition for the gluing), with α and $-\alpha$ not in the spectrum of the relevant operators. Consider an adapted bundle over the glued orbifold $(X_1 \sharp X_2, \Sigma_1 \sharp \Sigma_2, n)$. Then the additivity is expressed as follows.

Theorem 3.6. *The index is additive:*

$$ind\ D_{\mathbb{A}_1 \sharp \mathbb{A}_2} = ind^{(\alpha)}\ D_{\mathbb{A}_1} + ind^{(-\alpha)}\ D_{\mathbb{A}_2}.$$

Proof. Assume first that both operators $D_{\mathbb{A}_1}$ and $D_{\mathbb{A}_2}$ have no cokernel so that

$$ind\ D_{\mathbb{A}_i} = dimKer\ D_{\mathbb{A}_i},$$

and there are bounded right inverses

$$Q_i\colon L^{2,\alpha}(\Omega^0(\mathfrak{g}_{\mathbb{E}_k}) \oplus \Omega^+(\mathfrak{g}_{\mathbb{E}_k})) \to L_1^{2,\alpha}(\Omega^1(\mathfrak{g}_{\mathbb{E}_k})),$$

$D_{\mathbb{A}_i} Q_i = I$ such that

$$\| Q_i(\rho) \|_{L_1^{2,\alpha}} \leq C \| \rho \|_{L_1^{2,\alpha}} .$$

By conformal invariance, one can stretch the gluing region in $(X_1 \sharp X_2, \Sigma_1 \sharp \Sigma_2, n)$ to obtain an orbifold denoted $(X_1 \sharp_T X_2, \Sigma_1 \sharp_T \Sigma_2, n)$ for a real parameter T, gluing a cylinder of length $2T$ in the middle.

We first prove that for T large enough, there is an injection

$$\sigma\colon Ker\ D_{\mathbb{A}_1 \sharp \mathbb{A}_2} \to Ker\ D_{\mathbb{A}_1} \oplus Ker\ D_{\mathbb{A}_2}.$$

Indeed, let $\sigma(f) = (f_1, f_2) = (\phi_1 f - Q_1 D_{\mathbb{A}_1}(\phi_1 f), \phi_2 f - Q_2 D_{\mathbb{A}_2}(\phi_2 f))$ for functions on $(X_1 \sharp_T X_2, \Sigma_1 \sharp_T \Sigma_2, n)$ such that $\phi_1^2 + \phi_2^2 = 1$, where ϕ_i is supported on

$$(X_1 \sharp_T X_2, \Sigma_1 \sharp_T \Sigma_2, n) - (X_j \sharp_T (S^3, K, n) \times (0, T/2)),$$

for $i \neq j$ and $\| \nabla \phi_i \| = \epsilon(T)$ for $\epsilon(T) \to 0$ as $T \to \infty$. Then using the fact that $f \in Ker\ D_{\mathbb{A}_1 \sharp \mathbb{A}_2}$, we obtain

$$\begin{aligned} \| f_i - f_\infty \|_{L_1^{2,\alpha}} &= \| Q_i D_{\mathbb{A}_i}(\phi_i f) \| \\ &\leq C \| D_{\mathbb{A}_i}(\phi_i f) \| \\ &\leq C \| \nabla \phi_i f \| \\ &\leq C \epsilon(T) \| f \| . \end{aligned}$$

As $\phi_1^2 + \phi_2^2 = 1$, $\| f \|_{L^{2,\alpha}(X_1 \sharp X_2)} = \| (\phi_1 f, \phi_2 f) \|_{L^{2,\alpha}(X_1) \oplus L^{2,\alpha}(X_2)}$ and

$$\begin{aligned} | \| \sigma(f) \| - \| f \| | &= | \| \sigma(f) \| - \| (\phi_1 f, \phi_2 f) \| | \\ &\leq \| (f_1 - \phi_1 f, f_2 - \phi_2 f) \| \\ &\leq \sqrt{2} C \epsilon(T) \| f \| . \end{aligned}$$

If $\sigma(f) = 0$ and $\| f \| \neq 0$, then $\| f \| \leq \sqrt{2} C \epsilon(T) \| f \|$ so for $\epsilon(T) \leq 1/\sqrt{2}C$, σ has to be an injection.

Also $D_{\mathbb{A}_1 \sharp \mathbb{A}_2}$ is surjective if the $D_{\mathbb{A}_i}$ are. As above, there are right inverses Q_i to $D_{\mathbb{A}_i}$. Define

$$Q\colon L^{2,\alpha}(\Omega^0(\mathfrak{g}_{\mathbb{E}_k}) \oplus \Omega^+(\mathfrak{g}_{\mathbb{E}_k})) \to L_1^{2,\alpha}(\Omega^1(\mathfrak{g}_{\mathbb{E}_k}))$$

by

$$Q(\rho) = \phi_1^2 Q_1(\rho_1) + \phi_2^2 Q_2(\rho_2),$$

for ρ_i the restriction of ρ to the support of ϕ_i. Then

$$(D_{\mathbb{A}_1 \sharp \mathbb{A}_2} Q)\rho = \rho + \nabla(\phi_1^2) Q_1(\rho_1) + \nabla(\phi_2^2) Q_2(\rho_2)$$

as $D_{\mathbb{A}_i} Q_i = I$ and $D_{\mathbb{A}_1 \sharp \mathbb{A}_2} = D_{\mathbb{A}_i}$ over the support of ϕ_i. It follows that

$$\| (D_{\mathbb{A}_1 \sharp \mathbb{A}_2} Q - I)\rho \|_{L^{2,\alpha}} \leq 4C \| \rho \|_{L^{2,\alpha}} \| \nabla(\phi_i^2) \|_{L^\infty} .$$

Hence choosing ϕ_i such that $\nabla(\phi_i^2) = \epsilon^{'}(T)$, where $\epsilon^{'}(T) \to 0$ as $T \to \infty$, for T so large that $\epsilon^{'}(T) \leq 1/(4C)$ there is an inverse.

The final step, supposing surjectivity of the $D_{\mathbb{A}_i}$, is to show that for T large, there exists an injection

$$\sigma^{'} : Ker\ D_{\mathbb{A}_1} \oplus Ker\ D_{\mathbb{A}_2} \to Ker\ D_{\mathbb{A}_1 \sharp \mathbb{A}_2}.$$

This map is given by

$$\sigma^{'}(f_1, f_2) = \phi_1^2 f_1 + \phi_2^2 f_2 - Q D_{\mathbb{A}_1 \sharp \mathbb{A}_2}(\phi_1^2 f_1 + \phi_2^2 f_2).$$

As for the case of the construction of σ, $Q D_{\mathbb{A}_1 \sharp \mathbb{A}_2}(\phi_1^2 f_1 + \phi_2^2 f_2)$ is bounded by an arbitrarily small multiple of $\| (f_1, f_2) \|$. Then, as above,

$$| \| \sigma'(f_1, f_2) \| - \| (f_1, f_2) \| | \leq C' \epsilon'(T) \| (f_1, f_2) \|,$$

and anything in $Ker\ \sigma'$ is forced to be trivial if T is large enough. Therefore σ' is injective as required.

Using σ, one gets

$$dim\ Ker\ D_{\mathbb{A}_1 \sharp \mathbb{A}_2} \leq dim\ Ker\ D_{\mathbb{A}_1} + dim\ Ker\ D_{\mathbb{A}_2},$$

while using σ' one obtains

$$dim\ Ker\ D_{\mathbb{A}_1 \sharp \mathbb{A}_2} \geq dim\ Ker\ D_{\mathbb{A}_1} + dim\ Ker\ D_{\mathbb{A}_2}.$$

So the additive property is verified in the case where the operators are surjective.

In the case where the $D_{\mathbb{A}_i}$ are not necessarily surjective, choose maps

$$F_i \colon \mathbb{R}^n \to \Omega^0(\mathfrak{g}_{\mathbb{E}_k}) \oplus \Omega^+(\mathfrak{g}_{\mathbb{E}_k})$$

over each (X_i, Σ_i, n) with images supported in the interior so that

$$\tilde{D}_{\mathbb{A}_i} = D_{\mathbb{A}_i} \oplus F_i \colon \Omega^1(\mathfrak{g}_{\mathbb{E}_k}) \oplus \mathbb{R}^{n_i} \to \Omega^0(\mathfrak{g}_{\mathbb{E}_k}) \oplus \Omega^+(\mathfrak{g}_{\mathbb{E}_k})$$

are surjective. Forming $\tilde{D}_{\mathbb{A}_1\sharp\mathbb{A}_2} = D_{\mathbb{A}_1\sharp\mathbb{A}_2} \oplus F_1 \oplus F_2$, the gluing formula for the index in the surjective case may be used to get

$$ind\ \tilde{D}_{\mathbb{A}_1\sharp\mathbb{A}_2} = ind^{(\alpha)}\ \tilde{D}_{\mathbb{A}_1} + ind^{(-\alpha)}\ \tilde{D}_{\mathbb{A}_2}.$$

Now for

$$D_{\mathbb{A}_i} : \Omega^1(\mathfrak{g}_{\mathbb{E}_k}) \oplus \mathbb{R}^{n_i} \to \Omega^0(\mathfrak{g}_{\mathbb{E}_k}) \oplus \Omega^+(\mathfrak{g}_{\mathbb{E}_k}),$$

the index will be

$$ind^{(\alpha)}\ D_{\mathbb{A}_i} - n_i,$$

where $ind^{(\alpha)}\ D_{\mathbb{A}_i}$ is computed for

$$D_{\mathbb{A}_i} : \Omega^1(\mathfrak{g}_{\mathbb{E}_k}) \to \Omega^0(\mathfrak{g}_{\mathbb{E}_k}) \oplus \Omega^+(\mathfrak{g}_{\mathbb{E}_k}).$$

Moreover, by a homotopy $D_{\mathbb{A}_i} \oplus tF_i$, the operator

$$D_{\mathbb{A}_i} : \Omega^1(\mathfrak{g}_{\mathbb{E}_k}) \oplus \mathbb{R}^{n_i} \to \Omega^0(\mathfrak{g}_{\mathbb{E}_k}) \oplus \Omega^+(\mathfrak{g}_{\mathbb{E}_k})$$

and $\tilde{D}_{\mathbb{A}_i}$ have the same index. A similar argument applies to $D_{\mathbb{A}_1\sharp\mathbb{A}_2}$ and $\tilde{D}_{\mathbb{A}_1\sharp\mathbb{A}_2}$ to give finally

$$ind\ D_{\mathbb{A}_1\sharp\mathbb{A}_2} = ind\ \tilde{D}_{\mathbb{A}_1\sharp\mathbb{A}_2} - (n_1 + n_2) = ind^{(\alpha)}\ D_{\mathbb{A}_1} + ind^{(-\alpha)}\ D_{\mathbb{A}_2}.$$

q.e.d.

If a variation of α crosses the spectrum of L_A, then not all the operators in the homotopy are Fredholm, and the index could change. For simplicity, suppose that there is only one end, and let α^+ and α^- be respectively positive and negative such that $|\alpha^\pm| < \delta$, two weights for Sobolev spaces over the cylindrical end. The two indices $ind^{(\alpha^+)}\ D_{\mathbb{A}}$ and $ind^{(\alpha^-)}\ D_{\mathbb{A}}$ are well-defined and the relation between them is given by:

Theorem 3.7.

$$ind^{(\alpha^-)}\ D_{\mathbb{A}} - ind^{(\alpha^+)}\ D_{\mathbb{A}} = dimKer\ L_A.$$

Proof. The gluing result just proved may be used to reduce the computation of this difference to a computation of an index over the cylinder (with positive weight α^+ at one end and negative weight α^- at the other). Over the cylinder there is separation of variables as usual, and using an isometry argument as in Proposition 3.2 and Theorem 3.4, one reduces the argument to an operator

$$\frac{d}{dt} + (L_A + \sigma(t)) \colon L_1^2(\Omega^1(\mathfrak{g}_{\mathbb{E}_k})) \to L^2(\Omega^0(\mathfrak{g}_{\mathbb{E}_k}) \oplus \Omega^+(\mathfrak{g}_{\mathbb{E}_k})),$$

where $\sigma(t) \to \alpha^+$ as $t \to +\infty$ and $\sigma(t) \to \alpha^-$ as $t \to -\infty$. By splitting according to the spectrum of L_A, this operator may be reduced to an ODE operator on $\mathbb{R}$: $\frac{d}{dt} + k(t)$. For $k(t) = \lambda + \sigma(t)$, eigenvalues $\lambda \in Spec\ L_A$, the index contributions of this operator come only from the zero eigenspace. Therefore

$$ind\ (\frac{d}{dt} + L_A + \sigma(t)) = dimKer\ L_A.$$

q.e.d.

We now carry the discussion on the index of an adapted bundle over to the Floer setting: given an $SU(2)$-bundle E_k over a 3-orbifold (S^3, K, n), for each critical point of the Chern-Simons functional over E_k we wish to define an index. To do this we exploit the fact that there is only one reducible flat connection over E_k and, as in the case of Floer Homology for oriented homology 3-spheres, this will serve as a reference connection.

E_k can be pulled back to give a bundle over $(S^3, K, n) \times \mathbb{R}$, and let $\mathbb{E}_k$ be an adapted bundle over $(S^3, K, n) \times \mathbb{R}$ with a connection $\mathbb{A}$ such that the limiting flat data are the flat connections θ_k and A, where A is irreducible here. If $\mathbb{E}_k'$ is another such bundle, it was seen earlier that they are always equivalent as orbifold bundles but may be different as adapted bundles. For $\mathbb{E}_k$ and $\mathbb{E}_k'$ we have operators $D_{\mathbb{A}}$ and $D_{\mathbb{A}'}$ and, by the index theory developed, $ind^-\ D_{\mathbb{A}}$ and $ind^-\ D_{\mathbb{A}'}$. We should like to define the index of A to be $ind^- D_{\mathbb{A}}$, but this a priori depends not only on the limiting data θ_k and A, but also on the chosen connection on the cylinder. Moreover, the following removes any ambiguity if we reduce modulo 4.

Theorem 3.8. *The index of an adapted $\mathbb{E}_k$ over $(S^3, K, n) \times \mathbb{R}$ with limiting data θ_k and A depends only on A if it is reduced modulo 4.*

Proof. This essentially follows the approach in usual Floer Homology, by the gluing results introduced before, combined with the index theory of [32]. In the orbifold setting, the index formula of Kronheimer and Mrowka is as follows:

Theorem 3.9 ([32, Section 6]). *Let $\Sigma \hookrightarrow X$ be a closed embedded surface of genus g in a compact 4-manifold. Consider singular connections $\mathbb{A}$ with rational holonomy parameter $\alpha \in (0, 1/2)$ on a bundle with instanton and monopole numbers a and b. Then*

$$ind\ D_{\mathbb{A}} = 8a + 4b - 3(b^+(X) - b_1(X) + 1) - (2g - 2).$$

By this index formula for $D_{\mathbb{A}}$ on a closed orbifold, the standard argument yields the result. q.e.d.

This result partitions the set $\mathcal{M}^*_{flat}(E_k)$ of critical points of the Chern-Simons functional over a 3-orbifold (S^3, K, n) into four classes. We impose the following non-degeneracy condition in order to make the definition below.

Condition 3.10. For every $A \in \mathcal{M}_{flat}(E_k)$, A is a non-degenerate critical point of $CS\colon \mathcal{B}(E_k) \to \mathbb{R}/\mathbb{Z}$.

Definition 3.11. Let $\mathbb{A}$ be a connection on an adapted bundle over $(S^3, K, n) \times \mathbb{R}$ with limits θ_k and A.

1. The *Floer index* of an element $A \in \mathcal{M}^*_{flat}(E_k)$ is
$$\mu(A) \equiv ind^-\ D_{\mathbb{A}}\ (mod\ 4).$$
2. The *Floer chain groups* for E_k over (S^3, K, n),
$$C^{(k)}_i(S^3, K, n) \text{ for } i = 0, 1, 2, 3$$
are defined to be the free abelian groups generated by elements $A \in \mathcal{M}^*_{flat}(E_k)$ such that $\mu(A) = i$.

3.2 The Floer boundary operator

In analogy with Morse Theory, we wish to define a boundary operator between Floer chain groups, in order to define a homology complex, the Floer complex. This operator will, in fact, increase indices by 1:
$$\partial\colon C^{(k)}_*(S^3, K, n) \to C^{(k)}_{*+1}(S^3, K, n).$$
Just as Floer did, we define ∂ using finite energy instantons on the orbifold cylinder $(S^3 \times \mathbb{R}, K \times \mathbb{R}, n)$. Our luck is that these orbifold instantons correspond bijectively to singular instantons with holonomy parameter k/n over $S^3 \times \mathbb{R} - (K \times \mathbb{R})$, and therefore we can avail ourselves the detailed analysis of Kronheimer and Mrowka of the moduli spaces of such instantons. The new feature is that we are dealing with orbifolds which are not compact, but have cylindrical ends. However, analytically, the transition from the compact to the non-compact case in the orbifold setting corresponds exactly to the transition in the manifold case, as treated by Floer in [22].

The definition of the boundary operator in Floer Theory requires a great deal of analytical techniques. It is not clear how much detail should be given as we adapt such techniques to our setting. In many cases, results proved for manifolds in [16], [17] or [22] carry over directly to orbifolds. We shall largely follow the unpublished manuscript [17], essentially observing that the analytical results there extend to orbifolds, often by restricting to invariant connections and operators on a cyclic branched cover. As some of those are perhaps not readily available, we give them and the appropriate definitions, focusing on the extension to orbifolds as it is essential in generalising the Floer Homology. For more details on standard techniques, the reader is referred to [17], [40] or [45]. For clarity, we shall first state the main results of the section, then make a few comments on the techniques needed for the proofs of these results, and finally proceed to the actual proofs. As the reducible flat connections play a minor role here, in this section, unless we mention otherwise, we shall be working only with irreducible flat connections.

Let $\mathbb{E}_k$ be an $SU(2)$-bundle over $(S^3, K, n) \times \mathbb{R}$ (or more generally over an orbifold with such cylindrical ends). As usual, fix a metric on (S^3, K, n) and pull it back to the cylinder. We are interested in the following moduli spaces:

Definition 3.12. Let $\mathcal{M}_{ASD}(A, A^{'}, i)$ denote the moduli space of finite energy instantons $\mathbb{A}$ over on $\mathbb{E}_k$ with limits $A, A^{'} \in \mathcal{M}^{*}_{flat}(E_k)$ such that $i = ind\ D_{\mathbb{A}}$.

For convenience, we shall provisionally make the following transversality assumption:

Condition 3.13. For every $\mathbb{A}$ in $\mathcal{M}_{ASD}(A, A', i)$, $D_{\mathbb{A}}$ is surjective.

When this condition is satisfied, $\mathcal{M}_{ASD}(A, A', i)$ is said to be *regular*. Notice that there is an $\mathbb{R}$-action on this moduli space induced from translations along the cylinder. Denote by $\hat{\mathcal{M}}_{ASD}(A, A^{'}, i)$ the quotient $\mathcal{M}_{ASD}(A, A^{'}, i)/\mathbb{R}$. The main aim of this section is to prove the following theorem.

Theorem 3.14. *The following assertions hold if $\mathcal{M}_{ASD}(A, A', i)$ is regular.*

1. *$\mathcal{M}_{ASD}(A, A^{'}, i)$ is a smooth oriented manifold of dimension i, and the reduced moduli space $\hat{\mathcal{M}}_{ASD}(A, A^{'}, i) = \mathcal{M}_{ASD}(A, A^{'}, i)/\mathbb{R}$ is a smooth manifold of dimension $i - 1$.*

2. *For $\mu(A') - \mu(A) \equiv 1 \ (mod\ 4)$, $\hat{\mathcal{M}}_{ASD}(A, A^{'}, 1)$ is compact and*

therefore the algebraic number of points in $\hat{\mathcal{M}}_{ASD}(A, A^{'}, 1)$, $m(A, A^{'})$, *may be defined.*

3. *For* $\mu(A') - \mu(A) \equiv 2 \ (mod\ 4)$, $\hat{\mathcal{M}}_{ASD}(A, A^{'}, 2)$ *has an oriented compactification given by*

$$\partial \hat{\mathcal{M}}_{ASD}(A, A^{'}, 2) = \bigsqcup_{A^{''}} \hat{\mathcal{M}}_{ASD}(A, A^{''}, 1) \times \hat{\mathcal{M}}_{ASD}(A^{''}, A^{'}, 1),$$

   where the disjoint union is taken over all $A^{''}$*'s such that* $\mu(A^{''}) = \mu(A) - 1$.

4. *Define* $\partial \colon C_i^{(k)}(S^3, K, n) \to C_{i+1}^{(k)}(S^3, K, n)$ *by*

$$\partial(A) = \sum_{\mu(A^{'}) = \mu(A)+1} m(A, A^{'}) A^{'}.$$

   Then by (3) $\partial \circ \partial = 0$ *and so the Floer Homology groups of* (S^3, K, n) *may be defined as the homology groups of the complex* $(C_*^{(k)}(S^3, K, n), \partial)$.

The manifold property in clause (1) is obtained by proving the Fredholm property of the linearised ASD operator $D_{\mathbb{A}}$ and by showing that for $\mathbb{A} \in \mathcal{M}_{ASD}(A, A^{'}, i)$, $ind\ D_{\mathbb{A}} = i$. For the orientability, following [15], we need to show that the associated determinant line bundle is trivial. For clause (2), two distinct features come into play. The first one is present already in finite dimensional Morse Theory, where gradient lines between two critical points can be extended to broken trajectories, when the lines go through intermediate critical points. The second one is that in our gauge theoretic setting, "bubbles" may appear along the cylinder, so care is necessary. To prove the compactness of $\hat{\mathcal{M}}_{ASD}(A, A^{'}, 1)$, we need to exclude this possibility, and do so by adapting a general compactness principle given in [17] and, under another form, in [22]. For (3), to show the compactification property of $\hat{\mathcal{M}}_{ASD}(A, A^{'}, 2)$, the general compactness principle is again applied and, more to the point, a gluing construction for instantons is required. The latter is done using the Taubes construction.

Before we start, we mention two technical points. The first concerns the instanton and monopole numbers we shall be using. Recall that in the case of a closed orbifold, one has the instanton and monopole numbers a and b respectively. In dealing with an adapted bundle over an orbifold with cylindrical ends, things are slightly modified. This modification was explained in Section 3 of [31]. For an adapted bundle $\mathbb{E}_k$ over $(S^3, K, n) \times \mathbb{R}$ with limiting data A and A', the resulting instanton and monopole numbers will be denoted $\bar{a}$ and $\bar{b}$, in order to avoid confusion with the case of a closed orbifold.

The second technical point concerns the Sobolev spaces in which we work. We follow Floer and work with L^4_1-connections and L^4_2-gauge transformations. The independence of the construction with respect to this particular Sobolev setting is given by the following lemmas.

Lemma 3.15. *Given an instanton $\mathbb{A} \in L^p_1(\Omega^1(\mathfrak{g}_{\mathbb{E}_k}))$ such that $F_{\mathbb{A}} \in L^p(\Omega^2(\mathfrak{g}_{\mathbb{E}_k}))$, $(p \geq 2)$, there exist unique flat connections $A_{-\infty}$, A_∞ such that modulo gauge $\mathbb{A} = A_t$ is C^∞ convergent to $A_{-\infty}$ as $t \to -\infty$ and A_∞ as $t \to \infty$.*

Proof. Let B_T be a band of width 1, $(S^3, K, n) \times [T-1, T]$, and denote by $\mathbb{A}_T$ the restriction of $\mathbb{A}$ to B_T. As $\mathbb{A}$ is a finite energy instanton, $\| F_{\mathbb{A}_T} \| \to 0$ as T goes to infinity. By Uhlenbeck's weak compactness for orbifolds (see e.g. [32, Proposition 7.4]), for any sequence $T_i \to \infty$ and a flat connection A over B_T, after gauge transformation, $\mathbb{A}_{T'_i}$ converges to A in the C^∞ sense over compact subsets. In particular $\mathbb{A}_{T-1/2}$ converges in C^∞ to A. By Condition 3.10, A is isolated, and therefore the target flat connection $A_{\pm\infty}$ in the statement above does not depend on the choices of $\{T_i\}$ and its converging subsequence $\{T'_i\}$. q.e.d.

Lemma 3.16. *For $\mathbb{A}$ as in the previous lemma and δ smallest positive eigenvalue of $D_{\mathbb{A}}$, there exists a constant C such that*

$$|F_{y,t}(\mathbb{A})| \leq Ce^{\delta t}.$$

The proof is readily adapted from the manifold case (see Theorem 1 Section 4.1 in [17]) to the orbifold case. It is quite long but now well-known (see e.g. [40], [23]).

Lemma 3.17. *For any $\mathbb{A} \in L^p_1(\Omega^1(\mathfrak{g}_{\mathbb{E}_k}))$ with irreducible flat limits, $D_{\mathbb{A}}$ is Fredholm and has index ind $D_{\mathbb{A}}$.*

Proof. This is a straightforward adaptation of the material used to define the Floer index in Section 3.1. Notice that the result is true even if the flat limits are not irreducible, as long as weighted spaces are introduced. q.e.d.

Now we can proceed to the proof of (1). Exactly as in the standard Floer setting, in the Sobolev completions chosen, $\mathcal{G}(\mathbb{E}_k)$ has a Banach

Lie group structure and acts smoothly on $\mathcal{A}^*(\mathbb{E}_k)$, yielding $\mathcal{B}^*(\mathbb{E}_k) = \mathcal{A}^*(\mathbb{E}_k)/\mathcal{G}(\mathbb{E}_k)$ which has a Banach manifold structure locally modeled on $T_{\mathbb{A},\epsilon} = \{\mathbb{A} + a | d_{\mathbb{A}}^* a = 0, \| a \|_{L_1^p} \leq \epsilon\}$, provided $p > 2$. For manifolds, proofs of such facts are by now very standard, and as $\mathcal{A}(\mathbb{E}_k)$ and $\mathcal{B}(\mathbb{E}_k)$ can be seen in a $\mathbb{Z}_n$-invariant setting of objects defined for the cyclic branched cover $V_n(K)$, the proofs in Section 2 of [22] adapt readily.

Proposition 3.18. *Inside the Banach manifold $\mathcal{B}^*(\mathbb{E}_k)$ lies the instanton moduli space $\mathcal{M}_{ASD}(A, A^{'}, i)$, which is a manifold of virtual dimension ind $D_{\mathbb{A}}$.*

Proof. By Lemma 3.17 and the implicit function theorem, Condition 3.13 implies directly that the irreducible part in $\mathcal{M}_{ASD}(A, A', i)$ is a manifold whose dimension is indeed *ind* $D_{\mathbb{A}}$. The only obstacle is the possible presence of singularities corresponding to reducible instantons in the moduli space $\mathcal{M}_{ASD}(\mathbb{E}_k)$. For such points the local model given by the slices $T_{\mathbb{A},\epsilon}$ is not valid. But given a connection $\mathbb{A}$ on the cylinder whose limits are $A, A^{'}$, by Arondzan's Theorem (see Remark 3.2 in [23]) the following relation among the various stabilizers in the relevant gauge groups holds: $\Gamma_{\mathbb{A}} \subset \Gamma_A \cap \Gamma_{A'}$. So for each element $\mathbb{A} \in \mathcal{M}_{ASD}(A, A^{'}, i)$, $\Gamma_{\mathbb{A}}$ is central in $SU(2)$ by construction and therefore $\mathbb{A}$ is irreducible. q.e.d.

The proof of (1) is completed if $\mathcal{M}_{ASD}(A, A^{'}, i)$ is shown to be oriented. Here no differences arise in the method when we compare to ordinary Floer Homology. One essentially follows the construction given in [15]. The crucial fact to obtain is that the determinant line bundle $det(Ker\ D_{\mathbb{A}})$ be trivial, as this clearly implies the orientability of $\mathcal{M}_{ASD}(A, A^{'}, i)$.

Proposition 3.19. *The determinant line bundle $det(Ker\ D_{\mathbb{A}})$ over $\mathcal{B}(\mathbb{E}_k)$ is trivial.*

Proof. This is proved in Appendix I of [33], and we refer to this article for a detailed discussion. q.e.d.

To prove clause (2) of Theorem 3.14, we first have to derive general compactness properties of the moduli spaces of ASD connections. There is an adequate notion of weak convergence yielding desired compactness properties for the moduli spaces. With appropriate restrictions on the index of critical points, the compactness of $\hat{\mathcal{M}}_{ASD}(A, A^{'}, 1)$ may then be derived. In this we follow the exposition of [17].

First of all we gain some idea about which instantons over the cylinder $(S^3, K, n) \times \mathbb{R}$ belong to the spaces $\mathcal{M}_{ASD}(A, A^{'}, i)$ used to define ∂. The following lemma is a technical result and the proof, again, is a mild adaptation of material found in [17] (Lemma 2 in Section 5.1). Alternatively, one may refer to [40] (Theorem 4.0.1), where much greater generality is achieved.

Lemma 3.20. *Let E_k be a bundle over (S^3, K, n). There are $C, \delta \geq 0$ such that if $\{\mathbb{A}_\alpha\}$ is a sequence of ASD connections on $\mathbb{E}_k$ over $(S^3, K, n) \times [0, \infty)$ with*

$$\int_0^\infty |F_{\mathbb{A}_\alpha}|^2 \leq C,$$

then there is a subsequence $\{\mathbb{A}_{\alpha'}\}$, a flat connection A and an instanton $\mathbb{A} = A + \mathfrak{a}$ over the half-cylinder such that, up to gauge equivalence, $\mathbb{A}_{\alpha'} = A + \mathfrak{a}_{\alpha'}$ and for each $l \geq 0$ and $h, p > 0$

$$\int_h^\infty |\nabla_{\mathbb{A}}^{(l)}(\mathfrak{a}_{\alpha'} - \mathfrak{a})|^p e^{p\delta t} \to 0$$

as $\alpha^{'} \to \infty$.

Corollary 3.21. *The sequence of instantons $\{\mathbb{A}_\alpha\}$ converges in L_l^p, for all l, to a limit $\mathbb{A}$ if and only if $\{\mathbb{A}_\alpha\}$ converges on compact subsets and*

$$\int_h^\infty |F_{\mathbb{A}_\alpha}|^2 \to \int_h^\infty |F_{\mathbb{A}}|^2.$$

Proof. One way is easy: from convergence in L_l^p, in particular one has convergence on compact subsets and also

$$\int_h^\infty |F_{\mathbb{A}_\alpha}|^2 - \int_h^\infty |F_{\mathbb{A}}|^2 \leq \int_h^\infty |F_{\mathbb{A}_\alpha} - F_{\mathbb{A}}|^2 \to 0.$$

The converse statement follows from the previous lemma. q.e.d.

If we now consider moduli spaces on the whole cylinder, the previous results give that L_l^p convergence and convergence on compact subsets are equivalent:

Corollary 3.22. *The sequence $\{\mathbb{A}_\alpha\}$ on $\mathbb{E}_k$ over $(S^3, K, n) \times \mathbb{R}$ converges to $\mathbb{A}$ in the ASD moduli space of $\mathbb{E}_k$ if and only if $\{\mathbb{A}_\alpha\}$ converges to $\mathbb{A}$ in C^∞ on compact subsets of $(S^3, K, n) \times \mathbb{R}$.*

Proof. By elliptic regularity of the ASD equation, convergence in L_l^p gives C^∞ convergence on compact subsets. Conversely, by Chern-Weil applied to $\{\mathbb{A}_\alpha\}$ converging in the C^∞ sense to $\mathbb{A}$ on compact subsets,

$$\int_{(S^3,K,n)\times\mathbb{R}} |F_{\mathbb{A}_\alpha}|^2 = \int_{(S^3,K,n)\times\mathbb{R}} |F_{\mathbb{A}}|^2$$

and, using the previous corollary, get convergence in the L^p_l sense. q.e.d.

On the orbifold cylinder $(S^3, K, n) \times \mathbb{R}$, we can define translations

$$c_T : (S^3, K, n) \times \mathbb{R} \to (S^3, K, n) \times \mathbb{R}$$

by $c_T(x,t) = (x, t+T)$. These maps pullback on the level of forms and connections and we will be interested in the effect on the level of connections: $c_T^*(\mathbb{A})$. A *translation vector* $\vec{T}$ is a sequence of real numbers

$$T(1) \leq T(2) \leq ... \leq T(N)$$

for some integer N. For a sequence $\{\mathbb{A}_\alpha\}$ on an adapted bundle $\mathbb{E}_k$ and a sequence $\vec{T}_\alpha$ of translation vectors with $\vec{T}_\alpha(i) - \vec{T}_\alpha(i-1) \to \infty$ as $\alpha \to \infty$, we say that $\mathbb{A}_\alpha$ is $\vec{T}_\alpha$*-convergent* to connections $\mathbb{A}(1), \mathbb{A}(2), ..., \mathbb{A}(N)$ if each of the translates $c^*_{T_\alpha(i)}(\mathbb{A}_\alpha)$ converges on compact subsets to $\mathbb{A}(i)$. It should be pointed out that, because of bubbling, the ASD connections $\mathbb{A}(1), ..., \mathbb{A}(N)$ over $(S^3, K, n) \times \mathbb{R}$ may live on adapted bundles different from $\mathbb{E}_k$. However, by virtue of the flat limits providing the limiting data of these bundles, they must have the same isotropy as $\mathbb{E}_k$. Let $A_{-\infty}$, $A_{+\infty}$ be limiting flat connections on the adapted bundle $\mathbb{E}_k$. A *chain of bundles* $\vec{\mathbb{E}} = (\mathbb{E}(1), ..., \mathbb{E}(N))$ from $A_{-\infty}$ to $A_{+\infty}$ is a sequence of adapted bundles with limiting flat connections $A_-(i)$, $A_+(i)$ such that

$$A_{-\infty} = A_-(1), A_+(1) = A_-(2), ..., A_+(N-1) = A_-(N), A_+(N) = A_{+\infty}.$$

For such a chain $\vec{\mathbb{E}}$, we can define the instanton and monopole numbers to be

$$\bar{a}(\vec{\mathbb{E}}) = \sum_{i=1}^{N} \bar{a}(\mathbb{E}(i)),$$

$$\bar{b}(\vec{\mathbb{E}}) = \sum_{i=1}^{N} \bar{b}(\mathbb{E}(i)).$$

A sequence $\{\mathbb{A}_\alpha\}$ is said to be *chain convergent* if there is a sequence $\vec{T}_\alpha$ of translation vectors, a chain of bundles $\mathbb{E}(1), ..., \mathbb{E}(N)$ as above, and connections $\mathbb{A}(i)$ on $\mathbb{E}(i)$ such that $\{\mathbb{A}_\alpha\}$ is $\vec{T}_\alpha$-convergent to $\mathbb{A}(1), ..., \mathbb{A}(N)$. Similarly, *weak chain convergence* may be defined. The important feature is that there is a notion of convergence for any sequence. This is given by the following, which like the ideas is adapted from [17]:

Theorem 3.23. *Any sequence $\{\mathbb{A}_\alpha\}$ of connections on $\mathbb{E}_k$ has a weak chain convergent subsequence. If the limit $\vec{E}$ has $\bar{a}(\vec{E}) = \bar{a}(\mathbb{E}_k)$ and $\bar{b}(\vec{E}) = \bar{b}(\mathbb{E}_k)$, then the subsequence is chain convergent.*

Proof. The first thing to do is to find a positive "minimal energy number" which would be analogous to the energy of a "single instanton" in Floer's setting (see Proposition 3b.2 in [22]). By the Chern-Weil formula, a non-flat instanton must have a strictly positive instanton number or monopole number (and maybe both). As there are two invariants in our setting, there is no "single instanton", but in any case the energy of such an instanton is at least $8\pi^2(\frac{2k}{n})$ again by Chern-Weil (recall that $0 < 2k/n < 1$). This will be the "minimal energy number".

It may be supposed that any non-flat instanton over the cylinder $(S^3, K, n) \times \mathbb{R}$ has energy greater than C and that $C < 8\pi^2(\frac{2k}{n})$, where C is as in Lemma 3.20. We wish to construct a limiting chain for a sequence $\{\mathbb{A}_\alpha\}$. As the energy of $\mathbb{A}_\alpha$ is greater than C, there is a unique $T_\alpha = T_\alpha(1)$ such that

$$\int_{-\infty}^{T_\alpha} |F_{\mathbb{A}_\alpha}|^2 = C.$$

By replacing the $\mathbb{A}_\alpha$ by their translates, one may suppose that $T_\alpha = 0$. We know that we have weak convergence on compact subsets of the cylinder. Say that $\mathbb{A}_\alpha \to \mathbb{A}(1)$ on compact subsets, for $\mathbb{A}(1)$ on a bundle $\mathbb{E}(1)$. What is needed is to verify that no energy is lost at the ends of the cylinder (outside the compact subsets) when we do the limiting process. This is given by the following equality:

Lemma 3.24. *For any $h > 0$,*

$$\lim_{\alpha \to \infty} \int_{-\infty}^{-h} |F_{\mathbb{A}_\alpha}|^2 = \int_{-\infty}^{-h} |F_{\mathbb{A}(1)}|^2.$$

Proof. As $C < 8\pi^2(\frac{2k}{n})$, so the $\mathbb{A}_\alpha$ are converging strongly over

$(-\infty, -h)$. Now suppose that the equality to be found does not hold:

$$C^* = \lim_{\alpha\to\infty} \int_{-\infty}^{-h} |F_{\mathbb{A}_\alpha}|^2 - \int_{-\infty}^{-h} |F_{\mathbb{A}(1)}|^2 \neq 0.$$

First $C^* \leq C$ by definition of C, and on the other hand by Lemma 3.16 on exponential decay along the tube, C^* has to be positive. Define $S_\alpha \leq -h$ by

$$\int_{-\infty}^{S_\alpha} |F_{\mathbb{A}_\alpha}|^2 = C^*.$$

By strong convergence on compact subsets of $\mathbb{A}_\alpha$, one has $S_\alpha \to -\infty$. Let $\mathbb{A}'_\alpha$ be the translates of $\mathbb{A}_\alpha$ by S_α. Possibly after extracting a subsequence, $\{\mathbb{A}'_\alpha\}$ converges. But then the limit is an instanton with energy less or equal to $C^* \leq C$, which contradicts the minimality of C. q.e.d.

Now Corollary 3.21 may be applied, to get that $\{\mathbb{A}_\alpha\}$ converges in L^p_l over the negative end. Also it follows that the limiting flat connection for $\mathbb{A}(1)$ on $\mathbb{E}(1)$ is $A_{-\infty}$. If the equality in Lemma 3.24 extends to the whole cylinder, the other limiting flat connection for $\mathbb{A}(1)$ is $A_{+\infty}$. In that situation $\{\mathbb{A}_\alpha\}$ is chain convergent to a chain with only one term, $\mathbb{E}(1)$. The convergence is strong if moreover the terms in the Chern-Weil formula agree. Using the additivity of Chern classes under gluing of connections, this equality means that $\bar{a}(\mathbb{E}(1)) = \bar{a}(\mathbb{E}_k)$ and $\bar{b}(\mathbb{E}(1)) = \bar{b}(\mathbb{E}_k)$. That is, the convergence is strong if and only if $\mathbb{E}(1)$ is isomorphic to $\mathbb{E}_k$.

If the equality does not extend, suppose that

$$\bar{a}(\mathbb{E}_k) = \bar{a}(\mathbb{E}(1)) + \nu,$$

$$\bar{b}(\mathbb{E}_k) = \bar{b}(\mathbb{E}(1)) + \eta,$$

where $\nu + \frac{2k}{n}\eta$ is positive. Let $C' = min(C, \nu + \frac{2k}{n}\eta)$ and define $T_\alpha(2)$ by

$$\int_{-\infty}^{T_\alpha(2)} |F_{\mathbb{A}_\alpha}|^2 = C'.$$

In fact it turns out that $C' = C$ in the above. With $T_\alpha(2)$ define $\mathbb{A}_\alpha(2) = c^*_{T_\alpha(2)}(\mathbb{A}_\alpha)$. As above, $T_\alpha(2) \to +\infty$ as $\alpha \to +\infty$ and therefore taking a subsequence, $\{\mathbb{A}_\alpha(2)\}$ converges weakly on compact subsets to $\mathbb{A}(2)$ on a $\mathbb{E}(2)$, with $\bar{a}(\mathbb{E}_2) \leq \nu$ and $\bar{b}(\mathbb{E}(2)) \leq \eta$.

The process may be finished at this stage, if the energies of $\mathbb{A}(1)$ and $\mathbb{A}(2)$ use up all the energy of the $\mathbb{A}_\alpha$. Else there is a remainder which is larger than C. Iterating the process, after a finite number of steps, all the energy must be used up, as at each stage at least C units are lost, and the energy along the cylinder is bounded by definition.

The only thing left to verify is that one has indeed a chain as required. For this it suffices to show that $A_+(1)$ of $\mathbb{E}(1)$ is equal to $A_-(2)$ of $\mathbb{E}(2)$. By Lemma 3.16 and elliptic regularity, there is a $C'' > 0$ such that for any $T > 1$, any instanton over $(-T, T)$ with energy less than C'' may be represented as $A + \mathfrak{a}$ where A is a flat connection and $\mathfrak{a}$ satisfies the exponential decay conditions $|\nabla^{(l)}\mathfrak{a}| \leq C_l e^{-\delta(t-T)}$ for $|t| \leq T-1$. Then, as A is isolated, the exponential convergence of $\mathbb{A}_{\alpha,t}$ and $\mathbb{A}_{\alpha,-t}$ is towards the same connection; namely A above. Therefore

$$A_+(1) = A = A_-(2)$$

as required. q.e.d.

To prove compactness of $\hat{\mathcal{M}}_{ASD}(A, A', 1) = \mathcal{M}_{ASD}(A, A', 1)/\mathbb{R}$ and therefore complete the proof of clause (2) in Theorem 3.14, what we need is to strengthen Theorem 3.23, by restricting the index appropriately.

Theorem 3.25. *If $\mu(A) - \mu(A') \leq 4$, any sequence $\{\mathbb{A}_\alpha\}$ in $\mathcal{M}_{ASD}(A, A', i)$ has a chain convergent subsequence. Moreover if $\mu(A) - \mu(A') \leq 2$, then $\{\mathbb{A}_\alpha\}$ has a subsequence converging on a chain of length smaller or equal to 2 and with irreducible flat limits $A_-(j)$, $A_+(j)$.*

Proof. For the first part, all that is left to do is to show that bubbling off may be avoided. Applying the index formula to a chain of length N, one has

$$\begin{aligned}\mu(A) - \mu(A') = &\sum_{i=1}^{N} \mu(A_+(i)) - \mu(A_-(i)) \\ &+ \sum_{i=1}^{N} dim\ I_{A_\pm}(i) + 8\sum_{j=1}^{J} a_j + 4\sum_{j=1}^{J} b_j,\end{aligned}$$

where a_j, b_j are instanton and monopole numbers coming from J concentrated instantons on (S^4, S^2) arising from bubbling off. There is some information about these: $a_j \geq 0$ and $a_j + b_j \geq 0$ for $j = 1, ..., J$. It follows that

$$4\sum_{j=1}^{J}\{2a_j + b_j\} \geq 4\sum_{j=1}^{J} a_j$$

and hence by the index formula, if the index difference is less than or equal to 4, get $J = 0$, so that $\{\mathbb{A}_\alpha\}$ is not only weakly chain convergent, but chain convergent.

If, moreover, the index difference is less than or equal to 2, the index formula says that there is at most one intermediate flat limit $A(1)$ between $A_{-\infty}$ and $A_{+\infty}$ and the isotropy group $I_{A(1)}$ has to have dimension zero, implying the irreducibility of $A(1)$. And if the index difference is 1, then there is none. q.e.d.

In order to prove (3) in Theorem 3.14, another technical ingredient is needed: Taubes' gluing construction. This result is essential in defining the compactification of $\mathcal{M}_{ASD}(A, A^{'}, 2)$, using moduli spaces $\mathcal{M}_{ASD}(A, A^{''}, 1)$ and $\mathcal{M}_{ASD}(A^{''}, A^{'}, 1)$. We will deal mostly with points specific to Floer Homology, leaving out of the presentation some technical aspects which are well documented.

The first aspect to consider is the construction of an ASD connection $\mathbb{A}^\sharp$ over an orbifold connected sum $(X_1\sharp_T X_2, \Sigma_1\sharp_T\Sigma_2, n)$ from ASD connections $\mathbb{A}_i$ over the orbifolds with cylindrical ends (X_i, Σ_i, n). The connected sum is made with a cylinder of length T in the middle, for some large T, as discussed in [16], [17] or [23]. The solution involves three distinct steps which we outline. First estimates are used to show the existence of a connection $\mathbb{A}_0$ with small $\| F^+_{\mathbb{A}_0} \|$ obtained by modifying $\mathbb{A}_1$ and $\mathbb{A}_2$ over the gluing region. Then one shows that for T large enough, the linearised ASD operator is invertible. Finally, the contraction mapping principle for Banach spaces is used to find a solution. The relevant result concerning this is Theorem 7.2.24 in [16] and the case of cylinders is more straightforward, as seen in [22] and [17]; for orbifolds the same estimates hold, as does the contraction mapping principle.

A slight generalisation needed is the case where we have compact subsets $\mathcal{K}_i \subset \mathcal{M}_{ASD}(\mathbb{E}_i)$. Applying the construction above to each $\mathbb{A}_i \in \mathcal{K}_i$, for $T > 0$ large, there is a map

$$\tau_T\colon \mathcal{K}_1 \times \mathcal{K}_2 \to \mathcal{M}_{ASD}(\mathbb{E}^\sharp),$$

where $\mathbb{E}^\sharp$ is a bundle constructed over the connected sum. An important property of the map τ is the following.

Proposition 3.26. *If $T >> 0$, τ_T is a diffeomorphism to its image and the image consists of regular points in $\mathcal{M}_{ASD}(\mathbb{E}^\sharp)$.*

Proof. First show that the derivative $d\tau_T$ is an isomorphism if $T >> 0$. By the additivity of the index, it suffices to prove that $d\tau_T$ is injective. By definition of τ_T, one may restrict the problem to compact subsets of the (X_i, Σ_i, n) (around the gluing region), and then the result follows from the discussion in Section 7.2 of [16].

For regularity, $\mathbb{A}_0$, has surjective $d^+_{\mathbb{A}_0}$ with right inverse Q of bounded L^2-operator norm. Hence as $\tau_T(\mathbb{A}_1, \mathbb{A}_2) = \mathbb{A} = \mathbb{A}_0 + \mathfrak{a}$, with $\mathfrak{a}$ decaying exponentially with respect to T, the L^2-operator norm of $d^+_{\mathbb{A}} Q - I$ can be made as small as desired so long as T is large enough. Consequently $d^+_{\mathbb{A}}$ is surjective and $\mathbb{A}$ is regular, for $T >> 0$. q.e.d.

One still needs to show a result of the following kind: there is a constant $\epsilon > 0$ such that for $T >> 0$, any instanton $\mathbb{A} = \mathbb{A}_0 + \mathfrak{a}$ with $\| \mathfrak{a} \| < \epsilon$, over a connected sum is gauge equivalent to a $\tau_T(\mathbb{B}_1, \mathbb{B}_2)$, for $\mathbb{B}_i$ close to $\mathbb{A}_i$ in $\mathcal{M}_{ASD}(\mathbb{E}_i)$. This is proved in a standard way, using the method of continuity.

We still need to make more precise the notion of distance between $\mathbb{A}^\sharp$ and the $\mathbb{A}_i$. For our purpose, we have used the distance

$$d(\mathbb{A}^\sharp, (\mathbb{A}_i)) = \inf_{g \in \mathcal{G}(\mathbb{E}^\sharp)} (\| g \cdot (\mathbb{A}^\sharp - \mathbb{A}_0) \|_{L^4} + \| d^+_{\mathbb{A}_0}(g(\mathbb{A}^\sharp - \mathbb{A}_0)) \|_{L^2}).$$

Another notion of distance can be obtained by restricting our attention to the behaviour of connections on compact subsets $\mathcal{C}_i$:

$$d_{\mathcal{C}}(\mathbb{A}^\sharp, (\mathbb{A}_i)) = \sum_{i=1}^{2} \inf_{g \in \mathcal{G}(\mathbb{E}^\sharp)} \| g \cdot \mathbb{A}^\sharp - \mathbb{A}_i \|_{L^4(\mathcal{C}_i)} .$$

Proposition 3.27. *Let $\mathbb{A}_i$ be instantons over (X_i, Σ_i, n) as above. Then there are compact subsets $\mathcal{C}_i \subset (X_i, \Sigma_i, n)$ and constants δ, $C \geq 0$ such that for any instanton $\mathbb{A}^\sharp$ over $(X_1 \sharp_T X_2, \Sigma_1 \sharp_T \Sigma_2, n)$ for $T > 0$ large which satisfies $d_{\mathcal{C}}(\mathbb{A}^\sharp, (\mathbb{A}_i)) \leq \delta$, then*

$$d(\mathbb{A}^\sharp, (\mathbb{A}_i)) \leq C d_{\mathcal{C}}(\mathbb{A}^\sharp, (\mathbb{A}_i)).$$

Clearly $d_{\mathcal{C}}(\mathbb{A}^\sharp, (\mathbb{A}_i))$ is controlled by $d(\mathbb{A}^\sharp, (\mathbb{A}_i))$. One has a partial converse using exponential decay; see Lemma 3.16. To do this, one chooses $\mathcal{C}_i$ and δ such that any instanton satisfying $d_{\mathcal{C}}(\mathbb{A}^\sharp, \mathbb{A}_i) \leq \delta$ has very small energy on $(X_1 \sharp_T X_2, \Sigma_1 \sharp_T \Sigma_2, n) - \mathcal{C}_1 \cup \mathcal{C}_2$, and show that on the "neck" $(S^3, K, n) \times (-T/2, T/2) - \mathcal{C}_1 \cup \mathcal{C}_2$ one has exponential decay $|F_{A_t^\sharp}| \leq C' e^{-\delta(T-|t|)}$ (see [17] and also Proposition 2.d in [22]).

Then what was shown is that for $\epsilon > 0$ small enough, $T >> 0$ and $\mathcal{K}_i$ compact subsets of regular elements in $\mathcal{M}_{ASD}(\mathbb{E}_i)$, there are neighbourhoods $\mathcal{V}_i \subset \mathcal{K}_i$ and suitable compact subsets $\mathcal{C}_i$ of (X_i, Σ_i, n) for which

$$\tau_T \colon \mathcal{V}_1 \times \mathcal{V}_2 \to \mathcal{M}_{ASD}(\mathbb{E}^\sharp)$$

is a diffeomorphism to its image and consists of regular points with the following properties:

- $d_{\mathcal{C}}(\tau_T(\mathbb{A}_1, \mathbb{A}_2), (\mathbb{A}_i)) \leq \epsilon$ for all $\mathbb{A}_i$ in $\mathcal{V}_i$,
- any instanton $\mathbb{A}^\sharp$ with $d_{\mathcal{C}}(\mathbb{A}^\sharp, (\mathbb{A}_i)) \leq \epsilon$ for some $\mathbb{A}_i \in \mathcal{K}_i$ lies in the image $\tau_T(\mathcal{V}_1 \times \mathcal{V}_2)$.

We now have all the tools to complete the proof of (3) in Theorem 3.14. In general the moduli space $\hat{\mathcal{M}}_{ASD}(A, A', 2)$ is not compact. However, the ends of this moduli space may be identified with pairs $(\mathbb{A}(1), \mathbb{A}(2))$ on a 2-term chain, by Theorem 3.23. And conversely, given a pair $(\mathbb{A}(1), \mathbb{A}(2))$ by the gluing construction above, one can find an open subset $\mathcal{N}_T$ of $\hat{\mathcal{M}}_{ASD}(A, A', 2)$ with a single parameter $T >> 0$ such that any potentially divergent sequence in $\hat{\mathcal{M}}_{ASD}(A, A', 2)$ lies in $\mathcal{N}_T$. This gives the compactification that was needed.

Thus what was achieved in this section is the construction of the 3-orbifold Floer groups $HF_i^{(k)}(S^3, K, n)$ for $i = 0, 1, 2, 3$ associated to a knot K in S^3 and depending on a rational in $(0, 1/2)$. Up to now, nothing in our treatment allows us to claim that the Floer Homology is a knot invariant, as it may depend on the metric used on (S^3, K, n), upon which depends the construction of the Floer Homology. The invariance with respect to the metric will be proved in Section 4.3, when we discuss the functoriality property of $HF_*^{(k)}(S^3, K, n)$.

3.3 Perturbations

The goal of this section is to define an appropriate perturbation set-up to yield a well-defined Floer Homology for any knot in S^3. We will first perturb in order to remove degenerate elements in $\mathcal{M}_{flat}(E_k)$ and make sure that we have isolated points, to define the chain groups. Secondly, we will see how to use perturbations for the instanton equation to achieve the transversality for the definition of the boundary operator. This will remove the need for Conditions 3.10 and 3.13 in our construction. A discussion on the independence of the Floer Homology with respect to a generic perturbation concludes this section.

Before describing the construction, we make a digression. In the original case treated by Floer, when Y^3 is a homology sphere, the perturbations used to define $HF_*(Y^3)$ did not really cause problems. A similar picture emerged from Taubes' approach to Casson's invariant for homology spheres.

On the other hand, when considering rational homology spheres, the picture is much more subtle. This was explained by Walker in [47], where an extension of Casson's invariant was given. Here, one has to take into account non-trivial reducibles; as a result, this gives rise to Walker's correction term, to account for the dependence upon the perturbations used. In [35], Lee and Li have extended Floer's work to define the Floer Homology of 3-manifolds which are rational homology spheres, and the resulting Floer Homology depends on perturbations through Walker's correction term.

The type of perturbations we will use were introduced by Herald in his work on Chern-Simons theory for 3-manifolds with boundary (see [26]). These are, with minor modifications, those used by Floer. Our task is to extend their approach to the setting of the 3-orbifolds (S^3, K, n). First we describe Herald's construction and summarize the most important properties of the perturbed moduli space.

Consider an $SU(2)$-bundle E over a knot complement $S^3 - N_K$ and write $Y = S^3 - N_K$. Take a collection $\{\gamma_i\colon S^1 \times D^2 \to Y\}_{1\leq i\leq n}$ of embeddings of the solid torus in Y whose images are disjoint and also disjoint from the boundary torus of Y. Let $\eta\colon D^2 \to \mathbb{R}$ be a bump function on D^2 and define a function $h\colon \mathcal{A}(Y) \to \mathbb{R}$ by

$$h(A) = \sum_{i=1}^{n} \int_{D^2} h_i(tr Hol_A(\gamma_i(S^1 \times \{x\})))\eta(x)dx^2,$$

where $\{h_i\colon \mathbb{R} \to \mathbb{R}\}_{1\leq i\leq n}$ is a collection of smooth functions. The function h is called an *admissible perturbation function*. Given an admissible perturbation function h, the *perturbed flat moduli space* of Y is

$$\mathcal{M}_h(Y) = \{A \in \mathcal{B}(Y) \mid -\frac{1}{2\pi} * F_A + \nabla h(A) = 0\}.$$

This moduli space is then the critical point set of the functional $CS + h$. The following theorem gives the main properties of $\mathcal{M}_h(Y)$ which will be needed.

Theorem 3.28 ([26, Theorem 15 (c)]). *For a generic admissible perturbation function h, $\mathcal{M}_h(Y)$ is a compact stratified space*

$$\mathcal{M}_h(Y) = \mathcal{M}_h^{SU(2)} \bigsqcup \mathcal{M}_h^{U(1)} \bigsqcup \mathcal{M}_h^*,$$

where $\mathcal{M}_h^{SU(2)}$ (the central perturbed flat connections) consists of two points. $\mathcal{M}_h^{U(1)}$ (the reducible perturbed flat connections) is a smooth compact 1-manifold except for two ends limiting to $\mathcal{M}_h^{SU(2)}$. $\mathcal{M}_h^*$ (the irreducible perturbed flat connections) is an oriented 1-manifold which is compact, except for open ends which limit to points in $\mathcal{M}_h^{U(1)}$.

Let us now go to the orbifold setting. Given an $SU(2)$-bundle E_k over (S^3, K, n), it was already seen that $\mathcal{M}_{flat}(E_k)$ embeds in $\mathcal{M}_{flat}(Y)$ (in fact $\mathcal{B}(E_k) \hookrightarrow \mathcal{B}(Y)$). There is a natural way to adapt Herald's perturbations to the orbifold setting. The important observation is that Herald perturbs the flatness equation along solid tori embedded away from the boundary of Y. When one considers the orbifold case, the only remaining operation is to glue in an orbifold solid torus. An admissible perturbation function can be defined for the orbifold as previously, by using a collection of solid tori embedded in (S^3, K, n), away from the orbifold solid torus. This last condition is required in order to keep a direct relation to Herald's perturbations. Alternatively, one may consider the perturbed moduli space $\mathcal{M}_h(E_k)$ as the subset of elements $A \in \mathcal{M}_h(Y)$ such that $trHol_A(\mu) = 2cos(k/n2\pi)$. From this and Herald's theorem, one may conclude the following about a functional $CS + h$.

Proposition 3.29. *For a generic admissible perturbation function h, the set $\mathcal{M}_h^*(E_k)$ of irreducible critical points of $CS + h$ consists of finitely many points.*

Let us concentrate now on the reducible flat connection θ_k on E_k over (S^3, K, n). For this, first recall the deformation theory along the reducible arc in $\mathcal{M}_{flat}(Y)$. This arc is parameterised by the holonomy of the connection along a meridian $\mu \in \pi_1(Y)$. Up to gauge equivalence $Hol_A(\mu) \subset S^1 \subset SU(2)$. Let $\xi \in S^1$: $\xi = e^{i\alpha 2\pi}$. The *ξ-equivariant signature* of K, denoted $\sigma_\alpha(K)$, is the signature of the matrix $B_K(\xi) = (1-\xi)A_K + (1-\bar{\xi})A_K^T$, where A_K is an Alexander matrix for K and A_K^T its transpose. Also the *Alexander polynomial* of K is $\Delta_K(t) = det(A_K^T - tA_K)$.

Proposition 3.30 ([30, Theorem 19]). *Let A be a reducible connec-*

tion in $\mathcal{M}_{flat}(Y)$*. If* A *satisfies* $\Delta_K(Hol_A(\mu)^2) \neq 0$*, then* $H^1_A(Y) \simeq \mathbb{R}$*. Else, if* $\Delta_K(Hol_A(\mu)^2) = 0$*, then* $H^1_A(Y)$ *is such that*

$$dim\ H^1_A(Y) = 1 + 2dimKer\ B_K(Hol_A(\mu)^2).$$

Notice that this implies in particular that if a reducible connection A_0 is a limit point of some irreducible component of $\mathcal{M}^*_{flat}(Y)$, then $\Delta_K(Hol_{A_0}(\mu)^2) \neq 0$. A reducible flat connection, A, for which $Hol_A(\mu^2)$ is a root of the Alexander polynomial $\Delta_K(t)$ is called *exceptional*. And if an element $\beta \in S^1$ satisfies $\Delta_K(\beta^2) = 0$ we shall say, somewhat loosely, that β is a square root of the Alexander polynomial on S^1. We have the following obvious proposition.

Proposition 3.31. *Along the abelian arc in* $\mathcal{M}_{flat}(Y)$*, there are only finitely many exceptional connections.*

Therefore, for θ_k on E_k over (S^3, K, n), one has:

Proposition 3.32. *If* $\Delta_K(e^{i\frac{2k}{n}2\pi}) \neq 0$*,* $\theta_k \in \mathcal{M}_{flat}(E_k)$ *is non-degenerate and isolated. Otherwise,* θ_k *is an exceptional connection.*

A bundle E_k on which θ_k is exceptional we shall call an *exceptional bundle* for (S^3, K, n). We concentrate on the independence of perturbations of $HF^{(k)}_*(S^3, K, n)$ for non-exceptional bundles. In this case θ_k is isolated and non-degenerate and we are in a situation like that of Floer Homology for integral homology spheres: the critical point set may be perturbed compactly away from the reducible.

It is important to know whether there exists a pair (k, n) such that for all knots $K \hookrightarrow S^3$, E_k over (S^3, K, n) is not an exceptional bundle. For any knot K, the Alexander polynomial $\Delta_K(t)$ has the property that $\Delta_K(-1)$ is an odd integer. This implies the following:

Proposition 3.33. *For any knot* K *in* S^3*, the orbifold bundle* E_1 *over the 3-orbifold* $(S^3, K, 4)$ *is not an exceptional bundle.*

Proof. The reducible connection θ_1 on E_1 is such that

$$Hol_{\theta_1}(\mu) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

so $\Delta_K(e^{i\pi}) = \Delta_K(-1) \neq 0$ and therefore θ_1 is not exceptional and hence neither is E_1. q.e.d.

Thus for the pair $(1, 4)$, the Floer Homology $HF^{(1)}_*(S^3, K, 4)$ can always be defined and will be shown to be independent of perturbations.

More generally, the same will be true for a relatively prime pair (k, p^n), for any p prime and any n.

We now perturb so as to achieve the transversality of the instanton moduli spaces which we have imposed to state Theorem 3.14 (Condition 3.13). Using the interpretation of the instanton equation on the cylinder as the gradient flow of the Chern-Simons functional, we may perturb this instanton moduli space by simply perturbing the functional CS.

Given an admissible perturbation h, for any $A, A' \in \mathcal{M}_h(E_k)$, let the perturbed instanton moduli space connecting corresponding to $\mathcal{M}_{ASD}(A, A', i)$ be

$$\mathcal{M}_h(A, A', i) = \{\mathbb{A} = A_t \mid \frac{dA_t}{dt} + *F_{A_t} + \nabla h(A_t) = 0 \ , \ YM(\mathbb{A}) < \infty\},$$

in analogy with what was done in Section 3.2. Say that $\mathcal{M}_h(A, A', i)$ is *regular* if $D_{\mathbb{A}}$ is surjective for all $\mathbb{A} \in \mathcal{M}_h(A, A', i)$.

Now fix a metric on (S^3, K, n) (and hence on the cylinder as we take the product metric there) and an admissible perturbation h_0. We have to produce "enough" perturbations so that around h_0, a generic perturbation h will yield a regular moduli space $\mathcal{M}_h(A, A', i)$, for all $A, A' \in \mathcal{M}_h(E_k)$. As Herald's perturbations are essentially the ones used by Floer, we can adapt Floer's argument.

Proposition 3.34. *Given a fixed metric on (S^3, K, n) and an admissible perturbation h_0, the set of admissible perturbations for which $\mathcal{M}_h(A, A', i)$ is regular for all $A, A' \in \mathcal{M}_h(E_k)$ is of first Baire category near h_0.*

Proof. The proof follows that of Proposition 2c.2 in [22]. q.e.d.

Remark 3.35. The only point that could be unclear in the adaptation of Floer's proof in the orbifold setting is the use of the Sard-Smale Theorem to achieve transversality. The important observation is that in the context of 3-orbifolds, as explained in Section 2.3, one deals with connections which are $\mathbb{Z}_n$-invariant when seen on an equivariant bundle over the cyclic branched cover. This is a subspace of the equivariant space studied in [25] (see also Remark 2.12 in [25]). We do not require a $\mathbb{Z}_n$-manifold structure, but the weaker structure of an ordinary manifold. A similar situation occurred under different forms in [20], [32] and [9]. There is no transversality obstruction in this context, and in particular the Sard-Smale Theorem applies.

We summarize the results of this section, in the following theorem.

Theorem 3.36. *Given any knot K in S^3, there is a well-defined 3-orbifold Floer Homology $HF_*^{(k)}(S^3, K, n)$ for any pair (k, n), except for those pairs for which θ_k yields a square root of $\Delta_K(t)$. The Floer Homology is independent of perturbations when their use is required.*

Proof. Given K in S^3, choose a non-exceptional bundle E_k over a (S^3, K, n). On E_k, the reducible θ_k is isolated and non-degenerate. Then

$$\mathcal{M}_{flat}(E_k) = \mathcal{M}^*_{flat}(E_k) \bigsqcup \{\theta_k\},$$

and one needs only to worry about $\mathcal{M}_{flat}(E_k)$. Using Herald's perturbations adapted to our setting, for a generic admissible perturbation function h,

$$\mathcal{M}_h(E_k) = \mathcal{M}^*_h(E_k) \bigsqcup \{\theta_k\},$$

where $\mathcal{M}^*_h(E_k)$ is finite. Each $A \in \mathcal{M}^*_h(E_k)$ has a well-defined index $\mu(A) \in \mathbb{Z}_4$. Define the chain groups $C^{(k)}_{[h],*}(S^3, K, n)$ and the boundary operator as was previously done, possibly using perturbations to have regular instantons. The result is a Floer Homology $HF^{(k)}_{[h],*}(S^3, K, n)$. Floer's cobordism argument will then be applied (in Section 4.3) to obtain independence of the construction with respect to the generic perturbation function and metric used. q.e.d.

4. Properties of Floer homology

We start this section by giving some elementary properties of the Floer Homology developed in the last section, in particular noting how it generalises Floer's original construction. Then we develop a local deformation theory for the Floer Homology which enables us to generalise our construction from orbifold connections to singular connections with irrational holonomy parameter. Next, the functoriality of Floer Homology under orbifold cobordism is studied as, in particular, this is formally needed to prove invariance under change of metrics and perturbations. Finally, the Euler characteristic of the Floer Homology is related to a classical knot invariant, the equivariant signature of a knot.

4.1 Basic properties

The first example, a rather trivial one, is the unknot. Here there are no irreducible flat connections and is one reducible flat connection, θ_k, which is non-degenerate. Therefore the Floer Homology $HF_*^{(k)}(S^3, K, n)$ is always trivial. Similarly, we have

Proposition 4.1. *For any knot K in S^3, $HF_*^{(0)}(S^3, K, n) \equiv 0$; if n is even $HF_*^{(\frac{n}{2})}(S^3, K, n) \equiv 0$, and more generally there is an $\epsilon_K > 0$ such that if $0 < k/n < \epsilon_K$ or $1/2 - \epsilon_K < k/n < 1/2$, then $HF_*^{(k)}(S^3, K, n) \equiv 0$.*

Proof. As above, there are no irreducible flat $SU(2)$-connections in the case $k = 0$ or $k = n/2$. Using Theorem 3.31, it follows that $r(\mathcal{M}^*_{flat}(S^3 - N_K)) \cap S_{\frac{k}{n}} = \varnothing$ for k/n close enough to 0 or 1/2. q.e.d.

Up to now, we have focused on knots in S^3, but more generally we may consider knots $K \hookrightarrow Y^3$ in a homology sphere Y^3. There is no difficulty in developing a Floer Homology for 3-orbifolds (Y^3, K, n), as we have done for (S^3, K, n). Denote this Floer Homology $HF_*^{(k)}(Y^3, K, n)$. This Floer Homology is a generalisation of ordinary Floer Homology. Notice that for $k = 0$, there are not 4 but 8 Floer chain groups, and as E_0 and $\mathbb{E}_0$ have trivial isotropy, the connections involved extend across the singular loci (K and $K \times \mathbb{R}$) of the orbifolds ((S^3, K, n) and $(S^3, K, n) \times \mathbb{R}$). This readily implies the following:

Proposition 4.2. $HF_*^{(0)}(Y^3, K, n) \simeq HF_*(Y^3)$.

Next, we are interested in what happens when one applies an orientation reversing diffeomorphism to S^3. Given K in S^3, applying such a diffeomorphism changes K into its mirror image $\bar{K}$. There is a relation between the Floer Homology of K and that of $\bar{K}$, which is not difficult to obtain.

Proposition 4.3. $HF_*^{(k)}(S^3, \bar{K}, n) = HF_{-*-1}^{(k)}(S^3, K, n)$.

It follows that any knot K for which a Floer Homology satisfies

$$\begin{aligned} & HF_0^{(k)}(S^3, K, n) \neq HF_3^{(k)}(S^3, K, n) \\ \text{or} \quad & HF_1^{(k)}(S^3, K, n) \neq HF_2^{(k)}(S^3, K, n) \end{aligned}$$

will be distinguished from its mirror image.

4.2 Deformation theory

In this section we explore how the Floer Homology varies as one changes the pair (k, n) used to define $HF_*^{(k)}(S^3, K, n)$. One aim is to extend the construction of a Floer Homology for knots to cover the case of irrational holonomy parameter singular connections, which is excluded when working with orbifolds.

Start with a Floer Homology $HF_*^{(k)}(S^3, K, n)$, for a knot $K \hookrightarrow S^3$ and a pair of integers (k, n) such that E_k is not an exceptional bundle, and suppose that the elements in $\mathcal{M}^*_{flat}(E_k)$ which are generators of $C_*^{(k)}(S^3, K, n)$ are non-degenerate. Otherwise simply use an appropriate perturbation as done in Section 3.3.

Theorem 4.4. *There exists an $\epsilon_K > 0$ depending on the knot K and the pair (k, n), such that if (k', n') is another pair of integers for which $|k/n - k'/n'| < \epsilon_K$, then*

$$HF_*^{(k')}(S^3, K, n') \simeq HF^{(k)}(S^3, K, n).$$

Proof. By the non-degeneracy assumption, an element $A \in \mathcal{M}^*_{flat}(E_k)$ may be seen as lying in a one-dimensional component of $\mathcal{M}^*_{flat}(S^3 - N_K)$, which is locally an arc A_t centred about A, which may be parameterised by the trace of

$$Hol_{A_t}(\mu) = \begin{pmatrix} e^{i2t\pi} & 0 \\ 0 & e^{-i2t\pi} \end{pmatrix},$$

where $t \in (k/n - \epsilon/2, k/n + \epsilon/2)$ for a small enough $\epsilon > 0$ as to keep the local parameterisation. Then for any pair (k', n') such that k'/n' is in the above interval, there exists a corresponding $A' \in \mathcal{M}^*_{flat}(E_{k'})$ over the 3-orbifold (S^3, K, n'), and A' is well-defined. As there are finitely many elements in $\mathcal{M}^*_{flat}(E_k)$, take $\epsilon > 0$ small enough to have a one-to-one correspondence $A \leftrightarrow A'$, as above, for all $A \in \mathcal{M}^*_{flat}(E_k)$ and $A' \in \mathcal{M}^*_{flat}(E_{k'})$.

Now assemble these critical points into $C_*^{(k)}(S^3, K, n)$ and $C_*^{(k')}(S^3, K, n')$. For A and A' given by the construction above, their Floer indices $\mu(A)$ and $\mu(A')$ are equal. To prove this, one has to apply the deformation invariance of the index with respect to the variation in the holonomy parameter given in Section 6 of [32]. All connections along the arc A_t have to be non-degenerate: $H^1_{A_t} = 0$. Otherwise, the gluing formula for the index implies that there is a change in the index at an A_{t_0} where $H^1_{A_{t_0}}$ is non-zero. By the choice of ϵ, such points are avoided, and hence one gets a chain group isomorphism

$$C_*^{(k')}(S^3, K, n') \simeq C_*^{(k)}(S^3, K, n).$$

Now take into account the boundary operator in the two Floer Homologies. For this, the extended moduli space of singular instantons, parameterised by the holonomy parameter is needed. Adapting Proposition 2.8 in [32] to our setting, start with $\mathcal{M}_{ASD}(A_1, A_2)$ for flat connections on E_k, A_1, A_2. Take $\epsilon > 0$ as needed above to obtain corresponding $A_1^{'}$ and $A_2^{'}$ on $E_{k'}$. Consider $\mathcal{M}^t_{ASD}(A_1, A_2)$ as extended moduli space with holonomy parameter t such that at $t = k/n$ and $t = k^{'}/n^{'}$, one has $\mathcal{M}_{ASD}(A_1, A_2)$ and $\mathcal{M}_{ASD}(A_1^{'}, A_2^{'})$ respectively. This moduli space is a manifold of dimension $\mu(A_1) - \mu(A_2) + 1$, but taking into account the $\mathbb{R}$-translations along the cylinder, form the reduced moduli space $\hat{\mathcal{M}}^t_{ASD}(A_1, A_2)$, whose dimension will be $\mu(A_1) - \mu(A_2)$ and whose restriction to $t = k/n$ and $t = k^{'}/n^{'}$ gives the reduced instanton moduli spaces between A_1, A_2 and $A_2^{'}$, $A_2^{'}$ respectively. The case of interest for the boundary operator is when $\mu(A_1) - \mu(A_2) \equiv 1 \ (mod\ 4)$. Then $\hat{\mathcal{M}}^t_{ASD}(A_1, A_2)$ is a 1-dimensional manifold whose restrictions at each end gives the boundary operator ∂ and $\partial^{'}$ for $(C_*^{(k)}(S^3, K, n), \partial)$ and $(C_*^{(k^{'})}(S^3, K, n^{'}), \partial^{'})$. Let $\epsilon^{'} > 0$ be small enough so that, if $k/n \leq t \leq k/n + \epsilon^{'}$, $\hat{\mathcal{M}}^t_{ASD}(A_1, A_2)$ is a product $\hat{\mathcal{M}}_{ASD}(A_1, A_2) \times [k/n, k/n + \epsilon^{'}]$. Now take $\epsilon_K = min\{\epsilon, \epsilon^{'}\}$. Then the Floer chain groups and boundary operators coincide, and thus the Floer Homologies are naturally isomorphic. q.e.d.

Theorem 4.4 may be used to give a model for a general Floer Homology for knots built from singular connections over knot complements with irrational holonomy parameter α: we have a well-defined notion of limit of Floer groups with respect to rational parameters converging to an irrational parameter and hence we set:

Definition 4.5. If $\alpha \in (0, 1/2)$ is such that $\Delta_K(e^{i2\alpha 2\pi}) \neq 0$, define the Floer Homology

$$HF_*^{(\alpha)}(S^3, K) = \lim_{\frac{k}{n} \to \alpha} HF_*^{(k)}(S^3, K, n).$$

We may now give a reformulation of Theorem 4.4 in light of the general construction $HF_*^{(\alpha)}(S^3, K)$.

Corollary 4.6. *Let* $\mathcal{Z} = \{\alpha \in [0, 1/2] \mid \Delta_K(e^{i2\alpha 2\pi}) = 0\}$. *The Floer Homology* $HF_*^{(\alpha)}(S^3, K)$ *is constant on connected components of* $[0, 1/2] \backslash \mathcal{Z}$.

4.3 Functoriality property

In [22], Floer pointed out one of the novel aspects about his Homology Theory for homology spheres: this is the so-called functoriality property of $HF_*(Y)$. He considers a smooth cobordism X^4 between two 3-manifolds Y_1 and Y_2 and shows that there is an induced group homomorphism

$$\Psi_X \colon HF_i(Y_1) \to HF_{i+\xi}(Y_2),$$

where $\xi = 3(b^+(X) + b_1(X))$. Moreover, this construction yields the identity homomorphism if X is the product cobordism and is functorial with respect to composition of cobordisms: $\Psi_{X \circ X'} = \Psi_X \circ \Psi_{X'}$. On the other hand, in [35], where a Floer Homology for rational homology spheres was defined, Lee and Li pointed out that the properties described by Floer did not hold in general and gave an example where the map Ψ cannot be functorial. Therefore, in trying to understand the behaviour of Floer Homology for 3-orbifolds under cobordism, we have to worry about the functoriality property. First, a definition of orbifold cobordism is needed. Although there exists a general theory (see [18]), we shall be rather practical here: a 4-dimensional orbifold (X^4, Σ^2, n) is said to be a *cobordism* between 3-orbifolds (Y_1, K_1, n) and (Y_2, K_2, n) if X is a manifold cobordism between Y_1 and Y_2, with a smoothly embedded surface $\Sigma \hookrightarrow X$ such that $\Sigma \cap Y_i = K_i$ for $i = 1, 2$. We denote by $\chi(\Sigma)$ the Euler characteristic of the bordered surface Σ. Adjoin to (X, Σ, n) ends $(S^3, K_i, n) \times \mathbb{R}$ on each side to obtain an orbifold with cylindrical ends, which is also denoted (X, Σ, n). The situation is here identical to the one encountered when defining the boundary operator in the Floer Homology, except for the fact that we now allow Σ to be any surface of genus g rather than a standard product $K \times \mathbb{R}$. We consider a bundle $E_k^{(i)}$ $(k \neq 0)$ over (S^3, K_i, n), along with an adapted bundle $\mathbb{E}_k$ over (X, Σ, n). As in Section 3.2, using the appropriate analytical set-up, we can define $\mathcal{M}_{ASD}(A_1, A_2)$, the space of finite energy instantons on $\mathbb{E}_k$ limiting to $A_1 \in \mathcal{M}_{flat}(E_k^{(1)})$ and $A_2 \in \mathcal{M}_{flat}(E_k^{(2)})$. Then, generically, $\mathcal{M}_{ASD}(A_1, A_2)$ is a smooth oriented manifold whose dimension is computed by adapting the index computation mentioned in Theorem 3.9:

$$dim\ \mathcal{M}_{ASD}(A_1, A_2) = \mu(A_1) - \mu(A_2) - 3(b^+(X) - b_1(X)) + \chi(\Sigma).$$

We shall be concerned with knots in S^3, although a set-up for knots in homology spheres could be considered and our results extend to that situation. Moreover, as our interest is in the knots K_1 and K_2, we will work with $X = S^3 \times [0,1]$. Much in the same way as for the boundary operator, let

$$\psi_{(X,\Sigma,n)}\colon C_*^{(k)}(S^3, K_1, n) \to C_{*-\chi(\Sigma)}^{(k)}(S^3, K_2, n)$$

be defined by

$$\psi_{(X,\Sigma,n)}(A_1) = \sum_{A_2 \in \mathcal{M}^*_{flat}(E_k^{(2)})} n(A_1, A_2) \cdot A_2,$$

where $n(A_1, A_2)$ is the algebraic number of elements in $\mathcal{M}_{ASD}(A_1, A_2)$ (which is 0-dimensional by the above grading shift, and also compact following [22]). Notice that $\psi_{(X,\Sigma,n)}$ may depend on the metric chosen to obtain instantons, but what is important is that this does not carry over to the Floer Homology. Indeed, if $\psi_{(X,\Sigma,n)}$ induces $\Psi_{(X,\Sigma,n)}$ in homology, then we have the following result.

Theorem 4.7.

$$\Psi_{(X,\Sigma,n)}\colon HF_*^{(k)}(S^3, K_1, n) \to HF_{*-\chi(\Sigma)}^{(k)}(S^3, K_2, n)$$

depends only on the orbifold cobordism (X, Σ, n)*. For a composite cobordism* $(X \circ X', \Sigma \circ \Sigma', n)$*, one has* $\Psi_{(X\circ X',\Sigma\circ\Sigma',n)} = \Psi_{(X,\Sigma,n)} \circ \Psi_{(X,\Sigma,n)}$*. For the standard product cobordism, the map* $\Psi_{(S^3,K,n)\times[0,1]}$ *is the identity.*

Proof. We shall briefly discuss here as there is nothing really new in the orbifold case compared to Floer's case, and all the analytical techniques required were explained in the construction of Floer Homology. That $\Psi_{(X,\Sigma,n)}$ depends only on (X, Σ, n) is similar to the boundary property $\partial^2 = 0$ proved for ∂, and the argument given in Theorem 3 of [22] simply lifts to orbifolds.

As mentioned earlier when citing work of Lee and Li, the delicate point is the possibility that Ψ may not be functorial. First notice that the grading shift is compatible with functoriality as the Euler characteristic of a bordered surface is additive. The proof given in [22] could break down if one needs to perturb the Chern-Simons functional (and hence the ASD equation). Then it is possible that two different perturbations yield different critical points, or also the Floer index may be affected, as explained in [35]. However, in defining the Floer Homologies $HF_*^{(k)}(S^3, K_i, n)$, we have been careful to impose that each of the reducible connections $\theta_k^{(i)} \in \mathcal{M}_{flat}(E_k^{(i)})$ be non-degenerate. In this case, one can perturb compactly away from the reducible, and the situation described in [35, Example 5.1] cannot occur. Floer's argument then applies. Finally, that $\Psi_{(S^3,K,n)\times[0,1]}$ is the identity follows directly from the functoriality property. q.e.d.

With the functoriality property of the Floer Homology $HF_*^{(k)}(S^3,K,n)$, it is now a purely formal consequence that $HF_*^{(k)}(S^3,K,n)$ is independent of the metric used in its construction, as the product cobordism $(S^3,K,n)\times[0,1]$ (with any metric) induces the identity map on Floer Homology.

4.4 Relation to cyclic branched covers

We suppose in this section that n is odd and that the cyclic branched cover $V_n(K)$ is a homology sphere. The orbifold exact sequence yields, via holonomy, a pull-back map from $\mathcal{M}_{flat}(E_k)$ to $\mathcal{M}_{flat}(V_n(K))$, preserving irreducibility when $V_n(K)$ is a homology sphere. If each flat connection $A\in\mathcal{M}^*_{flat}(E_k)$ does induce a flat connection $\tilde{A}\in\mathcal{M}_{flat}(V_n(K))$, the converse is not true. Indeed, a flat connection over $V_n(K)$ is not necessarily $\mathbb{Z}_n$-invariant (and hence does not "push-down" to the quotient 3-orbifold). Our aim is to compare the relative index of generators in $C_*^{(k)}(S^3,K,n)$ with the relative index of the corresponding generators in $C_*(V_n(K))$, as this will be used later on.

Let $A\in C_*^{(k)}(S^3,K,n)$ be a generator in the 3-orbifold Floer complex, and let $\tilde{A}\in C_*(V_n(K))$ be the corresponding generator for the Floer complex of $V_n(K)$. The reducible $\theta_k\in\mathcal{M}_{flat}(E_k)$ corresponds to the trivial connection $\theta\in\mathcal{M}_{flat}(V_n(K))$. Choose an adapted connection $\mathbb{A}$ on $\mathbb{E}_k$ over $(S^3,K,n)\times\mathbb{R}$, limiting to θ_k and A at the ends. When seen as a $\mathbb{Z}_n$-invariant connection over $V_n(K)\times\mathbb{R}$, the connection $\mathbb{A}$ induces an adapted connection $\tilde{\mathbb{A}}$ on a bundle $\tilde{\mathbb{E}}$ over $V_n(K)\times\mathbb{R}$ with limits θ and $\tilde{A}$ at the ends. The Floer indices of A and $\tilde{A}$ are given respectively as

$$\mu(A) = ind\ D_{\mathbb{A}}\ (mod\ 4), \qquad \mu(\tilde{A}) = ind\ D_{\tilde{\mathbb{A}}}\ (mod\ 8)$$

for operators

$$D_{\mathbb{A}}\colon \Omega^1(\mathfrak{g}_{\mathbb{E}_k})\to\Omega^0(\mathfrak{g}_{\mathbb{E}_k})\oplus\Omega^+(\mathfrak{g}_{\mathbb{E}_k}),\quad D_{\tilde{\mathbb{A}}}\colon \Omega^1(\mathfrak{g}_{\tilde{\mathbb{E}}})\to\Omega^0(\mathfrak{g}_{\tilde{\mathbb{E}}})\oplus\Omega^+(\mathfrak{g}_{\tilde{\mathbb{E}}})$$

as defined in Section 3.1 (here we have removed the weighted Sobolev subscripts to simplify the notation). The following relates the Floer indices of A and $\tilde{A}$.

Proposition 4.8. $\mu(A) \equiv \mu(\tilde{A})$ $(mod\ 2)$.

Proof. A $\mathbb{Z}_n$-action on $V_n(K)$ induces an action on differential forms, yielding decompositions

$$\Omega^1(\mathfrak{g}_{\tilde{\mathbb{E}}}) = \bigoplus_{i=0}^{n-1} \Omega^1(\rho_i), \quad \Omega^0(\mathfrak{g}_{\tilde{\mathbb{E}}}) = \bigoplus_{i=0}^{n-1} \Omega^0(\rho_i), \quad \Omega^+(\mathfrak{g}_{\tilde{\mathbb{E}}}) = \bigoplus_{i=0}^{n-1} \Omega^+(\rho_i),$$

for representations $\rho_i \in \hat{\mathbb{Z}}_n$. In these decompositions, the contribution from the trivial representation ρ_0 is precisely the $\mathbb{Z}_n$-invariant subspaces. This gives the following expression for the operator $D_{\mathbb{A}}$: $D_{\mathbb{A}} = D_{\tilde{\mathbb{A}}_{|\Omega^1(\rho_0)}}$. Thus,

$$ind\ D_{\tilde{\mathbb{A}}} = \sum_{i=0}^{n-1} ind\ D_{\tilde{\mathbb{A}}_{|\Omega^1(\rho_i)}} = ind\ D_{\mathbb{A}} + \sum_{i=1}^{n-1} ind\ D_{\tilde{\mathbb{A}}_{|\Omega^1(\rho_i)}}.$$

As n was supposed to be odd at the beginning of this section, it follows from basic Representation Theory that the finite dimensional subspaces $Ker\ D_{\tilde{\mathbb{A}}_{|\Omega^1(\rho_i)}}$ and $Coker\ D_{\tilde{\mathbb{A}}_{|\Omega^1(\rho_i)}}$ are complex vector spaces for $i = 1, ..., n-1$. Consequently, $ind\ D_{\tilde{\mathbb{A}}_{|\Omega^1(\rho_i)}}$ is an even number if $i = 1, ...,$ $n-1$ and hence

$$\mu(\tilde{A}) \equiv ind\ D_{\tilde{\mathbb{A}}} = ind\ D_{\mathbb{A}} + \sum_{i=1}^{n-1} ind\ D_{\tilde{\mathbb{A}}_{|\Omega^1(\rho_i)}} \equiv ind\ D_{\mathbb{A}} \equiv \mu(A)\ (mod\ 2).$$

q.e.d.

4.5 Relation to equivariant signatures

Work of Taubes, by providing a gauge theoretic framework to Casson's invariant, shows that the Euler characteristic of the Floer Homology of a homology sphere Y^3 is twice the Casson invariant of Y^3. One is therefore naturally interested in a possible relation between the Euler characteristic of the Floer Homology $HF_*^{(k)}(S^3, K, n)$ and some known knot invariant. The appropriate invariant was introduced by Herald in [27], where he defined a Casson-type invariant for knots in homology spheres, generalizing ideas developed in [47]. The main result of [27] is to relate this Casson-type invariant to a classical knot invariant, the Tristram-Levine signature. We shall exploit this to interpret the Euler characteristic of $HF_*^{(k)}(S^3, K, n)$ in terms of such equivariant signatures of K, which were introduced in Section 3.3. The main result in this section is:

Theorem 4.9. *The Euler characteristic of* $HF_*^{(k)}(S^3, K, n)$ *is given by:*

$$\chi(HF_*^{(k)}(S^3, K, n)) = \frac{1}{2} \cdot \sigma_{2\frac{k}{n}}(K).$$

We shall prove this using Herald's work, and therefore we should first explain that briefly. Throughout it will be understood that we suppose $\mathcal{M}_{flat}(S^3 - N_K)$ to be non-degenerate (and we use a perturbation if this is not the case). $\mathcal{M}_{flat}(S^3 - N_K)$ consists of a union of three smooth strata corresponding to irreducible, abelian and central flat connections. Recall there is a restriction map $r\colon \mathcal{M}_{flat}(S^3 - N_K) \to \mathcal{M}_{flat}(T^2)$ and a flat connection A with $tr\ Hol_A(\mu) = 2cos(2\alpha\pi)$ ($\alpha \in (0, 1/2)$) corresponds to an intersection in $\mathcal{M}_{flat}(T^2)$ of $\mathcal{M}_{flat}(S^3 - N_K)$ with a vertical slice S_α. Consider the 2-fold covers of $\mathcal{M}_{flat}(S^3 - N_K)$, and $\mathcal{M}_{flat}(T^2) \supset S_\alpha$ along the two central flat connections: $\tilde{\mathcal{M}}_{flat}(S^3 - N_K)$, $\tilde{\mathcal{M}}_{flat}(T^2) \supset \tilde{S}_\alpha$ respectively. Then r is covered by

$$\tilde{r}\colon \tilde{\mathcal{M}}_{flat}(S^3 - N_K) \to \tilde{\mathcal{M}}_{flat}(T^2).$$

Herald's invariant, $\lambda_\alpha(K)$, is the oriented intersection number

$$\tilde{r}(\tilde{\mathcal{M}}^*_{flat}(S^3 - N_K)) \cdot \tilde{S}_\alpha,$$

and he proves the following:

Theorem 4.10 ([27, Corollary 3]). $\lambda_\alpha(K) = \frac{1}{2} \cdot \sigma_{2\alpha}(K)$.

Proof. We give a brief outline and refer to [27] for more details. First Herald introduces another number, $S(\alpha)$. At a bifurcation point A along the abelian arc in $\mathcal{M}_{flat}(S^3 - N_K)$, he shows that up to reparameterisation the Hessian of $A + t$ satisfies $H_{A+t} = \pm tI$. This gives a number associated to A, $\varsigma(A) = \pm 1$. He defines

$$S(\alpha) = \sum_A \varsigma(A),$$

where the sum is taken over all bifurcation points A along the abelian arc such that $tr\ Hol_A(\mu) = 2cos(2\alpha_A\pi)$ with $\alpha_A \in (0, \alpha)$. Herald first shows (Lemma 8 in [27]) that

$$\lambda_\alpha(K) = S(\alpha).$$

Now reducible connections in $\mathcal{M}_{flat}(S^3 - N_K)$ can be viewed as connections in $\mathcal{M}_{flat}(\bar{Y})$, where $\bar{Y}$ is the 3-manifold obtained by a 0-surgery on S^3 along K. In particular, let $\bar{A}_0$ and $\bar{A}_\alpha$ be such connections with holonomy parameter 0 and α respectively. There is a well-defined spectral flow $SF(\bar{A}_0, \bar{A}_\alpha)$ between the two connections (see [29] or [2] for details on spectral flow). Herald proves the following (Corollary 10 in [27]):

$$2S(\alpha) - 4 = SF(\bar{A}_0, \bar{A}_\alpha),$$

and completes the proof by comparing the spectral flow to an equivariant signature:

$$SF(\bar{A}_0, \bar{A}_\alpha) = \sigma_{2\alpha}(K) - 4.$$

q.e.d.

Proof. For the proof of Theorem 4.9 it suffices to show that

$$\chi(HF_*^{(k)}(S^3, K, n)) = S(k/n).$$

There are two types of critical points in $\mathcal{M}^*_{flat}(E_k)$: those lying in closed components (circles) of $\mathcal{M}_{flat}(S^3 - N_K)$ and those lying in irreducible arcs limiting to abelian connections in $\mathcal{M}_{flat}(S^3 - N_K)$. The first type generically comes in pairs of critical points. The claim is that the contribution of such a pair to the Euler characteristic of $HF_*^{(k)}(S^3, K, n)$ is trivial. To show this, all there is to know is that the index of the two critical points A_0 and A_1 satisfies

$$\mu(A_0) - \mu(A_1) \equiv 1 \ (mod\ 2).$$

First notice that by invariance of the index under deformation of the holonomy parameter , as one goes along the circle in $\mathcal{M}^*_{flat}(S^3 - N_K)$ (by varying the holonomy along a meridian), the index of the corresponding critical points is constant if no degenerate critical point is crossed. On the other hand, generically, a degenerate critical point appears when the holonomy parameter along the closed component has a non-degenerate critical point (doubles back on itself). By the independence of Floer Homology under perturbations, such double points cancel out in Floer Homology. Combining these two observations yields the result.

The contribution of the second type of critical points to the Euler characteristic of $HF_*^{(k)}(S^3, K, n)$ is as follows. Generically two cases arise: if on an arc there is an odd number of such points, all except one cancel out in the Euler characteristic, while, if on an arc there is an even number of such points, all these cancel out in the Euler characteristic. This follows from the index argument explained above. Therefore for the Floer Homology $HF_*^{(k)}(S^3, K, n)$, the contribution to the Euler characteristic is an algebraic number of bifurcation points along the abelian arc in $\mathcal{M}_{flat}(S^3 - N_K)$ with holonomy parameter in $(0, k/n)$. We need to show that this number agrees with Herald's $S(k/n)$. First notice that Herald's sign convention agrees, up to some universal sign, with the one used by Taubes in [44]. Indeed, this follows from Proposition 18 in [27] and the linearity of both Herald's and Taubes' invariants under the operation of connected sum. Also, the proof of Proposition 5.2 in [44] extends to the case of invariant connections (we shall not reproduce it here, as it is rather long). These two observations imply that our algebraic count may differ from Herald's by a universal sign. The matter will be settled by computing an example. With its standard orientation, the Poincaré sphere $\mathcal{P}^3$ is realised as a 5-fold cyclic branched covering of S^3 along a right-handed trefoil $K_{2,3}$. Casson's invariant for $\mathcal{P}^3$ is $\lambda(\mathcal{P}^3) = 1/2 \cdot \chi(HF_*(V_5(K_{2,3}))) = -1$ (this differs from [21] as they used $ind^+ \, D_{\mathbb{A}}$ rather than $ind^- \, D_{\mathbb{A}}$ for the Floer grading). Using Proposition 4.8, we have:

$$\chi(HF_*^{(1)}(S^3, K_{2,3}, 5)) = -1, \quad \chi(HF_*^{(2)}(S^3, K_{2,3}, 5)) = -1.$$

As a right-handed trefoil $K_{2,3}$ has non-positive equivariant signatures, the orientation conventions agree and this completes the proof that for any knot K,

$$\chi(HF_*^{(k)}(S^3, K, n)) = \frac{1}{2} \cdot \sigma_{2\frac{k}{n}}(K).$$

q.e.d.

5. Examples

The unknot was seen to have trivial Floer Homology (Section 4.1), and we first give here other examples of knots with trivial Floer Homology.

Proposition 5.1. *If $\Delta_K(t)$ is trivial, then $HF_*^{(k)}(S^3, K, n)$ vanishes for any pair (k, n). More generally the same holds for knots whose Alexander polynomial has no roots on $S^1 \subset \mathbb{A}$.*

Proof. Combine Corollary 4.6 and Proposition 4.1. q.e.d.

This gives quite a few examples of knots with trivial Floer Homology. The figure-8 knot has no roots of its Alexander polynomial along $S^1 \subset \mathbb{A}$, while (p,q,r)-pretzel knots (for p, q, r odd integers such that

$pr + rq + qp = -1$) or the Conway and Kinoshita-Terasaka knots have trivial Alexander polynomial. More generally any connected sum of such knots will satisfy the conditions of Proposition 5.1 (as $\Delta_{K_1 \sharp K_2}(t) = \Delta_{K_1}(t) \cdot \Delta_{K_2}(t)$), so will mutants of these knots (as mutation preserves the Alexander polynomial). These examples indicate that, like the knot signatures $\sigma_\alpha(K)$, the efficiency of $HF_*^{(k)}(S^3, K, n)$ as a knot invariant is in part limited by the Alexander polynomial.

The case we next study is that of torus knots. A torus knot $K_{p,q}$ of type (p,q) may be seen as the link of singularity $f^{-1}(0) \cap S^3_\epsilon$ of a polynomial $f\colon \mathbb{A}^2 \to \mathbb{A}$ given by $f(x_1, x_2) = x_1^p + x_2^q$. The n-fold cyclic branched covering of S^3 along $K_{p,q}$ is the Brieskorn sphere $\Sigma(p,q,n)$. The Floer Homology of Brieskorn spheres has been computed by Fintushel and Stern in [21]. This was rendered easier by the fact that the Floer boundary operator is always trivial because the Floer chain groups vanish in one parity. The computation of $HF_*(\Sigma(p,q,n))$ is therefore reduced to finding the Floer index of irreducible representations $\rho\colon \pi_1(\Sigma(p,q,n)) \to SU(2)$. For torus knots, we have:

Theorem 5.2. *For a torus knot $K_{p,q}$, the Floer Homology $HF_*^{(k)}(S^3, K_{p,q}, n)$ has trivial boundary operator and therefore*

$$HF_*^{(k)}(S^3, K_{p,q}, n) = C_*^{(k)}(S^3, K_{p,q}, n).$$

Proof. The case where n is odd and coprime with p and q follows directly from the result of Fintushel and Stern and Proposition 4.8. Otherwise, the argument is different. First notice that according to [30, Theorem 1], $\mathcal{M}^*_{flat}(S^3 - N_{K_{p,q}})$ consists of $\frac{1}{2}(p-1)(q-1)$ arcs limiting to abelian connections. Therefore any non-degenerate element in $\mathcal{M}^*_{flat}(E_k)$ over $(S^3, K_{p,q}, 2n)$ is on such a component in $\mathcal{M}^*_{flat}(S^3 - N_{K_{p,q}})$. Suppose $A_1, A_2 \in \mathcal{M}^*_{flat}(E_k)$ are such that

$$\mu(A_1) - \mu(A_2) \equiv 1 \ (mod\ 2),$$

so that A_1 and A_2 are not in the same parity in the Floer chain groups. By the deformation argument in the proof of Theorem 4.4, one may find a pair (k', n') where n' is odd and coprime with p and q, such that $C_*^{(k')}(S^3, K_{p,q}, n') \simeq C_*^{(k)}(S^3, K_{p,q}, 2n)$. Take $A'_1, A'_2 \in \mathcal{M}^*_{flat}(E'_k)$ corresponding to $A_1, A_2 \in \mathcal{M}^*_{flat}(E_k)$. Then

$$\mu(A'_1) - \mu(A'_2) \equiv 1 \ (mod\ 2),$$

and this is a contradiction with what was just proved. Therefore,

$$HF_*^{(k)}(S^3, K_{p,q}, 2n) = C_*^{(k)}(S^3, K_{p,q}, 2n).$$

q.e.d.

More generally, given a knot K in S^3 and a Floer chain complex $(C_*^{(k)}(S^3, K, n), \partial)$, the Chern-Simons functional CS is said to be *perfect* if the Floer chain groups vanish in even or odd dimensions. It would be be interesting to characterise classes of knots for which the Chern-Simons functional is perfect, as for such knots the computation of Floer Homology does not require knowledge of instantons used for the boundary operator. By Theorem 5.2 torus knots form such a class. It may be shown ([11]) that although torus knots happen to be fibred, non-amphichaeral and prime, these properties are not enough in general to yield perfectness of CS.

Let us then turn our attention to knots which arise as links of singularities, another property torus have. Recall that a knot $K \hookrightarrow S^3$ is said to be *algebraic* if it is the link of an isolated singularity $K = f^{-1}(0) \cap S^3_\epsilon$, for a polynomial $f: \mathbb{C}^2 \to \mathbb{C}$. This class of knots has been studied extensively, and a lot is known about them (see [19] or [34]). Our first aim is to find an expression of the Euler characteristic of the Floer Homology of an algebraic knot K.

For this, we relate algebraic knots to so-called *iterated torus knots*, as this gives us a way of inductively building the algebraic knot starting from torus knots (which are well understood). Given an algebraic knot K which is the link of $f: \mathbb{C}^2 \to \mathbb{C}$, by taking a Puiseux expansion, one sees that K is an iterated torus knot, say of type $((p_1, q_1), ..., (p_r, q_r))$, satisfying the conditions $p_i > 0$ and $q_{i+1} > p_i p_{i+1} q_i$. And conversely, an iterated torus knot of type $((p_1, q_1), ..., (p_r, q_r))$ is algebraic if $p_i > 0$ and $q_{i+1} > p_i p_{i+1} q_i$ (we refer the reader to [19] for details on this correspondence). Notice that, in particular, if K is algebraic then $p_i > 0$ and $q_i > 0$ for all i's. Thus, expressing an algebraic knot K as an iteration of torus knots K_1,...,K_r of type (p_1, q_1),..., $(p_r q_r)$, we have:

Theorem 5.3. *Let K be an algebraic knot of type $((p_1, q_1), ..., (p_r, q_r))$ and consider the Floer Homology $HF_*^{(k)}(S^3, K, n)$ and let k_i be an integer such that $0 \le k_i/n \le 1/2$ and $k_i \equiv kp_1...p_i (mod\ n)$. Then*

$$\chi(HF_*^{(k)}(S^3, K, n)) = \sum_{i=1}^{r} \chi(HF_*^{(k_i)}(S^3, K_i, n)),$$

where the K_i are torus knots of type (p_i, q_i).

Proof. We shall use inductively the equivariant signature computation for iterated torus knots provided by Litherland (Theorem 2 in [38]) and Theorem 4.9. Start with a torus knot $K_1 = (p_1, q_1)$ as above. Use another torus knot $K_2 = (p_2, q_2)$, and consider the iterated torus knot of type $((p_1, q_1), (p_2, q_2))$ $K_2(p_1, q_1)$. Then by [38],

$$\sigma_{2\frac{k}{n}}(K_2(p_1, q_1)) = \sigma_{2\frac{k}{n}p_1}(K_2) + \sigma_{2\frac{k}{n}}(K_1).$$

For an iterated torus knot $K^{1...r} = ((p_1, q_1), ..., (p_r, q_r))$, its equivariant signature $\sigma_{2\frac{k}{n}}(K^{1...r})$ will be computed inductively

$$\begin{aligned}\sigma_{2\frac{k}{n}}(K^{1...r}) &= \sigma_{2\frac{k}{n}p_1p_2...p_{r-1}}(K_r) + \sigma_{2\frac{k}{n}p_1p_2...p_{r-2}}(K_{r-1}) + ... + \sigma_{2\frac{k}{n}}(K_1)\\ &= \sum_{i=0}^{r-1} \sigma_{2\frac{k}{n}p_1...p_i}(K_i).\end{aligned}$$

The coefficients $kp_1...p_i$ have to lie between 0 and $n/2$ in order to evaluate the equivariant signature, so one obtains the k_i. Then Theorem 4.9 gives

$$\sigma_{2\frac{k}{n}p_1...p_i}((p_i, q_i)) = 2 \cdot \chi(HF_*^{(k_i)}(S^3, (p_i, q_i), n).$$

Now if K is algebraic, it will be an iterated torus knot $((p_1, q_1), ...(p_r, q_r))$ with extra conditions $p_i > 0$ and $q_i > 0$. These conditions are essential for the argument, as then all the torus knots K_i $(1 \leq i \leq r)$ used to obtain K are of the same "mirror type". That is, they all are either right-handed torus knots or left-handed torus knots. Therefore the contribution of each of these to the signature of the algebraic knot K is either always positive (for left-handed torus knots) or negative (for right-handed torus knots), proving the result. q.e.d.

We conclude this section conjecturing the perfectness of CS for algebraic knots, generalising Theorem 5.2.

Question 5.4. For an algebraic knot K in S^3, do the chain groups $C_*^{(k)}(S^3, K, n)$ all vanish either in even or in odd dimensions and therefore is

$$HF_*^{(k)}(S^3, K, n) = C_*^{(k)}(S^3, K, n)?$$

This conjecture is the analogue of one in ordinary Floer Homology, for 3-manifolds which arise as 3-dimensional links of complex singularities (see for example [21] or [24] for the Seifert fibred manifolds case).

6. Conclusion

In order to push further the theory presented in this article, there are many tools one may wish to develop aiming at a better understanding of the Floer Homology for knots and its role in Knot Theory. The possibilities below are being investigated.

We have seen that the Floer Homology for 3-orbifolds appears as a generalisation of ordinary Floer Homology, so one aspect to be considered is to generalise properties developed by Floer, Donaldson and others to the orbifold setting. From work done in this article, it is reasonable to expect that, at least in some cases, the generalisations pose no serious problems.

First comes to mind the surgery exact triangle developed by Floer (see [7]) which is a powerful tool for computations. To carry this out using 3-orbifolds, one needs other Floer homologies for knots which were introduced in [11], and adapted the arguments in [7] to the case of orbifolds.

Similarly, recall that in the case of 3-manifolds the Atiyah-Floer conjecture relates the instanton Floer Homology of a homology sphere, $HF_*(Y^3)$, with the symplectic Floer Homology of Y^3. We have briefly alluded to a symplectic Floer Homology for knots developed by Li in [36], and this has been generalised by Austin. It seems natural to look for a correspondence between the symplectic and instanton theories for knots, in light of the Atiyah-Floer conjecture.

Also, an analog of the relative Donaldson polynomials of 4-manifolds with boundary taking values in the Floer Homology of the boundary homology sphere should be found. This would yield relative versions of the polynomial invariants developed by Kronheimer and Mrowka for embedded surfaces in 4-manifolds, where the polynomials take values in a 3-orbifold Floer Homology. This should help in generalising results of Kronheimer and Mrowka about the 4-ball genus and unknotting numbers of knots.

For many recently developed knot invariants, skein relations play an important role in their definition or in elucidating some of their properties. Skein relations for $HF_*^{(k)}(S^3, K, n)$ would be good to have, in particular for explicit computations. This involves the generalisation to links outlined in Section 4.1 and an approach towards this is to obtain a surgery sequence in Floer Homology when performing a $+1$-surgery on S^3 along a small unknotted linking circle at a knot crossing. Ideas involved here are similar to those needed for the surgery exact triangle mentioned above. Notice that such skein relations could also yield the Atiyah-Floer conjecture for knots, generalising the original Atiyah-Floer conjecture.

As the 3-orbifolds (S^3, K, n) are closely related to cyclic branched covers of S^3, one expects some relationship between the ordinary Floer Homology $HF_*(V_n(K))$ and $HF_*^{(k)}(S^3, K, n)$, when $V_n(K)$ is a homology sphere. More generally one may compare $HF_*(V_m(K))$ and $HF_*^{(k)}(S^3, K, n)$ for m and n possibly different. In [13], N. Saveliev and the first author used such an approach to provide a geometric proof of the Fintushel-Stern formula relating the Casson invariant of a Brieskorn homology sphere to the signature of its Milnor fibre. More generally, the first author developed in [12] an equivariant Casson invariant via Floer Homology for knots which provides obstructions for generalisations of the Neumann-Wahl formula (see [41]) to hold.

Finally, it is worth looking at Floer Homology in relation to knot concordance. Indeed, $HF_*^{(k)}(S^3, K, n)$ generalises the equivariant signature $\sigma_{2\frac{k}{n}}(K)$, and it is a classical result of Tristram that equivariant signatures are topological concordance invariants. Among other reasons, the work contained in [32], [33] and [31] lead the first author to conjecture in [11] that $HF_*^{(k)}(S^3, K, n)$ is a smooth concordance invariant. If this turns out to be the case, then the Floer Homology would appear as a good candidate to quantitatively measure the difference between topological and smooth concordance of knots.

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