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WEIL-PETERSON CONVEXITY OF THE ENERGY FUNCTIONAL ON CLASSICAL AND UNIVERSAL TEICHMÜLLER SPACES

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1. Introduction

1.1. Weil-Petersson convexity and classical Teichmüller space. Suppose that we have a smooth compact Riemanninan manifold M^n of dimension n with a metric g, and a compact surface N^2 with a hyperbolic metric G. We assume that both M and N have no boundary. Let $\{x^i\}$ be a local coordinate for M^n, and $\{y^\alpha\}$ a local coordinate for N^2.

The following statements follow from the results of Eells, Sampson [3] and Hartman [8] as well as Al'bers [1].

Theorem. *Given a continuous map $\phi : M^n \to N^2$, there is a smooth harmonic map $u : M^n \to N^2$ homotopic to ϕ, and u is unique in the homotopy class, unless the image of the map is a point or a closed geodesic in N.*

Eells and Lemiare [4] have shown that as long as harmonic maps exist and are uniquely determined, when the image metric is varied smoothly along a curve G^t (with $G^0 = G$), the harmonic maps $u_t : (M,g) \to (N,G^t)$ vary smoothly without changing homotopy type in the parameter t for sufficiently small t. (In order to ensure the existence and uniqueness of u_t for all t, we require the negativity of the sectional curvature of G^t.) In particular, the energy functional;

$$\mathcal{E}(t) = \frac{1}{2}\int_{M^n} G^t_{\alpha\beta}(u_t) g^{ij} \frac{\partial u_t^\alpha}{\partial x^i}\frac{\partial u_t^\beta}{\partial x^j} d\mu_g$$

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is a smooth function of t. Here the energy functional is seen as a functional solely dependent on the image metric. Now we restrict our attention to the space $\mathcal{M}_{-1}$ of hyperbolic metrics on N^2. Then as the image metric varies within $\mathcal{M}_{-1}$, a harmonic map exists and is unique unless its image is a point or a geodesic, and the energy functional is smoothly dependent on the image metric G in $\mathcal{M}_{-1}$:

$$\mathcal{E} : M_{-1} \to \mathbf{R}.$$

Note that any diffeomorphism $f : N^2 \to N^2$ homotopic to the identity map is an isometry from (N, f^*G) to (N, G). Since an isometry preserves the energy, the energy functional defined on $\mathcal{M}_{-1}$ is invariant under the action of the identity component $\mathcal{D}_0$ of the diffeomorphism group $\mathcal{D}$. Hence the energy functional can be regarded as a functional well defined on the Teichmüller space $\mathcal{T} = \mathcal{M}_{-1}/\mathcal{D}_0$, that is

$$\mathcal{E} : \mathcal{M}_{-1}/\mathcal{D}_0 \to \mathbf{R}_+.$$

Notice that even when the image of the harmonic map is a closed geodesic, the map is unique up to rotation of the geodesic, under which the energy functional is invariant. Also when the image of the map is a point, it is unique up to translation of the point in N, and the energy in that case is zero. Hence as far as the energy functional is concerned, it is well defined for maps of all homotopy types, including the two degenerate cases.

A Teichmüller space, regarded as the space of conformal structures on a given Riemann surface of higher genus $(g > 1)$, has a Riemannian metric, a positive definite pairing between two tangent vectors, which are holomorphic quadratic differentials, called *Weil-Petersson* metric. On the other hand, the space of smooth metric $\mathcal{M}$ has a natural L^2 metric, which gives a pairing to two symmetric (0, 2)-tensors (see Freed-Groisser [5].) Fischer and Tromba showed in [7] that the Weil-Petersson metric is the same as the L^2-metric on $\mathcal{M}$ restricted to $\mathcal{M}_{-1}/\mathcal{D}$ (seen as a slice in $\mathcal{M}$).

Now suppose that $\sigma(t)$ is a Weil-Petersson geodesic parameterized by arc-length in the Teichmüller space $\mathcal{T}(N) = \mathcal{M}_{-1}/\mathcal{D}$. Then we can lift $\sigma(t)$ horizontally to $\mathcal{M}_{-1}$. The lift G_t is itself a geodesic in $\mathcal{M}_{-1}$ with its tangent vector h in $T_{G_t}\mathcal{M}$ satisfying the tracefree, transverse condition [6]:

$$\operatorname{tr}_G h = 0 \qquad \text{and} \qquad \delta_G h = 0. \tag{1}$$

Thus along the geodesic G_t, the energy functional $\mathcal{E}$ is a smooth function in t. We now state the main theorem.

Theorem 3.1.1 (Weil-Petersson convexity of the energy functional). *Under the assumptions above, the function $\mathcal{E}(t)$ is a strictly convex function in t, and hence the energy functional $\mathcal{E} : \mathcal{T}(N) \to \mathbf{R}$ is strictly convex along any Weil-Petersson geodesic in $\mathcal{T}(N)$.*

M. Wolf [26] studied the properties of harmonic maps into a higher genus Riemann surface as the *image* metric varies within the Teichmüller space. He showed that in the setting where the domain metric is a hyperbolic surface of the same genus as the image, and when the map is homotopic to the identity, the Weil-Petersson exponential map at the base point (the point representing the domain metric in the Teichnüller-space) can be approximated up to order two by the space of so-called Hopf differentials, or equivalently the space of Beltrami differentials. He has also shown that the energy functional is proper with respect to the topology induced by the Weil-Petersson distance, by constructing a diffeomorphism from the space of Hopf differentials to the Teichmüller space. His results thus suggested that at both local and global levels, the Teichmüller space of a Riemann surface with respect to the Weil-Petersson metric could be viewed as the space of harmonic maps as the metric of the image of the harmonic map is varied.

Note that the statement of the convexity theorem above is independent of the homotopy class of the harmonic map, as well as the domain (M, g).

For the case where the domain is (N^2, g) for some hyperbolic metric g, and the harmonic map $u : (N, g) \to (N, G)$ is homotopic to the identity map, the convexity has been proven by A. Tromba [24]. However, the second t derivative of the Weil-Petersson geodesic in Tromba's calculation [24] differs from that of the author. (See the Remark following Theorem 2.3.2.)

One can take an m copies of S^1, each is sent to the hyperbolic surface N^2 by a harmonic map u_i, $i = 1, 2, ..., m$ and the maps is arranged so that each copy of S_1 is mapped to a closed geodesic in N and the collection of the closed geodesics to "fill" the hyperbolic surface, that is, m is the smallest number such that the complement of the geodesics is a collection of simply connected open sets. Then using a result of S. Kerckhoff [11], one can show that $\mathcal{E}(G) = \sum_{i=1}^{m} E(u_i)$ is proper on $\mathcal{T}$, thus obtaining a functional both proper and convex, (for a sum of convex functionals is still convex) providing an exhaustion function of $\mathcal{T}(N)$ by its sublevel sets. S. Wolpert [29] showed the same result by means of the geodesic length functional instead of the energy functional, a major consequence of which was giving an alternative proof of the Nielsen Realization Problem originally resolved by Kerckhoff [11]

More generally we will show the following.

Proposition 3.1.1. *When* $u._* : \pi_1(M) \to \pi_1(N)$ *is surjective, then* $\mathcal{E}(G)$ *is proper on* $\mathcal{T}(N)$

Note that as in the previous example $M = \coprod^m S^1$, we do not require the domain manifold M to be connected.

Combining Theorem 3.1.1 and Proposition 3.2.1, one has

Theorem 3.2.1. *When* $u_* : \pi_1(M) \to \pi_1(N)$ *is surjective, then there is a hyperbolic metric* G *in* $\mathcal{T}(N)$ *which uniquely minimizes* $\mathcal{E}(G)$.

In the fourth section of this paper, the linear structure of the energy functional defined on the Teichmüller space is further studied. We will have an expression for the gradient vector θ of the energy functional $\mathcal{E}$.

When the domain manifold (M^{2n}, g) is Kähler, and if there exists a holomorphic map $u : (M^{2n}, g) \to (N^2, G)$, then we will show that the gradient vector θ of the energy functional $\mathcal{E}(\mathcal{G})$ vanishes, and therefore the hyperbolic metric G is characterized to be the unique energy minimizing metric within the homotopy class of u. As a consequence of Theorem 3.1.1 as well as the properties of the tensor θ, we have the following statement.

Corollary 4.4.1. *Suppose that a harmonic map* u *from a compact Kähler manifold* (M^{2n}, g) *to a compact hyperbolic Riemann surface* (N^2, G) *is holomorphic. Then the hyperbolic metric* G *minimizes the energy functional* $\mathcal{E} : \mathcal{T} \to \mathbf{R}$ *uniquely.*

1.2. Weil-Petersson geometry of universal Teichmüller sapce. In the second part of the paper, we will study the Weil-Petersson geometry of the universal Teichmüller space $\mathcal{UT}$. It is analogously constructed to the classical Teichmüller space $\mathcal{T}(N)$ of a compact Riemann surface N in the following sense.

The classical Teichmüller space can be regarded as the space of Riemannian surfaces $\mathbf{H}^2/\Gamma$ where the Poincaré disk $\mathbf{H}^2$ is the universal cover of N, and Γ (Fuchsian group) is the fundamental group of N represented as a subgroup of the isometry group $\mathrm{SL}(2, \mathbf{R})$ for $\mathbf{H}^2$. Two Fuchsian group Γ_1 and Γ_2 are related by a quasiconformal map $\phi : \mathbf{D} \to \mathbf{D}$ such that $\phi(\gamma_1(x)) = \gamma_2(\phi(x))$. Recall that two different conformal structures are equivalent in the Teichmüller space $\mathcal{T}$ if one is obtained by pulling back the other by a diffeomorphism homotopic to identity map of N. It can be shown (see [12]) that two different quasi-conformal maps $\phi_1, \phi_2 : \mathbf{D} \to \mathbf{D}$ are equivalent in this sense if

$$\phi_1\Big|_{S^1} = \phi_2\Big|_{S^1}$$

The universal Teichmüller space, first introduced by Bers [2], is defined to be the space obtained precisely as above, *except* one takes the Γ to be the identity in $\mathrm{SL}(2, \mathbf{R})$. Since a quasiconformal map from $\mathbf{D}^2$ onto itself restricted to $S^1 = \partial \mathbf{D}^2$ is quasisymmetric (i.e., one-dimensional dilation is uniformly bounded), the universal Teichmüller space $\mathcal{UT}$ can be identified with

$$\mathcal{QS}(S^1)/\,\mathrm{SL}(2, \mathbf{R}),$$

where $\mathcal{QS}(S^1)$ is the group of quasicsymmetric homeomorphisms of the unit circle.

It was shown by Nag-Veriovsky [16] that the Weil-Petersson metric defined on $\mathcal{UT}$ at the Poincaré metric on $\mathbf{D}^2$ is isometric to the pairing between two vector fields on S^1 given by the Sobolev norm of $H^{3/2}$. Hence we need to restrict our attention to the *smooth* part of $\mathcal{QS}(S^1)/\,\mathrm{SL}(2, \mathbf{R})$, i.e., a submanifold M of $\mathcal{UT}$ whose tangent space at the Poincaré metric G_0 is the space of the $L^2(\mathbf{H}^2)$-integrable symmetric $(0, 2)$-tensors, *or equivalently* the space of $H^{3/2}$-integrable vector fields. M is a Kähler submanifold on $\mathcal{UT}$, for the Weil-Petersson metric at G_0 is Kähler.

In the fifth section, we will show that one has an L^2-orthogonal decomposition on the tangent space at the identity G_0 (Theorem 5.0.2). Formally, the submanifold $M \subset \mathcal{UT}$ is a homogeneous space $\mathrm{Diff}\, S^1/\,\mathrm{SL}(2, \mathbf{R})$ (see [25] for the algebraic aspects of the space), yet we do not know the suitable regularity condition to be imposed on $\mathrm{Diff}\, S^1$ in order to make M a smooth manifold, except the Lie algebra of $\mathrm{Diff}\, S^1$ consists of vector fields on S^1 with $H^{3/2}$ Sobolev regularity. In particular the existence of geodesic given an initial velocity remains unknown. In this paper, we will assume the existence of Weil-Petersson geodesics near the identity.

The Weil-Petersson geometery of the classical Teichmüller space has been actively studied, and it is known that even though the Weil-Petersson metric is not complete [27], the Teichmüller space is geodesically convex [28], simply connected (see [10], first proven by Fricke), and sectional curvature is non-positive (Jost [9], Wolpert [27] and Tromba [23]).

Wolpert's proof [28] for $\mathcal{T}$ being Weil-Petersson-geodesically convex uses the convexity and properness of the length functional $\mathcal{L}(G)$ on $\mathcal{T}$, as well as the fact that $\mathcal{T}$ is negatively curved.

The functional $\mathcal{E}_{\overline{\partial}} : \mathcal{UT} \to \mathbf{R}$ we will look at is the integral of the anti-holomorphic part $|\overline{\partial} u|^2$ of the energy density of a quasiconformal harmonic maps $u : \mathbf{H}^2 \to \mathbf{H}^2$ with asymptotic Dirchlet conditions $f : S^1 \to S^1$, which is a sufficiently smooth quasisymmetric map. The existence and uniqueness of such maps, which substitutes the existence and uniqueness of Eells-Sampson [3], Hartman [8] and Al'bers [1] have been studied by Li-Tam [13], [14]. The functional is shown to be finite (Theorem 6.0.3) under the regularity condition that $[G_t] \in \mathcal{UT}$ represent points of twice differentiable diffeomorphisms in $\operatorname{Diff} S^1 / \operatorname{SL}(2, \mathbf{R})$.

The following theorem, which offers a convex functional locally defined near the identity in $\mathcal{UT}$, gives a hope that the universal Teichmüller space $\mathcal{UT}$ too is geodesically convex.

Theorem 6.0.3. *Given a family of harmonic maps*

$$u_t : (\mathbf{D}^2 G_0) \to (\mathbf{D}^2, G_0)$$

with G_t a horizontal lift of a Weil-Petersson geodesic in $\mathcal{UT}$, and with $u_t\Big|_{S^1} = \operatorname{Id}_{S^1}$, we have

$$\begin{aligned} \mathcal{E}_{\overline{\partial}}(G_0) &= 0, \\ \frac{d}{dt}\mathcal{E}_{\overline{\partial}}(G_t)|_{t=0} &= 0, \\ \frac{d^2}{dt^2}\mathcal{E}_{\overline{\partial}}(G_t)|_{t=0} &= \frac{1}{2}. \end{aligned}$$

In particular the anti-holomorphic energy functional defined on $\mathcal{UT} = \operatorname{Diff} S^1 / \operatorname{SL}(2, \mathbf{R})$ is convex with respect to the Weil-Petersson distance at $[G_0] \in \mathcal{UT}$ or equivalently at $[\operatorname{Id}_{S^1}] \in \operatorname{Diff} S^1 / \operatorname{SL}(2, \mathbf{R})$.

As seen above, the various regularity conditions imposed on $\operatorname{Diff} S^1$ leaves much to be desired in order to provide a well-defined geometry on M. Yet it is the author's hope to show that the submanifold $M = \operatorname{Diff} S^1 / \operatorname{SL}(2, \mathbf{R})$ of $\mathcal{UT}$ has a nice Weil-Petersson geometry similar to the one for the classical Teichmüller space $\mathcal{T}$

The first half of this paper contains a part of the author's doctoral dissertation [30] at Stanford University, and he wishes here to thank his thesis advisor Richard Schoen for his support and encouragement.

2. Weil-Petersson geodesic equation

2.1. L^2-metric and its Levi-Civita connection. The natural L^2-metric defined on the space of symmetric $(0,2)$-tensors at a Riemannian metric G on a manifold N is given as

$$\langle h, k\rangle_{L^2} = \int_{N^2} \langle h, g\rangle_{G(x)} d)\mu_G(x),$$

where

$$\begin{aligned}\langle h, g\rangle_{G(x)} &\equiv \mathrm{Tr}_{G(x)}(h,k)\\ &\equiv \mathrm{Tr}\Big((G^{-1}\cdot h)\cdot(G^{-1}\cdot k)\Big)\\ &\qquad\qquad (A\cdot B \text{ denotes matrix multiplication})\\ &\equiv \sum_{i,j,k,l=1}^{2} G^{ik}h_{kj}G^{jl}k_{li}.\end{aligned}$$

For this metric we will find an expression for the Levi-Civita connection D. Let h_1, h_2 and h_3 be constant $(0,2)$-tensor over N^2. Then we have $[h_i, h_j] \equiv 0$ for any i and j, where the bracket denotes the Lie derivative of h_j in the direction of h_i. These relations as well as the so-called six term formula (see [20] for example.) for the Levi-Civita connection give the following;

$$\begin{aligned}\langle D_{h_1}, h_2, h_3\rangle =& \frac{1}{2}\big\{h_1\langle h_2, h_3\rangle + h_2\langle h_1, h_2\rangle - h_3\langle h_1, h_2\rangle\\ &+\langle [h_1,h_2],h_3\rangle - \langle [h_1,h_3],h_2\rangle - \rangle [h_2,h_3],h_1\rangle\big\}\\ =& \frac{1}{2}\{h_1\langle h_2,h_3\rangle + h_2\langle h_1,h_3\rangle - h_3\langle h_1,h_2\rangle\}\end{aligned}$$

On the other hand, by definition we have

$$\begin{aligned}h_3\langle h_1, h_2\rangle =& \frac{d}{dt}\Big|_{t=0}\int_N \mathrm{Tr}(G_t^{-1}\cdot h\cdot G_t^{-1}\cdot k)d\mu_{G_t}(x)\\ &\qquad\qquad \text{where } G_t = G + \mathrm{th}_3\\ =& \int_N \mathrm{Tr}\Big(G^{-1}\cdot(-h_3)\cdot G^{-1}\cdot h_1\cdot G^{-1}\cdot h_2\Big)d\mu_G(x)\\ &+\int_N \mathrm{Tr}\Big(G^{-1}\cdot h_1\cdot G^{-1}\cdot(-h_3)\cdot G^{-1}\cdot h_2\Big)d\mu_G(x)\\ &+\int_N \mathrm{Tr}_G\Big(G^{-1}\cdot h_1\cdot G^{-1}\cdot h_2\Big)\frac{1}{2}(\mathrm{Tr}_G\, h_3)d\mu_G(x)\end{aligned}$$

$$
\begin{aligned}
&= - \int_N \operatorname{Tr}\Big(G^{-1} \cdot h_3 \cdot G^{-1} \cdot (h_1 \cdot G^{-1} \cdot h_2)\Big) d\mu_G(x) \\
&\quad - \int_N \operatorname{Tr}\Big(G^{-1} \cdot h_3 \cdot G^{-1} \cdot (h_2 \cdot G^{-1} \cdot h_1)\Big) \\
&\quad + \int_N \operatorname{Tr}_G \Big(G^{-1} \cdot h_1 \cdot G^{-1} \cdot h_2\Big) \frac{1}{2} (\operatorname{Tr}_G h_3) d\mu_G(x) \\
&= - \langle h_3, h_1 G^{-1} h_2 \rangle - (h_3, h_2 G^{-1} h_1 \rangle \\
&\quad + \Big\langle \Big(\frac{1}{4} \operatorname{Tr}_G h_3\Big) h_1, h_2 \Big\rangle + \Big\langle h_1, \Big(\frac{1}{4} \operatorname{Tr}_G h_3\Big) h_2 \Big\rangle.
\end{aligned}
$$

When h_i's are trace-free, note that $h_i \langle h_j, h_k \rangle$ is symmetric with respect to i, j and k. Then we have

Proposition 2.1.1. *For trace-free $(0,2)$-tensors h_1, h_2 and h_3, the Levi-Civita connection D for the L^2-metric can be written as*

$$\langle D_{h_1} h_2, h_3 \rangle = -\frac{1}{2} \langle h_1 G^{-1} h_2, h_3 \rangle - \frac{1}{2} \langle h_2 G^{-1} h_1, h_3 \rangle.$$

In particular we have

$$D_{h_1} h_2 = -\frac{1}{2} h_1 G^{-1} h_2 - \frac{1}{2} h_2 G^{-1} h_1. \tag{2}$$

2.2. L^2-Decomposition of the space of $(0,2)$-tensors. Suppose that G is a hyperbolic metric on N^2, i.e., the sectional curvature $K_G \equiv -1$. And let h be a $(0,2)$-tensor which induces a deformation of G preserving the curvature,

$$\frac{d}{dt} K_{G+\mathrm{th}} \Big|_{t=0} = 0.$$

Proposition 2.2.1. *h satisfies a partial differential equation globally on N:*

$$-(\Delta_G - 1) \operatorname{Tr}_G h + \delta_G \delta_G h = 0. \tag{3}$$

This is sometimes called the Lichnerowitz formula [6].

Proof. In dimension two, the Ricci tensor and the sectional curvature are related by $R_{\alpha\beta} = K G_{\alpha\beta}$. Using the Christoffel symbols we have the following expression for the Ricci tensor.

$$R_{\alpha\beta} = \sum_{\gamma,\delta} \{ \Gamma^{\gamma}_{\alpha\beta,\gamma} - \Gamma^{\gamma}_{\gamma\alpha,\beta} + \Gamma^{\gamma}_{\gamma\delta} \Gamma^{\delta}_{\beta\alpha} - \Gamma^{\gamma}_{\beta\alpha} \Gamma^{\delta}_{\gamma\alpha} \}$$

where

$$\Gamma^{\gamma}_{\alpha\beta} = \frac{1}{2} G^{\gamma\delta}(G_{\alpha\delta,\beta} + G_{\beta\delta,\alpha} - G_{\alpha\beta,\delta}).$$

When the metric varies in t so that $G^t = G + \mathrm{th}$,

$$\dot{\Gamma}^{\gamma}_{\alpha\beta} = \frac{1}{2} G^{\gamma\delta}(h_{\alpha\delta;\beta} + h_{\beta\delta;\alpha} - h_{\alpha\beta;\delta})$$

and therefore

$$\begin{aligned}\dot{R}_{\alpha\beta} =& \frac{1}{2} G^{\gamma\delta}(h_{\alpha\delta;\beta\gamma} + h_{\beta\delta;\alpha\gamma} - h_{\alpha\beta;\delta\gamma} \\ &\qquad\qquad - h_{\alpha\delta;\gamma\beta} + h_{\gamma\delta;\alpha\beta} - h_{\alpha\gamma;\delta\beta}) \\ =& - G^{\gamma\delta} h_{\alpha\beta;\gamma\delta} + \frac{1}{2} G^{\gamma\delta}(h_{\alpha\delta;\beta\gamma} + h_{\beta\delta;\alpha\gamma}).\end{aligned}$$

Since $2K = \sum_{\alpha,\beta} G^{\alpha\beta} R_{\alpha\beta}$, we have

$$\begin{aligned}2\dot{K} =& - \sum_{\alpha,\beta} h^{\alpha\beta} R_{\alpha\beta} + G^{\alpha\beta} \dot{R}_{\alpha\beta} \\ =& \operatorname{Tr}_G h - \Delta \operatorname{Tr}_G h + \delta_G \delta_G h \\ =& (-\Delta + 1) \operatorname{Tr}_G h + \delta_G \delta_G h = 0,\end{aligned}$$

where Δ is the Laplacian with respect to G. q.e.d.

Definition 2.2.1. For any $(0,2)$-tensor, we define a differential operator $\mathcal{L}_G$ as

$$\mathcal{L}_G h \equiv -(\Delta_G - 1) \operatorname{Tr}_G h + \delta_G \delta_G h.$$

Note that the kernel of the differential operator $\mathcal{L}_G$ is the tangent space of $\mathcal{M}_{-1}$ at the hyperbolic metric G.

We now obtain an expression for the adjoint operator $\mathcal{L}_G^*$ of $\mathcal{L}_G$. By definition,

$$\begin{aligned}
< \mathcal{L}_G^* f, h >= &< f, \mathcal{L}_G h > \\
&= \int_N f(x)\mathcal{L}_G h(x) d\mu_G(x) \\
&= \int_N f(x)(-\Delta_G + 1)\operatorname{Tr}_G h(x) d\mu_G(x) \\
&\quad + \int_N f(x)\delta_G\delta_G h(x) d\mu_G(x) \\
&= \int_N (-\Delta_G + 1) f(x) \operatorname{Tr}_G h(x) d\mu_G(x) \\
&\quad + \int_N \langle \operatorname{Hess}_G f, h\rangle_{G(x)} d\mu_G(x) \\
&= \int_N \langle \{(-\Delta_G + 1) f\} G, h\rangle_{G(x)} d\mu_G(x) \\
&\quad + \int_N \langle \operatorname{Hess}_G f, h\rangle_{G(x)} d\mu_G(x) \\
&= \int_N \langle \{(-\Delta_G + 1) f\} G + \operatorname{Hess}_G f, h\rangle_{G(x)} d\mu_G(x).
\end{aligned}$$

Proposition 2.2.2. *For a function f defined on N, the adjoint operator $\mathcal{L}^*$ of $\mathcal{L}$ is given as*

$$\mathcal{L}^* f \equiv \{(-\Delta_G + 1) f\} G + \operatorname{Hess}_G f.$$

We are now ready to state the theorem about the L^2-decomposition of symmetric $(0,2)$-tensors. It is a statement specific to two dimension, while in higher dimensional cases A. Fischer and J. Marsden [6] obtained a more general decomposition of the deformation space of a constant scalar curvature metric.

Theorem 2.2.1. *Suppose that G is a hyperbolic metric on N^2, and that h is a smooth symmetric $(0,2)$-tensor defined over N. Then there is a unique L^2 orthogonal decomposition of h as a tangent vector belonging to $T_G\mathcal{M}$ as follows:*

$$h = P_G(h) + L_X G + \mathcal{L}^* f, \tag{4}$$

where $L_X G$ is the Lie derivative of G in the direction of a vector field X on N, and X, f, P_G satisfy the following equations:

$$\delta_G(L_X h) = \delta_G h, \tag{5}$$

$$\mathcal{L}_G\mathcal{L}_G^* f = \mathcal{L}_G h, \tag{6}$$

$$P_G(h) = h - L_X G - \mathcal{L}^* h. \tag{7}$$

Proof. First note that the Lie derivative is the adjoint operator of δ_G up to a sign, that is,

$$< L_X G, h >=< X, -\delta_G h >=< -\delta_G^* X, h > .$$

Thus the operator $\delta_G L_X G = (-\delta_G \delta_G^* X)$ is self-adjoint with respect to the L^2-metric. It is of the second order and elliptic, with its kernel trivial, hence its cokernel is trivial as well. To see that the kernel is indeed trivial, note that

$$< \delta_G \delta_G^* X, X >=< \delta_G^* X, \delta_G^* X >= 0,$$

hence that $\delta\delta^* X = 0$ implies that $\delta_G^* X = -L_X G = 0$. But since there is no non-zero Killing vector field on a closed hyperbolic manifold, we know that $X = 0$. Therefore given h, there exists a smooth solution X to the equation 31 , and it is unique.

Similarly, the operator $\mathcal{L}_G\mathcal{L}_G^*$ is clearly self-adjoint, and a simple calculation shows that the principal symbol of the operator is Δ^2, and hence the operator is elliptic of the fourth order. Its kernel is also trivial. To see this, suppose that $\mathcal{L}_G^* f = 0$. Then

$$\begin{aligned}\mathcal{L}_G^* f =& \{(-\Delta_G + 1)f\}G + \mathrm{Hess}_G\, f\\ =& 0.\end{aligned}$$

Now take the trace of both sides with respect to the metric G, and obtain

$$\begin{aligned}2(-\Delta_G + 1)f + \Delta_G f =& -\Delta_G f + 2f\\ =& 0.\end{aligned}$$

Since Δ_G is a nonpositive operator, i.e., all the eigenvalues λ of $\Delta_G f = \lambda f$ are negative, $-\Delta_G f + 2f = 0$ implies that $f \equiv 0$. Thus, given $\mathcal{L}_G\mathcal{L}_G^* f = 0$, we have;

$$\begin{aligned}< \mathcal{L}_G\mathcal{L}_G^* f, f >=& < \mathcal{L}_G^* f, \mathcal{L}_G f >\\ =& 0.\end{aligned}$$

And hence $\mathcal{L}_G^* f = 0$. But we have just seen that this implies that $f \equiv 0$.

To show the orthogonality of the decomposed spaces, we first claim that $\delta_G \mathcal{L}_G^* f = 0$. To do so, choose a normal coordinate y^i so that $G_{ij} = \delta_{ij}$ and $G_{ij;k} = 0$ for all i, j, k, where ";" in the subscript denotes the covenant derivative. Then

$$\begin{aligned}\delta_G \mathcal{L}_G^* f &= \delta_G\{(-\Delta_G f + f)G + \mathrm{Hess}_G\, f\} \\ &= -\{(\Delta_G f)_j + f_j\}\delta_{ij} + f_{ij;j} \\ &= -(\Delta_G f)_i + f_i + f_{jj;i} + R_{ij} f_j \\ &= -(\Delta_G f)_i + f_i + (\Delta_G f)_i - f_i \\ &= 0.\end{aligned}$$

Here we have used the following fact:

$$f_{ij;j} = f_{jj;i} + R_{ij} f_j$$

where R_{ij} is the Ricci tensor. Given $R_{ij} = K_G G_{ij}$, we have $R_{ij} = -\delta_{ij}$.

Secondly we claim $\mathcal{L}_G L_X G = 0$. This is so since $L_X G$ is a $(0,2)$-tensor in $T_G\mathcal{M}_{-1}$, and since $T_G\mathcal{M}_{-1}$ is the kernel of $\mathcal{L}_G$.

Now the orthogonality follows formally:

$$\begin{aligned}< L_X G, \mathcal{L}_G f > &= < -\delta_G^* X, \mathcal{L}_G^* f > \\ &= < -X, \delta_G \mathcal{L}^* f > \\ &= 0.\end{aligned}$$

$$\begin{aligned}< P_G(h), L_X G > &= < h - L_X G - \mathcal{L}_G^* f, L_X G > \\ &= < h - L_X G - \mathcal{L}_G^* f, -\delta_G^* X > \\ &= < \delta_G h - \delta_G L_X G - \delta_G \mathcal{L}^* f, -X > \\ &= < \delta_G h - \delta_G L_X G, -X > \\ &= 0,\end{aligned}$$

$$\begin{aligned}< P_G(h), \mathcal{L}_G^* f > &= < \mathcal{L}_G P_G(h), f > \\ &= < (h - L_X G - \mathcal{L}_G^* f), \mathcal{L}_G^* f > \\ &= < \mathcal{L}_G h - \mathcal{L}_G L_X G - \mathcal{L}_G \mathcal{L}_G^* f, f > \\ &= < \mathcal{L}_G h - \mathcal{L}_G \mathcal{L}_G^* f, f > \\ &= 0.\end{aligned}$$

q.e.d.

We present one more calculation, which will be used in the next section.

Proposition 2.2.3. *The third component $\mathcal{L}^*f$ of the decomposition above has the following expression;*

$$\mathcal{L}_G^* f = \{-(\Delta - 2)^{-1}\mathcal{L}h\}G + \operatorname{Hess}_G f.$$

Equivalently

$$\{(\Delta - 1)f\}G = \{(\Delta - 2)^{-1}\mathcal{L}h\}G.$$

Proof. By definition we have

$$\begin{aligned}
\mathcal{L}\mathcal{L}^* f =& -(\Delta - 1)\operatorname{Tr}_G(\mathcal{L}^* f) + \delta_G\delta_G(\mathcal{L}^* f)\\
=& -(\Delta - 1)\operatorname{Tr}_G\{[(-\Delta - 1)f]G + \operatorname{Hess}_G f\}\\
& + \delta_G\delta_G\{[(-\Delta - 1)f]G + \operatorname{Hess}_G f\}\\
=& -(\Delta - 1)\{-2(\Delta - 1)f + \Delta f\} + \{(-\Delta f + f)\delta_{ij} + f_{ij}\}_{;ij}\\
=& -(\Delta - 1)\{-(\Delta - 2)f\} + \Delta(-\Delta f + f\} + f_{ij;ij}\\
=&(\Delta - 2)(\Delta - 1)f - \Delta(\Delta - 1)f + \Delta^2 f - \Delta f\\
=&(\Delta - 2)(\Delta - 1)f.
\end{aligned}$$

Since $\mathcal{L}\mathcal{L}^* f = \mathcal{L}h$, we have

$$(\Delta - 2)(\Delta - 1)f = \mathcal{L}h,$$

or equivalently (noting that $(\Delta - 2)$ is invertible)

$$(\Delta - 1)f = (\Delta - 2)^{-1}\mathcal{L}h.$$

Hence

$$\begin{aligned}
\mathcal{L}^* f =& -\{(\Delta - 1)f\}G + \operatorname{Hess}_G f\\
=& -\{(\Delta - 2)^{-1}\mathcal{L}h\}G + \operatorname{Hess}_G f.
\end{aligned}$$

q.e.d.

2.3. Weil-Petersson geodesic equation. Let $\mathcal{M}^s$ be the space of Riemannian metrics on N^2 whose components axe in $H^s(N, \mathbf{R})$, and denote the space of hyperbolic metrics by $\mathcal{M}^s_{-1} \in \mathcal{M}^s$. Then we will show

Theorem 2.3.1. *$\mathcal{M}^s_{-1}$ is a smooth submanifold of $\mathcal{M}$.*

Proof. The sectional curvature is a map

$$K : \mathcal{M}^s \to \mathbf{R},$$

and $\mathcal{M}^s_{-1} = K^{-1}\{-1\}$. By the implicit function theorem, it suffices to show that -1 is a regular value for K, i.e., that if $K(G) = -1$ for some $G \in \mathcal{M}^s_{-1}$, then

$$DK(G)(= \mathcal{L}_G) : T_G\mathcal{M}^s \to H^{s-2}(N, \mathbf{R})$$

is surjective. But in the previous section, we have shown that $\mathcal{L}^*_G$ is infective, which in turn implies that $\mathcal{L}_G$ is surjective. q.e.d.

Let $\mathcal{D}_0$ be the group of smooth diffeomorphisms on N^2 homotopic to the identity map: $N \to N$. Then we define the quotient space $\mathcal{M}_{-1}/\mathcal{D}_0$ to be the space of hyperbolic metrics with the following equivalent relation;

$$G \sim G' \text{ if exists some } \phi \text{ in } \mathcal{D}_0 \text{ such that } G' = \phi^* G.$$

As we have seen in the L^2-decomposition theorem in the previous section, any symmetric $(0,2)$-tensor h can be decomposed as

$$h = P_G h + L_X G + \mathcal{L}^*_G f$$

as a tangent vector in $T_G\mathcal{M}$. Note that $P_G h + L_G X$ is tangent to $T_G\mathcal{M}_{-1}$ by construction. Also, given a family of diffeomorphisms $\phi_t : N \to N$ with $\phi_0 = \mathrm{Id}_N$, $\phi^*_t G$ is clearly in $\mathcal{M}_{-1}$. Since by definition

$$L_X G = \left.\frac{d}{dt}\right|_{t=0} \phi^*_t G,$$

where $\left.\frac{d}{dt}\right|_{t=0} \phi_t$, it follows that $L_G X$ is an element of $T_G\mathcal{M}_{-1}$ as well. Hence $P_G h$ is also tangent to $T_G\mathcal{M}_{-1}$.

Remark. By construction, note that a symmetric $(0,2)$-tensor tangent to the Teichmüller space $T_G(\mathcal{M}_{-1}/\mathcal{D}_0)$ is in the kernel of the operator $\mathcal{L}_G$, as well as the kernel of the divergence operator δ_G. The first condition says that h satisfies

$$-(\Delta_G - 1)\operatorname{Tr}_G h + \delta_G \delta_G h = 0,$$

and the second condition is

$$\delta_G h = 0.$$

Thus h satisfies

$$-(\Delta_G - 1)\operatorname{Tr}_G h = 0.$$

But the Laplacian defined on N is a negative operator as we have seen, hence the equation above implies that

$$\operatorname{Tr}_G h \equiv 0.$$

Therefore we characterize the tangent space of the Teichmüller space as;

$$T_G(\mathcal{M}_{-1}\mathcal{D}_0) = \{h : \operatorname{Tr}_G h = 0 \text{ and } \Delta_G h = 0\}. \tag{8}$$

These are sometimes called *trace-free, transverse* conditions. It should be noted here that a deformation being tangent to the Teichmüller space is equivalent to saying that the deformation is the Lie derivative of the metric G in the direction of a vector field induced by a harmonic Beltrami differential. (See Wolpert's paper [27])

We now have three spaces: $\mathcal{M}, \mathcal{M}_{-1}$ and $\mathcal{M}_{-1}/\mathcal{D}_0$. The previous theorem says that $\mathcal{M}_{-1}$ is a smooth submanifold of $\mathcal{M}$. As for the relationship between $\mathcal{M}_{-1}$ and $\mathcal{M}_{-1}/\mathcal{D}_0$, we have so far shown the following:

Proposition 2.3.1. *The quotient map $\mathcal{P} : \mathcal{M}_{-1} \to \mathcal{M}_{-1}/\mathcal{D}_0)$ is a Riemannian subversion with respect to the L^2-metric where the tangent space $T_G\mathcal{M}_{-1}$ is decomposed into the horizontal subspace $T_G(\mathcal{M}_{-1}/\mathcal{D}_0)$ and the vertical subspace $T_G\mathcal{D}_0$.*

Fischer and Tromba [7] identified the L^2 metric restricted to $\mathcal{M}_{-1}/\mathcal{D}_0)$ with the metric historically called *Weil-Petersson* metric.

A standard result from differential geometry says (see [20] for example) that given a Riemannian submersion $\mathcal{P} : \mathcal{M}_{-1} \to \mathcal{M}_{-1}/\mathcal{D}_0)$, a Weil-Petersson geodesic $\sigma(t)$ in the Teichmüller space $\mathcal{M}_{-1}/\mathcal{D}_0)$ can be uniquely lifted to a geodesic G_t in $\mathcal{M}_{-1}$ once the initial point G_0 is specified in $\mathcal{M}_{-1}$ with $\pi(G_0) = \sigma(0)$. Moreover the length of the geodesic is preserved by the lifting.

Definition 2.3.1. The map $\prod_G : T_G\mathcal{M} \to (T_G\mathcal{M}_{-1})^\perp$ is defined to be the projection operator such that given the L^2 decomposition of a $(0,2)$-tensor h in $T_G\mathcal{M}$, i.e.,

$$h = P_G(h) + L_X G + \mathcal{L}_G^* f,$$

it is a projection onto the third component;

$$\prod_G(h) = \mathcal{L}_G^* f.$$

Now suppose that G_t is a horizontal lift of a Weil-Petersson geodesic $\sigma(t)$. One can consider G_t as a path of hyperbolic metrics in $\mathcal{M}$. Since G_t is a path in $\mathcal{M}_{-1}$, it satisfies the tangency condition:

$$\prod_{G_t}(\dot{G}_t t) = 0, \tag{9}$$

where the dot denotes the t-derivative. This says that the third component (of the L^2-decomposition 2.2.1) of the tangent vector to the path G_t is zero. Also since G_t is a horizontal lift of a path in the Teichmüller space $\mathcal{M}_{-1}/\mathcal{D}_0$, the tangent vector $\dot{G}_t$ is contained in the first component of the L^2-decomposition, i.e., it is orthogonal to both $T_{G_t}\mathcal{D}_0$ and $(T_{G_t}\mathcal{M}_{-1})^\perp$, or equivalently $P_{G_t}(\dot{G}_t) = \dot{G}_t$.

The fact that the path G^t is a geodesic in $\mathcal{M}_{-1}$ implies that the curvature of the curve G_t has to lie entirely in the direction

$$(T_{G_t}\mathcal{M}_{-1})^\perp \subset T_{G_t}\mathcal{M}.$$

Thus we must have

$$D_{\dot{G}_t}\dot{G}_t = \prod_{G_t}(D_{\dot{G}_t}\dot{G}_t), \tag{10}$$

where D stands for the covariant derivative for the L^2-metric.

We now state the main result of this chapter. Though it is highly technical, it will be crucial later in proving the main convexity theorem.

Theorem 2.3.2. *Given a horizontal lift G_t of a Weil-Petersson geodesic $\sigma(t)$, we have the following equality:*

$$\left.\frac{d^2}{dt^2}G_t\right|_{t=0} = \left(\frac{1}{4}\|\dot{G}_0\|^2 + \alpha\right) G_0 + L_{X+Y}G_0, \tag{11}$$

where $\alpha = \frac{1}{2}(\Delta_G - 2)^{-1}\|\dot{G}_0\|^2 \geq 0$ and X, Y are vector fields on N^2.

Remark. It should be rioted that A. Tromba in [24] has done a similar calculation to obtain an expression of the second time derivative of a horizontal lift of a Weil-Petersson geodesic, according to which,

$$\left.\frac{d^2}{dt^2}G_t\right|_{t=0} = \frac{1}{2}\|\dot{G}_0\|^2 G_0 + L_X G_0$$

with $\mathrm{tr}_{G_0}(L_X G_0) = 0$.

Proof. We may choose a coordinate system so that it is normal at a point, i.e., $G_{ij} = \delta_{ij}$ and $G_{ij;k} = 0$ for all i, j, and k. With these coordinates, the trace-free, transverse conditions for a tensor h are respectively written as:

$$h_{11} + h_{22} = 0, \tag{12}$$

$$h_{1j;j} + h_{2j;j} = 0 \quad \text{for } j = 1, 2. \tag{13}$$

Using the formula obtained previously for the Levi-Civita connection for the L^2-metric on $\mathcal{M}$, we have

$$\begin{aligned} D_{\dot G_t} \dot G \Big|_{t=0} &= \ddot G_0 + D_{\dot G_0} \dot G_0 \\ &= \ddot G_0 - \dot G_0 \cdot G_0^{-1} \cdot \dot G_0, \end{aligned}$$

and this,we can rewrite equation (10) as follows:

$$\begin{aligned} \ddot G_0 + D_{\dot G_0} \dot G_0 &= \prod_{G_0} (\ddot G_0 + D_{\dot G_0} \dot G_0) \\ &= \prod_{G_0} (\ddot G_0 + \prod_{G_0} (D_{\dot G_0} \dot G_0); \end{aligned}$$

equivalently,

$$\begin{aligned} \prod_{G_0} (\ddot G_0) &= \ddot G_0 + D_{\dot G_0} \dot G_0) - \prod_{G_0} (D_{\dot G_0} \dot G_0) \\ &= \ddot G_0 + \left(D_{\dot G_0} \dot G_0 \right)^{T_{G_0} \mathcal{M}_{-1}}, \end{aligned}$$

where the last term is the component of the symmetric $(0, 2)$-tensor $D_{\dot G_0} \dot G_0$, tangent to $\mathcal{M}_{-1}$ at G_0.

Now we will differentiate equation (9) with respect to t:

$$\begin{aligned} 0 &= \frac{d}{dt}\Big|_{t=0} \prod_{G_t} (\dot G_t) \\ &= \frac{d}{dt}\Big|_{t=0} \prod_{G_t} (\dot G_0) + \prod_{G_0} (\ddot G_0) \\ &= \frac{d}{dt}\Big|_{t=0} \prod_{G_t} (\dot G_0) + \ddot G_0 + \left(D_{\dot G_0} \dot G_0 \right)^{T_{G_0} \mathcal{M}_{-1}}, \end{aligned}$$

where the last equality is due to equation (14). Rearranging the terms gives

$$\ddot{G}_0 = -\frac{d}{dt}\Big|_{t=0} \prod_{G_t}(\dot{G}_0) - \left(D_{\dot{G}_0}\dot{G}_0\right)^{T_{G_0}\mathcal{M}_{-1}}.$$

Recall that $D_{\dot{G}_0}\dot{G}_0 = -\dot{G}_0 \cdot G_0^{-1} \cdot \dot{G}_0$. We will use the notation h and $\dot{G}_0$ interchangeably. Also G below is G_0. Note $h \cdot G^{-1} \cdot h = \sum_{k=1}^{2} h_{ik}h_{kj} = \frac{1}{2}\|h\|^2 G$. This is simply due to the observation that

$$\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} \\ h_{12} & -h_{11} \end{pmatrix}$$

and that

$$h \cdot G^{-1} \cdot h = h_{ik}h_{kj} = \begin{pmatrix} h_{11}^2 + h_{12}^2 & 0 \\ 0 & h_{11}^2 + h_{12}^2 \end{pmatrix} = \frac{1}{2}\|h\|^2 G.$$

Note that a symmetric $(0,2)$-tensor of the form fG where f is a function on N^2 (a deformation in a conformal direction) and a trace-free symmetric $(0,2)$-tensor is pointwise orthogonal with respect to the inner product $\langle\langle *, *\rangle\rangle_x$, and hence are L^2 orthogonal to each other.

In particular, since $P_G(h \cdot G^{-1} \cdot h)$ is trace-free, the tensor $h \cdot G^{-1} \cdot h = \frac{1}{2}\|h\|^2 G$ induces no deformation in the trace-free transverse direction, that is,

$$P_G(h \cdot G^{-1} \cdot h) = 0.$$

This in turn says that the term $\left(D_{\dot{G}_0}\dot{G}_0\right)^{T_{G_0}\mathcal{M}_{-1}}$, which appeared in equation (14), has no component tangent to the Teichmüller space, that is, $\left(D_{\dot{G}_0}\dot{G}_0\right)^{T_{G_0}\mathcal{M}_{-1}} = L_X G_0$. for some vector field X on N^2.

Putting the above observations together, we have so far shown

$$\begin{aligned} \ddot{G}_0 =& \frac{d}{dt} \prod_{G_t}(h)\Big|_{t=0} - \left(D_{\dot{G}_0}\dot{G}_0\right)^{T_{G_0}\mathcal{M}_{-1}} \\ =& \frac{d}{dt}\Big|_{t=0} \prod_{G_t}(h)\Big|_{t=0} - L_X G_0. \end{aligned}$$

Now we will calculate the term $\frac{d}{dt}\prod_{G_t}(h)\Big|_{t=0}$ above. Using the formula in Proposition 2.2.3 we obtain

$$
\begin{aligned}
\frac{d}{dt}\prod_{G_t}(h)\Big|_{t=0} =& \frac{d}{dt}\mathcal{L}^*_{G_t} f_t\Big|_{t=0} \\
=& -\left(\frac{d}{dt}\{(\Delta_{G_t}-2)^{-1}\mathcal{L}_{G_t}h\}G_t\Big|_{t=0}\right) \\
&+\left(\frac{d}{dt}\operatorname{Hess}_{G_t} f_t\Big|_{t=0}\right) \\
=& -(\Delta_G - 2)\left(\frac{d}{dt}\mathcal{L}_{G_t}h\Big|_{t=0}\right)G \\
&+\operatorname{Hess}_{G_0}\left(\frac{d}{dt}f_t\Big|_{t=0}\right).
\end{aligned} \tag{14}
$$

where the function f_t is the solution to $\mathcal{L}_{G_t}\mathcal{L}^*_{G_t}f_t = \mathcal{L}_{G_t}h$. To justify the last equality, we need to show

$$
\left(\frac{d}{dt}\operatorname{Hess}_{G_t} f_t\Big|_{t=0}\right) = \left(\frac{d}{dt}\operatorname{Hess}_{G_0} f_t\Big|_{t=0}\right),
$$

which follows from observing

$$
\begin{aligned}
\left(\frac{d}{dt}\operatorname{Hess}_{G_t} f_t\Big|_{t=0}\right) =& \lim_{t\to 0} 1/t(\operatorname{Hess}_{G_t} f_t - \operatorname{Hess}_{G_t} f_0 \\
&\qquad + \operatorname{Hess}_{G_t} f_0 - \operatorname{Hess}_{G_0} f_0) \\
=& \lim_{t\to 0} 1/t(\operatorname{Hess}_{G_t} f_t - \operatorname{Hess}_{G_t} f_0) \\
=& \operatorname{Hess}_{G_0}\frac{d}{dt}f_t\Big|_{t=0}.
\end{aligned}
$$

Now we proceed to calculate $\frac{d}{dt}\mathcal{L}_{G_t}h\Big|_{t=0}$.

Lemma 2.3.1. *In the given setting, we have*

$$
\frac{d}{dt}(\delta_{G_t}h)\Big|_{t=0} = -\frac{3}{4}\nabla\|h\|^2.
$$

Proof. By definition, we have

$$
\begin{aligned}
(\delta_G h)_i =& G^{jk}h_{ij;k} \\
=& G^{jk}(h_{ij,k} - h_{pj}\Gamma^p_{ik} - h_{ip}\Gamma^p_{jk}).
\end{aligned}
$$

where ";" stands for the covariant derivative, and , the partial derivative. Differentiating the expression at $t=0$ as G deforms in the direction h, we obtain

$$\begin{aligned}(\delta_{G_t}h)_i^\bullet =& -h^{jk}h_{ij;k}\\ &- G^{jk}h_{pj}\left(\frac{1}{2}G^{pq}(G_{iq;k}+G_{kq;i}-G_{ik;q})\right)^\bullet\\ &- G^{jk}h_{ip}\left(\frac{1}{2}G^{pq}(G_{jq;k}+G_{kq;j}-G_{jk;q})\right)^\bullet\\ =& -h^{jk}h_{ij;k}\\ &- \frac{1}{2}G^{jk}h_{pj}G^{pq}(h_{iq;k}+h_{kq;i}-h_{ik;q})\\ &- \frac{1}{2}G^{jk}h_{ip}G^{pq}(h_{jq;k}+h_{kq;j}-h_{jk;q}).\end{aligned}$$

We may assume that $G_{ij}=\delta_{ij}$. Then h being traceless transverse implies that $h_{ij;k}$ axe fully symmetric, i.e.,

$$\begin{aligned}h_{11;2}&=h_{12;1}=h_{21;1}=-h_{22;2},\\ h_{22;1}&=h_{21;2}=h_{12;2}=-h_{11;1},\end{aligned}$$

from which it follows that

$$\begin{aligned}(\delta_G h)_i^\bullet =& -\sum_{j,k}h_{jk}h_{jk;i}-\frac{1}{2}\sum_{j,p}h_{pj}h_{pj;i}\\ =& -\frac{3}{2}\sum_{p,q}h_{pq}h_{pq;i}\\ =& -\frac{3}{4}\|h\|^2_{;i}.\end{aligned}$$

Therefore we have $(\delta_G h)^\bullet = -\frac{3}{4}\nabla\|h\|^2$ q.e.d.

Now we observe

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}(\mathcal{L}_{G_t}h) =& [-(\Delta_{G_t}-1)\operatorname{Tr}_{G_t}h+\delta_{G_t}\delta_{G_t}h]^\bullet\\ =& -(\Delta_{G_0}-1)(\operatorname{Tr}_{G_t}h)^\bullet+(\delta_G\delta_{G_t}h)^\bullet\\ =& -(\Delta_{G_0}-1)(h^{ij}h_{ji})+\delta_{G_0}(\delta_{G_t}h)^\bullet\\ =& -(\Delta_{G_0}-1)\|h\|^2+\delta_{G_0}\left(-\frac{3}{4}\nabla\|h\|^2\right)\\ =& -(\Delta_{G_0}-1)\|h\|^2-\frac{3}{4}\Delta_{G_0}\|h\|^2\\ =& (\Delta_{G_0}-1)\,\|h\|^2.\end{aligned}$$

Here $G_0 = G$. Hence going back to equation (14), we have

$$
\begin{aligned}
\left(\frac{d}{dt}\prod_{G_t}\right)(h)\Big|_{t=0} &= -\left(\Delta_G - 2\right)^{-1}\left(\frac{d}{dt}\mathcal{L}_{G_t}h\Big|_{t=0}\right)G + \frac{d}{dt}\operatorname{Hess}_{G_t} f\Big|_{t=0} \\
&= -\left(\Delta_G - 2\right)^{-1}\left(\frac{1}{4}\Delta_G - 1\right)\|h\|^2 G \\
&\quad + \operatorname{Hess}_{G_0}\left(\frac{d}{dt}f_t\Big|_{t=0}\right) \\
&= -\left(\Delta_G - 2\right)^{-1}\frac{1}{4}(\Delta_G - 2 - 2)\|h\|^2 G \qquad (15) \\
&\quad + \operatorname{Hess}_G\left(\frac{d}{dt}f_t\Big|_{t=0}\right) \\
&= -\frac{1}{4}\|h\|^2 G + \frac{1}{2}\left(\Delta_G - 2\right)^{-1}\|h\|^2 G \\
&\quad + L_{\nabla\left(\frac{d}{dt}\big|_{t=0} f_t\right)} G_0,
\end{aligned}
$$

Denote the vector field $-\nabla\left(\frac{d}{dt}f_t\right)$ by Y.

Lemma 2.3.2. *If $(\Delta_G - 2)^{-1}f = u$ with $f \geq 0$, then $u \leq 0$.*

Proof. The hypothesis implies $\Delta_G u - 2u = f$. Suppose $\max u > 0$. Then at a point where the maximum is achieved, we have $\Delta_G u \leq 0$, and $-2u < 0$. Hence $f = \Delta_G u - 2u < 0$ which contradicts to the hypothesis $f \geq 0$. q.e.d.

Applying the lemma above when $f = \|h\|^2$, we know that $\frac{1}{2}(\Delta_G - 2)^{-1}\|h\|^2 \leq 0$, which together with the lemma yields

$$
\frac{d}{dt}\prod_{G_t}(h)\Big|_{t=0} = -\left(\frac{1}{4}\|h\|^2 + \alpha\right)G + L_Y G_0,
$$

where $\frac{1}{2}(\Delta_G - 2)^{-1}\|h\|^2 \geq 0$ and $Y = -\nabla\left(\frac{d}{dt}f_t\Big|_{t=0}\right)$.

Finally, going back to the Weil-Petersson geodesic equation (10) we get

$$\begin{aligned}\ddot{G}_0 &= -\frac{d}{dt}\Big|_{t=0}\prod_{G_t}(h) - L_X G_0\\ &= +\left(\frac{1}{4}\|h\|^2\right)G - \left(\frac{1}{2}(\Delta_G - 2)^{-1}\|h\|^2\right)G + L_Y G_0 - L_X G_0 \qquad (16)\\ &= \left(\frac{1}{4}\|h\|^2 + \alpha\right)G + L_{Y\ -X}\, G_0.\end{aligned}$$

q.e.d.

3. Convexity Theorem

3.1. Proof of the Convexity Theorem.

Theorem 3.1.1(Weil-Petersson convexity of the energy functional). *Under the assumptions above, the function $\mathcal{E}(t)$ is a strictly convex function in t, and hence the energy functional $\mathcal{E} : \mathcal{T}(N) \to \mathbf{R}_+$ is strictly convex along any Weil-Petersson geodesic in $\mathcal{T}(N)$.*

Remark. Here we regard the energy of the maps u_t as a well-defined functional on the Teichmüller space. It is well-defined; for if two hyperbolic metrics on $N^2 2$ differ by a diffeomorphism ϕ homotopic to identity $G_2 = \phi^* G_1$, then the map $\phi : (N, G_2) \to (N, G_1)$ is an isometry, hence if $u : (M, g) \to (N, G_i)$, $i = 1, 2$, are harmonic respectively, then $u_1 = \phi \circ u_2$ and $E(u_1) = E(u_2)$.

Proof. We want to show

$$\frac{d^2}{dt^2}\mathcal{E}(t) > 0.$$

First look at the first t derivative of $\mathcal{E}(t)$.

The energy for a family of harmonic maps $u_t : (M^n, g) \to (N^2, G_t)$ of a given homotopy type is written as

$$\begin{aligned}\mathcal{E}(u_t) =& \frac{1}{2}\int_M \mathrm{tr}_g(u_t^* G_t)d\mu_g\\ =& \frac{1}{2}\int_N g^{ij} G_{t\alpha\beta}\frac{\partial u_t^\alpha}{\partial x^i}\frac{\partial u_t^\beta}{\partial x^j}d\mu_g.\end{aligned}$$

Differentiating this expression with respect to t gives

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(u_t)\Big|_{t=t_0} =&\frac{1}{2}\int_M \mathrm{tr}_g\left(\frac{d}{dt}u_t^*G_t\right)d\mu_g\\ =&\frac{1}{2}\int_M \mathrm{tr}_g\left(u_{t_0}^*\frac{d}{dt}G_t\right)d\mu_g\\ &+\frac{1}{2}\int_M \mathrm{tr}_g\left(\frac{d}{dt}u_t^*G_{t_0}\right)d\mu_g\\ =&\frac{1}{2}\int_M \mathrm{tr}_g\left(u_{t_0}^*\dot{G}_{t_0}\right)d\mu_g\\ &+\frac{1}{2}\int_M \mathrm{tr}_g\left(u_{t_0}^*[L_{W_{t_0}}G_{t_0}]\right)d\mu_g,\end{aligned}$$

where

$$G_t = G_{t_0} + t\dot{G}_{t_0} + \frac{1}{2}t^2\ddot{G}_{t_0}$$

and

$$W_{t_0}(u_{t_0}(x)) = \frac{d}{dt}u_t\Big|_{t=t_0} \in u_{t_0}^*TN.$$

Note here that the second term is the first variation of the energy in the direction of $W_{t_0}(u_{t_0}(x))$ which is zero for u_{t_0} is harmonic. Therefore the first time-derivative of the energy functional is of the form

$$\frac{1}{2}\int_M \mathrm{tr}_g\left(u_{t_0}^*\frac{d}{dt}G_t\right)d\mu_g.$$

As for the second time-derivative, we have

$$\begin{aligned}\frac{d^2}{dt_0^2}\mathcal{E}(u_t)\Big|_{t=t_0} =&\frac{d}{dt_0}\left(\frac{1}{2}\int_M \mathrm{tr}_g(u_{t_0}^*\dot{G}_{t_0}\right)d\mu_g\Big|_{t_0=0}\\ =&\frac{1}{2}\int_M \mathrm{tr}_g(u_0^*\ddot{G}_0)\\ &+\frac{1}{2}\int_M \mathrm{tr}_g\left(u_0^*(L_{W_0}\dot{G}_0)\right)d\mu_g.\end{aligned}$$

As observed above, the term

$$\frac{1}{2}\int_M \mathrm{tr}_g\left(u_{t_0}^*(L_{W_{t_0}}G_{t_0})\right)d\mu_g$$

vanishes for all t_0 due to the fact the u_{t_0}, is harmonic. Differentiating this term with respect to t_0 at $t_0 = 0$ gives

$$\frac{1}{2}\int_M \mathrm{tr}_g\left(u_0^*(L_{W_0}(L_{W_0}G_0))\right) + \frac{1}{2}\int_M \mathrm{tr}_g\left(u_0^*(L_{W_0}\dot{G}_0)\right)d\mu_g = 0.$$

Since the first term above is the second variation of the enrgy $E(u_0)$ in the directions of W_0 and W_0, it follows that

$$\frac{1}{2}\int_M \mathrm{tr}_g \left(u_0^*(L_{W_0}\dot{G}_0)\right) d\mu_g = -\delta^2 E(u_0)(W_0, W_0).$$

Hence we have the following expression for the second time derivative of the energy functional $\mathcal{E}(u_t)$ at $t = 0$:

$$\frac{d^2}{dt^2}\mathcal{E}(u_t)\Big|_{t=0} == \frac{1}{2}\int_M \mathrm{tr}_g(u_0^*\ddot{G}_0) - \delta^2 E(u_0)(W_0, W_0).$$

We will now prove the following proposition.

Proposition 3.1.1. *Under the conditions previously set, we have*

$$\delta^2 E(G)(W_0, W_0) \leq \frac{1}{8}\int_N \|\dot{G}_0\|^2 G_{\alpha\beta} g^{ij} u_i^\alpha u_j^\beta d\mu_g.$$

Once we have this proposition, it follows that

$$\begin{aligned}\frac{d^2}{dt^2}\mathcal{E}(G_t) =& \frac{1}{2}\int_N (\ddot{G}_0)_{\alpha\beta} g^{ij} u_i^\alpha u_j^\beta d\mu_g - \delta^2 \mathcal{E}_{G_0}(W_0, W_0)\\ =& \frac{1}{2}\int_N \left(\frac{1}{4}\|\dot{G}_0\|^2 + \alpha\right) G_{\alpha\beta} g^{ij} u_i^\alpha u_j^\beta d\mu_g\\ &- \delta^2 \mathcal{E}_G(W_0, W_0)\\ \geq& \frac{1}{8}\int_N \left(\|\dot{G}_0\|^2 + 4\alpha\right) G_{\alpha\beta} g^{ij} u_i^\alpha u_j^\beta d\mu_g\\ &- \frac{1}{8}\int_N \|\dot{G}_0\|^2 G_{\alpha\beta} g^{ij} u_i^\alpha u_j^\beta d\mu_g\\ \geq& 0,\end{aligned}$$

which proves the convexity of the energy functional.

Note that the equality holds in the above inequality when α is zero as well as $\langle R^N(W, du)du, W\rangle_{L^2(G)}$ is zero. Since

$$\alpha = -\frac{1}{2}(\Delta_G - 2)^{-1}\|\dot{G}_0\|^2$$

and the operator $\frac{1}{2}(\Delta_G - 2)^{-1}$ has no kernel, $\alpha = 0$ implies $h \equiv 0$. When $\dot{G}_0 \equiv 0$, by definition $W_0 \equiv 0$, hence the quantity

$\langle R^N(W,du)du, W\rangle_{L^2(G)}$ vanishes. This implies that the energy functional is strictly convex. The strict convexity holds even when the image of the harmonic map is a closed geodesic. In that case, the term $\langle R^N(W,du)du, W\rangle_{L^2(G)}$ is zero, and thus the equality holds only when $\alpha = 0$ or equivalently $\dot{G}_0 = 0$. These observation establishes the *strict* convexity of the energy functional parameterized by the Weil-Petersson arc length.

Proof of Proposition 3.1.1. Going back to the equality previously obtained:

$$\frac{1}{2}\int_M \mathrm{tr}_g \left(u_0^*(L_{W_0}\dot{G}_0)\right) d\mu_g = \delta^2 E(u_0)(W_0, W_0).$$

Locally one can express the integral

$$\begin{aligned}\frac{1}{2}\int_M \mathrm{tr}_g \left(u_0^*(L_{W_0}\dot{G}_0)\right) =& \frac{1}{2}\int_N (\dot{G}_0)_{\alpha\beta,\gamma} W^\gamma g^{ij} u_i^\alpha u_j^\beta d\mu_G \\ &+ \int_N (\dot{G}_0)_{\alpha\beta} g^{ij} W_i^\alpha u_j^\beta d\mu_g.\end{aligned}$$

This is obtained by differentiating in t the expression

$$\frac{1}{2}\int_M \mathrm{tr}_g \left(u_t^*\dot{G}_0\right) d\mu_g = \frac{1}{2}\int_M g^{ij}(\dot{G}_0)_{\alpha\beta}(u_t(x))\frac{\partial u_t^\alpha}{\partial x^i}\frac{\partial u_t^\beta}{\partial x^j} d\mu_g.$$

Lemma 3.1.1. *In the current setting, the second variation of the energy has the following expression;*

$$\delta^2 \mathcal{E}_G(W_0, W_0) = -\frac{1}{2}\int_N (\dot{G}_0)_{\alpha\beta}\left(\nabla^E_{\partial/\partial x^i} W\right)^\alpha u_j^\beta g^{ij} d\mu_g, \tag{17}$$

where ∇^E is the covariant derivative on the bundle $E = u^{-1}(TN)$ induced by the connections defined on M and N.

Proof. Two things will be used in the proof; the harmonic map equation and the integration by parts. We now have the following series of equalities:

$$\begin{aligned}
\delta^2\mathcal{E}_G(W_0,W_0) =& -\frac{1}{2}\int_M (\dot{G}_0)_{\alpha\beta,\gamma}W^\gamma g^{ij}u_i^\alpha u_j^\beta d\mu_g \\
&+\int_M (\dot{G}_0)_{\alpha\beta}g^{ij}W_i^\alpha u_j^\beta d\mu_g \\
=& \frac{1}{2}\int_M (\dot{G}_0)_{\alpha\beta,\gamma}W^\gamma g^{ij}(x)u_i^\alpha u_j^\beta d\mu_g \\
&-\frac{1}{2}\int_M (\dot{G}_0)_{\alpha\beta,\gamma}u_i^\gamma g^{ij}W^\alpha u_j^\beta d\mu_g \\
&-\frac{1}{2}\int_M (\dot{G}_0)_{\alpha\beta}g^{ij}W^\alpha u_{ij}^\beta d\mu_g \\
&-\frac{1}{2}\int_M (\dot{G}_0)_{\alpha\beta}g^{ij}W^\alpha u_j^\beta d\mu_g \\
&-\frac{1}{2}\int_M (\dot{G}_0)_{\alpha\beta}g^{ij}W^\alpha u_j^\beta(\sqrt{g})_i dx \\
&+\frac{1}{2}\int_N (\dot{G}_0)_{\alpha\beta}g^{ij}W_i^\alpha u_j^\beta d\mu_g \\
=& -\frac{1}{2}\int_M (\dot{G}_0)_{\alpha\beta}W^\alpha\left(\frac{1}{\sqrt{g}}g^{ij}\sqrt{g}u_{ij}^\beta+\frac{1}{\sqrt{g}}g_i^{ij}\sqrt{g}u_j^\beta\right. \\
&\qquad\qquad\left.+\frac{1}{\sqrt{g}}g^{ij}(\sqrt{g}_i)u_j^\beta\right)d\mu_g \\
&+\frac{1}{2}\int_N (\dot{G}_0)_{\alpha\beta}g^{ij}W_i^\alpha u_j^\beta d\mu_g \\
=& -\frac{1}{2}\int_M (\dot{G}_0)_{\alpha\beta}W^\alpha\Delta_g u^\beta d\mu_g \\
&+\frac{1}{2}\int_M (\dot{G}_0)_{\alpha\beta}g^{ij}W_i^\alpha u_j^\beta d\mu_g \\
=& -\frac{1}{2}\int_M (\dot{G}_0)_{\alpha\beta}W^\alpha g^{ij}\Gamma_{\gamma\delta}^\beta u_i^\gamma u_j^\delta d\mu_g \\
&+\frac{1}{2}\int_M (\dot{G}_0)_{\alpha\beta}g^{ij}W_i^\alpha u_j^\beta d\mu_g \\
=& \frac{1}{2}\int_M (\dot{G}_0)_{\alpha\beta}g^{ij}(\nabla_i^E W)^\alpha u_i^\beta d\mu_g.
\end{aligned}$$

The first equality is due to what we have observed above, and the second one is obtained by integrating one half of the term $\int_M (\dot{G}_0)_{\alpha\beta}g^{ij}W_i^\alpha u_j^\beta d\mu_g$ of the previous line by parts.

The third equality implies that the first two terms of the previous step cancel each other, once one notes that the $\dot{G}_0 = h$ tensor is fully symmetric, i.e.,

$$\begin{aligned} h_{11} &= -\, h_{22}, \quad h_{12} = h_{21}, \\ h_{12,2} &= h_{21,2} = h_{22,1} = -h_{11,1}, \\ h_{12,1} &= h_{21,1} = h_{22,2} = -h_{11,2} \end{aligned}$$

with respect to $G_{\alpha\beta} = \delta_{\alpha\beta}$ since $\dot{G}_0 = h$ satisfies the tracefree-transverse condition.

The fourth equality follows from the harmonic map equation u satisfies;

$$\Delta_g u^\beta + g^{ij} \Delta^\beta_{\gamma\delta} u^\gamma_i u^\delta_j = 0.$$

Finally one obtains the last equality by noting that, with respect to $G_{\alpha\beta} = \delta_{\alpha\beta}$,

$$(\nabla^E_i W)^\alpha = W_i + \Gamma^\alpha_{\varepsilon\nu}(u) W^\varepsilon u^\nu_i$$

and also the symmetry of the Christoffel symbols, i.e.,

$$\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^1_{11} = -\Gamma^1_{22}$$

and

$$\Gamma^1_{12} = \Gamma^1_{21} = \Gamma^2_{22} = -\Gamma^2_{11}$$

as well as the relations $(\dot{G}_0)_{11} = h_{11} = -h_{22}, h_{12} = h_{21}$, and $g_{ij} = g_{ji}$.

q.e.d.

With respect to $G_{\alpha\beta} = \delta_{\alpha\beta}$, as well as $g_{ij} = \delta_{ij}$, we have the following inequality, which is due to the Cauchy-Schwarz inequality. Recall $h =$

$\dot{G}_0$ below.

$$\begin{aligned}
\delta^2\mathcal{E}_G(W_0,W_0) =&\sum_{i=1}^n\left(-\frac{1}{2}\int_M h_{\alpha\beta}(\nabla_i^{u^{-1}(TN)}W)^\alpha u_i^\beta d\mu_g\right)\\
=&\sum_{i=1}^n\left(-\frac{1}{2}\int_M\{h_{11}(\nabla_iW)^1+h_{12}(\nabla_iW)^2\}u_i^1d\mu_g\right.\\
&\qquad\left.-\frac{1}{2}\int_M\{h_{21}(\nabla_iW)^1+h_{22}(\nabla_iW)^2\}u_i^2d\mu_g\right)\\
=&\sum_{i=1}^n\left(-\frac{1}{2}\int_M\{h_{11}(\nabla_iW)^1+h_{12}(\nabla_iW)^2\}u_i^1d\mu_g\right.\\
&\qquad\left.-\frac{1}{2}\int_M\{h_{12}(\nabla_iW)^1-h_{11}(\nabla_iW)^2\}u_i^2d\mu_g\right)\\
\le&\sum_{i=1}^n\left(\frac{1}{2}\int_M\frac{1}{2}\left\{\frac{1}{2}(h_{11})^2(u_i^1)^2+\frac{1}{2}(h_{12})^2(u_i^1)^2\right\}d\mu_g\right.\\
&\qquad+\frac{1}{2}\int_M\frac{1}{2}\left\{\frac{1}{2}(h_{12})^2(u_i^2)^2+\frac{1}{2}(h_{11})^2(u_i^2)^2\right\}d\mu_g\\
&\qquad\left.+\frac{1}{2}\int_M\frac{1}{2}\{2[(\nabla_iW)^1]^2+2[(\nabla_iW)^2]^2\}d\mu_g\right)\\
=&\sum_{i=1}^n\left(-\frac{1}{8}\int_N\{(h_{11})^2+(h_{12})^2\}\{(u_i^1)^2+(u_i^2)^2\}d\mu_g\right.\\
&\qquad\left.+\frac{1}{2}\int_M[(\nabla_iW)^1]^2+[(\nabla_iW)^2]^2d\mu_g\right)\\
=&\frac{1}{16}\int_M\|\dot{G}_0\|^2G_{\alpha\beta}g^{ij}u_i^\alpha u_j^\beta d\mu_g+\frac{1}{2}\int_M\|\nabla W\|^2d\mu_g.
\end{aligned}$$

Now the standard second variation formula of the energy functional for harmonic maps says (see [9] for example) that

$$\delta^2\mathcal{E}_G(W,W)=\int_M\|\nabla W\|^2d\mu_g-\langle R^N(W,du)du,W\rangle_{L^2(G)}.$$

The second term on the left is the sectional curvature of the image N, hence it is always less than or equal to zero. Thus we have

$$\delta^2\mathcal{E}_G(W,W)\le\int_M\|\nabla W\|^2d\mu_g.$$

Hence the previous inequality gives

$$\delta^2\mathcal{E}_G(W,W)\le\frac{1}{16}\int_M\|\dot{G}_0\|^2G_{\alpha\beta}g^{ij}u_i^\alpha u_j^\beta d\mu_g+\frac{1}{2}\delta^2\mathcal{E}_G(W,W).$$

The statement of the proposition now follows, and hence the proof of the theorem is complete. q.e.d.

3.2. Applications of the Convexity Theorem. We will show that under a certain topological condition on the harmonic map $u : M \to N$, the energy functional defined on the Teichmüller space $\mathcal{T}(N)$ is not only strictly convex but also proper, hence one can find a unique minimizer G in $\mathcal{T}(N)$ of $\mathcal{E}$.

Theorem 3.2.1. *Suppose that we have harmonic maps $u : (M, g) \to (N, G)$ with varying hyperbolic metrics G in one homotopy class and with the induced map*

$$u_* : \pi_1(M) \to \pi_1(N)$$

surjective. Then the energy functional $\mathcal{E} : \mathcal{T}(N) \to \mathbf{R}$ is both strictly convex and proper, and hence one can find a unique energy-minimizing point $[G]$ in $\mathcal{T}$.

Remark. The domain manifold M need not be connected.

To prove this, what we need to show is

Proposition 3.2.1. *If the induced map $u_* : \pi_1(M) \to \pi_1(N)$ is surjective, then the energy functional $\mathcal{E} : \mathcal{T}(N) \to \mathbf{R}$ is proper.*

The properness of the energy functional has been shown by Michael Wolf [26] when the domain manifold is a hyperbolic surface (N, g) with the harmonic map homotopic to the identity. The energy minimizing metric in this case is simply g, the hyperbolic metric of the domain, and the harmonic map u is the identity on N.

When the domain manifold is a disjoint union of S^1, each copy of S^1 is mapped to a simple closed geodesic in N so that the image of S^1's "fill" the surface N, Steve Kerckhoff [11] has shown that the sum $\mathcal{L}(G)$ of the length $L(\gamma_i)$ of all the closed geodesics γ_i is a proper functional defined on $\mathcal{T}(N)$. Scott Wolpert [29] proved that the functional $\mathcal{L}(G)$ is convex along a Weil-Petersson geodesics, and used this result to demonstrate that even though the Teichmüller space is not geodesically complete wtih respect to the Weil-Petersson metric, one can show that for any two given points in $\mathcal{T}(N)$, one can find a unique length minimizing Weil-Petersson geodesic connecting the points. Note that the theorem above generalizes these two cases previously known.

Proof. We will look at the sublevel set $\mathcal{S}(C)$ of the energy functional within the space $\mathcal{M}_{-1}$ of hyperbolic metrics. Due to the result of Eells-Sampson [3], Hartman [8] and Al'bers [1], once a continuous map ϕ from

M to N is given with its induced homomorphism $\phi_* : \pi_1(M) \to \pi_1(N)$ surjective, then for any hyperbolic metric G there exists a unique smooth harmonic map $u : (M, g) \to (N, G)$ homotopic to $\mathcal{S}(C)$ which is defined as

$$\mathcal{S}(C) = \{G \in \mathcal{M}_{-1} : \mathcal{E}(G) \leq C < \infty\}.$$

Our goal is to show that $\mathcal{S}(C)/\mathcal{D}_0$ is sequentially compact in the Teichmüller space $\mathcal{M}_{-1}/\mathcal{D}_0$.

For our setting, the so called Böchner's formula (see [18]) combined with the standard elliptic estimate (the DiGeorgi-Nash-Moser iteration) gives the following estimate,

$$\sup_M e(u) \leq C \int_M e(u) d\mu_g,$$

where the constant C is dependent of the sectional curvature K_G, which is -1, but independent of the choice of the hyperbolic metric G on N. This says that on $\mathrm{SL}(K)$,

$$\sup_M e(u) \leq C < \infty.$$

Since the domian manifold is compact, having a uniform bound on the energy density ensures that the diameters of the images of the harmonic maps $u : (M, g) \to (N, G)$ are also uniformly bounded provided that G is in $\mathcal{S}(C)$.

We now claim that the infectivity radius $\mathrm{inj}(N, G)$ with G in $\mathcal{S}(C)$ is bounded below by some positive number ε_M. Suppose the contrary. Then there exists a sequence of hyperbolic metrics $\{G_i\}$ in $\mathcal{S}(C)$ such that $\mathrm{inj}(G_i)$ is going down to zero. This means that there is some nontrivial closed geodesic γ_1 in N, uniquely representing the homotopy class of $[\gamma_1]$ pinching off to a point as the hyperbolic metric on N^2 changes within $\mathcal{S}(C)$.

A classical theorem called the collar theorem (see [2] for example for a complete statement) states that there is a collar

$$C(\gamma_1) = \{p \in N : \mathrm{dist}(p, \gamma_1) \leq w(\gamma_1)\}$$

of width

$$w(\gamma_1) = \operatorname{arcsin} h \left\{ 1 / \sinh \left(\frac{1}{2} l(\gamma_1) \right) \right\},$$

which is isometric to the cylinder $[-w(\gamma_1), w(\gamma_1)] \times S^1$ with the Riemannian metric $ds^2 = dp^2 + l^2(\gamma_1) \cosh^2 \rho dt^2$.

As the length $l(\gamma_1)$ of the closed geodesic γ_1 goes down to zero, the width $w(\gamma_1)$ of the cylinder grows to infinity, which implies that the diameter of the hyperbolic surface is going to infinity. However, this contradicts to the fact that the diameter of the hyperbolic surface (N, G) is uniformly bounded as long as G belongs to $\mathcal{S}(C)$. Hence the claim that the infectivity radius $\mathrm{inj}(N, G)$ with G in $\mathcal{S}(C)$ is bounded below by some positive number ε_M has been verified.

Now we are ready to apply the following compactness result to our setting. (see Mumford's original paper [15], or [22].)

Lemma 3.2.1 (Mumford Compactness Theorem). *Let a set S consist of all the hyperbolic metrics defined on a closed Riemann surface N of genus greater than one, whose infectivity radii are bounded below by $\varepsilon > 0$. Then for any sequence $\{G_i\}$ in S, there is a subsequence $\{G_{i_k}\}$ and a sequence $\{f_{i_k}\}$ of diffeomorphisms of N, such that the pulled-back metric $f_{i_k}^* G_{i_k}$ converges to a hyperbolic metric G_∞ in S in the G^∞ sense.*

Applying this lemma to our setting with $\varepsilon = \varepsilon_M$, we know that when $\{G_i\}$ is a sequence of metrics in $\mathrm{SL}(K)$, then there is a subsequence $\{G_{i_k}\}$ and a sequence $\{f_{i_k}\}$ of diffeomorphisms of N such that

$$f_{i_k}^* G_{i_k} \to G_\infty$$

in $\mathcal{M}_{-1}$ in the G^∞ topology. For the sake of simplicity, we will be using index k instead of i_k from now on. Now consider the sequence of harmonic maps $u_k : (M, g) \to (N, G_k)$ which are all in the same homotopy class with its induced map $\pi_1(M) \to \pi_1(N)$ surjective. Note that since by construction the map $f_k^{-1} : (N, G) \to (N, f_k^*)$ is an isometry, the composite map

$$f_k^{-1} \circ u : (M, G) \to (N, f_k^* G)$$

is harmonic. Also note that the convergence $f_k^* G_k \to G_\infty$ is in C^∞, we have a C^∞-convergence of the maps $f_k^{-1} \circ u$ with respect to the metric g and G_∞. Hence there exists sufficiently large K such that for all $k > K$, the maps $f_k^{-1} \circ u_k : M \to N$ are in the same homotopy class as the harmonic map $u_\infty : (M, g) \to (N, G_\infty)$.

We now wish to show that for any i and j greater than K, f_i and f_j are homotopic. We have just shown that $f_k^{-1} \circ u_i$ and $f_k^{-1} \circ u_j$ are homotopic. Equivalently we can find a diffeomorphism $\psi_{ij} : N \to N$ such that ψ homotopic to identity, and satisfying

$$\psi_{ij} \circ f_i^{-1} \circ u_i(p) = f_j^{-1} \circ u_j(p).$$

Recall that u_i, u_j are homotopic. Hence there exists a diffeomorphism θ_{ij} of N homotopic to the identity, and satisfying $\theta_{ij} \circ u_i(p) = u_j(p)$. Let p in M be a base point for the homotopy group $\pi_1(M, p)$. Then we have the following commutative diagram:

$$\begin{tikzcd}
\pi_1(M,p) \arrow[r, "(u_i)_*"] \arrow[dr, "(u_i)_*"'] & (u_i)_*\pi_1(M,p) \arrow[r, "(f_i^{-1})_*"] \arrow[d, "(\theta_{ij})_*"] & (f_i^{-1})_*(u_i)_*\pi_1(M,p) \arrow[d, "(\psi_{ij})_*"] \\
 & (u_j)_*\pi_1(M,p) \arrow[r, "(f_j^{-1})_*"] & (f_j^{-1})_*(u_j)_*\pi_1(M,p)
\end{tikzcd}$$

Note that in the diagram, $(u_i)_*, (u_j)_*$ are suriective, $(\psi_{ij})_*, (\theta_{ij})_*$ are isomorphisms, and $(f_i^{-1})_*, (f_j^{-1})_*$ are bijective homomorpisms. We want to show that $(f_i^{-1})_*$ and $(f_j^{-1})_*$ are indeed isomorphic. Take any γ in $(f_j^{-1})_*(u_j)_*\pi_1(M,p)$. Then pull back γ to $(u_j)_*\pi_1(M,p)$ in two different ways in the commutative diagram to get

$$[f_j \circ \gamma \circ f_j^{-1}] = [\theta_{ij} \circ f_i \circ \psi_{ij}^{-1} \circ \psi_{ij} \circ f_j^{-1} \circ \theta_{ij}^{-1}]$$

in $(u_j)_*\pi_1(M,p) = \pi_1(N, u_j(p))$. It then follows that f_j^{-1} is homotopic to $\psi_{ij} \circ f_i^{-1} \circ \theta_{ij}^{-1}$. Since both ψ_{ij} and θ_{ij}^{-1} are diffeomorphisms homotopy to the identity map of N, we conclude that f_i is homotopic to f_j. This says that the sublevel set $\mathrm{SL}(K)$ is sequentially compact in $\mathcal{M}_{-1}/\mathcal{D}_0$. Thus the proposition is proved. q.e.d.

4. First variation of the energy functional

4.1. Linear functional $\mathcal{F}$ and its Riesz representation. Given a smooth harmonic map $u : (M^n, g) \to (N^2, G)$, with a hyperbolic metric G, define a linear functional $\mathcal{F}$ on the space $\Gamma^0(T_G\mathcal{M}(N))$ of continuous sections of the bundle $T_G\mathcal{M}(N)$, which is the bundle of symmetric $(0,2)$-tensors over N as follows:

$$\mathcal{F}(h) = \int_M \frac{1}{2} \operatorname{tr}_g(u^*h) d\mu_g.$$

By the Riesz representation theorem, there exists a measure λ on $\Gamma^0(T_G\mathcal{M}(N))$ such that

$$\mathcal{F}(h) = (h, \lambda),$$

where (,) gives the pairing between the space of continuous sections and its dual space, namely, the space of measures.

With a local coordinate system, one can locally write down $\mathcal{F}(h)$ as

$$\mathcal{F}(h) = \int \frac{1}{2} h_{\alpha\beta}(u(x)) g^{ij} u_i^\alpha u_j^\beta d\mu_g(x) = \int_N h_{\alpha\beta}(y) d\lambda_{\alpha\beta}(y).$$

Note that $\lambda_{\alpha]be}$ is symmetric as well as non-negative-definite, for the matrix $g^{ij}u_i^\alpha(x)u_j^\beta(x)$ is non-negative-definite pointwise.

Proposition 4.1.1. *The measure λ is divergence free in the weak sense, i.e., $\mathcal{F}(L_X G) = 0$ for any smooth vector field X on N.*

(Recall that the Lie derivative $L_X G$ is the dual operator of the divergence operator δ_G up to a sign.)

Proof. Let $\phi_t : N \to N$ be a family of diffeomorphisms of N with $\phi^0 = \mathrm{Id}_N$. Then $\phi_t : (N, G) \to (N, \phi_t^* G)$ is a family of isometrics, and we have

$$\mathcal{E}(u, G) = \mathcal{E}(\phi_t \circ u, \phi_t^* G).$$

Differentiate both sides with respect to t, and obtain

$$\begin{aligned}
0 =& \frac{d}{dt} \int_M \frac{1}{2} \operatorname{tr}_g((\phi_t^* \circ u)^*(\phi_t^* G)) d\mu_g \\
=& \frac{d}{dt} \int_M \frac{1}{2} \operatorname{tr}_g(u^*(L_X G)) d\mu_g + \int_M \frac{1}{2} \operatorname{tr}_g(u^*(L_X G)) d\mu_g \\
=& 2\mathcal{F}(L_X G) \\
=& 2(L_X G, \lambda),
\end{aligned}$$

where $X = \frac{d}{dt}\phi^t\Big|_{t=0}$. Since this holds for any smooth vector field X on N, it follows that λ is weakly divergent free. q.e.d.

4.2. Absolute continuity of the measure λ.

Theorem 4.2.1. *Suppose h is a continuous section of the bundle of symmetric $(0,2)$-tensors over N^2. Then the measure μ_h defined below is absolutely continuous with respect to the two-dimensional Lebesgue measure μ_G on N unless u maps M to a closed geodesic or a point in N:*

$$\mu_h(A) := \int_{A \subset N} h d\mu_G,$$

where A is any Lebesgue measurable set on N.

Proof. Suppose $A \subset N^2$ has zero Lebesgue measure. Then we claim that the set $u^{-1}[A]$ has zero measure with respect to the n-dimensional Lebesgue measure μ_g on M^n.

Suppose the contrary, i.e., $\mu_g(u^{-1}[A]) > 0$. We shall use the following results from [31].

Proposition 4.2.1. *The set of points $R_0 \subset M^n$, where the differential du of a smooth harmonic map $u : M^n \to N^k$ has rank zero, is of Hausdorff dimension at most $(n-2)$, unless u maps the whole M to a point in N.*

Proposition 4.2.2. *The set of points $R_1 \subset M^n$, where the differential of a smooth harmonic map $u : M^n \to N^k$ has rank one, is of Hausdorff dimension at most $(n-1)$, unless u maps the whole M to either a point or a geodesic in N.*

It thus follows that $\mu_g(R_0) = \mu_g(R_1) = 0$, so that

$$\mu_g(u^{-1}[A] \backslash (R_0 \cap R_1)) = \mu_g(u^{-1}[A]) > 0.$$

We can find a Lebesgue point p in $u^{-1}[A] \backslash (R_0 \cap R_1)$. At p the differential du is nondegenerate, and then $|\det(du)| > 0$. Thus there exist some $\varepsilon > 0$ and $r > 0$ such that $|\det(du)| \geq \varepsilon/2 > 0$ on the set

$$B_r(p) \cap \Big(u^{-1}[A] \backslash (R_0 \cap R_1)\Big).$$

We obtain

$$\liminf_{r \to 0} \left(\int_{(u^{-1}[A] \backslash (R_0 \cap R_1)) \cap B_r(p)} |\det(du)| d\mu_g \Big/ w_n r^n > \varepsilon/2 > 0, \right.$$

where w_n is the volume of the unit Euclidian ball in $\mathbf{R}^n$. On the other hand, the area formula implies that the numerator of the left hand side is equal to $\mu_g(A \cap u[B_r(p)]) < \mu_g(A) = 0$, a contradiction.

Hence we have shown that when A has μ_G-measure zero, then $u^{-1}[A]$ has μ_g-measure zero. Thus note that for a measurable set A in N^2, we have

$$\begin{aligned} \mu_h(A) &= \int_A h_{\alpha\beta} d\lambda_{\alpha\beta} \\ &= \int_{u^{-1}(A)} h_{\alpha\beta}(u(x)) g^{ij} \frac{\partial u^\alpha}{\partial x^i} \frac{\partial u^\beta}{\partial x^j} d\mu_g(x) \\ &= 0. \end{aligned}$$

This proves the absolute continuity of the measure μ_h on N. q.e.d.

We remark here that when one chooses the continuous section h of the bundle of $(0,2)$-tensor to be the image metric G, then μ_G can be regarded as the "push-forward measure" of the energy measure $e(u)d\mu_g$ on M.

4.3. The gradient vector of the energy functional. From now on, we restrict our attention back to the case in which the image is a closed hyperbolic surface of genus > 1. As seen in the previous section, the measure λ is absolute continuous with respect to the Lebesgue measure μ_G on N, and thus there is a set of L^1 functions $\tau_{\alpha\beta}$ on N such that

$$\begin{aligned}\mathcal{F}(h) =&(h,\lambda)\\ =&\int_N h_{\alpha\beta}d\lambda\\ =&\int_N h_{\alpha\beta}\tau_{\alpha\beta}d\mu_G\\ =&\langle h,\tau\rangle_{L^2(G)}.\end{aligned}$$

Recall, from the Section 2, the following differential operators

$$\begin{aligned}\mathcal{L}h =&(-\Delta+1)\operatorname{Tr}_G h+\delta_G\delta_G h,\\ \mathcal{L}^*f =&((\Delta+1)f)G+\operatorname{Hess}_G f.\end{aligned}$$

In Chapter Two, we have seen that $\mathcal{L}\mathcal{L}^*$ is an elliptic operator of order four whose principal symbol is Δ^2, and its pindex is zero. Thus $\mathcal{L}\mathcal{L}^*$ is a Fredholm operator from $H^s(N)$ to $H^{s-4}(N)$. Furthermore it was shown that the cokernel of the operator $\mathcal{L}\mathcal{L}^*$ is zero. So it follows that for a given function ϕ in $H^s(N)$, one can solve the equation

$$\mathcal{L}\mathcal{L}^*f=\phi$$

for f, and the solution is unique and is in $H^{s+4}(N)$.

Recall that for fixed α and β, $\tau_{\alpha\beta}$ is a L^1 function on N. We will show here that $\tau_{\alpha\beta}$ is in $H^{-1-\varepsilon}(N)$ for any $\varepsilon>0$. Set $\tau_{\alpha\beta}=f$. By the standard argument using a partition of unity when a function f defined on N is in $H^s(N)$, restricted to one chart f can be thought as a function in $H^s(\mathbf{R}^2)$ with a compact support. Hence we want to show that

$$\|f\|_{H^{1+\varepsilon}(\mathbf{R}^2)}=\int_{\mathbf{R}^2}\frac{|\hat{f}(\xi)|^2}{(1+|\xi|^2)^{1+\varepsilon}}d\xi<\infty.$$

By definition we have

$$\begin{aligned}\int_{\mathbf{R}^2} \frac{|\hat{f}(\xi)|^2}{(1+|\xi|^2)^{1+\varepsilon}} d\xi =& \int_{\mathbf{R}^2} \frac{|\int_{\mathbf{R}^2} f(x)e^{-ix\xi}dx|^2}{(1+\xi^2)^{1-\varepsilon}} d\xi \\ \leq& \int_{\mathbf{R}^2} \frac{(\int_{\mathbf{R}^2}|f|dx)^2}{(1+\xi^2)^{1-\varepsilon}} d\xi \\ \leq& \infty,\end{aligned}$$

since $\int |f|dx < \infty$, and $1/(1+\xi^2)^{1+\varepsilon}$ is integrable over $\mathbf{R}^2$.

Now consider the equation

$$\mathcal{L}\mathcal{L}^* f = \mathcal{L}\tau.$$

Since we know this is solvable, and when $\tau_{\alpha\beta}$ is in $H^{1+\varepsilon}(\mathbf{R}^2)$ as shown above, $\mathcal{L}\tau_{\alpha\beta}$ is in $H^{1-\varepsilon-2}$. Now $\mathcal{L}\mathcal{L}^*$ is an operator from $H^{1-\varepsilon}$to $H^{-3-\varepsilon}$, the solution f in the above equation lies in $H^{1-\varepsilon}$. Thus we have the following proposition.

Proposition 4.3.1. *The $(0,2)$-tensor τ as defined above is a section of the bundle of symmetric $(0,2)$-tensors whose components are in the Sobolev space $H^{-1-\varepsilon}$ for any $\varepsilon > 0$, and the solution f to the equation $\mathcal{L}\mathcal{L}^* f = \mathcal{L}\tau$ is in $H^{1-\varepsilon}(N)$.*

We will now show the following theorem.

Theorem 4.3.1. *The tensor $\theta = \tau - (\mathcal{L}^* f)$ is trace-less and transverse, and hence $\theta_{11} - i\theta_{12}$ is locally a holomorphic function.*

The theorem above says that in the space of symmetric $(0,2)$-tensors, τ is composed of two vectors; one is in the direction perpendicular to the space $\mathcal{M}_{-1}$ of hyperbolic metrics on N, whose components lies in $H^{-1-\varepsilon}$ and the other is tangent to the Teichmüller space whose components are smooth as real and imaginary parts of a quadratic holomorphic differential. Note here that when the functional $\mathcal{F}$ is restricted to the symmetric $(0,2)$-tensors of unit length which are trace-free, transeverse, i.e., unit tangent vectors to the Teichmüller space, θ gives the direction in which $\mathcal{F}$ is maximized. This clearly follows from the above decomposition, i.e.,

$$\begin{aligned}\mathcal{F}(h) =& \langle h, \tau \rangle_{L^2(G)} \\ =& \langle h, \theta + \mathcal{L}^* f \rangle_{L^2(G)} \\ =& \langle h, \theta \rangle_{L^2(G)},\end{aligned}$$

and the last expression is maximized when the tracefree transverse vector h is in the direction of θ.

It should be remarked that given a tracefree transverse deformation h of a hyperbolic metric $G, \mathcal{F}(h)$ is the directional derivative of the energy functional $\mathcal{E}(G)$ in the direction of $h \in T_G\mathcal{T}$.

Proof. For $\tau_{\alpha\beta}$ in $H^{-1-\varepsilon}$, we have a solution f to the equation $\mathcal{L}\mathcal{L}^* f = \mathcal{L}\tau$ where f is in $H^{1-\varepsilon}$. Note that this means that the equation is satisfied distrubutionally, i.e., for all ϕ in $C^\infty(N)$, we have

$$\int_N (\mathcal{L}\mathcal{L}^*\phi) f d\mu_G(y) - \int_N \langle \mathcal{L}^*\phi, \tau\rangle_{G(y)} d\mu_G(y) = 0. \tag{18}$$

Similarly $\mathcal{L}^* f$ is defined distributionally, i.e.,

$$\int_N \langle h, \mathcal{L}^* f\rangle_{G(y)} d\mu_G(y) = \int_N (\mathcal{L}h) f d\mu_G(y),$$

for any smooth symmetric $(0,2)$-tensor h. Recall that Proposition 4.1.1 implies that

$$\int_N \langle L_X G, \tau\rangle_{G(y)} d\mu_G(y) = 0 \tag{19}$$

for any smooth vector field X on N. Also note that for any vector field X on N, the decomposition theorem from Section 2 yields

$$\int_N \langle L_X G, \mathcal{L}^* f\rangle_{G(y)} d\mu_G(y) = \int_N \mathcal{L}(L_X G) f d\mu_G(y) = 0, \tag{20}$$

since $L_X G$ is in $T_G\mathcal{M}_{-1} = \ker \mathcal{L}$. Define $\theta = \tau - \mathcal{L}^* f$ which is at this point only weakly defined, since the regularity of τ is only $H^{-1-\varepsilon}$. Then from equations (19) and (20), we have

$$\begin{aligned}\langle L_X G, \theta\rangle_{L^2(G)} &= \langle L_X G, (\tau - \mathcal{L}^* f)\rangle_{L^2(G)} \\ &= \langle L_X G, \tau\rangle_{L^2(G)} - \langle L_X G, \mathcal{L}^* f\rangle_{L^2(G)} = 0\end{aligned}$$

for any smooth vector field X on N. Thus 0 is weakly divergence-free, i.e.,

$$\delta_G \theta = 0 \tag{21}$$

weakly.

Now we will show that $\mathrm{tr}_G\,\theta = 0$ weakly. Note that for any smooth function ϕ on N,

$$\begin{aligned}\int_N \phi(\mathcal{L}\theta)d\mu_G &= \int_N \phi(\mathcal{L}(\tau - \mathcal{L}^* f))d\mu_G \\ &= \int_N \phi(\mathcal{L}\tau - \mathcal{L}\mathcal{L}^* f)d\mu_G \\ &= \int_N \langle \mathcal{L}^*\phi, \tau\rangle_G d\mu_G - \int_N (\mathcal{L}\mathcal{L}^*\phi) f d\mu_G \\ &= 0,\end{aligned}$$

where the last equality is due to equation (18) above. Hence we have

$$\begin{aligned}0 &= \int_N \phi(\mathcal{L}\theta) \\ &= \int_N \langle \mathcal{L}^*\phi, \theta\rangle_G d\mu_G \\ &= \int_N \langle\{(-\Delta\phi + \phi)G + \mathrm{Hess}\,\phi\}, \theta\rangle_G d\mu_G \\ &= \int_N \langle\{(-\Delta\phi + \phi)G\}, \theta\rangle_G d\mu_G + \int_N \langle \mathrm{Hess}\,\phi, \theta\rangle_G d\mu_G \\ &= \int_N \langle\{(-\Delta\phi + \phi)G\}, \theta\rangle_G d\mu_G + \frac{1}{2}\int_N \langle L_{\Delta\phi} G, \theta\rangle_G d\mu_G \\ &= \int_N \langle\{(-\Delta\phi + \phi)G\}, \theta\rangle_G d\mu_G \\ &= \int_N (-\Delta\phi + \phi)\,\mathrm{Tr}_G\,\theta.\end{aligned}$$

Note that for a given smooth function ψ on N, we can always find a smooth function ϕ such that $-\Delta\phi + \phi = \psi$ since the operator $-\Delta + 1 : C^\infty(N) \to C^\infty(N)$ is invertible. Thus for any smooth function ψ on N,

$$\int_N \psi(\mathrm{Tr}_G\,\theta) = 0, \tag{22}$$

that is, $\mathrm{Tr}_G\,\theta = 0$ weakly.

Now fix a point p in N, and choose a coordinate chart U around p such that $G = \delta_{ij}$. We will show that the $(0,2)$-tensor $\theta = \theta_{ij}$ is locally smooth within the chart U.

Recall Weyl's lemma that if $\tilde{\theta}$ is a complex valued $L^1_{\mathrm{loc}}(U)$ function such that any complex valued compactly supported smooth function f

on U satisfies

$$\int_N (\partial_{\bar{z}} f)\tilde{\theta} = 0, \tag{23}$$

then $\tilde{\theta}$ is a holomorpic function.

Let $f = x_1 + ix_2$ where x_1 and x_2 are compactly supported smooth functions, and $\tilde{\theta} = \theta_{11} - i\theta_{12}$. Then knowing that θ is tracefree as well as transverse *weakly* we will show that equation (23) holds for any f, so that by Weyl's lemma, θ is holomorphic, and hence θ is trace-free, transverse strongly:

$$\begin{aligned}\int_U (\partial_{\bar{z}} f)\tilde{\theta} &= \int_U (\partial_1 f + i\partial_2 f)(\theta_{11} - i\theta_{12}) \\ &= \int_U \{(x_{1,1} - x_{2,2}) + i(x_{1,2} + x_{2,1})\}(\theta_{11} - i\theta_{12}) \\ &= \int_U (x_{1,1}\theta_{11} - x_{2,2}\theta_{11} + x_{1,2}\theta_{12} + x_{2,1}\theta_{12}) \\ &\quad + i\int_U (-x_{1,1}\theta_{12} - x_{2,2}\theta_{12} + x_{1,2}\theta_{11} + x_{2,1}\theta_{11}).\end{aligned} \tag{24}$$

Choose a smooth vector field X to be $x_1\partial_1$, i.e., with no ∂_2 component. Then the fact that θ is weakly divergence-free implies that

$$\int_U (L_X G)\theta = \int_U x_{1,1}\theta_{11} + x_{1,2}\theta_{12} = 0. \tag{25}$$

Similarly by choosing $X = x_2\partial_2$ one obtains

$$\int_U (L_X G)\theta = \int_U x_{2,1}\theta_{12} + x_{2,2}\theta_{22} = 0. \tag{26}$$

Since θ is tracefree, we have

$$\int_U x_{2,2}(\theta_{11} + \theta_{22}) = 0. \tag{27}$$

Putting the above three equations together, it follows that the real part of the last term in equation (24) is zero. The same method applies to show the imaginary part of the last term in equation (24) is also zero. Hence Weyl's lemma implies that $\theta_{11} - i\theta_{12}$ is holomorpic on $U \subset N$. This proves the regularity of θ. q.e.d.

4.4. Holomorphic maps and the $\mathcal{E}$-energy minimizing metrics. When the domain (M^{2n}, g) is a compact Kähler manifold and the harmonic map $u : (M^{2n}, g) \to (N^2, G)$ is holomorphic, the τ tensor has the following pointwise expression:

$$\tau_{\alpha\beta}(y) = \int_{u^{-1}(y)\subset M} g^{ij} u_i^\alpha u_j^\beta d\sigma_{g,y}(x),$$

where $u^{-1}(y)$ is an $(n-1)$-complex dimensional subvariety of M, and $d\sigma_{g,y}$ is the induced volume form on the subvariety $u^{-1}(y) \subset M$.

Since u is holomorphic, the linear map $du : T_x M^{2n} \to T_{u(x)} N^2$, restricted to the plane spanned by the two nondegenerate directions of du, is conformal. It then follows that the integrand $(g^{ij} u_i^\alpha u_j^\beta)(x)$ is conformal to $G_{\alpha\beta}, g(y)$ for every $x \in u^{-1}(y)$ and therefore τ is conformal to G.

This in turn implies that the tracefree transverse part θ of τ is identically zero on N. In other words, the gradient vector of the energy functional $\mathcal{E} : \mathcal{T} \to \mathbf{R}$ vanishes at G. Due to Theorem 3.1.1, we have the following statement.

Corollary 4.4.1. *Suppose that a harmonic map u from a compact Kähler manifold (M^{2n}, g) to a compact hyperbolic Riemann surface (N^2, G_0) is holomorphic. Then the hyperbolic metric G_0 minimizes the energy functional $\mathcal{E} : \mathcal{T} \to \mathbf{R}$ uniquely, while the hyperbolic metric G and thus the harmonic map u vary within the homotopy class $[u]$.*

Remark. The converse does not hold, for one can find a Riemann surface M with a certain conformal structure and with $g_M > g_N (g =$ genus), from which there is no nontrivial holomorpic map to N.

5. Universal Teicmüller space and its Weil-Petersson metric

Recall that the Teichmüller space $\mathcal{T}$ is constructed as the space $\mathcal{M}_{-1}$ of hyperbolic metrics on a compact Riemann surface N^2 of genus > 1 modulo the pull-back action of the group $\mathcal{D}_0$ of diffeomorphisms homotopic to the identity. With respect to the Weil-Petersson metric, $\mathcal{T}$ has non-positive sectional curvature, and though the space is not geodesically complete – a Weil-Petersson geodesic cannot be extended indefinitely – $\mathcal{T}$ is still geodesically convex, that is, every pair of points can be joined by a unique minimizing geodesic (S. Wolpert [29]). As an attempt to generalize those results to the cases over non-compact manifolds, we will consider the universal covering space of N^2, the hyperbolic two space $\mathbf{H}^2$. It can be modeled on the open disc $\mathbf{D}^2 = \{z \in \mathbf{R}^2 : |z| < 1\}$

with the standard Poincaré metric $G_0 = 4(1-|z|^2)^{-2}|dz|^2$. The uniformization theorem says that

$$\mathcal{M}_{-1}(\mathbf{H}^2) = \{G : G = \Phi^* G_0, \text{ where } \Phi \text{ a diffeomorphism of } \mathbf{D}^2\}.$$

As for the regularity of the diffeomorphisms, we will impose each diffeomorphism to be quasiconformal. This comes from the fact that each hyperbolic metric has a conformal structure obtained by pulling back the standard Euclidian structure by a solution of a Beltrami equation, a quasiconformal map from a disc to itself. The full isometry group of G_0 is all the Möbius transforms, which, using the upper half space model of $\mathbf{H}^2$, is $\mathrm{SL}(2,\mathbf{R})$. Thus one has the following identification:

$$\mathcal{M}_{-1}(\mathbf{H}^2) \cong \mathcal{QC}(\mathbf{D}^2)/\,\mathrm{SL}(2,\mathbf{R}), \tag{28}$$

where $\mathcal{QC}(\mathbf{D}^2)$ is the group of quasiconformal maps from $\mathbf{D}^2$ to itself. Now define the universal Teichmüller space $\mathcal{UT}$ to be

$$\mathcal{UT} = \mathcal{M}_{-1}(\mathbf{H}^2)/\mathcal{QC}(\mathbf{D}^2), \tag{29}$$

where $\mathcal{QC}(\mathbf{D}^2)$ is the group of quasi-conformal maps from $\mathbf{D}^2$ to itself fixing the boundary $\partial\mathbf{D}^2 = S^1$, or equivalently it can be seen as the group of diffeomorphisms of $\mathbf{H}^2$ inducing the trivial map $\mathrm{Id}_{S^1} : S^1 \to S^1$ on the geometric boundary of $\mathbf{H}^2$. (Once again at this point the diffeomorphisms are required to be quasiconformal.)

Noting here that the intersection between the two groups $\mathrm{SL}(2,\mathbf{R})$ and $\mathcal{QC}(\mathbf{D}^2)$ is the identity, we have following identifications.

$$\begin{aligned}
\mathcal{UT} =& \mathcal{M}_{-1}(\mathbf{H}^2)\mathcal{QC}(\mathbf{D}^2) \\
=& \mathcal{QC}(\mathbf{D}^2)/\,\mathrm{SL}(2,\mathbf{R})/\mathcal{QC}_0(\mathbf{D}^2) \\
=& \mathcal{QC}(\mathbf{D}^2)/\mathcal{QC}_0(\mathbf{D}^2)/\,\mathrm{SL}(2,\mathbf{R}) \\
=& \mathcal{QS}(S^1)/\,\mathrm{SL}(2,\mathbf{R}),
\end{aligned}$$

where $\mathcal{QS}(S^1)$ is the group of quasisymmetric maps from S^1 to itself.

In the tangent space $T_{G_0}\mathcal{M}_{-1}$ of $\mathcal{M}_{-1}$ at G_0, we need to restrict our attention to the space of L^2-integrable symmetric tensors, since we are to use the Weil-Petersson metric on $\mathcal{UT}$. Denote by M the Kähler submanifold of $\mathcal{UT}$ whose tangent space at $[G_0]$ is given by the L^2-integrable symmetric $(0,2)$-tensors, or equivalently M is the Kähler submanifold of $\mathcal{UT} = \mathcal{QS}(S^1)/\,\mathrm{SL}(2,\mathbf{R})$ whose tangent space is $H^{3/2}$ vector fields on S^1. Then we have the following L^2-decomposition theorem, almost identical to the decomposition theorem for the Teichmüller space $\mathcal{T}$ over a compact Riemann surface.

Theorem 5.0.1. *Suppose that G_0 is the Poincaré metric, i.e., $4(1-|z|^2)^{-2}(dx^2+dy^2)$ on $\mathbf{D}^2$, and that h is a smooth $L^2(\mathbf{H}^2)$-integrable symmetric $(0,2)$-tensor defined over $\mathbf{H}^2$. Then there is a unique L^2 orthogonal decomposition of h as a tangent vector belonging to $T_{G_0}\mathcal{M}$ as follows:*

$$h = P_{G_0}(h) + L_X G_0 + \mathcal{L}^* f, \tag{30}$$

where $L_X G_0$ is the Lie derivative of G_0 in the direction of a vector field X on N, and $X, P_{G_0}(h)$ satisfy the following equations:

$$\delta_{G_0}(L_X h) = \delta_{G_0} h, \tag{31}$$

$$\mathcal{L}_{G_0}\mathcal{L}^*_{G_0} f = \mathcal{L}_{G_0} h, \tag{32}$$

$$P_{G_0}(h) = h - L_X G_0 - \mathcal{L}^* h. \tag{33}$$

Proof. Recall that the space $\Gamma_0^\infty(T_{G_0}\mathcal{M})$ of compactly supported smooth sections of the bundle of symmetric $(0,2)$-tensors on $\mathbf{H}^2$ is dense in the space $\Gamma_{L^2}(T_{G_0}\mathcal{M})$ of L^2-integrable symmetric $(0,2)$-tensors on $\mathbf{H}^2$. Hence given a tensor h in $\Gamma_{L^2}(T_{G_0}\mathcal{M})$, there is a sequence $\{h_i\}$ in $\Gamma_0^\infty(T_{G_0}\mathcal{M})$, which converges to h in L^2-norm, namely,

$$\lim_{i\to\infty}\|h-h_i\|_{L^2} = \lim_{i\to\infty}\left(\int_{\mathbf{H}^2}\langle h-h_i, h-h_i\rangle_{G_0} d\mu_{G_0}(x)\right)^{\frac{1}{2}} = 0.$$

In the proof of the L^2-decomposition Theorem 2.2.1 of the space of sections of symmetric $(0,2)$-tensors, integrations by parts were used frequently to establish the orthogonality among the different components. In the current setting, the manifold in question is no longer compact, and we will need the following result, which would formally allow us to integrate by parts.

Lemma 5.0.1. *The differential operators $\mathcal{L}_{G_0}\mathcal{L}^*_{G_0}$, and $\delta_{G_0}\delta^*_{G_0}$, acting on functions in $C_0^\infty(\mathbf{H}^2)\cap H^2(\mathbf{H}^2)$ and vector fields in $X_0^\infty(\mathbf{H}^2)\cap H^1(\mathbf{H}^2)$ respectively, are both strictly coercive in the sense that*

$$\langle \mathcal{L}_{G_0}\mathcal{L}_{G_0} f, f\rangle_{L^2} \geq C\|f\|^2_{H^2}$$

and

$$\langle -\delta_{G_0}\delta^*_{G_0}X, X\rangle_{L^2} \geq C'\|X\|^2_{H^1}$$

for some $C, C' > 0$.

Proof. Since $f \in C_0^\infty(\mathbf{H}^2)$ is compactly supported, integration by parts induces no boundary term. Noting that $\mathcal{L}_{G_0}$ is self-adjoint, we have

$$\begin{aligned}
\int_{\mathbf{H}^2}(\mathcal{L}_{G_0}\mathcal{L}^*_{G_0}f)f\,d\mu_{G_0} =& \int_{\mathbf{H}^2}\langle \mathcal{L}^*_{G_0}f, \mathcal{L}^*_{G_0}f\rangle_{G_0}d\mu_{G_0}\\
=& \int_{\mathbf{H}^2}\langle [(-\Delta+1)f]G_0 + \operatorname{Hess}_{G_0} f, [(-\Delta+1)f]G_0\\
&\quad + \operatorname{Hess}_{G_0} f\rangle_{G_0}d\mu_{G_0}\\
=& \int_{\mathbf{H}^2}\|[(-\Delta+1)f]G_0\|^2 d\mu_{G_0}\\
&+ \int_{\mathbf{H}^2}\|\operatorname{Hess}_{G_0} f\|^2 d\mu_{G_0}\\
&+ 2\int_{\mathbf{H}^2}\langle (-\Delta+1)f]G_0, \operatorname{Hess}_{G_0} f\rangle_{G_0}d\mu_{G_0}\\
=& \int_{\mathbf{H}^2}\{2[(-\Delta+1)f]^2 + \|\operatorname{Hess}_{G_0} f\|^2\\
&\quad + 2(-\Delta+1)f(\Delta f)\}d\mu_{G_0}\\
=& \int_{\mathbf{H}^2}[-(\Delta f)f + 2f^2 + \|\operatorname{Hess}_{G_0} f\|^2]d\mu_{G_0}\\
=& \int_{\mathbf{H}^2}(\|\nabla f\|^2 + 2f^2 + \|\operatorname{Hess}_{G_0} f\|^2)d\mu_{G_0}\\
\leq& \int_{\mathbf{H}^2}\|\operatorname{Hess}_{G_0} f\|^2 d\mu_{G_0}\\
=& \|f\|^2_{H^2}.
\end{aligned}$$

As for the second part, it is similarly proven:

$$\begin{aligned}
&\int_{\mathbf{H}^2} \langle -\delta_{G_0}\delta^*_{G_0} X, X\rangle_{G_0} d\mu_{G_0} \\
&\quad = \int_{\mathbf{H}^2} \langle \delta^*_{G_0} X, \delta^*_{G_0} X\rangle_{G_0} d\mu_{G_0} \\
&\quad = \int_{\mathbf{H}^2} (\nabla_j X^i + \nabla_i X^j)^2 d\mu_{G_0} d\mu_{G_0} \\
&\quad = \int_{\mathbf{H}^2} (2\|\nabla X\|^2 + 2\nabla_j X^i \nabla_i X^j) d\mu_{G_0} \\
&\quad = \int_{\mathbf{H}^2} [2\|\nabla X\|^2 - 2(\nabla_i \nabla_j X^i) X^j] d\mu_{G_0} \\
&\quad = \int_{\mathbf{H}^2} \{2\|\nabla X\|^2 - 2[(\nabla_j \nabla_i X^i) + \mathrm{Ric}_{ij}\, X^i] X^j\} d\mu_{G_0} \\
&\quad = \int_{\mathbf{H}^2} \{2\|\nabla X\|^2 - 2[(\nabla_j \nabla_i X^i) - \delta_{ij} X^i] X^j\} d\mu_{G_0} \\
&\quad = \int_{\mathbf{H}^2} [2\|\nabla X\|^2 + 2(\delta_{G_0} X)^2 + 2\|X\|^2] d\mu_{G_0} \\
&\quad \geq \int_{\mathbf{H}^2} 2\|\nabla X\|^2 d\mu_{G_0} \\
&\quad = 2\|X\|^2_{H^1}.
\end{aligned}$$

q.e.d.

Having shown that those two operators are elliptic, self-adjoint, and coercive, from a standard argument it follows that given a smooth symmetric $(0,2)$-tensors h, which is compactly supported on $\mathbf{H}^2$, there are unique smooth solutions X and f to each of the equations

$$\delta_{G_0}\delta^*_{G_0} X = \delta_{G_0} h,$$
$$\mathcal{L}_{G_0}\mathcal{L}^*_{G_0} f = \mathcal{L}_{G_0} h.$$

Here as long as the given tensor h to be decomposed is compactly supported, then one can integrate by parts as in the proof of the L^2 decomposition of tensors over compact surfaces, and thus we have the L^2 orthogonal decomposition of

$$h = P_{G_0} + L_X G_0 + \mathcal{L}^*_{G_0} f,$$

where P_{G_0} belongs to $T_{G_0}\mathcal{UT}, L_X G_0$ to $T_{G_0}(\mathrm{Diff}_0\, \mathbf{D}^2)$ and $\mathcal{L}^*_{G_0} f$ to $(T_{G_0}\mathcal{M}_{-1})^{\perp}$.

Now recall that given a tensor h in $\Gamma_{L^2}(T_{G_0}\mathcal{M})$, we had a sequence of tensors $\{h_i\}$ in $\Gamma_0^\infty(T_{G_0}\mathcal{M})$ convergent to h in L^2-norm. For each i, solve the pair of equations

$$\begin{aligned}\delta_{G_0}\delta_{G_0}^* X_i &= \delta_{G_0} h_i,\\ \mathcal{L}_{G_0}\mathcal{L}_{G_0}^* f_i &= \mathcal{L}_{G_0} h_i,\end{aligned}$$

for X_i and f_i. We want to show next that $\{X_i\}$ and $\{f_i\}$ converge to some X and f, and to solve the pair of equations

$$\begin{aligned}\delta_{G_0}\delta_{G_0}^* X &= \delta_{G_0} h,\\ \mathcal{L}_{G_0}\mathcal{L}_{G_0}^* f &= \mathcal{L}_{G_0} h.\end{aligned}$$

To prove the convergence of X_i, note:

$$\begin{aligned}\|L_{X_i-X_j}G_0\|_{L^2}^2 &= \int_{\mathbf{H}^2}\langle L_{X_i-X_j}G_0, L_{X_i-X_j}G_0\rangle_{G_0} d\mu_{G_0}\\ &= \int_{\mathbf{H}^2}\langle h_i - h_j, L_{X_i-X_j}G_0\rangle_{G_0} d\mu_{G_0}\\ &\leq \left(\int_{\mathbf{H}^2}\langle h_i-h_j, h_i-h_j\rangle_{G_0} d\mu_{G_0}\right)^{\frac{1}{2}}\\ &\qquad \cdot \left(\int_{\mathbf{H}^2}\langle L_{X_i-X_j}G_0, L_{X_i-X_j}G_0\rangle_{G_0} d\mu_{G_0}\right)^{\frac{1}{2}}\\ &= \|h_i-h_j\|_{L^2}\|L_{X_i-X_j}G_0\|_{L^2}.\end{aligned}$$

Hence

$$\|L_{X_i-X_j}G_0\|_{L^2} \leq \|h_i-h_j\|_{L^2}.$$

Recall the statement of the previous lemma, which implies

$$2\|X_i - X_j\|_{H^1} \leq \|L_{X_i-X_j}G_0\|_{L^2}.$$

Combining them, we have

$$2\|X_i - X_j\|_{H^1} \leq \|h_i-h_j\|_{L^2}.$$

Since h_i is convergent, and hence Cauchy $\Gamma_{L^2}(T_{G_0}\mathcal{M})$, we know from the above inequality that X_i is Cauchy in the space of H^1 integrable vector fields on $\mathbf{H}^2$. Denote $\lim_{i\to\infty} X_i = X$.

As for the regularity of the vector field X, when h is smooth in the sense that h is in a Sobolev space H^s (i.e., the sth derivative of h is L^2-integrable), the convergence $h_i \to h$ occurs in H^s, and the convergence

$X_i \to X$ occurs in H^{s+1}, for the space of C^∞ tensors is dense in the space of H^{s+1} tensors. In particular, when h is a smooth tensor, then X is smooth satisfying the equation

$$\delta_{G_0}\delta^*_{G_0}X = \delta_{G_0}h.$$

Observe here the fact that X thus obtained is integrable implies that

$$\lim_{r\to\infty}\int_{\mathbf{H}^2\setminus B_r(0)} \|X\|^2_{G_0} d\mu_{G_0} = 0,$$

which in turn says that viewed as a vector field on the open unit disk, X has a trivial extension to the geometric boundary of $\partial\mathbf{D} = S^1$ (i.e., it vanishes on S^1). This is consistent with the description that $L_X G_0$ is an element of $T_{G_0}(\mathrm{Diff}_0\,\mathbf{D}^2)$.

Secondly we want to show the convergence of f_i. Very similar to the proceeding argument for X_i's, note that

$$\begin{aligned}
\|\mathcal{L}^*_{G_0}(f_i - f_j)\|^2_{L^2} &= \int_{\mathbf{H}^2} \langle \mathcal{L}^*_{G_0}(f_i - f_j), \mathcal{L}^*_{G_0}(f_i - f_j)\rangle_{G_0} d\mu_{G_0} \\
&= \int_{\mathbf{H}^2} \langle (h_i - h_j), \mathcal{L}^*_{G_0}(f_i - f_j)\rangle_{G_0} d\mu_{G_0} \\
&\le \left(\int_{\mathbf{H}^2} \langle (h_i - h_j), (h_i - h_j)\rangle_{G_0} d\mu_{G_0}\right)^{\frac{1}{2}} \\
&\qquad \cdot \left(\int_{\mathbf{H}^2} \langle \mathcal{L}^*_{G_0}(f_i - f_j), \mathcal{L}^*_{G_0}(f_i - f_j)\rangle_{G_0} d\mu_{G_0}\right)^{\frac{1}{2}} \\
&= \|h_i - h_j\|_{L^2}\|\mathcal{L}^*_{G_0}(f_i - f_j)\|_{L^2}.
\end{aligned}$$

Hence

$$\|\mathcal{L}^*_{G_0}(f_i - f_j)\|_{L^2} \le \|h_i - h_j\|_{L^2}.$$

From the proof of the above lemma, we have

$$\|f\|_{H^2} \le \|\mathcal{L}^*_{G_0} f\|_{L^2}.$$

Putting the two together yields

$$\|f_i - f_j\|_{H^2} \le \|h_i - h_j\|_{L^2}.$$

Since $\{h_i\}$ is convergent to h and is Cauchy, the above estimate tells us that $\{f_i\}$ is also Cauchy in H^2 . When h is a tensor with the regularity

H^s, then the function f is of H^{s+2} by the elliptic regularity theory. In particular, if h is smooth, then so is f, and satisfies the equation

$$\mathcal{L}_{G_0}\mathcal{L}^*_{G_0} f = \mathcal{L}_{G_0} h.$$

Summarizing what we have shown so far, we now have the decomposition of smooth L^2-integrable tensor h as

$$h = P_{G_0}(h) + L_X G_0 + \mathcal{L}^*_{G_0} f.$$

Recall that choosing a convergent sequence $\{h_i\}$ of smooth compactly supported symmetric $(0,2)$-tensor we have the L^2 orthogonal decomposition for each i

$$h_i = P_{G_0}(h_i) + L_{X_i} G_0 + \mathcal{L}^*_{G_0} f_i,$$

and the following convergences

$$\begin{aligned} \lim_{i\to\infty} X_i =& X \qquad \text{in } H^1, \\ \lim_{i\to\infty} f_i =& f \qquad \text{in } H^2, \end{aligned}$$

and therefore

$$\lim_{i\to\infty} P_{G_0}(h_i) = P_{G_0}(h) \qquad \text{in } L^2.$$

To show that the above decomposition of h is indeed orthogonal, simply note that

$$\langle L_X G_0, \mathcal{L}^*_{G_0} f\rangle_{G_0} = \lim_{i\to\infty} \langle L_{X_i} G_0, \mathcal{L}^*_{G_0} f_i\rangle_{G_0} = 0$$

and

$$\langle \mathcal{L}^*_{G_0} f, P_{G_0}(h)\rangle_{G_0} = \lim_{i\to\infty} \langle \mathcal{L}^*_{G_0} f_i, P_{G_0}(h_i)\rangle_{G_0} = 0.$$

q.e.d.

An immediate implication of this result, just as for the analogous result for the compact surfaces, is that the tensor $P_{G_0}(h)$ is trace-free and divergent-free pointwise on $\mathbf{H}^2$.

Recall that the space of complete hyperbolic metrics modeled on the open unit disk can be identified with the space of quasiconformal diffeomorphisms up to the Möbius transforms of the disk. Linearize this picture at the Poincaré metric:

$$\begin{aligned} T_{G_0}\mathcal{M}_{-1} =& \{L_Z G_0 : Z \text{ vector fields generating} \\ & \qquad\qquad \text{quasiconformal diffeomorphisms on } \mathbf{D}\}. \end{aligned}$$

The previous L^2 orthogonal decomposition states that given a tensor $h \in T_{G_0}\mathcal{M}$, the part tangent to the space $\mathcal{M}_{-1}$ consists of $P_{G_0}(h)$ and $L_X G_0$. Hence

$$L_Z G_0 = P_{G_0}(h) + L_X G_0$$

or equivalently

$$P_{G_0}(h) = L_{Z-X} G_0.$$

The fact that the tensor $P_{G_0}(h)$ is tracefree transverse is equivalent to the condition that the vector field $Y = Z - X$ satisfies the following linear PDE;

$$\partial_z \left(\frac{4}{(1-|z|^2)^2} \partial_{\bar{z}} Y \right) = 0.$$

The holomorphic quadratic differential $\overline{\left(\frac{4}{(1-|z|^2)^2} \partial_{\bar{z}} Y \right)} dz^2$ on $\mathbf{D}$ corresponds to the harmonic Beltrami differential $\overline{\left(\frac{4}{(1-|z|^2)^2} \partial_{\bar{z}} Y \right)} = \mu(z)$ (see Wolpert [27].)

6. Convexity theorem for the universal Teichmüller space

Before we proceed, we will digress in order to motivate the discussion to follow. Consider the case where the setting for the convexity Theorem 3.1.1 is that the domain manifold (M, g) is taken to be the topologically same as the target surface N of genus g with a fixed hyperbolic metric G. Fix then the homotopy type of the maps so that the harmonic maps $u_t : (N^2, G) \to (N^2, G_t)$ are all surjective mappings of degree-one. The degree one condition is preserved by the change in t because the hamonic maps u_t here are diffeomorphisms due to the result of Schoen-Yau [19] and Sampson [17]. Note then that the area functional $\mathcal{A}(G_t)$ is invariant in t, that is;

$$\begin{aligned}
\mathcal{A}(G_t) =& \int_N \frac{\sqrt{\det(u_t^* G_t)}}{\sqrt{\det G}} d\mu_G \\
=& (\det u_t) \operatorname{Vol}(N, G_t) \\
=& 1 \cdot \int_N d\mu_G = \int_N -K_G d\mu_G = 2g - 2.
\end{aligned}$$

Introducing the isothermal coordinates $ds^2_G = \lambda(z)|dz|^2$ on (N, G) and $ds^2_{G_t} = \rho(u)|du|^2$ on (N, G_t) in the complex coordinates $z = x + iy$

and $u = v + iw$, where we adapt the following notation:

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$
$$|\partial u|^2 = \frac{\rho(u(z))}{\lambda(z)}\left|\frac{\partial u}{\partial z}\right|^2, \qquad |\overline{\partial} u|^2 = \frac{\rho(u(z))}{\lambda(z)}\left|\frac{\partial u}{\partial \overline{z}}\right|^2.$$

Then we have the following expressions for the energy and area densities:

$$e(u_t) = \frac{1}{2}\operatorname{tr}_G(u_t^* G_t) = |\partial^t u_t|^2 + |\overline{\partial^t} u|^2,$$
$$J(u_t) = \frac{\sqrt{\det(u_t^* G_t)}}{\sqrt{\det G}} = |\partial^t u_t|^2 - |\overline{\partial^t} u|^2.$$

The preceeding observation that the area functional is invariant in t, implies that

$$\begin{aligned}\frac{d^2}{dt^2}\mathcal{E}(G_t) &= \frac{d^2}{dt^2}E(u_t, G_t)\\ &= \frac{d^2}{dt^2}\int_N \left(|\partial^t u_t|^2 + |\overline{\partial^t} u|^2\right) d\mu_G\\ &\quad - \frac{d^2}{dt^2}\int_N \left(|\partial^t u_t|^2 - |\overline{\partial^t} u|^2\right) d\mu_G\\ &= \frac{d^2}{dt^2}\int_N 2|\overline{\partial^t} u|^2 d\mu_G.\end{aligned}$$

Therefore the Weil-Petersson convexity of the energy functional 3.1.1 is equivalent to the the Weil-Petersson convexity of this new functional $\int_N 2|\overline{\partial^t} u_t|^2 d\mu_G$.

Definition 6.0.1. Define anti-holomorphic energy to be

$$E_{\bar{\partial}} = \int_N 2|\bar{\partial} u|^2 d\mu_G.$$

We would like to have the convexity result for the universal Teichmüller space setting. In particular, we will consider the following setting in analogy to the above setting we have just looked at.

Suppose that G_t is a horizontal lift of a Weil-Petersson geodesic in $\mathcal{UT}$ with G_0 the standard Poincaré metric modeled on the Euclidian unit disk $\mathbf{D}^2$. (It should be noted here that there is no satisfactory result concerning the existence of the geodesic in $\mathcal{UT}$, yet we will assume here that there is one.) Recall that the uniformization theorem says that $G_t = \phi^* G_0$ for some quasiconformal diffeomorphism $\phi : \mathbf{D}^2 \to \mathbf{D}^2$. Here G_t is regarded as a complete hyperbolic metric modeled on the open disk $\mathbf{D}^2$. Let u_t be a family of harmonic maps

$$u_t : (\mathbf{D}, G_0) \to (\mathbf{D}^2, G_t) = (\mathbf{D}^2, \phi_t^* G_0)$$

satisfying that u_t, viewed as a map from $\mathbf{D}^2$ to itself, has the trivial extension to the geometric boundary, that is $u_t \mid_{S^1} = \mathrm{Id}_{S^1}$. This condition is in place of specifying the homotopy type of the maps between compact surfaces.

We now have the following picture:

$$(\mathbf{D}^2, G_t) \xrightarrow{u_t} (\mathbf{D}^2, G_t = \phi_t^* G_0) \xrightarrow{\phi_t} (\mathbf{D}^2, G_t).$$

Since ϕ_t above is an isometriy, the composite map

$$\phi_t \circ u_t : (\mathbf{D}^2, G_0) \to (\mathbf{D}^2, G_0)$$

is a harmonic map with the asymptotic boundary condition

$$\phi_t \circ u_t \mid_{S^1} = \phi_t \mid_{S^1} \circ u_t \mid_{S^1} = \phi_t \mid_{S^1} \circ \mathrm{Id}_{S^1} = \phi_t \mid_{S^1} .$$

We would like to study the behavior of the energy functional as before in our new setting. However, the energy functional is not well-defined for it is infinite. So we will instead consider the anti-holomorphic energy functional discussed above, for which we have the following result.

Theorem 6.0.2. *Provided that $G = \phi^* G_0$ represents a point in $\mathcal{M}_{-1}$ with*

$$\phi_t \mid_{S^1} \in C^2(S^1; S^1), \mathcal{E}_{\overline{\partial}}(G)$$

is finite.

Proof. For a C^2 diffeomorphism $f : S^1 \to S^1$, which assumes that the differential does not vanish on S^1, we will show that one can find a harmonic map

$$u : (D, G_0) \to (D, G_0)$$

with the asymptotic boundary condition f such that the energy density $e(u)(z)$ with $z \in \mathbf{C}$ is an integrable function along the radial direction as $|z| \to 1$. For the sake of making the computations simple, we will model the hyperbolic space on the upper half space with the hyperbolic metric given as $ds^2 = (dx^2 + dy^2)/y^2$. Hence we have a proper map u from the upper half plane U to itself, which is harmonic with respect to the Poincaré metric G_0 with asymptotic boundary condition given by a C^2 diffeomorphism $f : \mathbf{R} \to \mathbf{R}$. Without loss of generality, assume the condition $f(0) = 0$, and $u(0) = 0$. We now define asymptotically harmonic map $h : (U, G_0) \to (U, G_0)$ satisfying

$$\begin{aligned} h^1(x,y) =& f(x) + o(y^2), \\ h^2(x,y) =& yf'(x) + o(y^2), \end{aligned}$$

where $f'(x) = \frac{d}{dx} f(x) > 0$. Then

$$\begin{aligned} h^1(0,0) = h^2(0,0) =& 0, \\ \partial h^1/\partial y \mid_{(00)} = \partial h^2/\partial x \mid_{(0,0)} =& 0, \\ \partial^2 h^1/\partial y^2 \mid_{(0,0)} = \partial^2 h^2/\partial y^2 \mid_{(0,0)} =& 0, \\ \partial^2 h^1/\partial x \partial y \mid_{(0,0)} =& 0, \end{aligned}$$

and the tension field

$$\tau^i(h) = 0(y^2) \quad (i = 1, 2),$$

where

$$\begin{aligned} \tau^1(h) &= y^2 \left(\Delta_0 h^1 - \frac{2}{h^2} \langle \nabla_0 u^1, \nabla_0 u^2 \rangle \right), \\ \tau^2(h) &= y^2 \left(\Delta_0 h^2 - \frac{1}{h^2} (|\nabla_0 u^1|^2 - |\nabla_0 u^2|^2) \right). \end{aligned}$$

Δ_0 and ∇_0 above are the Euclidian laplacian and the Euclidian gradient respectively.

The map h is also asymptotically conformal in the sense that with respect to the Euclidian coordinate $z = x + iy$,

$$\begin{aligned} \frac{\partial h}{\partial z} =& f'(0), \\ \frac{\partial h}{\partial \bar{z}} =& 0 \qquad \text{at } z = 0. \end{aligned}$$

Now by varying the map h by $y\phi(x, y)$ where ϕ is a smooth function with $\phi(0, 0) = (0, 0)$, we would like to have the tension field to have the asymptotic behavior of

$$\tau(h + y\phi) = O(y^3).$$

Using the asymptotic expansion at the point $(0,0)$ at the infinity, we will find conditions ϕ needs to satisfy.

Denote $v = h + y\phi$. Then the Taylor expansion of v at $(0,0)$ is

$$\begin{aligned} v^i =& h^i + (\partial_1 h^i(0))x + (\partial_2 h^i(0))y \\ &+ \frac{1}{2}(\partial_1^2 h^i(0))x^2 + \frac{1}{2}(\partial_2^2 h^i(0))y^2 \\ &+ (\partial_1\partial_2 h^1(0))xy + (\phi^i(0))y + (\partial_1\phi^i(0))xy \\ &+ (\phi^i(0))y + (\partial_1\phi^i(0))xy + (\partial_2\phi^i)y^2 + O(x^k y^l), \\ &\qquad\qquad\qquad\qquad k + l \geq 3. \end{aligned}$$

We proceed to write down the first derivatives of v as a function of y. They are evaluated on the line $x = 0$.

$$\begin{aligned} \partial_1 v^1 =& f'(0) + (\partial_1\phi^1(0))y + \frac{1}{2}(\partial_1\partial_2\phi^1(0))y^2 + O(y^3), \\ \partial_2 v^1 =& \frac{1}{2}(\partial_2^3 h^1(0))y^2 + 2(\partial_2\phi^1(0))y \\ &+ \frac{3}{2}(\partial_2^2 h^1(0))y^2 + O(y^3), \\ \partial_1 v^2 =& \{(\partial_1\partial_2 h^2(0)) + (\partial_1\phi^2(0))\}y \\ &+ \frac{1}{2}(\partial_1\partial_2\phi^2(0))y^2 + O(y^3), \\ \partial_2 v^2 =& f'(0) + \frac{1}{2}(\partial_2^3 h^2(0))y^2 + 2(\partial_2\phi^2(0))y \\ &+ \frac{3}{2}(\partial_2^2\phi^2(0))y^2 + O(y^3). \end{aligned}$$

We will also need to express the second derivatives for writing down the Laplacian term in the tension field.

$$\begin{aligned} \partial_1^2 v^i =& (\partial_1^2\phi^i(0))y + O(y^2), \\ \partial_2^2 v^i =& (\partial_2^3 h^i(0))y + 2(\partial_2\phi^i(0)) \\ &+ 3(\partial_2^2\phi^i(0))y + O(y^2). \end{aligned}$$

With those explicit terms, we now write down the tension field terms

$\tau^1(v), \tau^2(v)$:

$$\begin{aligned}\tau^1(v) =& y^2\Big(\Delta_0 h^1 - \frac{2}{h^2}\langle \nabla_0 u^1, \nabla_0 u^2\rangle\Big)\\ =& y^2\Bigg((\partial_1^2\phi^1(0))y + (\partial_2^3 h^1(0))y\\ &+ 2(\partial_2\phi^1(0)) + 3(\partial_2^2\phi^1(0))y\\ &- \frac{2}{v^2}\Big[\Big(f'(0) + (\partial_1\phi^1(0))y + \frac{1}{2}(\partial_1\partial_2\phi^1(0))y^2\Big)\\ &\cdot\Big(\{(\partial_1\partial_2 h^2(0)) + (\partial_1\phi^2(0))\}y + \frac{1}{2}(\partial_1\partial_2\phi^2(0))y^2\Big)\\ &+ \Big(\frac{1}{2}(\partial_2^3 h^1(0))y^2 + 2(\partial_2\phi^1(0))y + \frac{3}{2}(\partial_2^2 h^1(0))y^2\Big)\\ &\cdot\Big(f'(0) + \frac{1}{2}(\partial_2^3 h^2(0))y^2 + 2(\partial_2\phi^2(0))y\\ &+ \frac{3}{2}(\partial_2^2\phi^2(o))y^2\Big)\Big]\Bigg).\end{aligned}$$

It is here that $f : S^1 \to S^1$ is required to be C^2, namely $(\partial_1^2\phi^1(0))\frac{d^2}{dx^2}f(x)\Big|_{x=0}$ exists. For this to be of $O(y^3)$, we need the following:

$$\partial_2\phi^1(0) = 0, \tag{34}$$

$$\partial_1\partial_2 h^2(0) + \partial_1\phi^2(0) = 0, \tag{35}$$

where we have used the fact that $v^2 = O(y)$.

As for τ^2,

$$\begin{aligned}\tau^2(v) =& y^2\Big(\Delta_0 h^2 + \frac{1}{h^2}(|\nabla_0 u^1|^2 - |\nabla_0 u^2|^2)\Big)\\ =& y^2\Bigg((\partial_1^2\phi^2(0))y + (\partial_2^3 h^2(0))y\\ &+ 2(\partial_2\phi^2(0)) + 3(\partial_2^2\phi^2(0)))y\end{aligned}$$

$$
\begin{aligned}
-\frac{1}{v^2}\Big[&\Big(f'(0)+(\partial_1\phi^1(0))y+\frac{1}{2}(\partial_1\partial_2\phi^1(0))y^2\Big)^2\\
&+\Big(\frac{1}{2}(\partial_2^3 h^1(0))y^2+2(\partial_2\phi^1(0))y+\frac{3}{2}(\partial_2^2 h^1(0))y^2\Big)^2\\
&-\Big(\{(\partial_1\partial_2 h^2(0))+(\partial_1\phi^2(0))\}y+\frac{1}{2}(\partial_1\partial_2\phi^2(0))y^2\Big)^2\\
&-\Big(f'(0)+\frac{1}{2}(\partial_2^3 h^2(0))y^2+2(\partial_2\phi^2(0))y\\
&\qquad\qquad+\frac{3}{2}(\partial_2^2\phi^2(o))y^2\Big)^2\Big]\Big).
\end{aligned}
$$

In order to have $\tau^2(v)=O(y^3)$, it then is necessary to have

$$
\partial_2\phi^2(0) = 0, \tag{36}
$$

$$
\partial_1\phi^1(0) = 0, \tag{37}
$$

Note that the new asymptotically harmonic map $v(z)=h(z)+y\phi(z)$ approximates the harmonic map u in the sense that

$$
u(0+iy)-v(0+iy)=O(y^{1+\varepsilon}),\quad \varepsilon>0.
$$

For calculating the asymptotic expansions of the energy density $e(u)$ and the area density $J(u)$ evaluated along the line $x=0$ in the upper half space, using the expansions for the approximate map v; we obtain the following:

$$
\begin{aligned}
e(u) =&\frac{1}{2}\frac{y^2}{(u^2)^2}\sum_{i,j=1}^{2}(\partial_i u^j)^2=\frac{y^2}{(u^2)^2}([f'(0)]^2+O(y^2)),\\
J(u) =&\frac{y^2}{(u^2)^2}\det\{\partial_i u^j\}=\frac{y^2}{(u^2)^2}([f'(0)]^2+O(y^2)),
\end{aligned}
$$

as well as

$$
\frac{y^2}{(u^2)^2}=y^2\big(v^2+o(y)\big)^{-2}=\frac{1}{[f'(0)]^2}+O(y^2).
$$

Therefore

$$
\begin{aligned}
e(u) &= \left(\frac{1}{[f'(0)]^2}+O(y^2)\right)([f'(0)]^2,+O(y^2))=1+O(y^2),\\
J(u) &= \left(\frac{1}{[f'(0)]^2}+O(y^2)\right)([f'(0)]^2+O(y^2))=1+O(y^2).
\end{aligned}
$$

In particular

$$e(u) - J(u) = O(y^2),$$

and therefore it follows that

$$E_{\overline{\partial}}(u) = \int_{\mathbf{H}^2} (e(u) - J(u)) d\mu_{G_0} < \infty.$$

q.e.d.

We believe that the C^2 condition is not optimal, and one wishes to find a suitable regularity condition for the asymptotic boundary condition $\phi\big|_{S^1}$, to preclude all the points in $M \subset \mathcal{UT}$ that are finite Weil-Petersson distance away from the Poincaré metric G_0.

We can now state the main result of the section.

Theorem 6.0.3 (Weil-Petersson convexity of the anti-holomorphic energy functional). *Given a family of harmonic maps*

$$u_t : (\mathbf{D}^2, G_0) \to (\mathbf{D}^2, G_0)$$

with G_t a horizontal lift of a Weil-Petersson geodesic in $M \subset \mathcal{UT}$, and with $u_t\big|_{S^1} = \mathrm{Id}_{S^1}$, we have

$$\mathcal{E}_{\overline{\partial}}(G_0) = 0,$$

$$\frac{d}{dt}\mathcal{E}_{\overline{\partial}}(G_t)\Big|_{t=0} = 0,$$

and

$$\frac{d^2}{dt^2}\mathcal{E}_{\overline{\partial}}(G_t)\Big|_{t=0} = \frac{1}{2}.$$

In particular the anti-holomorphic energy functional defined on

$$M = \mathrm{Diff}\, S^1 / \mathrm{SL}(2, \mathbf{R}) \subset \mathcal{UT}$$

is convex with respect to the Weil-Petersson distance at $[G_0] \in M$, or equivalently at $[\mathrm{Id}_{S^1}] \in \mathrm{Diff}\, S^1 / \mathrm{SL}(2, \mathbf{R})$.

Remark.

$$M = \mathrm{Diff}\, S^1 / \mathrm{SL}(2, \mathbf{R}) \subset \mathcal{UT}$$

is formally a homogeneous space with a left invariant metric, given by the Weil-Petersson metric defined at $[G_0] \in \mathcal{UT}$ (or equivalently at $[\mathrm{Id}_{S^1}] \in \mathrm{Diff}\, S^1 / \mathrm{SL}(2, \mathbf{R})$). Hence the convexity of the functional at the identity can be translated to any point in

$$M = \mathrm{Diff}\, S^1 / \mathrm{SL}(2, \mathbf{R}) \subset \mathcal{UT}$$

by the group action of Diff S^1.

Proof. First of all, the identity map

$$u_0 : (\mathbf{D}^2, G_0) \to (\mathbf{D}^2, G_0)$$

is holomorpic, hence $|\overline{\partial} u| = 0$ and thus $\mathcal{E}_{\overline{\partial}}(G_0) = 0$. We want to show that

$$\begin{aligned}\frac{d^2}{dt^2}\mathcal{E}_{\overline{\partial}}(G_t)\Big|_{t=0} =& \frac{d^2}{dt^2}\int_{\mathbf{H}^2}(e(u_t, G_t) - J(u_t, G_t))d\mu_{G_0}\Big|_{t=0}\\ =& \frac{1}{2},\end{aligned}$$

where $e(u_t, G_t) = \frac{1}{2}\operatorname{tr}_{G_0}(u_t^* G_t)$ and $J(u_t, G_t) = \frac{\sqrt{\det(u_t^* G_t)}}{\sqrt{\det G_0}}$.

Before differentiating the integral, we will make the following observation.

Lemma 6.0.2. *The vector field* $W_0 \overset{def}{=} \frac{d}{dt}u_t\Big|_{t=0}$ *is identically zero.*

Remark. This statement corresponds, in the classical setting, to the observation made by Wolf [26], that the Weil-Petersson geodesic can be approximated by a path given by linearly extending a harmonic Beltrami coefficient up to order two.

Proof. Recall that for each t, we have a harmonic map

$$\tilde{u}_t \overset{def}{=} \phi_t \circ u_t : (\mathbf{D}^2, G_0) \to (\mathbf{D}^2, G_0)$$

with the geometric boundary condition

$$\tilde{u}_t\Big|_{S^1} = (\phi_t \circ u_t)\Big|_{S^1} = \phi_t\Big|_{S^1},$$

satisfying the harmonic map equation

$$\tilde{u}^t_{z\bar{z}} + \big(\log\lambda(\tilde{u}^t)\big)_{\tilde{u}^t}\tilde{u}^t_z\tilde{u}^t_{\bar{z}} = 0,$$

where $\lambda(z) = 4(1 - |z|^2)^{-2}$. Let us denote $\frac{d}{dt}\tilde{u}^t\Big|_{t=0} = V(z)$, where $V(z)$ is a vector field defined on $\mathbf{D}$. Differentiating the harmonic map equation above in t and evaluating it at $t = 0$, one gets

$$V_{z\bar{z}} + \frac{\lambda_z}{\lambda}V_{\bar{z}} = 0$$

or equivalently

$$\partial_z(\lambda\partial_{\bar{z}}V) = 0$$

for z in $\mathbf{D}$. This implies that the symmetric $(0,2)$ tensor given by the Lie derivative $L_V G_0$ is then tracefree divergence free.

On the other hand, differentiating $\phi_t \circ u_t$ with respect to t at $t = 0$, we have

$$\begin{aligned}\frac{d}{dt}\phi_t \circ u_t\Big|_{t=0} &= \frac{d}{dt}\phi_t\Big|_{t=0} + \frac{d}{dt}\tilde{u}_t\Big|_{t=0},\\ &= X(z) + W(z),\end{aligned}$$

where the vector field $\frac{d}{dt}\phi_t = X(z)$ induces a traceless transverse deformation $L_{G_0}X$ of G_0. That in turn yields that the vector field Z also satisfies the linerized harmonic map equation

$$\partial_z(\lambda\partial_{\bar{z}}X) = 0.$$

So we have

$$V(z) - X(z) = W(z),$$

where both V and X induce tracefree transverse deformations of the Poincaré metric G_0. Note then that the vector field $V - X$ on $\mathbf{H}^2$ satisfies the equation

$$\partial_z(\lambda\partial_{\bar{z}}(V - X)) = 0.$$

Also note that the vector field W vanishes on S^1, for u_t fixes S^1 for all t.

It was shown by P. Li and L.-F. Tam ([13]) that given a $C^{1,\alpha}$ asymptotic boundary condition $f : S^1 \to S^1$ on the geometric boundary S^1 of $\mathbf{H}^2$, there exists a unique proper harmonic map $u : \mathbf{H}^2 \to \mathbf{H}^2$ with $u\mid_{S^1} = f$. When $f = \mathrm{Id}_{S^1}$, the identity map $\mathbf{H}^2 \to \mathbf{H}^2$ is the unique harmonic map. In particular, it follows that there is no proper harmonic map between two copies of $\mathbf{H}^2$ with the trivial asymptotic boundary condition.

Note then that the vector field $V - X$ on $\mathbf{H}^2$ satisfies the equation

$$\partial_z\Big(\lambda(z)\partial_{\bar{z}}(V - X)(z)) = 0,$$

and hence generates a family of harmonic maps $u^t : \mathbf{H}^2 \to \mathbf{H}^2$ with $(V - X)\mid_{S^1} \equiv 0$. Yet this cannot occur due to the observation in the previous paragraph, provided that the vecotr fields V and X are sufficiently smooth functions on S^1 (e.g. $C^{1,\alpha}$). Therefore $W = V - X \equiv 0$, which proves the statement of the lemma. q.e.d.

Now we are to differentiate the anti-holomorphic functional $\mathcal{E}_{\overline{\partial}}(u_t)$ with respect to t and evaluate at $t = 0$. All the differentiations will occur within the integral, and all the calculation will be local.

First differentiate the energy density $e(u_t, G_t)$ in t, and get

$$\begin{aligned}
\frac{d}{dt}e(u_t,G_t)\Big|_{t=t_0} =& \frac{1}{2}\operatorname{tr}_{G_0}\left(\frac{d}{dt}\left[u_t^* G_t\right]\right)\\
=& \frac{1}{2}\operatorname{tr}_{G_0}\left(u_{t_0}^*\left[\frac{d}{dt}G_t\right]\right)\\
&+\frac{1}{2}\operatorname{tr}_{G_0}\left(\frac{d}{dt}u_t^* G_{t_0}\right)\\
=& \frac{1}{2}\operatorname{tr}_{G_0}(u_{t_0}^*\dot{G}_{t_0})\\
&+\frac{1}{2}\operatorname{tr}_{G_0}(u_{t_0}^*[L_{W_{t_0}}G_{t_0}]).
\end{aligned}$$

Note here that at $t_0 = 0$, $\frac{d}{dt}e(u_t, G_t) \equiv 0$ for $W \equiv 0$ as well as $\operatorname{tr}_{G_0}(\dot{G}_0) \equiv 0$. Differentiate this one more time to obtain

$$\begin{aligned}
\frac{d^2}{dt^2}e(u_t,G_t)\Big|_{t=t_0} =& \frac{1}{2}\operatorname{tr}_{G_0}(u_{t_0}^*[L_{\frac{d}{dt_0}W_{t_0}}G_{t_0}])\\
&+\frac{1}{2}\operatorname{tr}_{G_0}(u_{t_0}^*[L_{W_{t_0}}\dot{G}_{t_0}])\\
&+\frac{1}{2}\operatorname{tr}_{G_0}(u_{t_0}^*[L_{W_{t_0}}(L_{W_{t_0}}G_{t_0})])\\
&+\frac{1}{2}\operatorname{tr}_{G_0}(u_{t_0}^*[L_{W_{t_0}}\dot{G}_{t_0}])\\
&+\frac{1}{2}\operatorname{tr}_{G_0}(u_{t_0}^*\ddot{G}_{t_0}).
\end{aligned} \tag{38}$$

Secondly we now differentiate the area density $J(u_t, G_t)$ in t:

$$\begin{aligned}
\frac{d}{dt}J(u_t,G_t)\Big|_{t=t_0} =& \frac{d}{dt}\frac{\sqrt{\det(u_t^*G_t)}}{\sqrt{\det G_0}}\Big|_{t=t_0}\\
=& \frac{1}{2}\operatorname{tr}_{u_{t_0}^*}(u_{t_0}^*[L_{W_{t_0}}G_{t_0}])\sqrt{\det(u_{t_0}^*G_{t_0})}\\
&+\frac{1}{2}\operatorname{tr}_{u_{t_0}^*}(u_{t_0}^*\dot{G}_{t_0}])\sqrt{\det(u_{t_0}^*G_{t_0})}\\
=& \frac{1}{2}\operatorname{tr}_{u_{t_0}^*}(u_{t_0}^*[L_{W_{t_0}}G_{t_0}])\sqrt{\det(u_{t_0}^*G_{t_0})}.
\end{aligned} \tag{39}$$

The last equality holding for the symmetric $(0,2)$-tensor $\frac{1}{2}\operatorname{tr}_{u_{t_0}^* G_{t_0}}(u_{t_0}^* \dot{G}_{t_0}])$ is identically zero since $\dot{G}_{t=0}$ is tracefree with respect to G_{t_0}.

When $t_0 = 0$, $\frac{d}{dt}J(u_t, G_t) \equiv 0$ for $W \equiv 0$. Combined with the observation made above about the energy density $e(u_y, G_t)$, we can conclude that

$$\frac{d}{dt}\mathcal{E}_{\overline{\partial}}(G_t)\Big|_{t=t_0} = 0.$$

The second time derivative of the area density is given by

$$\begin{aligned}
\frac{d^2}{dt^2}J(u_t, G_t)\Big|_{t=t_0} = &- \langle L_{W_{t_0}} G_{t_0}, L_{W_{t_0}} G_{t_0}\rangle_{u^* G_{t_0}} \\
&- \langle L_{W_{t_0}} G_{t_0}, \dot{G}_{t_0}\rangle_{u^* G_{t_0}} \\
&+ \frac{1}{2}\operatorname{tr}_{u_{t_0}^* G_{t_0}}(u_{t_0}^*[L_{\frac{d}{dt_0}W_{t_0}} G_{t_0}]) \\
&+ \frac{1}{2}\operatorname{tr}_{u_{t_0}^* G_{t_0}}(u_{t_0}^*[L_{W_{t_0}}(L_{W_{t_0}} G_{t_0})]) \\
&+ \frac{1}{2}\operatorname{tr}_{u_{t_0}^* G_{t_0}}(u_{t_0}^*[L_{W_{t_0}} \dot{G}_{t_0}]).
\end{aligned} \tag{40}$$

Recall now the statement of the previous lemma which says $W_{t_0} \equiv 0$ when $t_0 = 0$ as well as $u_0 = \mathrm{Id}_{\mathbf{H}^2}$. Evaluating equations (38) and (40) at $t = 0$ gives

$$\frac{d^2}{dt^2}e(u_t, G_t)\Big|_{t=t_0} = \frac{1}{2}\operatorname{tr}_{G_0}(L_{\frac{d}{dt_0}W_{t_0}} G_0) + \frac{1}{2}\operatorname{tr}_{G_0}(\ddot{G}_0),$$

and

$$\frac{d^2}{dt^2}J(u_t, G_t)\Big|_{t=t_0} = \frac{1}{2}\operatorname{tr}_{G_0}(L_{\frac{d}{dt_0}W_{t_0}} G_0).$$

Integrating the difference of the terms $\frac{d^2}{dt^2}e(u_t, G_t)\Big|_{t=0}$ and $\frac{d^2}{dt^2}J(u_t, G_t)\Big|_{t=t_0}$, we have an expression for the second derivative of the antiholomorphic energy $\mathcal{E}_{\overline{\partial}}(G_t)$ with respect to the Weil-Petersson distance t at $t = 0$ as follows:

$$\begin{aligned}
\frac{d^2}{dt^2}\mathcal{E}_{\overline{\partial}}(G_t)\Big|_{t=0} &= \frac{d^2}{dt^2}\left(\int_{\mathbf{H}^2} e(u_t, G_t) - J(u_t, G_t)\right) d\mu_{G_0}\Big|_{t=0} \\
&= \frac{1}{2}\int_{\mathbf{H}^2} \operatorname{tr}_{G_0}(\ddot{G}_{t_0}) d\mu_{G_0}.
\end{aligned} \tag{41}$$

Knowing that the tangent vector $\dot{G}_t$ along the path of metrics G_t remains tracefree with respect to G_t implies that

$$\begin{aligned}\frac{d}{dt}\operatorname{tr}_{G_t}(\dot{G}_t)\Big|_{t=0} &= -\langle \dot{G}_0, \dot{G}_0\rangle_{G_0} + \operatorname{tr}_{G_0}(\ddot{G}_0)\\ &=0.\end{aligned}$$

Hence,

$$\operatorname{tr}_{G_0}(\ddot{G}_0) = \langle \dot{G}_0, \dot{G}_0\rangle_{G_0}.$$

Substituting in equation (41) yields the statement of the theorem:

$$\frac{d^2}{dt^2}\mathcal{E}_{\overline{\partial}}(G_t)\Big|_{t=0} = \frac{1}{2}\int_{\mathbf{H}^2}\langle \dot{G}_0, \dot{G}_0\rangle_{G_0} = \frac{1}{2},$$

for G_t is a geodesic parametrized by the Weil-Petersson arc length.

q.e.d.

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