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HARMONIC MAPS OF INFINITE ENERGY AND RIGIDITY RESULTS FOR REPRESENTATIONS OF FUNDAMENTAL GROUPS OF QUASIPROJECTIVE VARIETIES

JÜRGEN JOST & KANG ZUO

Abstract

We show the existence of harmonic maps associated with reductive homomorphisms of the fundamental group of a quasiprojective variety into a linear algebraic group over an archimedean or p-adic field. The map we construct may have infinite energy, but it satisfies suitable estimates at infinity, and it is pluriharmonic. We use this map to complete a previous result of Jost-Yau [21] on strong rigidity of nonuniform lattices in Hermitian symmetric spaces, and to drop a topological restriction in our previous theory ([24], [25]) of representations of fundamental groups of quasiprojective varieties.
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Introduction

Hodge theory represents cohomology classes by harmonic forms, and it is a fundamental tool in Kähler geometry. Homology or cohomology groups, however, do not contain the complete topological information about a manifold X, and, in particular, the fundamental group may be much more complicated than the first homology group, its abelianized version, indicates. Via the Albanese period mapping, the fundamental group acts as a lattice on some vector space. This can be considered as an abelian representation of $\pi_1(X)$, yielding the first homology group. It is therefore natural to study also nonabelian representations of $\pi_1(X)$, in order to obtain further information about the topology of X. In the same way as the Albanese period map is harmonic (which is essentially

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the Hodge theorem for 1-forms), one tries to associate to a nonabelian representation a harmonic map u as well and to study the properties of u. This has been done in work of Al'ber [1], [2], Eells-Sampson [10], Siu [46], Jost-Yau [19], [22], [23], Sampson [44], Diederich-Ohsawa [9], Corlette [5], Carlson-Toledo [7], Hitchin [13], Simpson [42], Zuo [50], [51], Gromov-Schoen [11], Mok-Siu-Yeung [37], and many others, in the case of a compact Riemannian, Kählerian, or algebraic manifold X. In the noncompact case, Iitaka [14] extended the Albanese map to quasiprojective varieties. The harmonic map theory has been extended to the quasiprojective case by Jost-Yau [20], [21], and Jost-Zuo [24], [25]. All these papers, however, work in a situation where one can produce a harmonic map of finite energy, and this requires some additional assumptions about the representations of $\pi_1(X)$ near infinity. Simpson [40] considered the case of complex dimension 1 and used harmonic maps with controlled behavior at infinity which he called "tame". In dimension 1, however, many of the analytical difficulties that we have to encounter in the present paper are not present. (For the one-dimensional analysis, see also the papers [48] and [32] that will be discussed in more details below.)

The main technical achievement of the present paper is to remove the restriction that the associated harmonic map must have finite energy, while working in arbitrary dimension. In our Theorem 1.1, we show that for any reductive representation ρ of π_1 of a smooth quasiprojective variety (or, more generally, for a Kähler manifold admitting a suitable compactification) into the isometry group of either a symmetric space of noncompact type or a locally compact Euclidean Tits building, we obtain a corresponding harmonic map u of possibly infinite energy that satisfies a precise growth estimate near infinity. (Here, a symmetric space is associated with a representation in a linear algebraic group defined over $\mathbb{R}$ or $\mathbb{C}$, whereas the Tits buildings arise from representations over p-adic ground fields.) In fact, the map is even pluriharmonic, making the foliation technique first discovered in [19] applicable. ("Pluriharmonic" means that the restriction of u to any subvariety of X is again harmonic.)

The prototype of such an infinite energy harmonic map with the typical growth behavior near the puncture is the map

$$u_0 : D^* := \{z \in \mathbb{C} : 0 < |z| \leq 1\} \to S^1$$

$$re^{i\vartheta} \mapsto \vartheta.$$

For the case of complex dimension 1, i.e., for holomorphic curves with cusps alias punctured Riemann surfaces, such a harmonic map was constructed by Wolf [48] and Lohkamp [32], controlling the growth of the energy density near the puncture through comparison with the prototype u_0. The same strategy also works in our situation, but the details become much harder.

It is conceivable that our existence result holds in more generality. Namely, one would like the target to be a more general space of nonpositive curvature. In our argument, besides the nonpositivity of the curvature, however, we need a decay condition for Jacobi fields in order to handle parabolic or quasihyperbolic elements in the image of ρ. (Wolf [48] and Lohkamp [32] only consider the case of hyperbolic (or elliptic) elements which suffices for their purposes.) Finally one would like to extend the result to nonlocally compact images, as in [15], [16], [18], [28].

For our present purposes, however, our existence result suffices, and we can essentially complete the nonabelian Hodge theory for quasiprojective manifolds.

Some partial existence results for harmonic maps on noncompact Kähler manifolds have recently obtained by J. Li [31]. It is not clear, however, whether the maps produced by his method can be shown to be pluriharmonic.

In §2, we complete strong rigidity for nonuniform lattices in bounded symmetric domains D in the holomorphic context. The result is that if a quasiprojective manifold X is homotopically equivalent to a quotient of an irreducible bounded symmetric domain by a lattice, then X is already biholomorphically equivalent to this quotient (with the standard exception where D is the hyperbolic plane). This is the holomorphic version of Margulis' rigidity theorem. In the compact case, it was shown by Siu [46], and in the noncompact case by Jost-Yau [20], [21] with an additional technical restriction on the compactification of X.

Our second application, presented in §3, extends our previous work [24], [25]. (In a subsequent paper, Katzarkov [27] stated results similar to those of [25], together with some applications to the Shafarevich conjecture.)

After recalling some preliminary constructions, we shall show two main results. The first one concerns Zariski dense representations ρ of $\pi_1(X)$ in a simple algebraic group G defined over $\mathbb{C}$, with fixed characteristic polynomials at infinity. If such a representation is not rigid, then it factors through a morphism $f : X \to X'$, with $\dim X' \leq \mathrm{rk}_{\mathbb{C}} G$.

In other words, any representation that does not factor through one obeying this rank restriction is rigid. The second result concerns the case where G is defined over a complete field K with a discrete valuation v. (We assume that the associated Tits building $\Delta(G(K_v))$ is locally compact.) If ρ is unbounded with respect to this valuation, i.e., if $\rho(\pi_1(X))$ is not contained in a compact subgroup of $G(K_v)$, i.e., in the stabilizer of a vertex of $\Delta(G(K_v))$, then ρ factors as before.

As explained in more detail in the introduction of [25], these results have far reaching consequences for archimedean and p-adic representations of fundamental groups of quasiprojective manifolds.

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We thank the referee for finding an error in the first version of our paper, and we should point out that the first Lemma in our announcement [26] is incorrect in the generality stated.

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1. The existence of pluriharmonic maps with controlled growth at infinity

1.1. Statement of the result

Theorem 1.1. *Let X be a Kähler manifold admitting a compactification $\bar{X}$ as a compact Kähler manifold for which $D := \bar{X} \setminus X$ is a divisor with simple normal crossings. Let Y be either*

a) a symmetric space of noncompact type

or

b) a locally compact Euclidean Tits building (with isometry group operating transitively on the set of vertices).

Let

$$\rho : \pi_1(X) \to I(Y)$$

be a homomorphism from the fundamental group of X into the isometry group of Y. Assume that ρ is reductive (see Def. below). Then there exists a ρ-equivariant pluriharmonic

$$u : \tilde{X} \to Y$$

from the universal cover of X to Y satisfying the following estimate:

For every noncompact holomorphic curve Σ in X, we represent a neighborhood of a node conformally as a punctured disc

$$D^* := \{z \in \mathbb{C} : 0 < |z| < 1\} \subset \mathbb{C}.$$

With respect to the (noncomplete) Euclidean metric on D^* and the natural metric of Y, we then have for the norm of the derivative of u

$$\|du(z)\|^2 \leq \frac{\text{const}}{|z|^2}.$$

(The constant depends only on the representation ρ, but not on the curve Σ; one may extract this from the subsequent proof, but we shall not go into the details. Note that in case b), u is only Lipschitz; the preceding estimate then holds a.e., and pluriharmonicity of u has to be interpreted in the sense of Gromov-Schoen [11].)

The following definition was suggested by Scot Adams.

Definition 1.1. Let Y be a complete, simply connected, locally compact space of nonpositive curvature with isometry group $I(Y)$. Let Γ be a group, and $\rho : \Gamma \to I(Y)$ a homomorphism.

ρ is called reductive if there exists a complete totally geodesic subspace Z of Y stabilized by $\rho(\Gamma)$ with the following property:

For every totally geodesic subspace Z' of Z that has no Euclidean factor, $\rho(\Gamma)$ does not fix any point in the sphere at infinity of Z', i.e., is not contained in a parabolic subgroup of the isometry group of Z'.

In geometric terms, this means that there does not exist an unbounded sequence $(y_n)_{n\in\mathbb{N}} \subset Z'$ with

$$\text{dist}\,(y_n, \gamma(y_n)) \leq c(\gamma)$$

for all $\gamma \in \rho(\Gamma)$, with a constant c depending on γ, but not on n. Note that any semisimple representation is reductive.

For simplicity of notation, we shall consider u in the sequel as a map from X into

$$N := Y/\rho\left(\pi_1\left(X\right)\right),$$

although N may be singular. Sometimes, it will be implicitly understood (and not explicitly mentioned) that for the proper meaning of a statement about a harmonic map one has to lift the universal covers in a ρ-equivariant manner.

In particular, the energy of u can be computed on a fundamental region F_X for X in $\tilde{X}$:

$$E(u) = \frac{1}{2}\int_{F_X} \|du(x)\|^2 \mathrm{dvol}(x).$$

Here, we assume that X is equipped with some Kähler metric which then lifts to $\tilde{X}$ and induces a volume form $\mathrm{dvol}(x)$, and the norm of the differential du is computed with respect to this metric and the one on Y.

1.2. Harmonic and pluriharmonic maps

In this section, we collect some known results about harmonic and pluriharmonic that will be employed in the proof of Theorem 1.1.

(H1) Let M be a compact Riemannian manifold, possibly with nonempty boundary, and Y be either

a) a complete simply connected Riemannian manifold of nonpositive curvature

or

b) a locally compact Euclidean Tits buildings, with isometry group $I(Y)$.

Let

$$\rho : \pi_1(M) \to I(Y)$$

be a reductive representation. In case $\partial M = \emptyset$, let g be smooth ρ-equivariant boundary values. Then there exists a ρ-equivariant harmonic map

$$u : \tilde{M} \to Y,$$

with $u|_{\partial \tilde{M}} = g$ in case $\partial M \neq \emptyset$. This was shown in [22] and [30] for a) building upon [1], [2], [10], [9], [8], [4], and for b) in [11]. Conversely, if such a ρ-equivariant harmonic map exists, then ρ is reductive; see [30]. Moreover, u minimizes energy in its class. In fact, instead of the compactness of M, one only needs to assume that there exists some finite energy map in the class under consideration.

(H2) If under the assumptions of (H1), u_1 and u_2 are ρ-equivariant harmonic maps, then there exists a family $u_t : \tilde{X} \to Y$ of ρ-equivariant harmonic maps, $1 \le t \le 2$, all of the same energy, and for each $x \in X$,

$$u_t(x) \ , \quad 1 \le t \le 2,$$

is a geodesic arc from $u_1(x)$ to $u_2(x)$ parametrized proportionally to arclength, with length independent of x. This was shown in [1], [2], [12], [11].

(H3) Let u be a harmonic map as in (H1). Then u is smooth in case a) and Lipschitz in case b) as shown in [11]. Furthermore, it is shown in [11] that u has sufficient regularity properties to justify the application of Bochner type identities that were previously developed for the smooth case; see (H4). Also, on regions of controlled geometry, a harmonic map satisfies estimates depending only on its energy. See e.g. [17] and the references contained therein.

(H4) Suppose that M is a compact Kähler manifold and that Y is a symmetric space of noncompact type. Let u be a harmonic map as in (H1). It was shown, by Siu [46] in the Hermitian symmetric case and by Sampson [44] in general, that u is pluriharmonic meaning that its restriction to any complex submanifold of M is again harmonic. Since harmonicity of a map on a holomorphic curve Σ is a property that does not depend on the choice of a (conformal) metric on Σ, also pluriharmonicity of u is a property that does not depend on the Kähler metric, but only on the complex structure of M. As noted in (H3), this result was extended by Gromov-Schoen to Euclidean Tits buildings Y. Likewise, the pluriharmonicity was extended to the finite energy case by Jost-Yau [20], [21].

1.3. Preliminary constructions

Let $\bar{X}$ be a compactification for which $D = \bar{X} \backslash X$ is a divisor with simple normal crossings as only possible singularities. We may also assume that each irreducible component D_λ of D is free from self intersections. Thus, at each intersection point, precisely two components of D meet. The irreducible components of D will be denoted by $D_1, \dots, D_l$.

For each irreducible component D_λ of D, we let σ_λ be a holomorphic section of $\mathcal{O}(\bar{X}, [D_\lambda])$ with a simple zero along D_λ. By multiplying σ_λ with a suitable constant, if necessary, we may assume that the derivative of σ_λ has no zeroes in the set $|\sigma_\lambda| \le 1$. We obtain a fibration of a neighborhood of D_λ in $\bar{X}$ by holomorphic disks $B = \{z \in \mathbb{C} : |z| < 1\}$ meeting D_λ transversally at $0 \in B$ by choosing local trivializations of the line bundle $[D_\lambda]$, and on each such disk, σ_λ defines a local coordinate through the trivialization. Since in general, the normal bundle of D_λ is nontrivial, these local coordinates are not globally defined, but coordinate changes preserve these local disks. Thus, for each $w \in D_\lambda$, we obtain such a transversal disk B_w. The fibrations corresponding to two different components of D meet transversally near the intersection locus of these components.

Likewise, for sufficiently small $\delta > 0$, $|\sigma_\lambda| = \delta$ defines the boundary Σ_δ^λ of such a tube around D_λ, fibered by circles. The intersection of two such boundaries $\Sigma_\delta^{\lambda_1}$, $\Sigma_\delta^{\lambda_2}$ is fibered by tori.

1.4. Hyperbolic and elliptic representations

Let Y be a simply connected complete metric space with distance function $d(\cdot,\cdot)$ of nonpositive curvature in the sense of Alexandrov, with isometry group $I(Y)$. We recall the following:

Definition 1.2. $g \in I(Y)$ is called

- elliptic if $\inf_{y\in Y} d(y, gy) = 0$ and there exists $y_0 \in Y$ with $gy_0 = y_0$,
- hyperbolic if $\inf_{y\in Y} d(y, gy) =: \lambda_g > 0$ and there exists $y_0 \in Y$ with $d(y_0, gy_0) = \lambda_g$,
- parabolic if $\inf_{y\in Y} d(y, gy) = 0$, but the infimum is not achieved,
- quasihyperbolic if $\inf_{y\in Y} d(y, gy) > 0$ and the infimum is not achieved.

Throughout this section, we assume:

(A) The representation
$$\rho : \pi_1(X) \to I(Y)$$
has the property that the image of every small loop around D is hyperbolic or elliptic.

Actually, the case of elliptic elements can be handled by the analysis of our previous papers [24], [25], because it does not cause infinite energy. Elliptic elements, however, also easily succumb to the treatment of hyperbolic elements that we are going to present.

Each hyperbolic element in $I(Y)$ can by represented by translation along some geodesic. In the quotient $N = Y/\varrho(\pi_1(X))$, this yields a closed geodesic c. Since each elliptic element has a fixed point, it corresponds to a point curve, i.e., a trivial closed geodesic in N.

We then map each circle in Σ_δ^λ onto the corresponding closed geodesic proportionally to arclength. This may be done smoothly on the family of circles defining Σ_δ^λ. On intersections $\Sigma_\delta^{\lambda_1} \cap \Sigma_\delta^{\lambda_2}$, this may be performed in a compatible, i.e., continuous manner because the two elements of $\varrho(\pi_1(X))$ corresponding to the circles in $\Sigma_\delta^{\lambda_1}$ and $\Sigma_\delta^{\lambda_2}$ commute, since the two circles themselves commute as elements of $\pi_1(X)$ near an intersection of D_{λ_1} and D_{λ_2}.

As explained above, we consider Σ_δ^λ as the boundary of a family of disks, $S_\delta^\lambda = \bigcup_{w \in D_\lambda} B_{w,\delta}$, all holomorphically identified with $\{z \in \mathbb{C} : |z| \leq \delta\}$, with $z = 0$ corresponding to $w \in D_\lambda$. Likewise, the boundary $\partial B_{w,\delta}$ is identified with the aforementioned circle in Σ_δ^λ. We also identify $X \cap B_{w,\delta} =: B_{w,\delta}^*$ with $B_\delta^* := \{z \in \mathbb{C} : 0 < |z| \leq \delta\}$, or with the semi-infinite cylinder $A_\delta := [\log \delta, \infty) \times S^1$.

The above map from Σ_1^λ to N induces a map from $\partial A_1 = \{0\} \times S^1$ to N, mapping ∂A_1 proportionally to arclength onto a closed geodesic γ in N. We denote this map by $\bar{g}_\gamma(\theta)$ ($\theta \in S^1$). We extend $\bar{g}_\gamma$ to all of A_1 by putting

$$g_\gamma(s, \theta) = \bar{g}_\gamma(\theta).$$

Obviously, this defines a harmonic map from A_1 onto the closed geodesic in N.

Performing this construction on all punctured disks $B_{w,1}^*$, we obtain a map v_1 from S_1^λ to N that is harmonic on each such punctured disk. Again, the construction is continuous in the intersection of two such sets $S_1^{\lambda_1}$, $S_1^{\lambda_2}$.

We shall perform a higherdimensional analogue of the construction of Wolf [48] and Lohkamp [32]. The essential point will be that v_1 is absolutely energy minimizing in its homotopy class on annulus type compact subsets of each disk $B_{w,1}^*$.

We recall the Poincaré metric on the punctured disk

$$-\frac{i}{2}\frac{\partial^2}{\partial z \partial \bar{z}} \log\log \frac{1}{|z|^2} dz \wedge d\bar{z} = \frac{i}{2}\frac{1}{|z|^2 \left(\log |z|^2\right)^2} dz \wedge d\bar{z}.$$

We put $\sigma := \prod_{\lambda=1}^{l} \sigma_\lambda$, choose $M > \sup_X |\sigma|$, and may then construct a Poincaré type Kähler metric on X via

$$\frac{i}{2}\gamma_{\alpha\bar\beta} dz^\alpha \wedge dz^{\bar\beta} = -\frac{i}{2}\frac{\partial^2}{\partial z^\alpha \wedge \partial z^{\bar\beta}} \log\log \frac{M^2}{|\sigma|^2} dz^\alpha \wedge dz^{\bar\beta} + c\omega_{\alpha\bar\beta} dz^\alpha \wedge dz^{\bar\beta}$$

where $\omega_{\alpha\bar\beta} dz^\alpha \wedge dz^{\bar\beta}$ is some Kähler metric on $\bar X$, and c is chosen so large that the above expression becomes positive definite. The resulting metric on X then is complete and of finite volume; see [6]. Also, when restricted to our above local disks near D, it behaves like the Poincaré metric.

We let

$$\gamma := \det\left(\gamma_{\alpha\bar\beta}\right).$$

We put

$$v_\delta := v_1|_{S_\delta^\lambda},$$

and consider the harmonic map

$$u_\delta : X \setminus S_\delta^\lambda \to N$$

with boundary values

$$u_\delta|_{\Sigma_\delta^\lambda} = v_\delta|_{\Sigma_\delta^\lambda}$$

and, of course, inducing the homomorphism $\varrho : \pi_1(X) \to I(Y)$. The existence of u_δ was shown by Schoen [45]. We shall show that the maps u_δ are equicontinuous on compact subsets of X as $\delta \to 0$ and induce a harmonic map $u : X \to N$ in the limit. In general, u will have infinite energy, but the blow-up of the energy density of u near D will be controlled sufficiently well in order to deduce that u is pluriharmonic.

We extend v_1 smoothly from S_1^λ to all of X as a map to N in the required homotopy class, and denote this extension by

$$v : X \to N.$$

We also put

$$A'_\delta := [0, \log\delta] \times S^1.$$

Thus

$$A_1 = A_\delta \cup A'_\delta.$$

Finally

$$S'^{\lambda}_\delta := S_1^\lambda \setminus S_\delta^\lambda\ , \quad S''_\delta := X \setminus \bigcup_{\lambda=1}^{l} S_\delta^\lambda\ , \quad S'_\delta := \bigcup_{\lambda=1}^{l} S'^{\lambda}_\delta.$$

Let $w \in D_\lambda$ be a regular point of D. Near w, we choose coordinates on $\bar{X}$ such that z^1 parametrizes $B_{w,1}$, and z^2 parametrizes D_λ. If $\dim_{\mathbb{C}} X > 2$, z^2 will have more than one component, but this will not affect our subsequent reasoning. In the sequel, the index 2 will always stand for all those z^2-directions together. In our local coodinates, w will be represented as $0 = (0,0)$. While the component $\gamma_{1\bar{1}}$ of the Kähler metric becomes singular at w, $\gamma_{2\bar{2}}$ extends smoothly to a neighborhood of w in D_λ. We write in our coordinates

$$\gamma_{2\bar{2}}(z^1, z^2) = \gamma_{2\bar{2}}(0, z^2) + O(|z^1|)$$

(see e.g. [21] for the extension property of $\gamma_{2\bar{2}}$).

For the subsequent estimates, we assume that D has only one connected component, because the case of several components is handled by the same type of reasoning, but requires a more complicated notation.

The energy of u_δ is

$$\begin{aligned} &E(u_\delta) \\ &= \int_{S''_\delta} \gamma^{1\bar{1}}(z^1, z^2) \frac{\partial u_\delta}{\partial z^1} \frac{\partial u_\delta}{\partial \bar{z}^1} \gamma(z^1, z^2) dz^1 \wedge d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^2 \\ &+ \int_{S''_\delta} \sum_{(\alpha,\beta)\neq(1,1)} \gamma^{\alpha\bar{\beta}}(z^1, z^2) \frac{\partial u_\delta}{\partial z^\alpha} \frac{\partial u_\delta}{\partial \bar{z}^\beta} \gamma(z^1, z^2) dz^1 \wedge d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^2. \end{aligned}$$

Since at each point, the metric tensor can be diagonalized through a choice of holomorphic local coodinates whose first component is tangential to $B_{w,1}$, we conclude that

$$\begin{aligned} E(u_\delta) &\geq \int_{S''_\delta} \gamma^{1\bar{1}}(z^1, z^2) \frac{\partial u_\delta}{\partial z^1} \frac{\partial u_\delta}{\partial \bar{z}^1} \gamma(z^1, z^2) dz^1 \wedge d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^2 \\ &= \int_{S''_\delta} \gamma_{2\bar{2}}(z^1, z^2) \frac{\partial u_\delta}{\partial z^1} \frac{\partial u_\delta}{\partial \bar{z}^1} dz^1 \wedge d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^2 \\ &\geq \int_{S'_\delta} \gamma_{2\bar{2}}(0, z^2) \frac{\partial u_\delta}{\partial z^1} \frac{\partial u_\delta}{\partial \bar{z}^1} dz^1 \wedge d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^2 \\ &\quad + \int_{S'_\delta} O(|z^1|) \frac{\partial u_\delta}{\partial z^1} \frac{\partial u_\delta}{\partial \bar{z}^1} dz^1 \wedge d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^2. \end{aligned}$$

Let $l = l(c)$ be the length of the image geodesic c. Then

$$\int_{A'_\delta} \gamma_{2\bar{2}}(0, z^2) \frac{\partial u_\delta}{\partial z^1} \frac{\partial u_\delta}{\partial \bar{z}^1} dz^1 \wedge d\bar{z}^1 \geq |\log \delta|\, l^2 \gamma_{2\bar{2}}(0, z^2).$$

On the other hand,

$$\begin{aligned} E\left(v|_{S''_\delta}\right) &= E\left(v|_{S'_\delta}\right) + E\left(v|_{X\setminus S_\delta}\right) \\ &\le E\left(v|_{S'_\delta}\right) + c_1 \\ &\quad \text{for some constant } c_1 \\ &= \int |\log\delta|\, l^2 \gamma_{2\bar{2}}(0, z^2) dz^2 \wedge d\bar{z}^2 + c_2 \\ &\quad \text{for some constant } c_2 \end{aligned}$$

using the same expansion as above and

$$E\left(v|_{A'_\delta}\right) = |\log\delta|\, l^2.$$

Here, the constants c_1, c_2 are independent of δ, and so will be all subsequent ones. Since u_δ as a harmonic map is energy minimizing among all maps on S''_δ with the same boundary values, we get

$$E(u_\delta) \le E\left(v|_{S''_\delta}\right).$$

Therefore

$$\int_{S''_\delta} \sum_{(\alpha,\beta)\neq(1,1)} \gamma^{\alpha\bar{\beta}}(z^1, z^2) \frac{\partial u_\delta}{\partial z^\alpha} \frac{\partial u_\delta}{\partial z^\beta} \gamma(z^1, z^2) dz^1 \wedge d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^2 \le c_3,$$

where c_3 again is independent of δ.

If $\delta < \eta < 1$, we may also write

$$E\left(v|_{S''_\delta}\right) = E\left(v|_{S''_\eta}\right) + E\left(v|_{S'_\delta \setminus S'_\eta}\right).$$

Since the energy of u_δ on $S'_\delta \setminus S'_\eta$ again is bounded from below by the corresponding one of v up to some additive constant c_4, from the energy minimizing property of u_δ we obtain

$$E\left(u_\delta|_{S''_\eta}\right) \le E\left(v_\delta|_{S''_\eta}\right) + c_5,$$

with c_5 independent of δ (but not of η).

Thus, the energy of u_δ on S''_η for fixed $\eta > \delta$ is bounded independently of δ. Therefore, as $\delta \to 0$, some subsequence of u_δ converges on S''_η to a harmonic map, using standard estimates for harmonic maps (see [17]). By a diagonal sequence construction, we then get a harmonic map

$$u : X \to N.$$

Here, the reductivity assumption prevents the limit u from disappearing at infinity. Furthermore, if we replace the coordinate z^1 by the coordinate (s, θ) on the cylinder A_1, we get a coordinate system in which all first derivatives of u are uniformly bounded. This results from the facts that the contribution of the energy of u_δ in the z^2-direction is bounded independently of u_δ and that the energy of u_δ on A'_η behaves like $|\log \eta|\, l^2$. Consequently, in our original (z^1, z^2) coordinates,

$$\begin{aligned} \left| \frac{\partial u}{\partial z^1}(z^1, z^2) \right| &\leq \frac{c_6}{|z^1|}, \\ \left| \frac{\partial u}{\partial z^2}(z^1, z^2) \right| &\leq c_7 \end{aligned}$$

for constants c_6, c_7.

Similar constructions apply near singularities of D, i.e., where two components of D meet. Here, we may use σ as our first coordinate and then obtain a cylindrical coordinate system by replacing σ by $\log \sigma$. We conclude that in the σ-direction, the derivative of u behaves like $\frac{1}{|\sigma|}$, whereas in directions normal to σ, it is bounded. (By construction, this is the behaviour of v, and we compare u_δ and v as before up to some bounded terms.)

Lemma 1.1. *u is pluriharmonic.*

Proof. By Siu's formula [46], using local coordinates on N and denoting the corresponding metric tensor by (g_{ij}), we obtain

$$\partial\bar{\partial}\left(g_{ij}\bar{\partial}u^i \wedge \partial u^j\right) \wedge \omega^{m-2} = f\omega^m$$

($\omega =$Kähler form of some Kähler metric on X, $m = \dim_{\mathbb{C}} X$), for some nonnegative[1] function f, provided Y has nonpositive curvature operator, a condition satisfied in our applications. If Y is a Euclidean Tits building, the preceding formula holds in a weak sense, but the computations can be justified by the analysis of Gromov-Schoen [11]. If $f \equiv 0$, then u is pluriharmonic.

On a compact manifold, the left hand side of Siu's formula integrates to 0, and one readily concludes $f \equiv 0$. In the present noncompact case,

[1]The signs in [46] are not quite correct, but this will be irrelevant for the structure of the argument.

we shall verify this with the help of a cut-off argument. For simplicity of notation, we shall only consider the case $m = 2$, and the higher dimensional case is completely analogous.

If φ is a cut-off function that vanishes near D, we have

$$\int_X \varphi \partial\bar\partial \left(g_{ij}\bar\partial u^i \wedge \partial u^j\right) = \int_X \partial\bar\partial\varphi \wedge g_{ij}\bar\partial u^i \wedge \partial u^j.$$

For $0 < \epsilon < 1$, we choose the cut-off function

$$\varphi_\epsilon(z) := \begin{cases} \left(1 - \frac{\log|\sigma(z)|^2}{\log \epsilon^2}\right)^2 & \text{for } \epsilon \le |\sigma(z)| \le 1, \\ 0 & \text{for } |\sigma(z)| < \epsilon, \\ 1 & \text{for } |\sigma(z)| > 1. \end{cases}$$

$\varphi_\epsilon(z)$ is of class $C^{1,1}$ near $|\sigma(z)| = \epsilon$. It is not of class $C^{1,1}$ at $|\sigma(z)| = 1$, but the derivative of φ_ϵ near $|\sigma(z)| = 1$ tends to 0 as ϵ tends to 0, and therefore, this defect can be remedied by a straightforward interpolation that is left out.

We compute

$$\begin{aligned} \frac{\partial}{\partial z^\alpha}\varphi_\epsilon &= -2\left(1 - \frac{\log|\sigma(z)|^2}{\log \epsilon^2}\right)^2 \frac{1}{\log \epsilon^2}\frac{\sigma_{z^\alpha}}{\sigma} \\ \frac{\partial^2}{\partial z^\alpha \partial \bar z^\beta}\varphi_\epsilon &= \frac{1}{(\log \epsilon^2)^2}\frac{\sigma_{z^\alpha}\bar\sigma_{\bar z^\beta}}{|\sigma|^2}. \end{aligned}$$

We then have

$$\begin{aligned} &\int \partial\bar\partial\varphi_\epsilon \wedge g_{ij}\bar\partial u^i \wedge \partial u^j \\ &\qquad \le c_8 \int_{|\sigma(z)|=\epsilon}^{|\sigma(z)|=1} \frac{1}{(\log \epsilon^2)^2}\frac{1}{|\sigma|^2} \end{aligned}$$

(since the derivative of u is bounded in the directions normal to σ)

$$\begin{aligned} &\le c_9 \int_\epsilon^1 \frac{1}{(\log \epsilon^2)^2}\frac{1}{r^2} r\,dr \\ &\qquad \text{(using polar coordinates } \sigma = re^{i\theta}) \\ &= \frac{-c_9 \log \epsilon}{4(\log \epsilon)^2}, \end{aligned}$$

and this tends to 0 as $\epsilon \to 0$.

Therefore,

$$\int_X \partial\bar{\partial}\left(g_{ij}\bar{\partial}u^i \wedge \partial u^j\right) = \lim_{\epsilon\to 0}\int_X \partial\bar{\partial}\varphi_\epsilon \wedge g_{ij}\bar{\partial}u^i \wedge \partial u^j = 0.$$

Thus, in Siu's formula, $f \equiv 0$, and u is pluriharmonic as desired. q.e.d.

Remark. Instead of Siu's Bochner identity, we might as well employ one of the other Bochner type identities that have been shown to apply in the present situation; see e.g. [44] or [47].

Theorem 1 for the case of a Euclidean Tits building now follows from

Lemma 1.2. *Let Y be a locally compact Euclidean Tits building, with isometry group operating transitively on the vertices. Then every isometry of Y is elliptic or hyperbolic.*

Proof. Let $\gamma \in I(Y)$, and let $(y_n)_{n\in\mathbb{N}} \subset Y$ be a sequence with

$$\lim_{n\to\infty} d(y_n, \gamma y_n) = \inf_{y\in Y} d(y, \gamma y).$$

If (y_n) is bounded, a subsequence converges, and the infimum is realized so that γ is elliptic or hyperbolic. If (y_n) is unbounded, we perform the following construction. We choose $p_0 \in Y$ and let $c_n(t)$ be the geodesic from p_0 to y_n, parametrized by arclength $t \geq 0$. $c'_n(t) := \gamma c_n(t)$ then is the geodesic from γp_0 to γy_n. Since (y_n) is unbounded, after selection of a subsequence, we obtain geodesic rays $c(t)$ and $c'(t) = \gamma c(t)$. These two rays stay within bounded distance of each other. Therefore, they are asymptotically contained in a flat F. Thus, they are geodesic rays in the Euclidean space F, and they then have to be parallel. In particular, we may choose p_0 so that

$$d(p_0, \gamma p_0) = d(y_n, \gamma y_n)$$

for all n, and p_0 realizes the infimum. Thus, γ has to be hyperbolic.

q.e.d.

1.5. Arbitrary representations

In this section, we make the following assumption:

(B) Let $c_1(t)$, $c_2(t)$ be geodesic rays in Y parametrized by arclength $t \in \mathbb{R}^+$ with

$$\lim_{t\to\infty} d(c_1(t), c_2(t)) = \alpha.$$

Then there exist constants $\beta > 0$ and γ with

$$d(c_1(t), c_2(t)) \le \alpha + \gamma e^{-\beta t} \quad \text{for all } t > 0. \tag{1.1}$$

Assumption (B) is satisfied if Y is a symmetric space of noncompact type. Therefore, the considerations of the present section will prove Theorem 1.1 in that case.

Actually, under the assumption (B), the case of parabolic elements can be handled by the analysis of our previous paper [24] because it does not cause infinite energy. Parabolic elements, however, also easily succumb to the treatment of quasihyperbolic elements that we are going to present. (Namely, our reasoning will not need that the constant λ_g introduced below is positive.)

We wish to proceed as in the preceding section, but we need to construct the comparison map $v_r : \Sigma_r \to N$ somewhat differently. In the notation of the preceding section, we let $\{0\} \times S^1 \subset A_1$ correspond to $g \in \pi_1(X)$, and choose a sequence $(y_n)_{n\in\mathbb{N}} \subset Y$ with

$$\lim_{n\to\infty} d(y_n, \rho(g)y_n) = \inf_{y\in Y} d(y, \rho(g)y) =: \lambda_g.$$

We also choose $y_0 \in Y$. If $\rho(g)$ is not elliptic or hyperbolic, $(y_n)_{n\in\mathbb{N}}$ diverges, and after selection of a subsequence, the geodesic arcs from y_0 to y_n converge to a geodesic ray $c_0(t)$, t being the arclength parameter. The geodesic arcs from $\rho(g)y_0$ to $\rho(g)y_n$ converge to the geodesic ray $c_1(t) := \rho(g)c_0(t)$. We let $\gamma_t(s)$ be the geodesic arc from $c_1(t)$ to $c_2(t)$ with parameter $s \in [0, 2\pi]$ proportional to arclength (the proportionality factor depends on t). Using polar coordinates on our local disks transversal to D, on

$$\{(\rho, \vartheta) : 0 \le \rho \le r,\ 0 \le \vartheta \le 2\pi\},$$

we put

$$v_r(\rho, \vartheta) := \gamma_{\mu \log(\rho+1)}(\vartheta), \tag{1.2}$$

where passing to the quotient N is implicitly understood, and $\mu > 0$ is chosen in such a way that $\mu\beta > 1$ with β as in (B). We have

$$\left\| \frac{\partial v_r}{\partial \rho} \right\| \le c \frac{1}{\rho+1} \quad \text{for some constant } \ c, \tag{1.3}$$

and

$$\left\| \frac{\partial v_r}{\partial \vartheta} \right\| \le \frac{1}{2\pi} \left(\lambda_g + \gamma e^{-\beta\mu \log \rho} \right) = \frac{1}{2\pi} \left(\lambda_g + \gamma \frac{1}{\rho^{\beta\mu}} \right), \tag{1.4}$$

because of assumption (B). Since $\beta\mu > 1$ by choice of μ,

$$E(v_r) \leq \frac{1}{2} r \lambda_g^2 + \kappa_1 \tag{1.5}$$

for some constant κ_1, and we may then proceed as in the previous section to conclude the proof. q.e.d.

1.6. Harmonic bundles

In order to formulate a corollary, we need to recall some constructions of Hitchin [13], Corlette [4], and Simpson [42]. Let V be a $Gl(n, \mathbb{C})$ bundle with a flat connection D. Introducing a metric g leads to the decomposition

$$D = D_g + \vartheta,$$

where D_g preserves the metric. Let $\rho : \pi_1(X) \to Gl(n, \mathbb{C})$ be the representation defined by the flat bundle V. A metric on V can be considered as a ρ-equivariant map

$$h : \tilde{X} \to Gl(n, \mathbb{C})/U(n).$$

We have $dh = \vartheta$, and h is harmonic iff

$$D_g^* \vartheta = 0.$$

In this case, one says that the metric is harmonic. We then decompose into types:

$$\begin{aligned} D_g &= D' + D'', \\ \vartheta &= \vartheta^{1,0} + \vartheta^{0,1}. \end{aligned}$$

We have $(D'')^2 = 0$, and

$$E := \big(V, D''\big)$$

then is a holomorphic bundle. (The complex structure on E is different from the complex structure on V induced by the flat connection D, unless h is constant, i.e., ρ is a $U(n)$ representation.) Thus $\big(E, \vartheta^{1,0}\big)$ is a Higgs bundle. A Higgs bundle on X consists of a holomorphic bundle E together with

$$\vartheta^{1,0} : E \to \Omega^{1,0}(X) \otimes E$$

satisfying the integrability condition

$$\vartheta^{1,0} \wedge \vartheta^{1,0} = 0.$$

(It was shown by Sampson [44] that the harmonicity of h implies this integrability condition.) Let $(E, \vartheta^{1,0})$ be a Higgs bundle on X. The analogue of the $\bar\partial$-operator on E now is

$$D^2 = \bar\partial + \vartheta^{1,0}.$$

A metric on E then defines

$$D^1 = \partial + \vartheta^{0,1}$$

with

$$\left(\vartheta^{0,1} e, f\right) = \left(e, \vartheta^{0,1} f\right) \quad \text{for all } e, f.$$

The metric again is preserved by

$$D_g = \partial + \bar\partial \ ,$$

and the metric is harmonic iff D is flat. These two conditions are inverses of each other.

Corollary 1.1.

(i) Let (V, D) be a flat bundle on X defined by a reductive representation ρ. Let u of Theorem 1.1 define the corresponding harmonic metric g, and write

$$D = D_g + du = \partial + \bar\partial + \vartheta^{1,0} + \vartheta^{0,1}$$

as above. Then the curvature of the metric connection $D_g = \partial + \bar\partial$ is bounded by a Poincaré type Kähler metric. (In particular, the curvature is in L^1.)

(ii) Let $(E, \vartheta^{1,0})$ be a Higgs bundle on X with u of Theorem 1.1 defining a harmonic metric as before. Then the metric connection associated to the holomorphic structure $\bar\partial + \vartheta^{1,0}$, namely $\partial + \bar\partial + \vartheta^{1,0} + \vartheta^{0,1}$, has curvature again bounded by a Poincaré type Kähler metric.

Proof. If we restrict everything to a curve transversal to D in the direction defined by σ, from the proof of Theorem 1.1, we have the estimate

$$\|du(z)\|^2 \le \frac{\text{const}}{|z|^2} \quad \text{(using } z = \sigma \text{ as a coordinate on the curve)}$$

on that curve, while $\|du\|$ is locally bounded in the other directions. The argument of Simpson (see [40, Section 2; main estimate; Thm. 1]) then implies that the curvatures R of the connections under consideration locally satisfy an estimate

$$\|R\| \leq \frac{\text{const}}{|z|^2 (\log |z|)^2}.$$

q.e.d.

2. Strong rigidity of lattices in Hermitian symmetric spaces

Theorem 2.1. *Let Y be an irreducible Hermitian symmetric space of noncompact type other than the hyperbolic plane. Let Γ be a lattice in Y, i.e., a discrete subgroup of $I(Y)$ with a quotient $\Gamma\backslash Y$ of finite volume. Let X be a smooth quasiprojective variety (or, more generally, a Kähler manifold X that can be compactified as a Kähler manifold $\bar{X}$ such that $\bar{X} \setminus X$ is a divisor with at most simple normal crossings as singularities — note that in the quasiprojective case such a compactification always exists by Hironaka's theorem) with contractible universal cover of dimension (larger or) equal to the dimension of Y and with $\pi_1(X)$ isomorphic to Γ. Then the universal cover $\tilde{X}$ of X is $\pm$biholomorphically[2] equivalent to Y, and $\pi_1(X)$ is conjugate to Γ as a group of automorphisms of Y.*

Remark. We have formulated Theorem 2.1 here for lattices without fixed points only. It is known, however, that the general case can easily be reduced to that case by taking suitable finite covers. Therefore Theorem 2.1 continues to hold if X is a finite quotient of a smooth quasiprojective variety.

In the case where X is also locally Hermitian symmetric, the result is due to Mostow [36] (X compact), Prasad [38] (Y of rank 1) and Margulis [33]. If X is compact, the result was proved by Siu [46] with harmonic map techniques. In the general case, the result was shown by Jost-Yau [21] under an additional technical assumption on the projective compactification $\bar{X}$ of X.

Before giving the actual proof of Theorem 2.1, we should describe the essential difficulty encountered in it. This difficulty stems from the

[2]this means biholomorphically or antibiholomorphically

fact that while X and $\Gamma\backslash Y$ are assumed to be homotopically equivalent, the compactifications of these spaces may a priori be quite different. Let us consider the simple example $N := S^1 \times S^1 \times [0,1]$. This space may be considered as either

$$N_1 := S^1 \times \{z\epsilon\mathbb{C} : 0 < |z| \leq 1\} \text{ or } N_2 := \{z\epsilon\mathbb{C} : 0 < |z| \leq 1\} \times S^1,$$

and thus it may be compactified by adding either the first or the second S^1 factor times $\{0\}$. If we equip $\{z\epsilon\mathbb{C} : 0 < |z| \leq 1\}$ with its complete hyperbolic metric, and N_1 and N_2 with the resulting product metrics, N_1 and N_2 become complete Riemannian manifolds with boundary $S^1 \times S^1$. The identity of N obviously induces a proper homotopy equivalence between N_1 and N_2. In N_1, the first S^1 factor corresponds to a hyperbolic element of the fundamental group, and the second S^1 factor to a parabolic one, while for N_2 these roles are exchanged. Consequently, the harmonic map between N_1 and N_2 which is homotopic to the identity produced by our method will have infinite energy because a parabolic element is mapped to a hyperbolic one.

Since, as mentioned, in the situation of Theorem 2.1, at the beginning we cannot identify the two compactifications of X and $\Gamma\backslash Y$, we have to deal with such maps of infinite energy although in the end one of the conclusions of the proof has to be that the harmonic map under consideration after all does have finite energy, and the two spaces do possess isomorphic compactifications.

If Y has rank 1, then $\Gamma\backslash Y$ may be compactified by adding finitely many cusp points. Consequently, all elements of the fundamental group, i.e., of Γ, that can be deformed to infinity are parabolic. Therefore, in particular, the images of all parabolic elements of $\pi_1(X)$ are parabolic again, and one may construct a harmonic map of finite energy. This is the case treated in [20]. Thus, in the sequel, we may assume that Y has rank at least 2, and by the theorem of Margulis (see [34]), Γ is arithmetic. This fact then gives additional information about possible compactifications of $\Gamma\backslash Y$.

In [21], the situation was considered where one can produce a finite energy harmonic map in spite of the possibility that a parabolic element may be mapped to a hyperbolic or quasihyperbolic one. Namely even if that occurs and the energy density of any homotopy equivalence then has to blow up towards infinity, this might still be compensated by a sufficiently fast decay of the volume form of the domain near infinity so that the overall energy of some suitable map might still turn out to be finite. This requires an additional assumption on the compactification of X. Here, we do not wish to make such an assumption.

The reason for wanting to have a finite energy harmonic map was to make sure that the Bochner type identity of Siu is satisfied. In §1, however, we have produced a map of possibly infinite energy that still satisfies this Bochner identity. For this reason, the proof of Theorem 2.1 may proceed essentially as in [21], and we may therefore be somewhat brief in certain places.

After this discussion, let us now start with the

Proof of Theorem 2.1. Since $\pi_1(X)$ is isomorphic to the lattice Γ, we obtain a reductive homomorphism

$$\rho : \pi_1(X) \to I(Y) \quad \text{with } \rho\left(\pi_1\left(X\right)\right) = \Gamma.$$

Theorem 1.1 yields a ρ-equivariant pluriharmonic map

$$u : \tilde{X} \to Y.$$

u can be considered as a pluriharmonic map

$$u : X \to \Gamma \setminus Y.$$

Moreover, in the course of proof of Theorem 1.1 we have shown that u satisfies Siu's Bochner identity. By our topological assumptions, u is a homotopy equivalence.

The work of Borel-Serre [3] implies that the $\Gamma \setminus Y$ has nontrivial cohomology of degree $2\dim_{\mathbb{C}} Y - \mathbb{Q}\text{-}rank(\Gamma)$, which is at least $2\dim_{\mathbb{C}} Y - rank(Y)$. Therefore, the maximal rank of the differential du of u is at least this number. This is large enough for Siu's analysis [47] to apply. Thus we may conclude that u is $\pm$holomorphic.

We equip X with a Poincaré type metric as in §1.4. By a result of Cornalba-Griffiths [6], such a metric has Ricci curvature bounded from below, and therefore Royden's version [39] of Yau's Schwarz Lemma [49] may be applied to u as u is $\pm$holomorphic.

This Schwarz lemma says that

$$\|du(z)\|^2 \leq \frac{k_1}{k_2},$$

where k_1 is a lower bound for the Ricci curvature of X, and k_2 is an upper bound for the holomorphic sectional curvature of Y. With the standard normalizations, we have in fact $k_2 = -1$. In particular, u has bounded energy density, and since the Poincaré type metric on X has finite volume (see [6], or also [21]), u has finite energy after all. Therefore, the rest of the proof may proceed essentially as in [21]. Nevertheless, we shall describe the key steps now.

The first step is the properness of u. The details of this step are provided in [21, §3d]. The essential idea is the following: Let us consider a loop around a component of $\bar{X} \setminus X$ that is contractible in $\bar{X}$ but not in X (at least not in a neighborhood of $\bar{X} \setminus X$). We call such a loop short, because it is homotopic to loops of arbitrarily small length in X. The Schwarz lemma implies that its image under u is likewise short in this sense, i.e., it corresponds to a parabolic element of Γ. Consequently, if we take a sequence of such loops in a given homotopy class in X that contract to a point on $\bar{X} \setminus X$, the images of these loops have to go to infinity in $\Gamma \setminus Y$. From this observation, properness is deduced.

(In passing, we observe that it follows from Remmert's proper mapping theorem that the image of X under the proper $\pm$holomophic map u now can be seen to be an analytic subvariety of $\Gamma \setminus Y$.)

Also, one shows as in [21, §3d] that u extends as a continuous map from $\bar{X}$ to the Baily-Borel compactification of $\Gamma \setminus Y$, essentially because the latter is the minimal compactification. Therefore, u has a well defined degree.

As u is a proper $\pm$holomorphic map, the preimage of a point in $\Gamma \setminus Y$ has to be zerodimensional, as otherwise it would be a compact subvariety, hence represent a nontrivial homology class in X. As u is a homotopy equivalence, such a class cannot be contracted to a point.

Consequently, u has nonzero degree. It remains to show that the absolute value of the degree is 1. For that purpose, one proceeds as in [21, §3e] and considers a proper homotopy equivalence $g : \Gamma \setminus Y \to X$ such that $u \circ g$ is homotopic to the identity of $\Gamma \setminus Y$.

As above, we may deform $u \circ g$ into a harmonic homotopy equivalence h which then again turns out to be proper and $\pm$holomorphic.

The Schwarz lemma implies

$$|deg h| \leq 1,$$

either directly by observing that the constants in the Schwarz lemma of Yau-Royden are sharp, or by considering the iterates h^n which would violate the inequality of the Schwarz lemma for sufficiently large n if $|deg h|$ were larger than 1.

Since $degh = degu \cdot degg$, we also have

$$|degu| \leq 1.$$

Altogether, $|degu| = 1$.

Finally, as in [46] and [21, §3e] one verifies that u is ±biholomorphic. Namely, let V be the set of points where the Jacobian of u vanishes. If V were not empty, it would be a complex subvariety of complex codimension 1 whereas $u(V)$ would have codimension at least 2 as u is of degree ±1.

Thus, the preimage of a generic point in $u(V)$ would be a nontrivial analytic subvariety, and compact as u is proper.

Again, this is not compatible with u being a homotopy equivalence. We conclude that V is empty. Thus, u is ±biholomorphic. q.e.d.

Remark. The proceeding proof made heavy use of the fact that Γ is known to be arithmetic by the work of Margulis. Ideally, the harmonic map approach should not use this result, but rather deduce it as a corollary. We thus obtain a generalization of the superrigidity theorem of Margulis, but not a new proof of the original version. In this sense, the harmonic map approach to superrigidity in the noncompact case is still not completely satisfactory.

3. Some applications to representations of π_1 of algebraic varieties

3.1. Holomorphic 1-forms and spectral coverings

We now study the holomorphic 1-forms arising from the harmonic map $u : \tilde{X} \to \Delta$. Let $\tilde{X}_r$ be the set of regular points of u. A point $x_0 \in \tilde{X}$ is called a *regular* point of u if there exist a ball $B(x_0, \delta_0)$ of radius $\delta_0 > 0$ and a $\mathrm{rk}_{K_p} G$-flat $F \subset \Delta(G)$ with $B(x_0, \sigma_0) \subset F$ (see [11, p.68]). And $S(u) := \tilde{X} \setminus \tilde{X}_r$ is called the singular set of u. Let A ($\simeq R^r$)be an apartment of Δ and $W = Z^r \times \bar{W}$ be the affine Weyl group of $\delta(G)$. Here, $\bar{W}$ is the usual Weyl group of $G(K_p)$ which operates on A as a finite linear subgroup generated by reflections, and Z^r acts on A as the usual translations.

The main point in this section is to construct the spectral covering of u. The spectral covering for the Higgs bundle case has been intensively studied by R. Donagi [8] and C. Simpson [42]. For the case where u is a harmonic map into a building we have a similar construction. The basic idea is to construct the coefficients of the characteristic polynomial directly from the differential of u.

Let $\mathcal{R} = \{\beta_1, \dots, \beta_l\}$ be the root system of $\bar{W}$, where β_i are normalized vectors in R^r and $\beta^\perp$ are the reflection hyperplanes. We can consider β_i as coordinate functions on A by orthogonal projection from A to β_i. By taking the differential, we obtain a collection of differential 1-forms $\{d\beta_1, \dots, d\beta_l\}_A$ on A. It has the property that on the common part of any two apartments A and A' these collections of 1-forms $\{d\beta_1, \dots, d\beta_l\}_A$ and $\{d\beta_1, \dots, d\beta_l\}_{A'}$ coincide as sets, and the indices of the 1-forms in these two collections differ by a permutation from $\bar{W}$. Here, a differential form on Δ means that its restriction to each apartment is a usual differential form.

Let $s_1, \dots, s_l$ be the basic symmetric polynomials of l variables. We set $\sigma_1 = s_1(\beta_1, \dots, d\beta_l)$, $\dots, \sigma_l = s_l(\beta_1, \dots, d\beta_l)$. Since they are invariant under the $\bar{W}$-action, they piece together into differential forms $\sigma_1, \dots, \sigma_l$ on Δ in the symmetric tensor product. It is clear that $\{d\beta_1, \dots, d\beta_l\}$ are roots of the polynomial $t^l + \sigma_1 t^{l-1} + \cdots + \sigma_l$.

We then take the complexified pull back $u^{*c}(\sigma_1), \dots, u^{*c}(\sigma_l)$ via the differential du. Their $(1,0)$-parts are holomorphic forms on $\tilde{X}$ in the symmetric tensor product which are called again $\sigma_1, \dots, \sigma_l$. This can be seen as follows. Clearly, $\sigma_1, \dots, \sigma_l$ are holomorphic on $\tilde{X}^r = \tilde{X} \setminus S$. Since u is Lipschitz, du is bounded near S and $\operatorname{codim} S \geq 2$, they can be extended over S. Because u is equivariant, $\sigma_1, \dots, \sigma_l$ is $\pi_1(X)$-invariant, they descend to some holomorphic forms on X that we call again $\sigma_1, \dots, \sigma_l$. Because of the controlled growth at infinity, $\sigma_1, \dots, \sigma_r$ have at most log poles along D_∞.

Now the characteristic polynomial $t^l + \sigma_1 t^{l-1} + \cdots + \sigma_l$ defines a subvariety Q in the total space T^*_X of the holomorphic cotangent bundle of X (see [42] for details). It is called the spectral variety of u and has the property that the restriction of the projection to Q induces a ramified covering (maybe nonreduced) $p : Q \to X$ and the preimage $p^{-1}(x)$ coincides with the roots of the polymonial $t^l + \sigma_1(x) t^{l-1} + \cdots + \sigma_l(x)$. From this property we see that the embedding of $Q \subset T^*_X$ defines tautologically a holomorphic 1-form ω_1 on Q such that ω_1 is a root of the polymonial $t^l + p^*\sigma_l t^{l-1} + \cdots + p^*\sigma_l$.

Furthermore we take the Galois closure of the function field extension $K(Q)/K(X)$, and obtain a Galois covering $\sigma : X^s \to X$ and l 1-forms $\omega_1, \dots, \omega_l \in \Gamma(X^s, \sigma^*\Omega^1_X)$ having at most log-poles along D_∞

such that they are the roots of the polynomial $t^l + \sigma^*\sigma_l t^{l-1} + \cdots + \sigma^*\sigma_l$, and the ramification divisor $R \subset X^s$ of σ is contained in the union of the loci $\bigcup_{\omega_i \neq \omega_j}(\omega_i - \omega_j)_0$. Vie the construction we see that the pull back $\sigma^* u$ is the equivariant pluriharmonic map for $\sigma^* p$, and $d'(\sigma^* u)\beta_1, \ldots, d'(\sigma^* u)\beta_l$ glue together and yield $\omega_1, \ldots, \omega_l$. The image of the singular set $u(S(u))$ is contained in the closed faces of all $\mathrm{rk}_{K_p} G$-simplices of $\Delta(G)$. By taking sequences along normal directions going to faces and using the estimate (1), we obtain the following.

Lemma 3.1.

1) *There exists a finite ramified Galois covering $\sigma : X^s \to X$ so that the differentials $d'(u\sigma)\beta_1, \ldots, d'(u\sigma)\beta_l$ of the coordinate functions on all apartments of $\Delta(G(K_p))$ chosen as above piece together and yield l holomorphic 1-forms $\omega_1, \ldots, \omega_l \in \Gamma(X^s, \sigma^*\Omega^1_{\bar{X}})$ having at most log poles along D_∞ and the ramification divisor $R \subset X^s \subset \bigcup_{\omega_i \neq \omega_j}(\omega_i - \omega_j)_0$.*

2) *The singular set $\sigma^* S(u)$ of the harmonic map $\sigma^* u$ is contained in the union of the zero loci of some holomorphic 1-forms which are linear combinations of $\omega_1, \ldots, \omega_l$.*

3.2. The quasi Albanese map, related fibrations and a Lefschetz-type theorem

For a quasi projective smooth variety $X = \bar{X} \setminus D_\infty$, Iitaka [14] has defined the quasi Albanese map

$$\psi : X \xrightarrow{\int_{x_0}^{x} \omega} \mathrm{Alb}(X)$$

by taking integrals of holomorphic 1-forms of X which have at most log poles along D_∞. $\mathrm{Alb}(X)$ is a semi abelian variety which is a group extension of $\mathbb{C}^{*d}$ by $\mathrm{Alb}(\tilde{X})$. The morphism ψ extends to a rational map

$$\bar{\psi} : \bar{X} \to \overline{\mathrm{Alb}(X)},$$

where $\overline{\mathrm{Alb}(X)}$ is an algebraic P^d-fibre bundle over $\mathrm{Alb}(\bar{X})$. Using the map $\bar{\psi}$, one obtains (as in [51], [24], and [25])

Lemma 3.2 ([24, Lemma 6.1]). *Given r 1-forms*

$$\omega_j \in H^0\left(\bar{X}, \Omega^1_{\bar{X}}\left(\log D_\infty(j)\right)\right),$$

$1 \le j \le r$, suppose that the pulled back forms satisfy $\mathrm{d}i_{B_j}\omega_j = 0$ *along an effective divisor $i_{B_j} : B_j \hookrightarrow \bar{X}$ and $B_j \cap D_\infty(j) = \emptyset$. Then there exists the following diagram*

$$\begin{tikzcd} \bar{Z} \arrow[r, "\tau"] \arrow[d, "g"'] & \bar{X} \\ \bar{W} & \end{tikzcd}$$

where τ is a surjective, generically finite Galois map, g is a fibration over $\bar{W}$ with $\dim \bar{W} \le r$, *and $G^{-1}(\omega) \cap \left(\sum_{1\le j\le r} B_j\right) = \emptyset$ for generic ω.*

Given again r holomorphic 1-forms ω_j on $\bar{X}$, which have at most log poles at infinity, we let $\tilde{X} \to X$ be the universal covering of X, and consider the integrals

$$\tilde{X} \xrightarrow{\left(\int_{x_0}^{x} \omega_1, \dots, \int_{x_0}^{x} \omega_r\right) =: h} \mathbb{C}^r .$$

Lemma 3.3 ([25], Section 3, Theorem 3.3). *If $\forall\ 1 \le k \le r$ there is no collection of k linearly independent 1-forms in the span of $\omega_1, \dots, \omega_r$ that factor through a surjective morphism $g : X \to W$ of* $\dim W \le k$, *then h defined as above is surjective and all fibres of h are connected.*

Lemma 3.4. *Let $f : X \to Y$ be a surjective morphism with connected fibres, and $\rho : \pi_1(X) \to G$ be a Zariski dense representation into a simple algebraic group G. Suppose that $f^{-1}(y)$ is a smooth fibre. Then either*

a) the restriction $\rho|_{f^{-1}(y)}$ is again Zariski dense, or

b) $\rho|_{f^{-1}(y)} = \{1\}$.

c) If $\varrho|_{f^{-1}(y)} = \{1\}$, then after passing to a blowingup and a finite etale covering $e : X' \to X$, the pull back $e^*\varrho$ factors through the Stein-factorization of fe.

Proof. The proof of a) and b) is very simple, just using the fact that $\pi_1(f^{-1}(y))$ is a normal subgroup in $\pi_1(X)$. We only need to show c).

The main point in the proof is to use the fact that any finitely generated group of matrices is residually finite. Recall that a group is said to be residually finite if the intersection of all subgroups of finite index consists of the identity element alone.

We adopt some arguments used in [29] Section 4, and [43] Lemmata 6 and 6.4. Let $Y_0 \subset Y$ be the open subset over which f is regular. By blowing up we may assume that $Y \setminus Y_0 = \bigcup B_j$ is a divisor with normal crossings only. On the regular part, the exact homotopy sequence

$$\pi_1(F) \to \pi_1(f^{-1}(Y_0)) \to \pi_1(Y_0) \to 1$$

implies that ϱ factors through a representation $\tau : \pi_1(Y_0) \to GL_r$.

Claim 3.1. *Let $f^*(B_j) = \sum_{i=1}^{m_j} b_{ij} B_{ij}$ and γ_j be a short loop wich goes around B_j. Then $\tau(\gamma_j)$ has finite order, and the order n_j satisfies $n_j | b_{ij}$, $1 \le i \le m_j$.*

Proof. Let Δ_j be a small disc transversal to B_j at the generic point, and let $\Delta_{ij} \subset f^{-1}(\Delta_j)$ be a small disc in a small neighborhood of the generic point of $B_{ij} \cap f^{-1}(\Delta_j)$. Choose coordinates x on Δ_{ij} and z on Δ_j such that $f : \Delta_{ij} \to \Delta_j$ is given by $z = x^{b_{ij}}$. Take a short loop γ_{ij} which lies in Δ_{ij}^* and goes around B_{ij}. Thus we have $f_*(\gamma_{ij}) = \gamma_j^{b_{ij}}$, and $\tau(\gamma_j)^{b_{ij}} = \varrho(\gamma_{ij}) = 1$. Hence Claim 3.1 is complete.

For a point $z \in \bigcup B_j$, let U_z denote a small ball in Y centered in z, and let Υ_z denote the fundamental group of $U_z \setminus \bigcup B_j$. It is known that Υ_z is a free abelian group generated freely by the short loops which go around the components of $\bigcup B_j$ passing through z.

Claim 3.2.

1) *There exists a subgroup $\Gamma \subset \tau\pi_1(Y_0)$ of finite index such that $\Gamma \cap \tau(\Upsilon_z) = 1 \ \forall z \in \bigcup B_j$.*

2) *Let $e : Y_0' \to Y_0$ be the finite etale covering corresponding to the finite index subgroup $\tau^{-1}\Gamma \subset \pi_1(Y_0)$, Then there is a smooth completion $Y' \supset Y_0'$ such that the extended map $e : Y' \to Y$ is a branched covering with the branching order n_j along B_j, and $e^*\tau$ can be extended across Y'.*

Proof. 1) Since $\bigcup B_j$ has only finitely many components, only finitely many images $\{\Upsilon_z \to \pi_1(Y_0), z \in \bigcup B_j\}$ can occur. Note that $\tau\Upsilon_z$ is finite by Claim 3.1. Taking a decreasing sequence of finite index subgroups $\{\Gamma_i\}$ of $\tau\pi_1(Y_0)$ with $\bigcap_i \Gamma_i = 1$, we find a Γ_l such that $\Gamma_l \cap \tau\Upsilon_z = 1\ \forall z \in \bigcup B_j$.

2) Let $e : Y_0' \to Y_0$ be the finite etale covering corresponding to $\tau^{-1}\Gamma_l$. For each point $z \in \bigcup B_j$, let $B_1, \dots, B_d$ be the components of $\bigcup B_j$ which contain z. We may take a small neighborhood of z in the form $\Delta^d \times \Delta^{k-d}$ and with coordinates $z_1, \dots, z_d, z_{d+1}, \dots, z_k$, such that $z_j = 0$ defines B_j, $1 \le j \le d$. The local covering

$$e : e^{-1}(\Delta^{*d} \times \Delta^{k-d}) \to \Delta^{*d} \times \Delta^{k-d}$$

corresponds to the subgroup $\tau^{-1}(\Gamma_l) \cap \Upsilon_z$, which is generated by the loops $\gamma_1^{n_1}, \dots, \gamma_d^{n_d}$. Thus, the corresponding covering is given by

$$e : \Delta^{*d} \times \Delta^{k-d} \to \Delta^{*d} \times \Delta^{k-d}$$

with

$$z_1 = x_1^{n_1}, \dots, z_d = x_d^{n_d}, z_{d+1} = x_{d+1}, \dots, z_k = x_k.$$

The completion Y' is smooth, and the extended map $e : Y' \to Y$ has the branching order n_j along B_j via its construction.

Now $\pi_1(Y')$ is the quotient of $\pi_1(Y_0')$ divided by the short loops around the components of the preimage $e^{-1}(\bigcup B_j)$. Let γ_j' be a short loop around a component of $e^{-1}(B_j)$, Then $e_*(\gamma_j') = \gamma_j^{n_j}$. This shows that $e^*\tau$ factors through the quotient $\pi_1(Y')$. We are done with Claim 3.2.

Finally, we take the normalization of the fibre product

$$e : X' =: (X \times_Y Y')^{nor} \to X.$$

Since $f^*(B_j) = \sum_{i=1}^{m_j} b_{ij} B_{ij}$ and the covering $e : Y' \to Y$ has the branching order n_j along B_j with $n_j | b_{ij}$, $1 \le i \le m_j$ by Claim 3.1, it follows that $e : X' \to X$ is etale. Let f' be the Stein-factorization of fe. Then it is clear that $f'^* e^* \tau = e^* \varrho$. c) is proved. q.e.d.

The following lemma is well known.

Lemma 3.5 [25]. *Suppose that K is a complete field with respect to a discrete valuation and F is a flat of positive dimension in the Tits building $\Delta(G(K))$. Then the isotropy group $I_F \subset G(K)$ of F is not Zariski dense.*

3.3. Factorization theorems for nonrigid and unbounded representations

Theorem 3.1. *Let G be a simple algebraic group over $\mathbb{C}$. If $\rho \in \chi_\infty^{-1}(t)$ is a Zariski dense representation, and ρ is nonrigid in χ_∞^{-1}, then ρ factors through a morphism $f : X \to Y$ with $\dim Y \leq \mathrm{rk}_{\mathbb{C}}\, G$.*

Remark. The Zariski density of ρ is not a serious restriction. In general we consider a semisimple representation ρ, and take its Zariski closure in G. It is a direct product of almost simple algebraic groups.

Theorem 3.2. *Let G be a simple algebraic group over a complete field K with a discrete valuation. Suppose that $\Delta(G(K))$ is locally compact. If ρ is a Zariski dense representation, unbounded with respect to this valuation, then ρ factors through a morphism $f : X \to Y$ with* $\dim Y \leq \mathrm{rk}_K(G)$.

Remark. Without the local compactness of $\Delta(G(K))$, but assuming that ρ is stabilizing at infinity, Theorem 3.2 is Theorem 2 of [25]. For $G = \mathrm{SL}_{r+1}(\mathbb{C})$, there are two proofs for Theorem 3.1. The first proof uses Higgs bundles as dicussed below, and the second one reduces it to Theorem 2 in [25] for SL_{r+1} over a function field. This latter proof works for an arbitrary simple group.

Proof of Theorem 3.2. Let $\omega_1, \ldots, \omega_r$, $r \leq \mathrm{rk}_K\, G$, be a base of the holomorphic 1-forms from the harmonic map in section 3.1. Using Lemma 3.4 and Lemma 3.3, we obtain the following property (see [25], Claim 4.2):

There are two possibilities: either

1) There is a decomposition of the span $\langle \omega_1, \ldots, \omega_r \rangle = U_1 \oplus U_2$ with $\dim U_2 > 0$, and a subvariety $V \subset X^s$ of $\dim V \geq \dim X^s - \dim U_1$ such that the holomorphic map
$$\tilde{V} \xrightarrow{\int U_2 := h} \mathbb{C}^{\dim U_2}$$
is surjective and with connected fibres and $\tau\rho|_V$ is Zariski dense;

or

2) $\omega_1, \ldots, \omega_r$ all factor through a morphism $g : X^s \to W$ with $\dim W \leq r$ and g has connected fibres.

Case 1) in fact cannot happen (see 3.1). This can be seen as follows. The map $\mathrm{Re}\, h$
$$\tilde{V} \xrightarrow{h} \mathbb{C}^{\dim U_2} \xrightarrow{\mathrm{Re}} \mathbb{R}^{\dim U_2}$$

is surjective and has connected fibres. Since the image of the singular part $h(S(u\tau))$ of $u\tau : \tilde{V} \to \Delta(G)$ is an analytic subvariety of $\mathbb{C}^{\dim U_2}$ of codimension at least 1 and any fibre of $\mathrm{Re} : \mathbb{C}^{\dim U_2} \to \mathbb{R}^{\dim U_2}$ is a real vector space $\mathbb{R}^{\dim U_2}$ in $\mathbb{C}^{\dim U_2}$ after a translation on $\mathbb{C}^{\dim U_2}$, the fibres of Re can not be contained in $h(S(u\tau))$. Hence,

$$(\mathrm{Re}\, H)^{-1}(x) \cap S(u\tau)$$

is a measure zero subset of $(\mathrm{Re}\, h)^{-1}(x)$. Since U_2 comes from the complexified differential of $u\tau$, $d(\mathrm{Re}\, h)$ and $d(u\tau)$ coincide of the regular part $\tilde{V}_r$. This implies that a fibre of $\mathrm{Re}\, h$ is a connected component of a fibre of $u\tau$. Therefore, we have a factor map:

$$\begin{tikzcd} \widetilde{V} \arrow[r, "\mathrm{Re}\, h"] \arrow[dr, "u\tau"'] & \mathbb{R}^{\dim U_2} \arrow[d, "\phi"] \\ & \Delta(G) \end{tikzcd}$$

The differential $d\phi$ is equal to 1 with respect to some isometric coordinate systems on $\mathbb{R}^{\dim u_2}$ and $\Delta(G)$. Hence, ϕ is an isometric submersion and its image is a flat submanifold $T \times \mathbb{R}^l$ where T is a torus. Since it is the image of the pluriharmonic map $u\tau$, if one direction is bounded, then this direction vanishes completly. Therefore, $\phi(\mathbb{R}^{\dim U_2}) = \mathbb{R}^{\dim U_2}$, a flat in $\Delta(G)$. Because $\tau^*\rho|_V$ fixes this flat, it cannot be Zariski dense. A contradiction.

We want to show that $\tau^*\rho$ factors through g in Case 2). Pulling g back to the universal coverings, $u\tau$ factors through $\tilde{g}$,

$$\begin{tikzcd} \widetilde{X}^s \arrow[r, "\tilde{g}"] \arrow[dr, "u\tau"'] & \widetilde{W} \arrow[d, "v"] \\ & \Delta(G) \end{tikzcd}$$

and the action of $\tau^*\rho$ on $u\tau(\tilde{X}^s)$ factors through $\pi_1(W)$. Let $\Delta_{u\tau}$ be the minimal convex subcomplex containing $u\tau(\tilde{X}^s)$. Since any $\gamma \in \pi_1(g^{-1}(\omega))$ goes to $1 \in \pi_1(W)$, $\tau^*\rho(\pi_1(g^{-1}(\omega)))$ fixes $u\tau(\tilde{X}^s)$ and hence

$\Delta_{u\tau}$. Since $u\tau(\tilde{X}^s)$ is unbounded, $\Delta_{u\tau}$ contains at least a geodesic line L, and $\tau^*\rho(\pi_1(g^{-1}(\omega)))$ fixes L. By Lemma 3.5, $\tau\rho(\pi_1(g^{-1}(\omega)))$ cannot be Zariski dense. Since $\tau^*\rho$ is Zariski dense, Lemma 3.4 says that $\tau\rho$ factors through g. We again take the Shafarevich map with respect to $\tau^*\rho$. It descends to an $f : X \to Y$ with $\dim Y \leq \dim W \leq \mathrm{rk}_K(G)$, and ρ factors through f.

Proof of Theorem 3.1. The algebraic group $G \subset SL_n$ is in fact defined over a number field K after some conjugations. Hence, the morphism $Rep(G) \to M_B(G)$ from the space of representations of π_1 into G to its moduli space is defined over $Spec(\mathcal{O}_K)$ by Seshadri's geometric reductivity theorem over an arbitrary base. We take a completion $K_p \supset K$ for a prime o, and let $\mathcal{O}_{K_p}$ be the ring of the algebraic integers of K_p. The $\mathcal{O}_{K_p}$-valued points in $M_B(G)$ correspond to the representations into $G(\mathcal{O}_{K_p})$ after conjugations. If ϱ is not an isolated point in $M(G)$, we may find an irreducible algebraic curve S in $M(G)$ passing through ϱ and containing infinitely many nonintegral point $\{\varrho_s\}_{s\in S_0}$, valued in some finite extensions $E_s \supset K_p$. Since the subset of Zariski dense representations into a semisimple algebraic group is Zariski open, we may assume that all $\{\varrho_s\}_{s\in S_0}$ are also Zariski dense.

Let $u_s : \tilde{X} \to \Delta(G(E_s))$ be the equivariant harmonic map for ϱ_s. Since $\mathrm{rk}\, G(E_s) \leq \mathrm{rk}\, G(\bar{K}_p)$ where $\bar{K}_p$ is the Galois closure of K_p, the cardinal numbers of the Weyl groups $\bar{W}_s$ of $G(E_s)$ are bounded, hence the cardinal numbers of the root systems of $\bar{W}_s$ are also bounded. This implies that the characteristic polynomials $p_s(t)$ for u_s (see Lemma 3.1) have bounded degrees. So, we may assume that all $p_s(t)$ for $s \in S_0$ have the same degree d.

Let $\Gamma := \bigoplus_{i=1}^d \Gamma(X, Sym^i\Omega_X^1)$. Then the subset of all characteristic polynomials of degree d is an algebraic subvariety $\mathcal{W} \subset \Gamma$ (see [42] Section "Hitchin proper map"). By taking the Zariski closure

$$V := \{p_s(t)\}_{s\in S_0} \subset \mathcal{W},$$

we obtain an algebraic family of coverings $\{\sigma_v : X_v^s \to X\}_{v\in V}$ constructed in Lemma 3.1. Using Lemma 3.3 in the relative case, the constructions in the proof of Theorem 3.2 can be performed relative to V, in particular, the fibrations $\{f_v^s : X_v^s \to Y_v^s\}_{v\in V}$ constructed in the proof of Theorem 3.2 are relative to V. Clearly, we may find a Zariski open subset $V_0 \subset V$ so that the generic fibres of f_v for $v \in V_0$ are homotopy equivalent. Furthermore, since $\{p_s(t)\}_{s\in S_0}$ is Zariski dense, there are infinitely many $s \in S_0$ such that $p_s(t) \in V_0$; we call this subset S_0'.

Fixing one fibre $f_{v_0}^{s-1}$, $v_0 \in V_0$, then $\sigma^* \varrho_s|_{f_{v_0}^{s-1}} = 1$ for $s \in S_0'$ by a) and b) in Lemma 3.4. Since $\{f_v^s : X_v^s \to Y_v^s\}_{v \in V}$ is $Gal(\{X_v^s/X\}_{v\in V})$-equivariant, it descends to a family of fibrations $\{f_v : X \to Y_v\}_{v\in V}$ with $\varrho_s|_{f_{v_0}^{-1}} = 1$ for $s \in S_0'$. Because the subset $\{\varrho_s|_{f_{v_0}^{-1}} = 1 | s \in S\}$ is closed in the Zariski topology on S and S is irreducible, this implies $\varrho_s|_{f_{v_0}^{-1}} = 1$, $\forall s \in S$. In particular, $\varrho_s|_{f_{v_0}^{-1}} = 1$. Finally, by c) in Lemma 3.4, ϱ factors through f_{v_0} after passing to a finite etale covering and a blowingup.

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Max-Planck-Institute for Mathematics in the Sciences
University Kaiserslautern, Germany