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EXACT LAGRANGE SUBMANIFOLDS, PERIODIC ORBITS AND THE COHOMOLOGY OF FREE LOOP SPACES

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Abstract

We prove that there are obstructions to the existence of an exact Lagrange embedding from a closed manifold L to T^*N. This may be seen as an extension of Gromov's theorem as formulated by Lalonde and Sikorav, showing that no such embedding exists for N open. For example we answer positively a question by Lalonde and Sikorav on the non-existence of exact Lagrange embeddings from T^2 into T^*S^2. Our obstruction is in terms of the cohomology of the loop space of L and N and the map induced by the embedding in the cohomologies of these loop spaces. In particular, we give obstructions to the existence of an exact Lagrangian embedding inducing a degree-zero map from L to N. As another application of our method, we prove the Weinstein conjecture in cotangent bundles of simply connected manifolds (removing an assumption in a previous joint paper with H. Hofer). A number of these results had been announced in [48] and [49]

0. Introduction

Let N be a manifold, T^*N its cotangent bundle, endowed with the standard symplectic form, $\omega = d\lambda$ where $\lambda = \sum_{i=1}^n p_i dq^i$ in local coordinates (the q^i are coordinates on N, and the p_i the dual coordinates).

An embedding from a manifold L of dimension $n = dim(L) = dim(N)$ to T^*N is said to be Lagrange if ω vanishes on the tangent space to L, and exact (Lagrange) if λ induces an exact form on L.

It is one of the striking results of [19], that there are no exact Lagrange embeddings from a compact manifold L into $M = V \times \mathbb{R}$, and in fact, as noticed by Lalonde and Sikorav ([28]), Gromov's argument

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extends to the cotangent bundle of any non-compact manifold. From this they conclude that if L is an exact Lagrange submanifold in T^*M, the projection p of L on M must be onto. It is then tempting to make the following:

Conjecture. *p has nonzero degree,*

as Arnold did in [1]. Unfortunately, right now very little is known on the question. To be precise the above conjecture has been proved in the following cases ([28], [34]):

1. $L = M$ and $H^n(M)$ is generated by elements of $H^1(M)$ for instance if $M = T^n$.
2. L and M are surfaces, and $(L, M) \neq (T^2, S^2)$.

More generally, we may consider the easier question as to whether $p^*(\alpha)$ is zero for some cohomology class α on M. In fact Gromov's statement is equivalent to the following:

$$\text{if } M = V \times S^1 \text{ then } p^*(1 \otimes d\theta) \neq 0.$$

This we shall generalize as

Proposition 0.1. *Let $M = V \times W$ and W be a product of spheres and complex projective spaces. Then $p^* : H^*(W) \to H^*(L)$ is injective. In particular if $V = \{pt\}$, p has nonzero degree.*

For simplicity, we assume throughout the paper that all manifolds are orientable. We leave to the reader the task of figuring out how these methods could be adapted to the nonorientable case.

For the pair (L, M), we shall investigate whether exact Lagrange embeddings of L into T^*M satisfy one of the following properties:

(A) p has nonzero degree.
(B) p is cohomologically nontrivial.
(C) p is homotopically nontrivial.

We just remind the reader that the injection of the constant loops $c : M \to \Lambda M$ and evaluation map $e : \Lambda M \to M$ induce maps in cohomology, and that e^* yields an injection $H^*(M) \to H^*(\Lambda M)$. We shall denote by μ_M the top dimensional generator of $H^*(M)$ that we identify with its image by e^* in $H^*(\Lambda M)$. Finally if $f : L \to M$ is a map, we denote by $\Lambda f : \Lambda L \to \Lambda M$ the induced map on loop spaces. We shall need the following:

Definition 0.2. Let $z \in H^*(\Lambda M)$ and $\alpha \in H^*(M)$. We shall say that z is *tied* to α, if for any map $f : L \to M$, $(\Lambda f)^*(z) \neq 0$ implies $f^*(\alpha) \neq 0$.

Remark 0.3. We also used the term "tied" in the paper [46]. The term "tied" is used here in a sense slightly weaker than that of "weakly tied" in [46] according to which for z to be weakly tied to μ_M we would need the implication to hold for all maps from X to ΛM and not only maps of the type Λf. We refer to [46] for methods to construct tied classes, in particular through the use of Massey products, as well as for applications of this notion to the calculus of variations.

We may now state:

Theorem 0.4. *Assume that $H^*(\Lambda M)$ contains an element z such that $\mu_M \cdot z$ is nonzero. Then for all product manifolds $V \times M$, p is homotopically nontrivial. Moreover, let z be such that $\mu_M \cdot z \neq 0$, and z is tied to $\alpha \in H^*(M)$. Then $p^*(\alpha) \neq 0$.*

We will also get obstructions to the existence of exact Lagrange embeddings with nonzero degree. In fact we have:

Proposition 0.5. *Let j be such that p has non-zero degree. Then $(\Lambda j)^*$ is an injection.*

Remark 0.6. Note that it is not true in general that if f^* is injective then so is $(\Lambda f)^*$. As a counter example, consider f, a degree-one map from T^{2n-1} to S^{2n-1}. The map f^* is then obviously injective, while $(\Lambda f)^*$ cannot be injective since $H^k(\Lambda T^{2n-1}) = 0$ for $k > 2n-1$ while $H^{2nr-1}(\Lambda S^{2n-1}) \neq 0$.

Thus according to Proposition 0.5, we get:

Corollary 0.7.

There is no exact embedding of T^n into T^*S^n.

There is no exact Lagrange embedding from $S^{2k+1} \times S^{2l}$ into $T^*S^{2(k+l)+1}$.

The first statement for $n = 2$ answers positively a question by Lalonde and Sikorav in [28].

The ring $H^*(\Lambda M, \mathbb{Q})$ may be computed using the minimal model of Sullivan, as in [38]. Its structure has been studied for a while, and we shall make use of the results of Burghelea, Goodwillie and Vigué-Poirrier ([5],[6], [7],[18], etc...). Proposition 0.1 then follows from 0.4 and the structure of $H^*(\Lambda S^{2n-1}, \mathbb{Q})$ (for a sphere (A), (B) and (C) are equivalent).

The results of this paper are based on the following:

MAIN THEOREM. *To the exact Lagrange embedding $j : L \to T^*M$ we may associate a map $(\Lambda j)! : H^*(\Lambda L) \to H^*(\Lambda M)$ such that the following hold:*

1. *$(\Lambda j)!$ is a group homomorphism, and if j and k are respectively Lagrange embeddings of L into T^*M and M into T^*N, we have $\Lambda(k \circ j)! = (\Lambda k)! \circ (\Lambda j)!$.*
2. *$(\Lambda j)!(x \wedge (\Lambda j)^* y) = (\Lambda j)!(x) \wedge y$.*
3. *If $j!$ is the usual transfer homomorphism, e_L the evaluation map $\Lambda L \to L$, and c_L the injection of the constant loops (identified to L) in ΛL, then we have a commutative diagram:*

$$\begin{array}{ccc} H^*(\Lambda L) & \xrightarrow{(\Lambda j)!} & H^*(\Lambda M) \\ \downarrow c^* \uparrow e^* & & \downarrow c^* \uparrow e^* \\ H^*(L) & \xrightarrow{j!} & H^*(M) \end{array}$$

The same holds if we replace H^* by $H^*_{S^1}$ the equivariant Borel theory (see [3]), however the map e^* is not defined, and must be replaced by the map B^*e^* where B^* is the map from $H^*(X)$ to $H^{*-1}_{S^1}(X)$ in the standard Gysin exact sequence connecting $H^*(X)$ and $H^*_{S^1}(X)$. Moreover all maps are now $H^*(BS^1)$ module homomorphisms. Also we have for Massey products $(\Lambda j)^! < (\Lambda j)^*(x), y, (\Lambda j)^*(z) > \subset < x, (\Lambda j)^!(y), z >$.

The proof of this is the main goal of this paper, and a proof is sketched at the end of this introduction. In the next paragraphes, we shall state and prove some results based on this main theorem.

Note also that, apart from the main theorem, most of our results are just examples of applications of this theorem. It would be easy to construct many more examples of obstructions to the existence of Lagrange embedding. However, as we do not have any simple characterisation of manifolds such that $H^*(\Lambda M)$ contains a class z tied to a class α in $H^*(M)$, with $\mu_M \wedge z \neq 0$, we prefer to give a number of significant examples, rather than a necessarily incomplete list of mostly "exotic" cases.

To begin with, let us give proofs of Theorem 0.4 and Propositions 0.1 and 0.5. Note that for a map $j : L \to M$, we have $j^!(1) = deg(j) \cdot 1$ and $j^!(\mu_L) = \mu_M$.

Proof of 0.4. We have that

$$(\Lambda j)^!(\mu_L \wedge (\Lambda j)^*(z)) = (\Lambda j)^!(\mu_L) \wedge z = \mu_M \wedge z \neq 0.$$

Thus $(\Lambda j)^*(z) \neq 0$ which implies that j is not homotopically trivial.

If moreover z is tied to α, we get that $(\Lambda j)^*(\alpha) = j^*(\alpha) \neq 0$. q.e.d.

Proof of 0.1. This follows from Sullivan's theory of minimal models.

Let $F(a_\alpha, b_\beta, c_\gamma, \dots)$ be the "free graded algebra" with generators $a_\alpha, b_\beta, c_\gamma, \dots$ of degrees $\alpha, \beta, \gamma, \dots$. By "free graded algebra", we mean the tensor product of the exterior algebra in generators of odd degrees, with the symmetric algebra in generators of even degrees. Adding a graded differential d to F makes it into a differential graded algebra, with cohomology denoted by $H^*(F, d)$.

Now the cohomologies of loop spaces are easy to express in terms of cohomologies of a free graded algebra called the "minimal model". For instance as the minimal model of:

ΛS^{2k+1} we have $F(x_{2k+1}, \bar{x}_{2k})$, $d = 0$, and $\mu_M = x_{2k+1}$,

ΛS^{2k} we have $F(x_{2k}, \bar{x}_{2k-1}, y_{4k-1}, \bar{y}_{4k-2})$, $dx_{2k} = d\bar{x}_{2k-1} = 0$, $dy_{4k-1} = x_{2k}^2$, $d\bar{y}_{4k-2} = x_{2k} \cdot \bar{x}_{2k-1}$, and the cohomology class μ_M corresponds to x_{2k},

ΛCP^{2k} we have $F(x_2, \bar{x}_1, y_{2k+1}, \bar{y}_{2k})$, $dx_2 = d\bar{x}_1 = 0$, $dy_{2k+1} = x_2^{k+1}$, $d\bar{y}_{2k} = x_2^k \cdot \bar{x}_1$, and the cohomology class μ_M corresponds to x_2^k.

In all three cases, there is a class z tied to μ_M.

Indeed for S^{2k+1}, $z = \bar{x}_{2k}$ will do. For we have that $\bar{x}_{2k} \cdot x_{2k+1}$ is nonzero in the cohomology ring of ΛS^{2k+1}, and since for any map j, from L to M, $(\Lambda j)^*$ commutes with the degree -1 map, β that sends "unbarred" generators to "barred" ones (the map β corresponds on ΛM to interior product with $\frac{\partial}{\partial\theta}$, see [7]), we have that $(\Lambda j)^*(\bar{x}_{2k}) \neq 0$ implies that $(\Lambda j)^*(x_{2k+1}) = j^*(x_{2k+1}) \neq 0$, and thus $\bar{x}_{2k}$ is tied to x_{2k+1}.

In the case of S^{2k} and CP^{2k} (the exponent denotes the dimension of the space) we need to use Massey products.

In ΛS^{2k}, we have that $< \bar{x}_{2k-1}, x_{2k}, \bar{x}_{2k-1} >= \bar{y}_{4k-2}\bar{x}_{2k-1}$ and is thus nonzero. Using again that $(\Lambda j)^*$ commutes with β, we have that if $(\Lambda j)^*(\bar{x}_{2k-1}) \neq 0$ we also have $(\Lambda j)^*(x_{2k}) = j^*(x_{2k}) \neq 0$.

We may then conclude as follows: assume we have a Lagrange embedding of L into T^*S^{2k}, inducing a map $j : L \to S^{2k}$. Since we have that

$$\begin{aligned}(\Lambda j)!(< (\Lambda j)^*(\bar{x}_{2k-1}), \mu_L, (\Lambda j)^*(\bar{x}_{2k-1}) >)\\ \subset (< \bar{x}_{2k-1}, (\Lambda j)!(\mu_L), \bar{x}_{2k-1} >)\\ =< \bar{x}_{2k-1}, x_{2k}, \bar{x}_{2k-1} >= \bar{y}_{4k-2}\bar{x}_{2k-1} \neq 0,\end{aligned}$$

we must have $(\Lambda j)^*(\bar{x}_{2k-1}) \neq 0$, hence j has nonzero degree.

Finally, in CP^{2k}, we may use a similar argument. In ΛCP^{2k}, we have that

$$< \bar{x}_1, x_2^k, \bar{y}_{2k}^q \bar{x}_1 >= \bar{y}_{2k}^{q+1} \bar{x}_1.$$

If $j^*(x_2^k) = 0$, than $(\Lambda j)^*(\bar{y}_{2k}\bar{x}_1) = 0$. Now for j induced by a Lagrange embedding,

$$\begin{aligned}(\Lambda j)!(< (\Lambda j)^*(\bar{x}_1), \mu_L, (\Lambda j)^*(\bar{y}_{2k}\bar{x}_1) >) &\subset < \bar{x}_1, (\Lambda j)!(\mu_M), \bar{y}_{2k}\bar{x}_1 > \\ &=< \bar{x}_1, x_2^k, \bar{y}_{2k}\bar{x}_1 >= \bar{y}_{2k}^2\bar{x}_1 \neq 0,\end{aligned}$$

which implies that $(\Lambda j)^*(\bar{y}_{2k}\bar{x}_1) \neq 0$, hence $j^*(x_2^k) \neq 0$. q.e.d.

Proof of 0.5. Indeed, if j has nonzero degree, then $j!(1) = d \neq 0$, and therefore $(\Lambda j)!(1 \cup (\Lambda j)^*(u)) = (\Lambda j)!(1) \cup u = j!(1) \cup u = d \cdot u$. Hence $(\Lambda j)^*(u) = 0$ implies $d \cdot u = 0$, that is u is 0. Thus $(\Lambda j)^*$ is injective as claimed. q.e.d.

Proof of 0.7. This follows again from the structure of the cohomology of $H^*(\Lambda S^r)$, and Propositions 0.1 and 0.5. Let us start with the first statement.

It is easy to show that $H^*(\Lambda T^n) = \bigoplus_{x \in \mathbb{Z}^n} H^*(T^n)$ because for each connected component in the free loop space of the n-torus, parametrized in an obvious way by $\mathbb{Z}^n$, the set of free loops in this homotopy class has the homotopy type of the n-torus. Then according to 0.1, j cannot have zero degree. Thus according to 0.5, $(\Lambda j)^*$ must be an injection of $H^*(\Lambda S^n)$ into $H^*(\Lambda T^n)$, but this is clearly impossible, since $H^q(\Lambda S^n)$ is nonzero for arbitrary large values of q, while $H^q(\Lambda T^n)$ vanishes for $q \geq n+1$.

Similarly, for the second statement j cannot have zero degree according to 0.1; thus using 0.5, it must induce an injective map

$$\begin{aligned}(\Lambda j)^* : H^*(\Lambda S^{2(k+l)+1}) &\to H^*(\Lambda(S^{2k+1} \times S^{2l})) \\ &= H^*(\Lambda S^{2k+1}) \otimes H^*(\Lambda S^{2l}).\end{aligned}$$

But this is impossible by the following argument. Let $z_{2(k+l)+1}, \bar{z}_{2(k+l)}$ be the generators of $H^*(\Lambda S^{2k+1})$, $x_{2k+1}, \bar{x}_{2k}$ those of $H^*(\Lambda S^{2k+1})$, and $H^*(\Lambda S^{2l})$ be generated by $y_{2l}, \bar{y}_{2l-1}, u_{4l-1}, \bar{u}_{4l-2}$ with the relations $du_{4l-1} = y_{2l}^2, d\bar{u}_{4l-2} = y_{2l}\bar{y}_{2l-1}$. Since the map $(\Lambda j)^*$ commutes with the map β, we have that $\bar{z}_{2(k+l)}$ goes to an element of the type $\bar{x}^p\bar{u}^q\bar{y}$ $(p, q \geq 0)$, because it follows easily from the computations in [5, p. 65] that this is the only element in $\ker(\beta)$. Hence equality of degrees implies that $2k \cdot p + (4l-2) \cdot q + 2l - 1 = 2(k+l)$ or else

$$2k \cdot p + (2l-1) \cdot (2q+1) = 2(k+l).$$

This is clearly impossible. q.e.d.

Remark 0.8. Let L (respectively M) have an exact embedding into T^*M, (respectively T^*N) but we do not assume here that M or N is compact. Then, using Weinstein's theorem we see that L has an exact embedding into T^*N such that the associated projection is homotopic to the composition of the projections $L \to M$ and $M \to N$.

We thus get alternatives of the following type: given a manifold M, either there is no exact embedding of S^n into T^*M with zero degree, or of M into T^*S^n with zero degree.

The same holds for maps of nonzero degree, unless $H^*(\Lambda M)$ is isomorphic to $H^*(\Lambda S^n)$. We may also recover the fact that for M open there is no exact embedding of L in T^*M. Indeed it is sufficient to prove this for $L' = L \times T^k$ and $M' = M \times T^k$. Now let X be a compact manifold such that there exists a map $f : M \to X$ such that $f^*(TX) = TM \oplus \varepsilon_{\mathbb{R}}^{k-1}$, where $\varepsilon_{\mathbb{R}}$ means some trivial real line bundle. It is easy to find X by taking a Grassmannian manifold of sufficiently large dimension and f to be the classifying map of the normal bundle to TM. Since $f^*(TX) \otimes \mathbb{C} = TM \otimes \mathbb{C} \oplus \epsilon_{\mathbb{C}}^{k-1}$, using Gromov's h-principle it is known that there exists a Lagrange immersion of $M' = M \times T^k$ into $T^*(X \times \mathbb{R})$. Because M' is open, we may even assume that the immersion is in fact an exact embedding. By composition, we find an exact embedding of L' into $T^*(X \times \mathbb{R})$, a contradiction.

Remark 0.9. All our arguments are based on the study of the rational minimal model of the manifolds. Thus our proofs apply to any manifold with the same rational homotopy type as those we considered. On the other hand we have not used torsion information at all. Unfortunately, to our knowledge very little is known on this for loop spaces.

We now turn to a different problem, that of finding conditions on the Maslov class of an exact Lagrange embedding. This problem is in particular relevant in studying invariant Lagrange manifolds for Hamiltonian flows (see [21], [2], [54]). It is also of a more "positive nature" since it gives information on the Maslov class of existing embeddings, and not restrictions on embeddings that are conjectured to be trivial.

As a consequence of the above methods, we get:

Proposition 0.10. *Let L be a manifold having the homotopy type of an Eilenberg-MacLane space, and M be a simply connected manifold. Then there is no exact Lagrange embedding from L into T^*M. Moreover* *given a (non-exact) embedding from L to T^*M, the Maslov class μ of the embedding satisfies that for some γ in $H_1(L)$, $1 \leq < \mu, \gamma > \leq n+1$ and $0 << \lambda, \gamma >$*

Proof. According to the equivariant version of the Main theorem, we have a commutative diagram:

$$\begin{array}{ccc} H^*_{S^1}(\Lambda L) & \xrightarrow{(\Lambda j)!} & H^*_{S^1}(\Lambda M) \\ \Big\downarrow c^* & & \Big\downarrow c^* \\ H^*(L) \otimes H^*(BS^1) & \xrightarrow{j! \otimes Id} & H^*(M) \otimes H^*(BS^1) \end{array}$$

According to Goodwillie's theorem (see Section 3 for a precise statement), the right-hand side vertical arrow is zero into $H^n(M)$ after localization, while the left-hand one is onto in all dimensions (in fact the inclusion map from the constants to null homotopic loops induces a homotopy equivalence, in this case). Thus let ξ in $H^n_{S^1}(\Lambda L)$ be an element going to $\mu_L \otimes 1$ in $H^n(L) \otimes H^*(BS^1)$. Then its image by $j^! \otimes 1$ is nonzero in $H^n(M) \otimes H^*(BS^1)$ (note that if $f : L \to M$ is a map, $f^!(\mu_L) = \mu_M$, independently of the degree of f). So on one hand $c^* \circ (\Lambda j)!$ is zero, since the right-hand side c^* is zero in degree n, while $j^! \circ c^*$ sends ξ to μ_M. This contradicts the commutativity of the diagram.

The second part of the proposition is similar to the proof of 0.12 in Section 7, and is left to the reader.

Similar theorems are as follows (see Section 7 for the proofs)

Proposition 0.11. *Assume M is a manifold such that $\pi_1(M)$ has a center Z of finite index. Let j be an exact Lagrange embedding of T^n into T^*M. Then* $\operatorname{rank} Z = n$ *and the image of $\pi_1(T^n)$ in Z is injective. Moreover if we do not assume j exact anymore and* $\operatorname{rank} Z \neq n$, *there is a loop γ on T^n such that*

1. $\int_\gamma pdq > 0$,
2. $(\mu(j), \gamma) \in [2, n+1]$.

This generalizes an earlier result from [42], dealing with the case $L = M = T^n$.

More generally we have:

Proposition 0.12. *Let M satisfy the first assumption of Theorem 0.4 and let $j : L \to T^*M$ be a Lagrange embedding such that* $\deg(p) = 0$. *Then there exists c in $H_1(L)$ such that:*

(i) $c \in \ker(p)$,

(ii) $< \lambda, c >> 0$, *so in particular* $c \notin \mathrm{Ker}\lambda$,

(iii) $< \mu(j), c > \leq d_M$, *where* d_M *does depend not on* L *but only on* M.

This had been proved for $L = M = T^n$ (cf. [42], and [33] for the case $n = 2$), and manifolds of negative curvature. More precise estimates for the Maslov class are due to Y.G. Oh in particular for monotone tori in $\mathbb{R}^{2n}$ (cf. [32]). In a certain number of cases, this implies that when $\deg(p) \neq 0$, the Maslov class of L vanishes.

Remark 0.13. It is not hard to verify that our proof of the MAIN THEOREM still holds if we replace ordinary cohomology by any cohomological theory (e.g. K-theory, stable homotopy, etc.) provided it has a Thom isomorphism. The same also holds for an equivariant cohomology theory, but besides the existence of Thom isomorphism, we need that the S^1 equivariant theory be in some sense determined by the knowledge of the $\mathbb{Z}_k$ equivariant one. However the existence of the map B^* is not granted, thus the commutative diagram in (3) does not necessarily exist.

Finally we give a proof of the Weinstein conjecture in a cotangent bundle, provided M is simply connected. We refer to Section 3 for an introduction to the subject, and detailed statement of the theorems. This is obtained as a byproduct of our method that we shall now describe.

Sketch of the proof for the MAIN THEOREM:

Even though our proof is based on a finite dimensional approach, inspired by the work of Chaperon, Laudenbach-Sikorav, Givental (see [8],[29], [16], [17]), we can easily give a heuristic description of it in terms of Floer cohomology, making it, we hope, much easier to understand.

Note that a Floer cohomology proof is indeed possible as in [48], where we construct the analogue of the map $(\Lambda j)^!$ at the level of Floer cohomology, satisfying the same properties as in our main theorem. Note that this does not really simplify the proofs of the present paper, since we need the isomorphism between the Floer cohomology of a cotangent bundle, and the cohomology of the corresponding loop space, and this requires more or less the same methods as those used in the first sections.

The discretization is carried out in Section 1.

Given a symplectic manifold W with contact type boundary, we may consider the Floer cohomology associated to a Hamiltonian, H, which goes to infinity near the boundary. In other words, $FH^*(H)$ is the cohomology of a complex having one generator for each periodic orbit of period 1 of X_H, the Hamiltonian vector field of H, and coboundary operator obtained by counting the number of solutions of an elliptic partial differential equation on an infinite cylinder, asymptotically converging to the periodic orbits.

This corresponds also to the relative cohomology of the level sets for a finite dimensional reduction of the action functional

$$A_H(q,p) = \int_{S^1} [p\dot{q} - H(q,p)]dt.$$

That is, for a finite dimensional reduction, $A_{H,N}$ of A_H, we have an isomorphism (up to a shift in grading though) between $H^*(A^c_{H,N}, A^{-c}_{H,N})$, for c large enough, and $FH^*(H)$, where $A^c_{H,N}$ is the sublevel set of $A_{H,N}$.

If W is the unit disk bundle of T^*N, and the Hamiltonian is zero in the interior of W (see Section 1 for details), then $FH^*(H)$ is isomorphic to $H^*(\Lambda N)$, where ΛN is the free loop space of N. Now let L be an exact Lagrange submanifold of T^*N. According to Weinstein's theorem, we may consider a tubular neighborhood U of L, and identify it with the unit disk bundle in T^*L.

Now choose a Hamiltonian K on T^*N such that the following hold:

-it is zero inside U,

-grows very fast as we reach the boundary of U (in other words K is almost equal to H on U),

-is constant outside U, up to the boundary of a unit disc bundle W of T^*N,

-grows very fast as we reach the boundary of W.

In Sections 2 and 4 we prove that the contribution of the characteristics contained in U, the neighbourhood of L, to $FH^*(K)$ is again given by $H^*(\Lambda L)$.

In Section 5 we show that, in computing $FH^*(K)$, and choosing appropriately some parameters, the critical points corresponding to closed characteristics near L are at a higher level than the other characteristics. This yields a map from $FH^*(H)$ to $FH^*(K)$, that is, up to a limiting process, the map $(\Lambda j)!$ we are looking for.

The other arrows in the diagram are obtained in Sections 5 and 6 by restricting this map to the set of constant loops. Here we use the fact that with our choice of the Hamiltonians, constants correspond to the lowest critical levels

Let us now explain where the assumption that the Lagrange submanifold be exact is used in our proof. If ΛL has several connected components, each of these contribute to the cohomology of $FH^*(H)$, but each of the components is on a different level for A_H. The level is in fact approximately given by $< \lambda, \gamma >$ (for γ any loop in the connected component). Now if there is no trajectory of the gradient flow of A_H (or rather $A_{H,N}$) from one component to the other, each connected component of ΛL will contribute to the total cohomology by just adding each contribution. This is of course the case, when λ is exact on L, since then all components of ΛL will be on the same level, and there can be no gradient trajectory connecting two components.

Finally in Section 7, we show more precisely that, unless the Liouville and Maslov class satisfy some restriction, the trajectories of the gradient map above will not modify the above picture in the relevant cohomological degrees, and the above obstructions to Lagrange embedding will still hold. This gives Maslov type obstructions to Lagrange embeddings.

One more comment on our main theorem. We will show in this paper how algebraic topology of the free loop space yields obstructions to the existence of Lagrange embeddings. But one may wonder in general, given a map $f : L \to M$ whether there exists a map $(\Lambda f)! : H^*(\Lambda L) \to H^*(\Lambda M)$ satisfying the conditions of the main theorem. We essentially proved that there are certain obstructions to the existence of such a map. However the following question seems natural to us:

Question. Is there a functorial subring $F^*(\Lambda X)$ of $H^*(\Lambda X)$ such that $(\Lambda f)!$ is well defined from $F^*(\Lambda L)$ to $F^*(\Lambda M)$?

Also we are ashamed to confess that in spite of all the above results about Lagrange embeddings the simple question
"Is there a (non-exact!) Lagrange embedding of the Klein bottle in $\mathbb{R}^4$"
is still unanswered.

Since this paper was written, Eliashberg and Polterovich proved that any Lagrange torus homologous to the zero section in T^*T^2 is isotopic to the zero section (but their proof does not say whether it is Lagrange isotopic), and the analogous result for S^2 in T^*S^2.

Also Hofer proved that all exact Lagrange tori in T^*T^2 are isotopic. The methods seems to be purely 4-dimensional though.

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1. Discretization of Hamiltonian flows

Let L be a (not necessarily compact) manifold, and $\widetilde{g}$ a complete metric on L. The metric $\widetilde{g}$ defines a fiberwise quadratic map on TL, and by duality, a fiberwise quadratic map on T^*L, that we shall denote by g. In local coordinates, if $\widetilde{g} = \Sigma \widetilde{g}_{ij} dq^i dq^j$, we have

$$g = \Sigma g^{ij} dp_i dp_j \text{ , where } (g_{ij}) = (\widetilde{g}_{ij})^{-1} .$$

In this section we denote by ω the canonical symplectic form on T^*L, and by H_0 a Hamiltonian on T^*L of the form $H_0(q,p) = h(|p|)$, where $|p| = (g(p)p,p)^{\frac{1}{2}}$, and $h : \mathbb{R} \to \mathbb{R}_+$ satisfies:

(a) h is smooth, convex,

(b) $h(0) = h'(0) = 0$,

(c) $h(u) = h_\infty \cdot u$ for u large enough.

Let X_0 be the Hamiltonian vector field associated to H_0, i.e.,

$$\omega(X_0, \xi) = dH_0 \cdot \xi,$$

and φ_t^0 its flow. It is clear that φ_t^0 is a reparametrization of the geodesic flow of g. Note that if we consider 1-periodic orbits of φ_t^0, they will be in one to one correspondence with closed geodesics of length less than h_∞.

The Hamiltonians which we shall consider in this paper will always coincide with H_0 outside some compact set. Let H be such a Hamiltonian, X_H, φ_t its associated vector field and flow.

We now explain how to discretize φ_t.

Let $\overline{T^*L}$ denote the symplectic manifold $(T^*L, -\omega)$, and

$$E_r = (\overline{T^*L})^r \times (T^*L)^r$$

for some integer r. Let $\sigma : (T^*L)^r \to (T^*L)^r$ be the shift map $\sigma(z_1, \ldots, z_r) = (z_2, \ldots, z_r, z_1)$, and $\Phi = \sigma \circ (\varphi_{1/r} \times \ldots \times \varphi_{1/r})$, that is

$$\Phi(z_1, \ldots, z_r) = (\varphi_{1/r}(z_2), \ldots, \varphi_{1/r}(z_r), \varphi_{1/r}(z_1)) .$$

Since σ and $\varphi_{1/r}\times\ldots\times\varphi_{1/r}$ preserve the symplectic structure of $(T^*L)^r$, the same is true for Φ. The graph $\Gamma(\Phi)$ of Φ is then a Lagrange submanifold of E_r (endowed with the obvious symplectic structure).

For convenience we shall often use the following notation or conventions :

— for $z \in (T^*L)^r$

$$z = (z_j)_{j\in\mathbb{Z}/r} \text{ so that } \sigma(z_j) = (z_{j+1}) \text{ and } z_j = (q_j, p_j),$$

— for $(z, Z) \in E_r$

$$z = (z_j)_{j\in\mathbb{Z}/r}, (z, Z) = ((z_j, Z_j))_{j\in\mathbb{Z}/r} \text{ so that } \Gamma(\Phi) = (z_j, \varphi_{1/r}(z_{j+1})),$$

— the diagonal $\Delta = \Delta_r = (\Delta_{T^*L})^r \subset E_r$ is the product of the diagonals in $\overline{T^*L} \times T^*L$.

Our first task will be to define a region of E_r such that the portion of $\Gamma(\Phi)$ contained in this region will be a graph over Δ_r. We first make precise our identification of a neighbourhood of Δ_r in E_r with a neighbourhood of the zero section in $T^*\Delta_r$.

Lemma 1.1. *Assume the injectivity radius of $\widetilde{g}$ is bounded from below by ε_0. Let $\varepsilon < \varepsilon_0$ small enough and $U_\varepsilon \subset \overline{T^*L} \times T^*L$ be the set $U_\varepsilon = \{(q,p,Q,P) | d(q,Q) \le \varepsilon\}$. Then there is a proper symplectic embedding $i : U_\varepsilon \to T^*(T^*L)$ such that on U_0 we have*

(1) $i(q,p,q,P) = (q,P,P-p,0)$,

(2) $Di(q,p,q,P)(\delta q,\delta p,\delta Q,\delta P) = (\delta q,\delta P,\delta P-\delta p,\delta q-\delta Q)$.

The proof follows immediately from the Darboux-Weinstein theorem. This tells us that if K is a submanifold of a symplectic manifold (M,ω), and (M',ω') is some other symplectic manifold, then, given a map $\varphi : K \to M'$ such that $\varphi^*\omega'_{|K} = \omega_{|K}$, and an extension f of $d\varphi$ to a symplectic fibre map from $T_K M$ to TM', there exists a symplectic extension $\widetilde{\varphi}$ of φ to a neighbourhood of K such that $d\widetilde{\varphi}_{|K} = f$.

Using this, the proof is easy and left to the reader. q.e.d.

Remark. We shall denote by "$P-p$" the coordinate dual to q in $T^*(T^*L)$. Let us point out that it *is not* the difference of "P" and "p" !

Similarly we may define "$q-Q$". Using the symplectic map i, we may pull back these coordinates on U_ε. Then, both coincide with some "naive" definition up to a term of order ε for "$P-p$" and ε^2 for "$Q-q$".

By a "naive" definition of "$P-p$" we mean for example the sum of $-p$ (in T_q^*L) and the image of P by the parallel transport from Q to q along the unique minimizing geodesic. Similarly, one may define a "naive" $Q-q$ by using $X \in T_qL$ such that $\exp_q X = Q$.

Let $(U_\varepsilon)^r \subset E_r$, and consider the symplectic embedding $i^r : U_\varepsilon^r \to T^*(T^*L^r)$,

$$\Gamma_\Phi = i^r(\Gamma(\Phi) \cap U_\varepsilon^r)\,.$$

We first examine Γ_{Id}. Since

$$\Gamma(\mathrm{Id}) = \big\{(q_j, p_j, q_{j+1}, p_{j+1}) | (q_j, p_j) \in T^*L\big\}\,,$$

we have

$$\Gamma(\mathrm{Id}) \cap (U_\varepsilon)^r = \{(q_j, p_j, q_{j+1}, p_{j+1}) | d(q_j, q_{j+1}) \leq \varepsilon\}\,.$$

We thus see that Γ_{Id} has a projection on the base of $T^*(T^*L^r)$ " essentially" given by $(q_j, p_j, q_{j+1}, p_{j+1}) \to (q_j, p_{j+1})$ (again, we use parallel transport to define this).

We may in fact assert that Γ_{Id} is a graph over

$$\mathcal{U}_{r,\varepsilon} = \big\{(q_j, P_j) \in T^*L^r | d(q_j, q_{j+1}) \leq \varepsilon/2\big\}.$$

Since $\Gamma(\Phi)$ is close to Γ_{Id} (for the distance $\sup_j d(z_j, z_j'))$ provided r is large enough, the same will hold for Γ_Φ. Because φ_t is Hamiltonian, it is easy to see that $\Gamma(\Phi)$ and thus Γ_Φ are exact Lagrange submanifolds.

We may now conclude that over $\mathcal{U}_{r,\varepsilon}$, Γ_Φ is the graph of $d\mathcal{S}_\Phi$ for some function $\mathcal{S}_\Phi : \mathcal{U}_{r,\varepsilon} \to \mathbb{R}$.

The rest of this section is devoted to computing the Conley index (read further for the definition) of $\mathcal{U}_{r,\varepsilon}$ for some pseudo-gradient vector field ξ_Φ of $\mathcal{S}_\Phi$, when $H = H_0$ (the next section is devoted to more general cases).

We remind the reader of some results from Conley's book [9].

Let U be a manifold with boundary (and possibly corners) ∂U, and ξ a vector field on U. The Conley index of ξ of U, denoted by $I^*(U, \xi)$ is the homotopy type of the quotient $U/\partial^- U$, where $\partial^- U$ is the exit set of ξ (i.e., $\big\{x | (\xi(x), \nu(x)) > 0\big\}$, where $\nu(x)$ is the outward normal on ∂U).

Let C be the maximal invariant set in U. Then, provided $C \subset U - \partial U$ (in Conley's terminology, U is an isolating block for C) we have that $I^*(U, \xi)$ only depends on C (and of course on ξ), not on U.

In case ξ is a pseudo gradient vector field for some function f (i.e., $df(x)\cdot\xi(x)\le 0$ with equality if and only if $df(x)=0$ and then, also ξ vanishes), C will be the union of the critical points of f, and the heteroclinic trajectories of ξ connecting them.

If we set $U^a=\{x\in U|f(x)\le a\}$, and if U is an isolating block, than U^b-U^a is also an isolating block for the set C_a^b of critical points with critical value in $[a,b]$ and heteroclinic trajectories connecting them (provided a and b are regular values of f).

Note that this has a straightforward generalization to the case where U is endowed with some group action, and everything is equivariant. The equivariant Conley index, denoted by $I_G^*(U,\xi)$ is then the equivariant homotopy type of U/∂^-U.

We now return to our original problem, and point out that it has an obvious $\mathbb{Z}/r$ symmetry, and that everything will indeed be equivariant, even if it is not specified.

To be able to compute $I^*_{\mathbb{Z}/r}(\mathcal{U}_{r,\varepsilon},\xi_\Phi)$, we first have to define ξ_Φ !

Let $E_\Phi(q)=\sup\limits_P \mathcal{S}_\Phi(q,P)$ in $\mathbb{R}\cup\{+\infty\}$, and

$$\Lambda^a_{r,\varepsilon}=\big\{(q_j)\in\Lambda_{r,\varepsilon}|E_\Phi(q)\le a\big\}.$$

We shall need:

Lemma 1.2. *For $|P|\le R(\varepsilon,r)$, the map $P\to\mathcal{S}_\Phi(q,P)$ is strictly concave. For $a<a(\varepsilon,r)$, and $q\in\Lambda^a_{r,\varepsilon}$ the map $P\to\mathcal{S}_\Phi(q,P)$ has a unique critical point, which is a maximum.*

Moreover $R(\varepsilon,r)$ and $a(\varepsilon,r)$ go to $+\infty$, as $r\varepsilon^2$ goes to zero.

Proof. Let us write coordinates in $\mathcal{U}_{r,\varepsilon}$ as (q_j,P_j,X_j,Y_j) so that Γ_Φ is given by

$$X_j=\frac{\partial\mathcal{S}_\Phi}{\partial q_j},\qquad\qquad Y_j=\frac{\partial\mathcal{S}_\Phi}{\partial P_j},$$

and we have that

$$Y_j=q_{j+1}-q_j-\frac{1}{r}\frac{\partial H}{\partial p}(q_j,P_j)+\eta(q_{j+1}-q_j,P_j),$$

where $\eta(0,P)=D\eta(0,P)=0$.

Note that Y_j only depends on q_j,P_j,q_{j+1},P_{j+1}. Now

$$\frac{\partial Y_j}{\partial P_j}=-\frac{1}{r}\frac{\partial^2 H}{\partial p^2}(q_j,P_j)+\frac{\partial\eta}{\partial P_j}(q_{j+1}-q_j,P_j),$$

and $\frac{\partial \eta}{\partial P}$ goes to zero with $q_{j+1} - q_j$, while $\frac{1}{r}\frac{\partial^2 H}{\partial p^2} > \frac{C_0}{r}$ for $\sup |p_j| \leq R(\varepsilon, r)$. Thus $\frac{\partial Y_j}{\partial P_j} < -\frac{C_0}{r} + C\varepsilon^2$.

Since

$$\begin{aligned} \frac{\partial Y_j}{\partial P_k} &= 0 && \text{for } k \neq,\ j+1, \\ &= \frac{\partial}{\partial p_{j+1}} \eta(q_{j+1} - q_j, P_j) && \text{for } k = j+1\,, \end{aligned}$$

we have

$$\begin{aligned} \frac{\partial^2 \mathcal{S}_\Phi}{\partial P_j^2} &= \frac{\partial Y_j}{\partial P_j} < -\frac{C_0}{r} + C\varepsilon^2, \\ \left| \frac{\partial^2 \mathcal{S}_\Phi}{\partial P_j \partial P_{j+1}} \right| &= \left| \frac{\partial Y_j}{\partial P_{j+1}} \right| < C\varepsilon^2, \\ \frac{\partial^2 \mathcal{S}_\Phi}{\partial P_j \partial P_k} &= 0 && \text{for } k \neq j,\ j+1. \end{aligned}$$

As a result, provided $r \leq \frac{1}{2}\frac{C_0}{C}\varepsilon^{-2}$, we have that $P \to \mathcal{S}_\Phi(q, P)$ is concave.

Note that C_0 goes to zero as $R(\varepsilon, r)$ increases, that is, as $r\varepsilon^2$ goes to zero we may let C_0 decrease to zero (and still have $r\varepsilon^2 < \frac{1}{2}\frac{C_0}{C}$), hence $R(\varepsilon, r)$ goes to infinity.

Let us set $\Lambda_{r,\varepsilon} = \{(q_j) \in L^r | d(q_j, q_{j+1}) \leq \varepsilon/2\}$ and $\pi_r : \mathcal{U}_{r,\varepsilon} \to \Lambda_{r,\varepsilon}$ be the obvious projection. Using Lemma 1.2, we see that the restriction of $\mathcal{S}_\Phi$ to $\pi_r^{-1}((q_j))$ is either unbounded from above (and there are no critical points) or has a unique maximum $P = P(q)$.

From the properties of $\mathcal{S}_\Phi$ it follows that

$$\widetilde{\Lambda}^a_{r,\varepsilon} = \{(q, P) \in (T^*L)^r | \frac{\partial}{\partial P}\mathcal{S}_\Phi(q, P) = 0, \qquad (q) \in \Lambda^a_{r,\varepsilon}\}$$

is contained in $\mathcal{U}_{r,\varepsilon}$, and is a graph over $\Lambda^a_{r,\varepsilon}$. We set ξ_P to be minus the gradient of the restriction of $\mathcal{S}_\Phi$ to $\pi_r^{-1}(q)$, that is, $-\nabla_p \mathcal{S}_\Phi(q, P)$. Then, in $\mathcal{V}^a_{r,\varepsilon} = \pi_r^{-1}(\Lambda^a_{r,\varepsilon})$, ξ_P is a pseudo-gradient for $\mathcal{S}_\Phi$, except on $\widetilde{\Lambda}^a_{r,\varepsilon}$.

We must now modify ξ_P near $\widetilde{\Lambda}^a_{r,\varepsilon}$ to get a pseudo-gradient everywhere. This is easily achieved, using ξ_0, an extension of minus the gradient of $\mathcal{S}_\Phi$ restricted to $\widetilde{\Lambda}^a_{r,\varepsilon}$. We may set

$$\xi_\Phi = (1 - \alpha(|P - P(q)|)\xi_P + \alpha(|P - P(q)|)\xi_0,$$

where α is a real nonincreasing function with values in $[0,1]$, such that $\alpha = 1$ near 0, and $\alpha = 0$ outside a neighbourhood of 0.

It is clear that ξ_Φ is a pseudo-gradient for $\mathcal{S}_\Phi$ in $\mathcal{V}^a_{r,\varepsilon}$.

We now denote by $N^a_{r,\varepsilon}$ a tubular neighbourhood of $\widetilde{\Lambda}^a_{r,\varepsilon}$, and by $\mathcal{V}^a_{r,\varepsilon}(R), \mathcal{U}^a_{r,\varepsilon}(R)$ the intersection of $\mathcal{V}^a_{r,\varepsilon}, \mathcal{U}^a_{r,\varepsilon}$ with $\big\{(q,P) \,|\, \sup|P_i| \leq R\big\}$.

We first prove

Lemma 1.3.

$$I^*_{\mathbb{Z}/r}(\mathcal{V}^a_{r,\varepsilon}(R), \xi_\Phi) = I^*_{\mathbb{Z}/r}(N^a_{r,\varepsilon}, \xi_\Phi)\,.$$

Proof. We shall take for $N^a_{r,\varepsilon}$ the disk bundle over $\widetilde{\Lambda}^a_{r,\varepsilon}$ given by $\alpha(|P - P(q)|) > 0$.

To prove the lemma, we only have to prove that the maximal invariant set for ξ_Φ in $\mathcal{V}^a_{r,\varepsilon}(R)$ is already contained in $N^a_{r,\varepsilon}$. We must thus show that:

(a) $N^a_{r,\varepsilon}$ contains all the critical points in $\mathcal{V}^a_{r,\varepsilon}(R)$,

(b) $N^a_{r,\varepsilon}$ contains all the heteroclinic orbits in $\mathcal{V}^a_{r,\varepsilon}(R)$.

The first statement is trivial, since at a critical point of $\mathcal{S}_\Phi$, $\frac{\partial}{\partial P}\mathcal{S}_\Phi$ vanishes, i.e., all critical points are in $\widetilde{\Lambda}^a_{r,\varepsilon}$.

As for (b), it follows from the concavity of $\mathcal{S}_\Phi$ as a function of P. Indeed, outside $N^a_{r,\varepsilon}$, $\xi_\Phi = \xi_P$ is $-\nabla_P \mathcal{S}_\Phi$. Hence if some orbit exits from $N^a_{r,\varepsilon}$, it does so through a point where $\alpha(|P - P(q)|) = 0$ (at the other points of $\partial N^a_{r,\varepsilon}$, ξ_Φ enters $N^a_{r,\varepsilon}$, see Figure 1.1).

FIGURE 1.1

But this orbit is then in a region where $\mathcal{S}_\Phi < a$, hence it cannot reenter $N^a_{r,\varepsilon}$, since it would have to reach a region where $\mathcal{S}_\Phi = a$ (ξ_Φ enters $N^a_{r,\varepsilon}$ only where $\mathcal{S}_\Phi = a$). This concludes our proof. q.e.d.

It is also easy to prove

Lemma 1.4.

$$I^*_{\mathbb{Z}/r}(\mathcal{U}^a_{r,\varepsilon}(R), \xi_\Phi) = I^*_{\mathbb{Z}/r}(\mathcal{V}^a_{r,\varepsilon}(R), \xi_\Phi)\,.$$

Proof. The proof is the same as for 1.3.

Finally, we have to compute $I^*_{\mathbb{Z}/r}(N^a_{r,\varepsilon}, \xi_\Phi)$. We first need a definition. Let $E \to X$ be a vector bundle. We denote by $\Sigma_E X$ the Thom space $D(E)/S(E)$ where $D(E)$ and $S(E)$ are respectively the disk and sphere bundles associated to E. More generally, for A in X,

$$\Sigma_E(X/A) = D(E)/(S(E) \cup D(E_{|A})).$$

The Thom isomorphism tells us that, provided E is orientable, $H^*(\Sigma_E(X/A)) = H^{*-k}(X/A)$, and the same is true in equivariant cohomology (see [10]).

Remark. From our proof of 1.3 it follows that if U is any bounded open manifold with boundary, containing $N^a_{r,\varepsilon}$ and not other critical points than those contained in $N^a_{r,\varepsilon}$, then

$$I^*_{\mathbb{Z}/r}(U, \widetilde{\xi}_\Phi) = I^*_{\mathbb{Z}/r}(N^a_{r,\varepsilon}, \xi_\Phi)$$

for any pseudo-gradient $\widetilde{\xi}_\Phi$ of $\mathcal{S}_\Phi$ coinciding with ξ_Φ on $N^a_{r,\varepsilon}$. In fact this is even true for any pseudo-gradient of $\mathcal{S}_\Phi$, since our argument only used the properties of the restriction of $\mathcal{S}_\Phi$ on the boundary of $N^a_{r,\varepsilon}$.

Lemma 1.5.

$$I^*_{\mathbb{Z}/r}(N^a_{r,\varepsilon}, \xi_\Phi) = \Sigma_N(\Lambda^a_{r,\varepsilon}),$$

where N is the normal bundle of Λ_r in Δ_r.

Proof. Since $\widetilde{\Lambda}^a_{r,\varepsilon}$ is a graph over $\Lambda^a_{r,\varepsilon}$, we may as well replace the right-hand side of 1.5 by $\Sigma_N(\widetilde{\Lambda}^a_{r,\varepsilon})$. Now $N^a_{r,\varepsilon}$ may be identified with the disk bundle associated to N. Since we saw that the exit set of ξ_Φ on $N^a_{r,\varepsilon}$ is $S(N) \cap N^a_{r,\varepsilon}$, the lemma follows immediately.

To summarize our finding we proved

Proposition 1.6.

$$I^*_{\mathbb{Z}/r}(\mathcal{U}^a_{r,\varepsilon}(R),\xi_\Phi) = I^*_{\mathbb{Z}/r}(\mathcal{V}^a_{r,\varepsilon}(R),\xi_\Phi) = \Sigma_N(\Lambda^a_{r,\varepsilon})\,.$$

The same proof yields

Proposition 1.7. *For $a(r,\varepsilon) < a < b < b(r,\varepsilon)$*

$$I^*_{\mathbb{Z}/r}(\mathcal{U}^b_{r,\varepsilon} - \mathcal{U}^a_{r,\varepsilon},\xi_\Phi) = I^*_{\mathbb{Z}/r}(\mathcal{V}^b_{r,\varepsilon} - \mathcal{V}^a_{r,\varepsilon},\xi_\Phi) = \Sigma_N(\Lambda^b_{r,\varepsilon}/\Lambda^a_{r,\varepsilon})\,.$$

Here $a(r,\varepsilon)$ goes to $-\infty$ and $b(r,\varepsilon)$ goes to $+\infty$ as $r\varepsilon^2$ goes to 0.

Remark. In the relative case we may replace $\mathcal{U}_{r,\varepsilon}(R)$ by $\mathcal{U}_{r,\varepsilon}$, since

$$\mathcal{U}^b_{r,\varepsilon} - \mathcal{U}^b_{r,\varepsilon}(R) = \mathcal{U}^a_{r,\varepsilon} - \mathcal{U}^a_{r,\varepsilon}(R)\,.$$

Note that for b and r going to $+\infty$, and ε going to 0, we have that $\Lambda^b_{r,\varepsilon}$ converges to $\Lambda^\gamma N = \{$loops in N with length less than $\gamma\}$, where $\gamma = h_\infty$ is the slope of h at infinity.

2. Generalization and localization of the results of the previous section

In this section, we shall be again interested in computations of Conley indices $I^*(U,\xi)$ with ξ a pseudo-gradient for $\mathcal{S}_\Phi$.

Here Φ is associated to a Hamiltonian H as in Section 1, but we do not assume H to be convex in p. We shall consider the following two cases

(1) $H(q,p) = H_0(q,p)$ for $|p| \geq R$.

(2) $H(q,p) = H_0(q,p)$ for $|p| \leq R$.

We start with the first case, which is based on the following idea. We consider a family H_τ of Hamiltonians such that

(a) $H_\tau = H_0$ for $|p| \geq R$,

(b) the fixed points for the time-one flow φ^τ of H_τ have their action in some interval J of $\mathbb{R}$, that is, if $\varphi^\tau_1(x_\tau) = x_\tau$, then $x_\tau(s) = \varphi^\tau_s(x_\tau)$ is such that $\int_0^1 [p\dot{q} - H_\tau(x_\tau(s))]ds$ is in J.

We first want to prove :

Proposition 2.1. *Let H_τ satisfy (a) and (b) above, and assume J is contained in $]a,b[$. Then there is a pseudo-gradient ξ_τ of $\mathcal{S}_{\Phi_\tau}$ on $\mathcal{U}_{r,\varepsilon}$ such that $\xi_0 = \xi_{\Phi_0}$, and*

$$I^*(\mathcal{U}^b_{r,\varepsilon}(\tau) - \mathcal{U}^a_{r,\varepsilon}(\tau), \xi_\tau)$$

does not depend on τ.

Proof. The idea is that since we may add a constant to $\mathcal{S}_\Phi$, so that the critical value of $\mathcal{S}_\Phi$ associated to a critical point (which corresponds to a fixed point of φ) coincides with the action of the periodic orbit (cf. [40], [41]), if $J \subset]a,b[$, then no critical point of $\mathcal{S}_{\Phi_\tau}$ will "interfere" with $\mathcal{U}^b_{r,\varepsilon}(\tau)$ or $\mathcal{U}^a_{r,\varepsilon}(\tau)$, hence $I^*(\mathcal{U}^b_{r,\varepsilon}(\tau) - \mathcal{U}^a_{r,\varepsilon}(\tau), \xi_\tau)$ will not depend on τ.

We first consider the following abstract situation. Let f_τ be a family of functions on U, such that

(i) $f_{\tau|\partial U}$ has no critical value in $[\lambda, \mu]$,

(ii) f_τ has λ and μ as regular values.

Let ξ_τ be a pseudo-gradient for f_τ such that on $\partial U \cap f_\tau^{-1}([\lambda,\mu])$, ξ_τ is tangent to ∂U. Then $I^*(U^\mu(\tau) - U^\lambda(\tau), \xi_\tau)$ does not depend on τ.

The proof of this statement follows from the standard properties of the Conley index, provided we remark that $I^*(U^\mu(\tau) - U^\lambda(\tau), \xi_\tau) = U^\mu(\tau)/U^\lambda(\tau)$ because ξ_τ does not exit on $\partial U \cap f_\tau^{-1}([\lambda,\mu])$, and that (ii) implies that the homotopy type of $U^\mu(\tau)/U^\lambda(\tau)$ does not depend on τ.

To prove our proposition, we will show that we are in the above situation, with $\mathcal{U}_{r,\varepsilon}$ and $\mathcal{S}_{\Phi_\tau}$ replacing U and f_τ.

Because as we said before, the critical values of $\mathcal{S}_{\Phi_\tau}$ are the actions of the periodic orbits, (b) implies that $\mathcal{S}_{\Phi_\tau}$ has a and b as regular values. Property **(ii)** is then satisfied, and we only have to check that $\mathcal{S}_{\Phi_\tau}$ restricted to $\partial\mathcal{U}_{r,\varepsilon}$ has no critical value in $[a,b]$, for ε small enough.

Arguing by contradiction, we see that otherwise, we would have a solution of $\frac{\partial}{\partial P}\mathcal{S}_{\Phi_\tau}(q,P) = 0$, $\mathcal{S}_{\Phi_\tau}(q,P) \in [a,b]$ for $(q,P) \in \partial\mathcal{U}_{r,\varepsilon}$. Now if $(q,P) \in \partial\mathcal{U}_{r,\varepsilon}$, we must have $d(q_j, q_{j+1}) = \varepsilon$ for at least one j in $\mathbb{Z}/r$. Now, up to higher order terms, $\frac{\partial}{\partial P}\mathcal{S}_\Phi = q_j - q_{j+1} - \frac{1}{r}\frac{\partial H}{\partial P}(q_j, P_j)$. If this vanishes, we have $\varepsilon = d(q_j, q_{j+1}) < \frac{1}{r}\left|\frac{\partial H}{\partial p}\right| \leq \frac{C}{r}$ where C is a bound for $\left|\frac{\partial H}{\partial p}\right|$ on T^*L. This implies $r \leq \frac{C}{\varepsilon}$, so it cannot hold for r large enough (remember that we assumed $r \leq \frac{1}{2}\frac{C_0}{C}\varepsilon^{-2}$, which is however compatible with the above condition). Thus assumption (i) is satisfied. q.e.d.

Corollary 2.2. *Assume that $H = H_0$ outside some compact set, and that the quantity $p\frac{\partial H}{\partial p} - H(q,p)$, which is compact supported, is bounded by the real numbers $a < 0 < b$. Then $I^*_{\mathbb{Z}/r}(\mathcal{U}^b_{r,\varepsilon} - \mathcal{U}^a_{r,\varepsilon}, \xi) = \Sigma_N(\Lambda^b_{r,\varepsilon})$, where ξ is a pseudo-gradient for $\mathcal{S}_\Phi$ tangent to the boundary of $\mathcal{U}_{r,\varepsilon}$.*

Proof. Set $H_\tau = (1-\tau)H_0 + \tau H$. Then provided

$$a < p\frac{\partial H_0}{\partial p} - H_0(q,p) < b,$$

and the same holds for H, it will hold for H_τ. Since the action of a periodic orbit of H_τ is given by $\int_0^1 p\frac{\partial H_\tau}{\partial p} - H_\tau$ (note that for a solution $\dot{q} = \frac{\partial H_\tau}{\partial p}$), τ is thus in $[a,b]$. q.e.d.

We now consider case (2).

Since $H = H_0$ in $|P| \le R$, we see that $\mathcal{S}_\Phi = \mathcal{S}_{\Phi_0}$ on $\mathcal{U}_{r,\varepsilon}(R)$. Hence we may assume that $\xi_\Phi = \xi_{\Phi_0}$ in this set, and as a result

$$I^*(\mathcal{U}_{r,\varepsilon}(R), \xi_\Phi) = I^*(\mathcal{U}_{r,\varepsilon}(R), \xi_{\Phi_0})\,,$$

and the same holds for $\mathcal{U}_{r,\varepsilon}(R)$ replaced by $\mathcal{U}^b_{r,\varepsilon}(R) - \mathcal{U}^a_{r,\varepsilon}(R)$.

Here we shall be interested in a case where we may drop the "R" in $\mathcal{U}_{r,\varepsilon}(R)$. This indeed happens if $\mathcal{S}_\Phi$ has no critical point with critical value in $[a,b]$ outside $\mathcal{U}_{r,\varepsilon}(R)$, or else X_H has no 1-periodic orbit with action in $[a,b]$ outside $\{(q,p) \,|\, |p| \le R\}$. We may summarize this in

Proposition 2.3. *Assume $H(q,p) = H_0(q,p)$ for $|p| \le R$. Then*

$$\begin{aligned} I^*_{\mathbb{Z}/r}(\mathcal{U}^b_{r,\varepsilon}(R) - \mathcal{U}^a_{r,\varepsilon}(R), \xi_\Phi) =& I^*_{\mathbb{Z}/r}(\mathcal{U}^b_{r,\varepsilon}(R) - \mathcal{U}^a_{r,\varepsilon}(R), \xi_{\Phi_0}) \\ =& \Sigma_N(\Lambda^b_{r,\varepsilon}/\Lambda^a_{r,\varepsilon})\,. \end{aligned}$$

Moreover if the 1-periodic orbits of X_H outside $\{(q,p) \,|\, |p| \le R\}$ have their action outside $[a,b]$, then

$$I^*_{\mathbb{Z}/r}(\mathcal{U}^b_{r,\varepsilon} - \mathcal{U}^a_{r,\varepsilon}, \xi_\Phi) = \Sigma_N(\Lambda^b_{r,\varepsilon}/\Lambda^a_{r,\varepsilon})\,.$$

3. Periodic orbits on hypersurfaces of T^*N

The aim of this section in to use the results of the first two sections, in order to find a closed characteristic for a hypersurface Σ_0 in T^*N.

We shall assume that Σ_0 is the boundary of a **compact** submanifold U_0 of T^*N. Moreover, we assume that a neighbourhood of Σ_0 is foliated by hypersurfaces Σ_t, t in $[-\varepsilon, \varepsilon]$, and we set $\Sigma_t = \partial U_t$.

If for any positive δ, there exists t with $|t| < \delta$ such that Σ_t has a closed characteristic, we shall say that the foliation Σ_t has property (QE) (= quasi existence of periodic orbits). If this holds for any foliation of a neighbourhood of Σ_0, we shall say Σ_0 has property (QE).

Note that if Σ_0 is a hypersurface of contact type (see [53] or [41]), then (QE) for Σ_0 actually implies that Σ_0 itself carries a closed characteristic, since in this case we may choose the Σ_t to be conformally equivalent to Σ_0.

This means that if (QE) holds for all hypersurfaces in a certain class, then the Weinstein conjecture will hold for the contact hypersurfaces in the same class. For instance, in [23] we proved that if U_0 contains the zero section and N is compact, then (QE) and thus the Weinstein conjecture holds. This assumption is rather strange, since it was proved in [41] and [25] that (QE) holds in $\mathbb{R}^{2n}$ and as pointed out by M. Chaperon that this implies that if N is compact and has a Lagrange embedding in $\mathbb{R}^{2n}$, then (QE) holds with no need of such a condition in T^*N.

We shall prove in this section that (QE) holds provided the image of $\pi_1(U_0)$ in $\pi_1(N)$ is finite.

Theorem 3.1. *Assume the image of $\pi_1(U_0)$ in $\pi_1(N)$ to be finite. Then property (QE) holds in T^*N for $\Sigma_0 = \partial U_0$. In particular if $\pi_1(N)$ is finite, the Weinstein conjecture and (QE) hold in T^*N.*

Remark. In [36], Michael Struwe proved that in $\mathbb{R}^{2n}$, (QE) may be replaced by the stronger property (AE) (almost existence) : for almost all t in $[-\varepsilon, \varepsilon]$, Σ_t has a closed characteristic. In the sequel one may replace (QE) by (AE).

The rest of this section is devoted to the proof of 3.1.

We first assume that $\pi_1(N) = 0$ (hence N is orientable).

We may in fact assume N to be compact since U is, and (QE) only depends on U. So if we assume that $U \subset T^*V$ for V bounded in N, and we carefully choose V so that its double W is simply connected, then we may replace N by W.

As is now classical, the problem may be reduced to finding a 1-periodic solution for a Hamiltonian flow, defined as follows:

(i) $H = 0$ on $U_{-\varepsilon}$.

(ii) $H = a$ in $B(R) - U_\varepsilon$.

(iii) $H(\Sigma_t) = k(t)$ with k as in Figure 3.1.

(iv) $H(q,p) = g(|p|)$ with g such that g is increasing, convex, satisfies $g'(u)\cdot u - g(u) < 0$, $g(u) = a$ for u near R, and $\lim_{u\to\infty} g'(u) = g_\infty$.

FIGURE 3.1

We easily see from the above assumptions on H, that there are no periodic orbits with positive action outside $B(R)$.

We now assume that Σ_t has no closed characteristic, implying that H has no nonconstant periodic orbit inside $B(R)$.

Thus the 1-periodic orbits of X_H fall in three classes:

(I) The constants inside U_0, with action zero.

(II) The constants outside U_0, with negative action ($H > 0$ implies $\int p\dot{q} - H < 0$).

(III) The nonconstant periodic orbits outside $B(R)$, corresponding to closed geodesics of length bounded by some function γ of g_∞. Because $g'(u)\cdot u - g(u) < 0$, the action of such orbits is negative.

Let us point out that provided we sufficiently increase a, the constant g_∞ may be taken arbitrarily large.

Now, let $\mathcal{S}_\Phi$ be associated as in Section 1 to the flow of H. Then we know that for b large enough $H^*_{\mathbb{Z}/r}(\mathcal{U}^b_{r,\varepsilon}, \mathcal{U}^{-b}_{r,\varepsilon}) \simeq H^{*-d}_{\mathbb{Z}/r}(\widetilde{\Lambda}^b_{r,\varepsilon})$, since $\widetilde{\Lambda}^{-b}_{r,\varepsilon}$ is empty, $\mathcal{S}_\Phi$ being bounded from below.

On the other hand, in $\mathcal{U}^{\delta}_{r,\varepsilon} - \mathcal{U}^{-\delta}_{r,\varepsilon}$, for δ small enough, the only critical points of $\mathcal{S}_\Phi$ are given by the constants in U_0. A small perturbation easily yields that

$$H^*_{\mathbb{Z}/r}(\mathcal{U}^{\delta}_{r,\varepsilon}, \mathcal{U}^{-\delta}_{r,\varepsilon}) \simeq H^{*-d'}(U_0, \partial U_0) \otimes H^*(B\mathbb{Z}/r),$$

because the critical points considered are all fixed points for the $\mathbb{Z}/r$ action.

Now since $\mathcal{S}_\Phi$ has no critical value in $[\delta, b]$, we may identify $H^*_{\mathbb{Z}/r}(\mathcal{U}^{b}_{r,\varepsilon}, \mathcal{U}^{-b}_{r,\varepsilon})$ with $H^*_{\mathbb{Z}/r}(\mathcal{U}^{\delta}_{r,\varepsilon}, \mathcal{U}^{-b}_{r,\varepsilon})$, and write down the cohomology exact sequence of the triple $(\mathcal{U}^{\delta}_{r,\varepsilon}, \mathcal{U}^{-\delta}_{r,\varepsilon}, \mathcal{U}^{-b}_{r,\varepsilon})$, that is,

Remember that $\gamma = h_\infty$. Let μ_{2n} be the generator of $H^{2n}(U_0, \partial U_0)$. We want to consider the image of $\mu_{2n} \otimes 1$ in $H^*_{\mathbb{Z}/r}(\mathcal{U}^{\delta}_{r,\varepsilon}, \mathcal{U}^{-b}_{r,\varepsilon})$. This will lead to a contradiction.

First of all we set $H^*(B\mathbb{Z}/r, \mathbb{Z}/r) \simeq \Lambda(\alpha) \otimes \mathbb{Z}/r[u]$ where $\Lambda(\alpha)$ is $\mathbb{Z}[\alpha]/(r, \alpha^2)$, $\deg \alpha = 1$, and $\deg u = 2$.

We first have

Lemma 3.2. *Provided g_∞ is large enough, the image of $\mu_{2n} \otimes u^j$ in $H^*_{\mathbb{Z}/r}(\mathcal{U}^{\delta}_{r,\varepsilon}, \mathcal{U}^{-b}_{r,\varepsilon}) \simeq H^*_{\mathbb{Z}/r}(\widetilde{\Lambda}^{\gamma}_{r,\varepsilon})$ is zero for j large enough, independent of g_∞.*

We shall prove that for a class σ in $H^{2n}_{\mathbb{Z}/r}(\widetilde{\Lambda}^{\gamma}_{r,\varepsilon})$, provided σ has zero projection on $H^*_{\mathbb{Z}/r}(\mathrm{pt})$ (the projection $H^*_{\mathbb{Z}/r}(\widetilde{\Lambda}^{\gamma}_{r,\varepsilon}) \simeq H^*_{\mathbb{Z}/r}(\Lambda^{\gamma}N) \to H^*_{\mathbb{Z}/r}(\mathrm{pt})$ being induced by the inclusion $\mathrm{pt} \to \Lambda N$) we have that $\sigma u^j = 0$ for j large enough ($j \geq n+1$ is sufficient). This is based on the following theorem

Theorem (Goodwillie [18]). *Let ΛN be the free loop space of N. If σ in $H^k_{S^1}(\Lambda N, \mathbb{Q})$ has zero projection on $H^k_{S^1}(pt, \mathbb{Q})$, then we have $u^{k+1}\sigma = 0$.*

The Lemma may now be proved as follows.

Proof of Lemma. According to Appendix at the end of this paper, we have that for r large enough,

$$H^*_{\mathbb{Z}/r}(\Lambda^\gamma_{r,\varepsilon},\mathbb{Z}/r) \simeq H^*_{S^1}(\Lambda^\gamma N,\mathbb{Z}) \otimes H^*(S^1,\mathbb{Z}/r),$$
$$H^*_{S^1}(\Lambda^\gamma N,\mathbb{Q}) \simeq H^*_{S^1}(\Lambda^\gamma N,\mathbb{Z}) \otimes \mathbb{Q}\,.$$

If we choose r prime, large enough so that $H^*_{S^1}(\Lambda^\gamma N,\mathbb{Z})$ has no $\mathbb{Z}/r$ torsion, then any $\sigma \in H^*_{\mathbb{Z}/r}(\Lambda^\gamma_{r,\varepsilon},\mathbb{Z}/r)$ may be written as $\widetilde{\sigma}\otimes\beta$ with $\widetilde{\sigma} \in H^*_{S^1}(\Lambda^\gamma N,\mathbb{Z})$, $\beta \in H^*(S^1,\mathbb{Z}/r)$ and σu^j corresponds to $\widetilde{\sigma}u^j\otimes\beta$. According to Goodwillie's result, $\widetilde{\sigma}u^j$ is torsion, since it vanishes after tensoring with the rationals. Because there is no r-torsion, $\sigma u^j \sim \widetilde{\sigma}u^j\otimes\beta$ vanishes.

The proof of our lemma will be complete if we show that the image of μ_{2n} in $H^*_{\mathbb{Z}/r}(\Lambda^\gamma_{r,\varepsilon},\mathbb{Z}/r)$ has zero projection on the submodule $H^*_{\mathbb{Z}/r}(\mathrm{pt},\mathbb{Z}/r) \simeq H^*(B\mathbb{Z}/r)$; embedded in $H^*_{\mathbb{Z}/r}(\Lambda^\gamma_{r,\varepsilon})$ through the constant map $\widetilde{\Lambda}^\gamma_{r,\varepsilon} \to \mathrm{pt}$.

We first need to remind the reader of the definition of the localization. Let S be a multiplicative subset of a ring R (i.e., $a,b\in S \Rightarrow a\cdot b \in S$). Let M be an R module. We denote by $S^{-1}M$ the quotient module $S\times M/\sim$ where $(s_1,m_1)\sim(s_2,m_2)$ if and only if there exists t in S, such that $ts_1m_1 = ts_2m_2$. Then $S^{-1}M$ is an $S^{-1}R$ module, called the localization of M (at S). Note that localization commutes with exact sequences (i.e., it is an exact functor).

If $S = H^*(BG) - H^0(BG)$, and X, Y are G spaces, with fixed point sets F_X, F_Y, then given an equivariant map $f : X \to Y$ we may consider $f^*_G : H^*_G(Y) \to H^*_G(X)$. Thus a classical result states that $S^{-1}H^*_G(Y)$, $S^{-1}H^*_G(X)$ coincide with $H^*(F_X)\otimes S^{-1}H^*(BG)$ and $H^*(F_Y)\otimes S^{-1}H^*(BG)$, and $S^{-1}f^*_G$ is induced by $f^*_{|F}$ where $f_{|F}$ is the restriction of f to the fixed point sets (see [10]).

In our case the map

$$H^*(U_0,\partial U_0)\otimes H^*(B\mathbb{Z}/r) \to H^*_{\mathbb{Z}/r}(\widetilde{\Lambda}^\gamma_{r,\varepsilon})$$

localizes to a map induced by the obvious map

$$H^*(U_0,\partial U_0) \to H^*(B(R),\partial B(R)) \simeq H^{*-n}(N)\,.$$

This map sends μ_{2n} to λ_n the generator of $H^n(N)$. Because N is compact, λ_n is nonzero.

Consider the diagram

this is the localization of the diagram:

In the above diagrams

$$\mathcal{D} = \{(p, P) | q = q_0 \ , \ |P| \leq R\},$$
$$\partial\mathcal{D} = \{(q, P) | q = q_0 \ , \ |P| = R\},$$

and $D^n, \partial D^n$ are the fixed point sets of $\mathcal{D}, \partial\mathcal{D}$ for the $\mathbb{Z}/r$ action. q.e.d.

Now $H^*(U_0, \partial U_0) \to H^*(D^n, \partial D^n) \simeq H^{*-n}(\mathrm{pt})$ maps μ_{2n} to zero. Hence the image of λ_n in $H^*(B\mathbb{Z}/r)$ is zero. We may thus apply Lemma 3.3 to get a contradiction : the image of $\mu_{2n} u^k$ is zero but it is nonzero after localizing. This concludes the proof of Theorem 3.1. q.e.d.

Remark. We assumed here that

$$H^p(\Lambda L) = \lim_{\leftarrow} \quad H^p(\Lambda^c L).$$

According to [20, p. 410], it is enough to check that the inverse limit of cochain complexes which we are considering satisfies the Mittag-Leffler condition, that is:

1) $\Omega^p(\Lambda^c L)$ whose inverse limit is $\Omega^p(\Lambda L)$ is such that the image of $\Omega^p(\Lambda L)$ in $\Omega^p(\Lambda^c L)$ does not depend on c, for c large enough.

2) For each fixed a, the image of $H^p(\Lambda^c L)$ in $H^p(\Lambda^a L)$ does not depend on c, for c large enough (depending of course on a).

The first property is obvious, since any form on $\Lambda^c L$ extending to $\Lambda^{c'} L$ actually extends to ΛL. Since the $H^p(\Lambda^a L)$ are finite dimensional for each a, the second property is trivially satisfied.

We also have used that $\Lambda^{\gamma}_{r,\varepsilon}$ has the $\mathbb{Z}/r$ equivariant homotopy type of Λ^{γ} for which we refer to [31].

Remark. The above proof has in fact a wider range than the theorem. Indeed Goodwillie's theorem states that the localization of $H^*_{S^1}(\Lambda N)$ only depends on $\pi_1(N)$. Now we only needed that the inclusion $N \to \Lambda N$ induces a map $H^*_{S^1}(\Lambda N) \to H^*(N) \otimes H^*(BS^1)$ which does not contain $H^n(N) \otimes H^*(BS^1)$ in its image.

In particular this will be satisfied if $\pi_1(N)$ is abelian of rank $< n = \dim N$, because we may compute $H^*_{S^1}(\Lambda N)$ for $N = T^k \times \mathbb{R}^{n-k}$. We thus get

Proposition 3.3. *Assume that the image of $\pi_1(U_0)$ in $\pi_1(N)$ has a finite index subgroup which is abelian of rank strictly less than n. Then (QE) holds for $\Sigma_0 = \partial U_0$.*

4. The case of a Lagrange submanifold in T^*N

Let L be a Lagrange submanifold in T^*N. Using Weinstein's theorem by a symplectic map we may identify a neighbourhood of L with $B_\rho = \{(q,p) \in T^*L \mid |p| \leq \rho\}$.

Let H be a Hamiltonian on T^*N such that:

- in $B_\rho, H(z) = h(|p|)$, h is convex and will be made precise later,
- in $D_R - B_\rho$, H is constant equal to a,
- outside D_R, $H(z) = \gamma(H_0)$ with γ convex and $\lim\limits_{t\to\infty} \gamma'(t) = \gamma_\infty$.

The reader will be careful to distinguish between (q,p) coordinates in B_ρ and (q,p) coordinates in T^*N.

We consider the map Φ_1 associated to the flow φ_u of H, as in the previous sections, and $\Gamma(\Phi_1)$ its graph in $(\overline{T^*N} \times T^*N)^r$.

In the subset $(\overline{B_\rho} \times B_\rho)^r$, $\Gamma(\Phi_1)$ coincides with $\Gamma(\Phi_2)$, where Φ_2 is associated to the Hamiltonian K defined on T^*L by $K(q,p) = h(|p|)$.

Note that $\Gamma(\Phi_2) \subset (\overline{T^*L} \times T^*L)^r$. We may then identify its intersection with $(\overline{B_\rho} \times B_\rho)^r$ to a subset of $(\overline{T^*N} \times T^*N)^r$, using Weinstein's theorem.

One may then think that this implies $\mathcal{S}_{\Phi_1} = \mathcal{S}_{\Phi_2}$ on some subset. But this is wrong for the following reason : the maps i_1 and i_2 obtained by applying Lemma 1.1 to T^*N and T^*L define vertical foliations in $\mathcal{U}_\varepsilon$ which do not coincide (see Figure 4.1).

FIGURE 4.1

FIGURE 4.2

The goal of this section is to analyze the contribution of the neighbourhood of L to the variational picture of $\mathcal{S}_\Phi$. In the sequel we shall assume r and ε to be fixed. Before stating the main proposition of this section, we remind the reader that any loop in L has a Liouville number $\ell(\gamma)$ and a Maslov number $m(\gamma)$, and these numbers only depend on the free homotopy class of γ. Set $\widetilde{\Lambda}_2^b = \widetilde{\Lambda}_{r,\varepsilon}^b L$, then $\widetilde{\Lambda}_2$ may be identified with a subset of the free loop space of L. In particular, to every connected component of $\widetilde{\Lambda}_2^b$, we may associate its Liouville and Maslov numbers ℓ and m.

We may now state

Proposition 4.1. *Let $a < b$ be such that φ_u has no periodic orbit with action in $[a, b]$ outside B_ρ. Then there exist $\mathbb{Z}/r$ bundles M_1, M_2*

such that

$$\Sigma_{M_2} I^*_{\mathbb{Z}/r}(\mathcal{U}^{1,b} - \mathcal{U}^{1,a}) = \Sigma_{M_1} I^*_{\mathbb{Z}/r}(\mathcal{U}^{2,b+\ell}_\rho - \mathcal{U}^{2,a+\ell}_\rho)$$
$$= \Sigma_{M_1 \oplus N}(\widetilde{\Lambda}_2^{b+\ell}/\widetilde{\Lambda}_2^{a+\ell}),$$

where ℓ is the Liouville number of the connected component of $\widetilde{\Lambda}_2$ which we are in, M_1, M_2 are $\mathbb{Z}/r$ vector bundles such that $rk(M_1) - rk(M_2) = m(\gamma)$ is the Maslov class of the connected component, and N is the bundle from Lemma 1.5.

This proposition tells us that up to a suspension and shift in levels, the change in the topology of the sets $\{\mathcal{S}_{\Phi_1} \leq \nu\}$, for ν in $[a, b]$, is the same as the change in topology of the sets $\{\mathcal{S}_{\Phi_2} \leq \nu\}$. Nothing changes outside B_ρ, and what happens there has been described in Section 2.

The proof of the above theorem will take up most of this section.

We first consider the following abstract situation.

Let W be a manifold with boundary ∂W, χ_t a Hamiltonian isotopy of T^*W, such that $\chi_t = \mathrm{Id}$ on the union of 0_W and a neighbourhood of $T^*_{\partial W}W$, and L_0 the graph of df_0 for $f_0 : W \to \mathbb{R}$ a smooth function.

We denote by L_t the Lagrange submanifold $\chi_t(L_0)$.

Lemma 4.2. *There are g.f.q.i. S_0 and S_1 of L_0 and L_1 which are diffeomorphic, i.e.; $S_0 \circ G = S_1$, with G a diffeomorphism of $W \times \mathbb{R}^k$. The diffeomorphism G may be assumed to be the identity over ∂W, but it is not, in general, fiber preserving.*

Proof. Let S_t be a g.f.q.i. of L_t, we may assume that $S_t \equiv S_0$ over a neighbourhood of $\partial W \times \mathbb{R}^k$, using the fact that $W \cup \partial W \times [0, 1]$ is a deformation retract of $W \times [0, 1]$. We then look for the flow G_t of a vector field X_t such that $S_t \circ G_t = S_0$. By Moser's lemma, this is equivalent to solving $dS_t \cdot X_t = -\frac{\partial}{\partial t} S_t$.

Whenever $dS_t(x, \xi) \neq 0$, the above equation is trivial to solve, so we only have to consider the neighbourhood of critical points of S_t. Such points correspond to the intersection points of L_t with the zero section, and the order of the critical point corresponds to the order of the contact of L_t with 0_W. Now since χ_t is the identity on 0_W, the points of $L_t \cap 0_W$ and their order of contact do not depend on t.

Assume now that (x_0, ξ_0) is a critical point of S_t, and that $S_t(x_0, \xi_0)$ does not depend on t. Then $\frac{\partial}{\partial t} S_t(x_0, \xi_0)$ must vanish to the same order as $dS_t(x_0, \xi_0)$, so we may solve $dS_t(x, \xi) X_t(x, \xi) = -\frac{\partial}{\partial t} S_t(x, \xi)$ near (x_0, ξ_0).

To conclude our proof, we just have to show that $t \mapsto S_t(x_0, \xi_0)$ does not depend on t. This is a priori not correct, but we have that for (x_0, ξ_0), (x_1, ξ_1) two critical points of S_t, $S_t(x_1, \xi_1) - S_t(x_0, \xi_0)$ does not depend on t. Indeed, this quantity is given by $\int_{\gamma_t} pdx$, where γ_t is a path in L_t connecting $(x_0, 0)$ to $(x_1, 0)$ (cf [40]).

If we take $\gamma_t = \chi_t(\gamma_0)$, the fact that χ_t is symplectic and preserves 0_W, and that $\int_{\gamma_t} pdx$ may be identified with the symplectic area of a disk with boundary $\gamma_t \circ \overline{\gamma_t}^{-1}$, where $\overline{\gamma}_t$ is a path in 0_W connecting $(x_0, 0)$ and $(x_1, 0)$, readily imply that $\int_{\gamma_t} pdx$ does not depend on t.

As a result we may replace S_t by $S_t(x, \xi) + c(t)$ such that the critical values of this function do not depend on t, and thus conclude our proof.

q.e.d.

Remark. The assumption that L_0 in the graph of df_0 may be removed. We only need that L_0 has a g.f.q.i.

By uniqueness theorem for generating functions quadratic at infinity, abbreviated as g.f.q.i. ([45], Section 1) we have that $S_0 = (f_0 + Q_0) \circ F_0 + c_0$ where F_0 is a fiber preserving diffeomorphism, Q_0 a nondegenerate quadratic form on $\mathbb{R}^k$ and c_0 some constant (actually this is only true provided we also add a nondegenerate quadratic form in new variables to S_0, that we may always do).

Now assume that L_1 is the graph of df_1. Then $S_1 = (f_1 + Q_1) \circ F_1 + c_1$.

Set $E = W \times \mathbb{R}^k$,

$$
\begin{aligned}
E_i^a &= \{(x, \xi) \in E | S_i(x, \xi) \leq a\} , \\
W_i^a &= \{x \in W | f_i(x) \leq a\} .
\end{aligned}
$$

By the above argument, $E_i^b / E_i^a = \Sigma^{k_i}(W_i^{b-c_i} / W_i^{a-c_i})$ and from Lemma 4.2, we infer that $E_1^a / E_1^b \simeq E_0^a / E_0^b$. We may thus state

Corollary 4.3. *Under the assumptions of Lemma 4.2, we have*

$$
\Sigma^{k_1}(W_1^{b-c_1} / W_1^{a-c_1}) = \Sigma^{k_0}(W_0^{b-c_0} / W_0^{a-c_0}) .
$$

Similarly if η_i is a pseudogradient vector field for f_i such that $\eta_1 = \eta_0$ near ∂W, then we have

Corollary 4.4.

$$\Sigma^{k_1} I^*(W_1^{b-c_1} - W_1^{a-c_1}, \eta_1) = \Sigma^{k_0} I^*(W_0^{b-c_0} - W_0^{a-c_0}, \eta_0).$$

The proof is the same.

The reader may wonder how this is related with the proof of 4.1. To see this, one should think of L_t as a fixed Lagrange submanifold, while the vertical foliation of T^*W is moved by χ_t^{-1}. The above corollary thus holds if f_0 and f_1 are generating functions of L_0 with respect to two distinct but isotopic vertical foliations, that coincide over ∂W (see Figure 4.2).

Except for this last condition (and for the fact that we did not deal with the $\mathbb{Z}/r$ action) this is exactly our situation : $L_0 = \Gamma(\Phi_1) = \Gamma(\Phi_2)$, and the vertical foliations are determined by i_1 and i_2. These foliation are Hamiltonianly isotopic, at least in some neighbourhood of the diagonal. Indeed it is a general fact, that any two vertical foliations are Hamiltonianly isotopic near the zero section. This just means that any two tubular neighbourhoods of the zero section are isotopic. But $\Gamma(\Phi_1)$ and $\Gamma(\Phi_2)$ are contained in a neighbourhood of the diagonal.

We now go back to our abstract situation, and shall consider the case where χ_t does not coincide with the identity near $T^*_{\partial W}W$.

Let $U \subset W$ be some open subset, and assume that $\chi_t(L_0 \cap T^*U) \cap T^*_{\partial W}W = \phi$. Then we may modify our isotopy χ_t into $\widetilde{\chi}_t$ such that $\widetilde{\chi}_t = \chi_t$. The image by $\widetilde{\chi}_1$ of L_0 coincides over U with $\chi_1(L_0)$, hence if $\widetilde{S}_1$ is the g.f.q.i. of $\widetilde{\chi}_1(L_0)$ on W, we may assume that $\widetilde{S}_1(x,\xi) = f_1(x) + Q_1(\xi)$ over U.

If U is such that for some pseudogradient vector field η_1 of f_1, U is an isolating block for the maximal invariant set of η_1 in W, then $E_1^b/E_1^a \simeq \Sigma^{k_1} W_1^b/W_1^a \simeq \Sigma^{k_1} U_1^b/U_1^a$.

Since $E_1^b/E_1^a \simeq E_0^b/E_0^a$, we conclude $\Sigma^{k_1} I^*(U_1^b - U_1^a, \eta_1) = \Sigma^{k_0} I^*(U_0^b - U_0^a, \eta_0)$. This may be summarized as

Proposition 4.5. *Let us assume that the following hold:*

(1) $\chi_t(L_0 \cap T^*U) \cap T^*_{\partial W}W = \phi$.

(2) *U is an isolating block for the maximal invariant set of a pseudo-gradient η_1 of f_1 in W.*

Then $\Sigma^{k_0} I^*(W_0^b - W_0^a, \widetilde{\eta}) = \Sigma^{k_1} I^*(U^{b-c} - U_1^{a-c}, \eta_1)$ *where* $\widetilde{\eta}_1 = \eta_1$ *on* ∂W.

Now we apply this to $W = \mathcal{U}^1(R)$, $U = \mathcal{U}^2_\rho$, $L_0 = i_1^{-1}(\Gamma_{\Phi_1}) = i_2^{-1}(\Gamma_{\Phi_2})$, and χ_t has been described previously.

Then (1) is satisfied since L_0 is arbitrarily close to the zero section (for the sup norm) and (2) is satisfied according to Proposition 2.3.

Now Proposition 4.1 follows from 4.5.

5. The variational picture

We start again with the situation of Section 4, but we make our choice of H more precise.

We assume the following:

(1) Outside D_R, $p\frac{\partial H}{\partial p} - H < 0$. This is possible provided R/a is small enough (see Section 3).

(2) h is as on Figure 5.1, depending on the parameters σ, δ, c, a (c is the slope of the dotted line) and we have

- **(i)** $|h'(u)| \leq c$,
- **(ii)** $uh'(u) - h(u) \leq \delta$,
- **(iii)** $h(u) = a$ for $u > \rho$.

The critical points of $\mathcal{S}_{\Phi_1}$ corresponding to the periodic orbits of the flow may be gathered in families as follows:

FIGURE 5.1

(I) Periodic orbits in B_σ, corresponding to closed geodesics of L with length less than c.

(II) Periodic orbits in $B_\rho - B_\sigma$.

(III) Constants in $D_R - B_\rho$.

(IV) Periodic orbits outside D_R.

To describe the associated critical values, we associate to a loop (q,p) in B_ρ the value $\langle \lambda, q\rangle$ of the Liouville class on the projection of the loop on L.

Now the critical values are given

$$
\begin{array}{lll}
\text{(I)} & |p|\, h'(|p|) - h(|p|) + \langle \lambda, q\rangle & \text{in } [0,\delta] + \langle \lambda, q\rangle\,, \\
\text{(II)} & |p|\, h'(|p|) - h(|p|) + \langle \lambda, q\rangle & \text{in }]-\infty, 0[+ \langle \lambda, q\rangle\,, \\
\text{(III)} & & \text{in }]-\infty, -a[, \\
\text{(IV)} & & \text{in }]-\infty, 0[.
\end{array}
$$

We see that if $[\lambda] = 0$, the " type (I)" orbits are above the other critical points.

Thus, there is a map

$$I^*(\mathcal{U}^{1,\delta} - \mathcal{U}^{1,\beta}, \xi_{\Phi_1}) \to I^*(\mathcal{U}^{1,\delta} - \mathcal{U}^{1,\alpha}, \xi_{\Phi_1}),$$

where α is less than any critical value, and $\beta < 0$ is greater than the negative critical values.

Now, according to Proposition 4.1, the left-hand side may be identified with

$$H^*(I^*(\mathcal{U}^{2,\delta} - \mathcal{U}^{2,\beta}, \xi_{\Phi_2})) \simeq \bigoplus_c H^{*-k-m(c)}(I^*(\Lambda_c^\delta L)),$$

where $c \in \pi_0(\Lambda L)$, and $\Lambda_c L$ is the connected component of c. $\Lambda_0 L$ will be the connected component of constant paths, and k is equal to $rk(M_1)$ as in 4.1.

On the other hand, since $\mathcal{S}_{\Phi_1}$ has no critical value outside the interval $[\alpha, \delta]$, we have

$$H^*(I^*(\mathcal{U}^{1,\delta} - \mathcal{U}^{1,\beta}, \xi_{\Phi_1})) \simeq \bigoplus_{c_1} H^{*-k}(\Lambda_{c_1} N).$$

Thus we get a map, denoted by $(\Lambda j)!$,

$$\bigoplus H^{*-m(c)}(\Lambda_c L) \to \bigoplus H^*(\Lambda_c N)\,.$$

For $c = 0$, this restricts to a map

$$H^*(\Lambda_0 L) \to H^*(\Lambda_0 N)\,.$$

The goal of this section is to prove the following.

Theorem 5.1. *If L has an exact Lagrange embedding into T^*N, there is a map*

$$(\Lambda j)! : H^*(\Lambda_0 L) \to H^*(\Lambda_0 N)$$

satisfying the following property:

if $(\Lambda j)^*$ is the natural map $H^*(\Lambda_0 N) \to H^*(\Lambda_0 L)$, induced by the inclusion $\Lambda j : \Lambda_0 L \to \Lambda_0 N$, then

$$(\Lambda j)!(\xi \cup (\Lambda j)^*(\eta)) = (\Lambda j)!(\xi) \cup \eta.$$

The above statements still holds if we replace cohomology by S^1-equivariant cohomology.

Remark. It is crucial to have an exact Lagrange embedding as exactness will prevent interaction between the different factors in $H^*(\mathcal{U}^{2,\delta} - \mathcal{U}^{2,\beta}, \xi_{\Phi_2})$. Indeed such interaction would mean that there are trajectories of the gradient flow connecting $H^*(\Lambda_0 L)$ and $H^*(\Lambda_c L)$ for some $c \neq 0$. This will then imply that the isolating blocks are one below the other, hence $\langle \lambda, c \rangle \neq 0$.

Remark. If $f : M \to N$ is a map between manifolds, we may define a cohomological push-forward $f^! : H^*(M) \to H^{*+k}(N)$ ($k = \dim N - \dim M$) satisfying $f^!(x \cup f^*(y)) = f^!(x) \cup y$. We may write f (up to homotopy) as the composition of an embedding and a fibration. Then, $f^!$ corresponds to integration over the fibers in the case of a fibration. If f is an embedding, the Thom isomorphism sends $H^*(M)$ to $H_c^{*+k}(U)$ where U is a tubular neighbourhood of M in N, while there is an obvious map $H_c^{*+k}(U) \to H^{*+k}(N)$. Composing these two maps yields $f^!$.

It seems impossible to extend the definition of f to maps like Λj while preserving nontriviality and the identity $f^!(x \cup f^*(y)) = f^!(x \cup y)$.

On the submodule $H^*(L_i)$ of $H^*(\Lambda_0 L_i)$ ($i = 1, 2$) we would like that $(\Lambda j)!$ and $j!$ coincide. We shall see in Section 6 that this is true if j

is an exact Lagrange embedding. In fact we shall see shortly that we derive the non existence of certain exact Lagrange embeddings from the algebraic impossibility of finding $(\Lambda j)!$ with these property.

If $j : L \to N$ is a map, we denote by Λj the induced map $\Lambda L \to \Lambda N$.

Theorem 5.1 is based on the following:

Proposition 5.2. *With the assumptions of Theorem 5.1, there is for each positive c a map $(\Lambda^c j)! H^*(\Lambda_0^c L) \to H^*(\Lambda_0^* N)$ satisfying*

$$(\Lambda j^c)!(\xi \cup (\Lambda j^c)^* \eta) = (\Lambda j^c)!(\xi) \cup \eta.$$

Moreover the same holds if we replace H^* by $H^*_{\mathbb{Z}/r}$.

The map $(\Lambda j^c)!$ has been constructed above. We still have to prove that $(\Lambda j)!$ satisfies the required identity. In fact, in our framework, the map $(\Lambda j)!$ is obvious, while we have to construct $(\Lambda j)^*$.

Let us transpose once more our situation in an abstract setting. Let E be a vector bundle over a space B, f a function on E, and ξ a pseudogradient vector field for f.

If B (hence E) has a boundary, we assume that the flow of ξ preserves E.

Now if f is quadratic negative definite outside a compact set, we have $H^*(E^c, E^a) \simeq H^{*-i}(B)$ where i is the index of the quadratic form. The above isomorphism is induced by the Thom isomorphism of E.

We now assume that in $E^c - E^b$, there is a normally hyperbolic manifold, P, so that

$$H^*(E^c, E^b) \simeq H^{*-j}(P),$$

where j is the codimension of P (P is a repeller). Again the isomorphism is the Thom isomorphism of the normal bundle of P.

We may now consider the inclusion

$$\alpha : (E^c, E^a) \to (E^c, E^b)$$

inducing a diagram:

$$\begin{tikzcd} H^*(E^c, E^a) & H^*(E^c, E^b) \arrow[l, "\alpha^*"'] \\ H^{*-i}(B) \arrow[u, "\sim"] & H^{*-j}(P) \arrow[u, "\sim"] \arrow[l, dotted, "\beta!"] \end{tikzcd}$$

We denote the dotted map by $\beta!$. We now prove

Lemma 5.3. *Let γ be the restriction to P of the projection $E \xrightarrow{\pi} B$. Then we have for any w in $H^*(P)$, y in $H^*(B)$,*

$$\beta!(w \cup \gamma^*(y)) = \beta!(w) \cup y\,.$$

Remark. We also have the following result. Let z in $H^*(B)$. Then

$$\beta!(\langle \gamma^*(y), w, \gamma^*(z)\rangle) = \langle y, \beta!(w), z\rangle\,,$$

where $\langle a, b, c\rangle$ denotes Massey's triple product. Similar results hold for generalized Massey products (see [30, pp.290-297] and [27]).

Proof. The map γ defines an $H^*(B)$-module structure on $H^*(P)$. This structure is the same one induced by the natural map $H^*(E) \to H^*(P)$; note that $H^*(E) \approx H^*(B)$. Now all the maps in the above diagram are $H^*(E)$-module homomorphisms. It is clear for α^*, and also for both Thom isomorphisms, since they are restrictions of the Thom isomorphisms associated to the inclusions $B \hookrightarrow E$ and $P \hookrightarrow E$.

From this we conclude that $\beta!$ must be an $H^*(B)$ module homomorphism. q.e.d.

Proof of 5.2. We apply Lemma 5.3 to $E = \mathcal{V}^{1,\delta}$, and get

$$B = \Lambda^\delta N\,,\ P = \Lambda^\delta L\,,\ c = \delta\,, b = \alpha\,, a = \beta\,.$$

This tells us that the map $(\Lambda^\delta j)!$ is a $(\Lambda^\delta j)^*$ module homomorphism, that is what we wanted to prove. q.e.d.

Finally to prove 5.1, we have to show that all the maps $(\Lambda^\delta j)^*$ which we obtained are somehow compatible as we change H.

Let us consider a Hamiltonian H_1 satisfying the assumptions of Section 4, as well as (1) and (2) of this section. In particular outside D_R, we have that $H_1 = \gamma_1 H_0$.

Now we set H_2 to be equal to H_1 in $D_{R'/2}$ for $\frac{R'}{2} > R$, and $H_2(z) = \widetilde{\gamma}_2(H_0)$ where $\widetilde{\gamma}_2$ is a function such that $\widetilde{\gamma}_2$ is convex, with derivative increasing from γ_1 to γ_2. We shall assume that $\widetilde{\gamma}_2(H_0) = \gamma_2 \cdot H_0$ outside $D_{R'}$.

Now as we go from H_1 to H_2, we introduce new periodic orbits in $D_{R'} - D_{R'/2}$, corresponding to closed geodesics with length in $[\gamma_1, \gamma_2]$.

We may choose $\widetilde{\gamma}_2$ (and R') so that all these critical values lie above those of the critical points inside $B_{R'/2}$. Indeed, outside B_R, $H_2(q,p) =$

$h_2(|p|)$, and the critical value corresponding to a periodic orbit on $|p| = t$ of length $h'(t)$ is given by $h'(t)t - h(t)$, that is the ordinate of the intersection of the tangent to the graph of h at t. Now we assume that for $R \le t \le \frac{R'}{2}$, $h'(t) = \gamma_1$ is not the length of any geodesic.

FIGURE 5.2

We may in fact assume that this still holds for γ in $]\gamma_1 - \varepsilon, \gamma_1 + \varepsilon[$ so that we only have to consider the quantities $h'(t) \cdot t - h(t)$ for $h'(t) \ge \gamma_1 + \varepsilon$. If h' goes from γ_1 to γ_2 as t goes from t_0 to $t_0 + \tau$, then for $h'(t) \ge \gamma_1 + \varepsilon$, we have $h(t) \le a + \gamma_1 t_0 + (\gamma_1 + \varepsilon)(t - t_0)$ and hence $h'(t) \cdot t - h(t) \ge -a + \varepsilon t_0$. For t_0 large enough, this will be arbitrarily large.

Now, let b separate the two families of critical values which associated to geodesics inside and outside $D_{R'/2}$. We thus have, according to Proposition 2.3,

$$I^*\Big(\mathcal{U}^{1,b}(\frac{R'}{2}) - \mathcal{U}^{1,a}(\frac{R'}{2}), \xi_\Phi\Big) \simeq \Sigma_N \Lambda^b N,$$
$$I^*\Big(\mathcal{U}^{1,c}(R') - \mathcal{U}^{1,a}(R'), \xi_\Phi\Big) \simeq \Sigma_N \Lambda^c N\,.$$

Now the map

$$\mathcal{U}^{1,b}/\mathcal{U}^{1,a} \to \mathcal{U}^{1,c}/\mathcal{U}^{1,a}$$

induces a map $\Sigma_N \Lambda^b L \to \Sigma_N \Lambda^c L$.

From the definitions it is easy to check that this map is in fact induced by the inclusion $\Lambda i_b^c : \Lambda^b L \to \Lambda^c L$.

On the other hand, we have the maps constructed in 5.2,

$$H^*(\Lambda^b L) \to H^*(\Lambda^\delta N)$$

and

$$H^*(\Lambda^c L) \to H^*(\Lambda^\delta N)\,.$$

We claim that the following diagram is commutative:

$$\begin{tikzcd}
H^*(\Lambda^b L) & & H^*(\Lambda^c L) \arrow[ll, "(\Lambda i_b^c)^*"'] \\
& H^*(\Lambda^\delta N) \arrow[ul, "(\Lambda^b j)!"] \arrow[ur, "(\Lambda^c j)!"'] &
\end{tikzcd}$$

The map $\Lambda^b L \to \Lambda^\delta N$ is obtained using the Hamiltonian H_1, while the map $\Lambda^c L \to \Lambda^\delta N$ is obtained using H_3, where H_3 looks like H_2, except for the fact that all the new critical values are below zero; this is easy to achieve; see Figure 5.2.

We shall denote by $\mathcal{U}_i^c$ $(i = 1, 2, 3)$ the sets $\mathcal{U}^{1,c}$ associated to H_i. We claim that the following canonical isomorphisms hold:

$$\mathcal{U}_2^b/\mathcal{U}_2^a \simeq \mathcal{U}_1^b/\mathcal{U}_1^a,$$

and

$$\mathcal{U}_2^c/\mathcal{U}_2^a \simeq \mathcal{U}_3^c/\mathcal{U}_3^a \quad \text{for } a \text{ small enough},$$
$$\mathcal{U}_3^c/\mathcal{U}_3^{-\delta} \simeq \mathcal{U}_1^c/\mathcal{U}_1^{-\delta} \simeq \mathcal{U}_1^b/\mathcal{U}_1^{-\delta}\,.$$

Indeed, the first isomorphism follows from 2.3 since $H_1 = H_2$ in D_R, and the periodic orbits of H_2 (and of any linear interpolation between H_1 and H_2) outside D_R have action above b.

As for the second one, the same argument applies, except for the fact that the assumptions of 2.3 will be satisfied only if a is small enough, since we deform H_2 to H_3, the action of the periodic orbits outside $B_{R'/2}$ moves down from $[b, c]$ to a compact subset of $]-\infty, 0[$. The maps $(\Lambda^b j)!$, $(\Lambda^c j)!$, $(\Lambda i_b^c)^*$ are induced by the maps :

$$\mathcal{U}_1^b/\mathcal{U}_1^a \to \mathcal{U}_1^b/\mathcal{U}_1^{-\delta} \quad \text{with } \delta > 0 \text{ small, and a small enough},$$
$$\mathcal{U}_3^c/\mathcal{U}_3^a \to \mathcal{U}_3^c/\mathcal{U}_3^{-\delta},$$
$$\mathcal{U}_2^b/\mathcal{U}_2^a \to \mathcal{U}_2^c/\mathcal{U}_2^a\,.$$

As we identify $\mathcal{U}_2^b/\mathcal{U}_2^a$ with $\mathcal{U}_1^b/\mathcal{U}_1^a$ and $\mathcal{U}_2^c/\mathcal{U}_2^a$ with $\mathcal{U}_3^c/\mathcal{U}_3^a$, we may compose the last two maps to a map $\mathcal{U}_1^b/\mathcal{U}_1^a \to \mathcal{U}_3^c/\mathcal{U}_3^{-\delta}$. Finally as we identify $\mathcal{U}_3^c/\mathcal{U}_3^{-\delta}$ with $\mathcal{U}_1^b/\mathcal{U}_1^{-\delta}$, we get the map $\mathcal{U}_1^b/\mathcal{U}_1^a \to \mathcal{U}_1^b/\mathcal{U}_1^{-\delta}$ inducing $\Lambda^b j$, and thus prove $(\Lambda i_b^c)^*(\Lambda_j^c)! = (\Lambda^b j)!$.

As a result we get a map

$$H^*(\Lambda^b L) \to H^*(\Lambda N)\,.$$

Using a similar argument, we may replace $\Lambda^b L$ by ΛL, and hence get a map $(\Lambda j)!$. That this map is a $H^*(\Lambda N)$ module homomorphism (using $(\Lambda j)^*$) follows from the fact that this is true for $(\Lambda^c j)!$. q.e.d.

We conclude this section with a remark. Let $j : L \to N$. Then we have a map $j! : H^*(L) \to H^*(N)$.

On the other hand we have the maps $c_i : L_i \to \Lambda L_i$ associating to a point in L_i the constant loop at this point. We claim

Proposition 5.4. *The following diagram is commutative*

$$\begin{tikzcd} H^*(L) \arrow[r, "j!"] & H^*(N) \\ H^*(\Lambda L) \arrow[u, "c_2^*"'] \arrow[r, "(\Lambda j)!"] & H^*(\Lambda N) \arrow[u, "c_1^*"'] \end{tikzcd}$$

There is also a commutative diagram for S^1 equivariant cohomology, where $H^*(\Lambda L_i)$ is replaced by $H^*_{S^1}(\Lambda L_i)$, and $H^*(L_i)$ by

$$H^*(L_i) \otimes H^*(BS^1).$$

Proof. This follows immediately from the equality

$$(\Lambda i_b^c)^*(\Lambda^c j)! = (\Lambda^b j)!\,.$$

If H_1 has small C^2 norm, there are no periodic orbits but the constant ones, and then $(\Lambda^b j)! \simeq j!$ and $(\Lambda i_b^c)^*$ corresponds to c_1^*. Thus, associated to H_1, we have a commutative triangle:

$$\begin{tikzcd} H^*(L) \arrow[r] \arrow[dr, dotted] & H^*(N) \\ & H^*(\Lambda N) \arrow[u, "c_1^*"'] \end{tikzcd}$$

The other triangle is obtained similarly. q.e.d.

6. Loops spaces, iterations, applications

In the previous section we proved that if $L_2 \to T^*L_1$ is an exact Lagrange embedding, there is a map $(\Lambda j)! : H^*(\Lambda L_2) \to H^*(\Lambda L_1)$ such that the following hold:

(i) $(\Lambda j)!(x \cup (\Lambda j)^*(y) = (\Lambda j)!(x) \cup y$.

(ii) If c_i is the natural embedding from $L_i \to \Lambda L_i$ induced by the inclusion of constants, we have $j!c_2^* = c_1^*(\Lambda j)!$.

The main difficulties in finding obstructions to the existence of such a map are due to the fact that for a general manifold M, very little is known about the algebra $H^*(\Lambda M)$. The theory of minimal models, due to Sullivan, only deals with rational cohomology of simply connected spaces (or at least with nilpotent fundamental group). However, even in this case, very few general properties of the above cohomology ring are known.

One of the results that we shall be using is due to Burghelea, Fiodorowicz and Gajda, it describes how the map $e(k)$ induced by iterating k times a loop, acts on this cohomology.

Theorem. ([6]) *All cohomologies being intended with rational coefficients, we have that $H^*(\Lambda M)$ decomposes, under the action of $e(k)$ into subspaces $H^{*,i}(\Lambda M)$, eigenspaces for the eigenvalue k^i. These subspaces do not depend on k. Moreover we have $H^{*,0}(\Lambda M) = e^*(M)$, where e is the evaluation map.*

The aim of this section is to prove:

Proposition 6.1. *All cohomologies being intended with rational coefficients, we have*

$$(\Lambda j)!e^* = e^*j!$$

We shall need the following lemma.

Lemma 6.2. *Let $S : M \times N \times \mathbb{R}^k \to \mathbb{R}$ be a g.f.q.i. for $L \subset T^*(M \times N)$. Let $L_M = L \cap \{(x, y, \xi, 0)\}/\{y\} = L \cap T^*(M \times N)_{|M}/N$. Then L_M is a Lagrange submanifold of T^*M, provided L is transverse to ν^*N the conormal to N. This implies that for each (x, ρ) the* *equation* $\frac{\partial S}{\partial y} = 0$ *has a unique nondegenerate solution* $y(x,\rho)$. *Then* $S_M(x,\rho) = S(x, y(x,\rho), \rho)$ *is a g.f.q.i. for* L_M.

The proof is obvious.

Let $\phi = \psi^k$ be close to the identity map, and S_Φ be the generating function of $\Gamma_\Phi \subset \{(z_j, \phi(z_{j+1}) \mid j \in \mathbb{Z}/r\mathbb{Z}\}$. Let $\tilde{e}_r(k)$ be the map $(z_j, Z_j) \to (z_{kj}, Z_{kj})$. Then $\tilde{e}_r(k)(\Gamma_{\Psi,k}) \simeq \Gamma_\Phi$,

$$(\xi_j, \psi^k(\xi_{j+k}) \to (\xi_{kj}, \psi^k(\xi_{kj+k})) \simeq (\eta_j, \psi^k(\eta_{j+1}))$$

We now claim that $S_{\psi,k}$ is equivalent to S_ψ. Indeed let us write Γ_Ψ as

$$\{(\zeta_l, \psi(\zeta_{l+1}) \mid l \in \mathbb{Z}/kr\mathbb{Z}\};$$

we may consider its reduction through

$$z_{kj} = Z_{kj},\ z_{kj+1} = Z_{kj+1},\ \dots,\ z_{kj+k-1} = Z_{kj+k-1},$$

that is we make

$$\zeta_{kj} = \psi(\zeta_{kj+1}),\ \zeta_{kj+1} = \psi(\zeta_{kj+2}),\ \dots,\ \zeta_{kj+k-1} = \psi(\zeta_{kj+k}),$$

or else

$$\zeta_{kj} = \psi^k(\zeta_{kj+k}),\ \zeta_{kj+i} = \psi^{k-i}(\zeta_{kj+k})\ \ i = 0, \dots, k.$$

Using the coordinates $(q_j, P_j, p_j - P_j, Q_j - q_j)$ we get

$$\{q_{kj}, P_{kj}, p_{kj} - P_{kj}, Q_{kj} - q_{kj}) \mid (Q_{kj}, P_{kj}) = \psi^k(q_{kj+k}, p_{kj+k})\}.$$

So we have S_Φ as a generating function, and $S_{\psi,k} \simeq S_\Phi$ according to the above lemma. The map $\tilde{e}_r(k)$ which we obtain induces on the base $\mathcal{U}_{r,\epsilon}$ a map $\xi_j \to \xi_{kj}$, that is $e_r(k)$. Through this map, $S_{\psi,k}$ and S_Φ coincide so that the map $\tilde{e}_r(k)$ induces $e_r(k)$ on $\Lambda_r N$.

Since $f^* j_2^! = j_1^! f^*$, we have:

$$\begin{array}{ccc} W_1 A & \xrightarrow{f} & W_2 \\ \uparrow{\scriptstyle j_1} & & \uparrow{\scriptstyle j_2} \\ P_1 & \xrightarrow{k} & P_2 \end{array}$$

Using the results from Section 5, it follows that $e_r(k)^*$ and $(\Lambda j)!$ commute, so that $(\Lambda j)!$ preserves $H^{*,0}(\Lambda M)$. Because $c^*(\Lambda j)! = j!c^*$, we see that e^*c^* is the projection on the subspace $H^{*,0}(\Lambda M)$. This concludes the proof of our proposition.

Remark. We have used here the fact that $e^*(H^*(M)) = H^{*,0}(\Lambda M)$. This follows from [6], because e^* is injective, its image is obviously contained in $H^{*,0}(\Lambda M)$, and we have equality of dimensions.

Remark. We shall also use the following generalization of the identity (i) of this section. Denoting the Massey's triple product by $< u, v, w >$, we have

$$(\Lambda j)!(< (\Lambda j)^*(x), y, (\Lambda j)^*(z) >) \subset (< x, (\Lambda j)!(y), z >),$$

which again follows from a similar property of the Thom isomorphism.

This concludes our proof of the MAIN THEOREM. The equivariant generalization is left to the reader, since it offers no special difficulty, and we only used it in the proof of Proposition 0.10.

7. The Maslov class of embedded Lagrange submanifolds

Let $j : L \to T^*M$ be a Lagrange embedding, L and M being compact manifolds, and $p : L \to M$ the composition of j with the projection $T^*M \to M$. From the results obtained in the previous sections, we may conclude that if $\deg(p) = 0$, there is some obstruction to the exactness of j, and moreover this obstruction lives in $H^{\leq d}(\Lambda M)$ for some finite d (corresponding to the degree of the tied class z plus n). This means that even in the case where j is not exact, the cohomology of $I^*(\mathcal{U}^{1,\delta} - \mathcal{U}^{1,\beta}, \xi)$ may not coincide with $H^*(\Lambda M)$ in degree less than d. Now when j is not exact, the cohomology of $I^*(\mathcal{U}^{1,\varepsilon} - \mathcal{U}^{1,\beta}, \xi)$ is given as follows (see Section 5). $H^*(\Lambda_0 L)$ becomes

$$\oplus_{p(c)=0;<\lambda,c>=0} H^{*-m(c)}(\Lambda_c L),$$

where c is a free homotopy class of loops on L, and the condition $p(c) = 0$ means that c is contractible as a loop in T^*M. Also, only the homotopy classes with $< \lambda, c >= 0$ occur, since for the other ones, either the critical level obtained will be either above ε or below β, provided we take these two numbers close enough to 0.

The really new fact in the nonexact case is that there may be critical levels above β, corresponding to connected components of $\Lambda_c L$ such that $< \lambda, c >> 0$. Each of this contributes to the cohomology of

$I^*(\mathcal{U}^{1,l+\varepsilon} - \mathcal{U}^{1,l-\varepsilon}, \xi)$ ($l =< \lambda, c >$) by $H^{*-m(c)}(\Lambda_c L)$. But we see that this vanishes in degrees less than $m(c)$. If the $m(c)$ such that $< \lambda, c >> 0$ are all larger than $d+1$, the contributions of the $H^*(\Lambda_c L)$ to the total cohomology in degree less than d of $I^*(\mathcal{U}^{1,\delta} - \mathcal{U}^{1,\beta}, \xi)$ will vanish, so that the total contribution is given only by the contractible loops. Thus we may repeat the argument of Section 6, and deduce from this that the main theorem still holds in degree less than d. As a result, our obstructions to the existence of exact Lagrange embedding still hold in the nonexact case, provided the proof only involves cohomology classes of degree less than d, and the Maslov class satisfies

$$< \lambda, c >> 0 \Longrightarrow m(c) > d+1.$$

In particular, the proof of Theorem 0.4 does not involve cohomology classes of degree larger than $d_M = \deg(z) + dim(M)$.

We hence have proved:

Proposition 0.12. *Let M satisfy the first assumption of Theorem 0.4 and let $j : L \to T^*M$ be a Lagrange embedding such that* $\deg(p) = 0$. *Then there exists c in $H_1(L)$ such that:*

(i) $c \in \operatorname{Ker}(p)$,

(ii) $< \lambda, c >> 0$ *(so in particular* $c \notin \operatorname{Ker}\lambda$*),*

(iii) $< \mu(j), c > \leq d_M$ *(d_M depends not on L, but only on M).*

Corollary. *With the assumptions of the above proposition, we have that λ is not in the image of $p^1 : H^1(M) \to H^1(L)$.*

Appendix
S^1 and $\mathbb{Z}/k$ equivariant cohomologies

Let M be a S^1 space. Let S^∞ be the unit sphere in a Hilbert space endowed with the S^1 action $e^{i\theta}(z_j)_{j\in\mathbb{N}} = (e^{i\theta} z_j)$.

Then $M \times S^\infty$ is a free S^1 space for the diagonal action $e^{i\theta}(m, z) = (e^{i\theta} m, e^{-i\theta} z)$, and we denote by M_{S^1} the quotient space $M \times S^\infty / S^1$. We then define $H^*_{S^1}(M)$ to be $H^*(M_{S^1})$. We refer the reader to [3] for more details, but remark that most natural constructions from ordinary cohomology carry over trivially to the equivariant case.

Let us point out that we have two fibrations associated to the above construction.

First the fibration $M \to M_{S^1} \to S^\infty/S^1 = \mathbb{C}P^\infty$ yields a spectral sequence with $E_2^{p,q} = H^p(\mathbb{C}P^\infty) \otimes H^q(M)$, converging to $H^*_{S^1}(M)$.

For the second fibration $S^1 \to M \times S^\infty \to M_{S^1}$, we notice that $M \times S^\infty$ has the homotopy type of M since S^∞ is contractible. This yields a spectral sequence with $E_2^{p,q} = H^p_{S^1}(M) \otimes H^q(S^1)$ converging to $H^*(M)$. Since $H^q(S^1)$ is nonzero only if $q = 0$ or 1, this last sequence has $d_k = 0$ for $k \geq 3$. It is determined by

$$d_2 : H^p(M_{S^1}) \otimes H^1(S^1) \to H^{p+2}(M_{S^1}) \otimes H^0(S^1).$$

Because $H^0(S^1) \simeq H^1(S^1) \simeq R$, d_2 is determined by some map $d : H^*(M_{S^1}) \to H^{*+2}(M_{S^1})$. It is a classical fact that d is the multiplication by $c \in H^2(M_{S^1})$, the first Chern class of the above S^1 bundle. Because $E_3^{p,q} \simeq H^{p+q}(M)$ and $E_3^{p,\cdot}$ is the cohomology of $E_2^{p,q}, d_2)$, these information may be summarized by the Gysin exact sequence

$$\to H^{p+1}(M_{S^1}) \xrightarrow{p^*} H^{p+1}(M) \underset{p!}{\to} H^p(M_{S^1}) \xrightarrow{\cup c} H^{p+2}(M_{S^1}),$$

where p is the projection $M \times S^\infty \to M_{S^1}$, and $p!$ the map induced in cohomology by integration over the fibers of p.

Since $E_\infty^{p,q} \simeq E_3^{p,q}$, $H^*(M)$ decomposes into two subspaces corresponding to $H^*(M_{S^1}) \otimes H^0(S^1)$ and $H^*(M_{S^1}) \otimes H^1(S^1)$. The map p^* has for image the first factor (i.e., $E_3^{p,0}$), while $p!$ corresponds to the projection on the second one $E_3^{p,1}$ identified to the kernel of multiplication by c in $H^p_{S^1}(M)$.

We would like to consider the following problem. Let M be a S^1 space; we may of course consider M as a $\mathbb{Z}/k$ space for any $k > 1$, if we look at $\mathbb{Z}/k$ as the subgroup of k-roots of 1 in S^1. We wish to understand whether the knowledge of $H^*_{\mathbb{Z}/k}(M)$ determines $H^*_{S^1}(M)$.

Now $M_{\mathbb{Z}/k}$, that we shall denote by M_k, is the quotient $(M \times S^\infty)/(\mathbb{Z}/k)$. Thus $M_{S^1} = M_k/S^1$, and we have the following diagram of fibrations

$$\begin{tikzcd}
S^1 \arrow[r] \arrow[d] & M \times S^\infty \arrow[r] \arrow[d] & M_{S^1} \arrow[d] \\
S^1 \arrow[r] & M_k \arrow[r] & M_{S^1}
\end{tikzcd}$$

the vertical map from S^1 to S^1 corresponds to $z \mapsto z^k$.

This diagram induces a map between Gysin exact sequences:

$$\begin{tikzcd}
H^*_{S^1}(M) \arrow[r] \arrow[d, "\mathrm{id}"] & H^*(M_k) \arrow[r] \arrow[d] & H^{*-1}_{S^1}(M) \arrow[r, "\cup kc"] \arrow[d, "\cdot k"] & H^{*+1}_{S^1}(M) \arrow[r] \arrow[d, "\mathrm{id}"] & {} \\
H^*_{S^1}(M) \arrow[r] & H^*(M) \arrow[r] & H^{*-1}_{S^1}(M) \arrow[r, "\cup c"] & H^{*+1}_{S^1}(M) \arrow[r] & {}
\end{tikzcd}$$

Note that if p divides k, then

$$H^*_{\mathbb{Z}/k}(M;\mathbb{Z}/p) \simeq H^*(M_{S^1};\mathbb{Z}/p) \otimes H^*(S^1;\mathbb{Z}/p).$$

The map between $H^*_{\mathbb{Z}/k}(M;\mathbb{Z}/p)$ and $H^*_{\mathbb{Z}/kd}(M;\mathbb{Z}/p)$ induced by the map $M_k \to M_{kd}$ corresponds to the map

$$id \otimes \nu_d : H^*(M_{S^1};\mathbb{Z}/p) \otimes H^*(S^1;\mathbb{Z}/p) \to H^*(M_{S^1};\mathbb{Z}/p) \otimes H^*(S^1;\mathbb{Z}/p)$$

where $\nu_d : H^*(S^1;\mathbb{Z}/p) \to H^*(S^1;\mathbb{Z}/p)$ is the identity on H^0 and multiplication by d on H^1.

Note that if p divides d, the map $id \otimes \nu_d$ has both kernel and image isomorphic to $H^*(M_{S^1};\mathbb{Z}/p)$. Moreover for p such that $H^*(M_{S^1};\mathbb{Z})$ has no p-torsion, we have

$$H^*(M_{S^1};\mathbb{Z}/p) = H^*(M_{S^1};\mathbb{Z}) \otimes \mathbb{Z}/p,$$
$$H^*(M_{S^1};\mathbb{Q}) = H^*(M_{S^1};\mathbb{Z}) \otimes \mathbb{Q}.$$

Thus $H^*(M_{S^1},\mathbb{Q})$ determines the free part of $H^*(M_{S^1};\mathbb{Z})$, which in turn determines $H^*(M_{S^1};\mathbb{Z}/p)$ and hence $H^*_{\mathbb{Z}/k}(M;\mathbb{Z}/p)$.

Vice versa, $H^*(M_{S^1};\mathbb{Z}/p)$ may be recovered from $H^*(M_k;\mathbb{Z}/p)$ and $H^*(M_{kp};\mathbb{Z}/p)$, as the image of $H^*(M_{kp},\mathbb{Z}/p^\ell) \to H^*(M_k,\mathbb{Z}/p)$, the map being induced by the $\mathbb{Z}/p$ covering $M_k \to M_{kp}$.

This may be summarized in

Proposition. *Let p divide k, and assume $R = H^*(M_{S^1};\mathbb{Z})$ has no p-torsion. Then $R \otimes \mathbb{Q} = H^*(M_{S^1};\mathbb{Q})$ and*

$$H^*_{\mathbb{Z}/k}(M,\mathbb{Z}/p) = R \otimes H^*(S^1,\mathbb{Z}/p).$$

Moreover $R \otimes \mathbb{Z}/p$ may be identified with the image of $H^*_{\mathbb{Z}/kp}(M,\mathbb{Z}/p)$ in $H^*_{\mathbb{Z}/k}(M,\mathbb{Z}/p)$.

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