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GRAFTING, HARMONIC MAPS AND PROJECTIVE STRUCTURES ON SURFACES

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Abstract

Grafting is a surgery on Riemann surfaces introduced by Thurston; it connects hyperbolic geometry and the theory of projective structures on surfaces. ([4], [7]) We will discuss the space of projective structures in terms of the Thurston's geometric parametrization given by grafting. From this approach we will prove that on any compact Riemann surface with genus greater than 1 there exist infinitely many projective structures with Fuchsian holonomy representations. In course of the proof it will turn out that grafting is closely related to harmonic maps between surfaces.

1. Introduction

A projective structure (or a $\mathbf{C}P^1$-structure) on a surface is a coordinate system modelled on the projective space $\mathbf{C}P^1$ such that the transition maps are projective homeomorphisms (and hence the restriction of elements of $\mathrm{PSL}(2,\mathbf{C})$). For an oriented closed surface Σ_g of genus $g \geq 2$, it is well known that the space of projective structures P_g on Σ_g is parametrized by the bundle of holomorphic quadratic differentials on Riemann surfaces $\pi : Q_g \to T_g$ over the Teichmüller space: for each projective structure on Σ_g, taking the Schwarzian derivative of the developing map we have a quadratic differential which is holomorphic with respect to the underlying complex structure of the projective structure. As this parametrization is dealing with projective or complex analytic mappings and manifolds, a lot of researches have been developed from

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the viewpoint of complex analysis. (As for this parametrization, see Hejhal [6] for example.)

The connection between projective structures on surfaces and hyperbolic geometry was revealed by W. Thurston (unpublished). He showed that the space P_g of projective structures is parametrized by the product of the Teichmüller space and the space of measured laminations. His idea is to see a projective structure as a structure obtained by bending a hyperbolic 2-space in the hyperbolic 3-space along a measured geodesic lamination. (Here, note that $\partial\mathbf{H}^3 = \mathbf{C}P^1$ where $\mathbf{H}^3$ is a hyperbolic 3-space.) Bending along a measured geodesic lamination is in some sense conjugate to the earthquake deformation along the lamination (see Epstein-Marden [2] for details). W. Thurston also defined a surgery called grafting, which is an equivalent concept with bending.

In this paper, we will study projective structures and their underlying complex structures from this geometric viewpoint. Especially, we will investigate the underlying complex structures of projective structures with discrete holonomy representations whose developing maps are not covering maps. The existence of such projective structures was shown by Maskit [10], Hejhal [6] and Goldman [4], while it was unknown on which complex structure such projective structures exist. We will show that on *any* complex structure on Σ_g there are infinitely many projective structures with Fuchsian holonomy representations. To prove this fact, we will define a mapping on the Teichmüller space to itself by grafting.

We will prove our results in Section 3 after describing bending, grafting and the Thurston's parametrization theorem in Section 2.

In course of arguments, we will see that harmonic maps are involved in the proofs: when we consider a projective structure as a bent hyperbolic structure, the bent surface is a generalization of a pleated surface for the holonomy representation, which is not necessarily discrete (see Section 2). In fact, when the holonomy representation is discrete, the bent surface is a pleated surface of the quotient 3-manifold. On the other hand, pleated surfaces in hyperbolic 3-manifolds are the limits of the images of harmonic maps (See Minsky [12] and Thurston [14].) We will see that the inverse of bending can be seen as mappings from Riemann surfaces to the generalized pleated surfaces, so that grafting is naturally related to harmonic maps, in view of [12] (see Remark 1 after Theorem 3.4).

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2. Bending, grafting and geometric parametrization of projective structures

In this section we sketch Thurston's geometric parametrization theorem. This geometric description of projective structures is given by two equivalent concepts, *bending* or *grafting*, which we will describe in this section. A bent surface plays a role similar to that of pleated surfaces for hyperbolic 3-manifolds. Roughly speaking, bending is the way to see a projective structure as a hyperbolic structure bent in the hyperbolic 3-space, and grafting is the observation of bending on the sphere at infinity.

2.1. Thurston metric

We begin with a metric introduced by Thurston which is a powerful tool to understand projective structures.

Recall that every complex structure on a compact oriented surface Σ_g of genus g admits a unique hyperbolic structure. This fact provides two different approaches for Teichmüller theory: the Teichmüller space T_g is the space of complex structures and, at the same time, the space of hyperbolic structures on a compact surface Σ_g. Now, for any complex structure $X \in T_g$ the set of projective structures on X are parametrized by the space of holomorphic quadratic differentials on X, which is a $(3g-3)$-dimensional complex vector space. As the complex structures under these projective structures are all the same, the hyperbolic metric does not distinguish them. The metric structure which characterizes a projective structure is defined by a very natural analogue of the definition of hyperbolic metrics.

Definition 2.1 (Thurston (pseudo-)metric). Let M be a $\mathbf{C}P^1$-manifold. For each point $x \in M$ and each tangent vector $v \in T_xM$, define the length of the vector v by

$$t_M(v) = \inf_{f:\Delta\to M} \rho_\Delta(f^*v),$$

where the infimum is taken over all projective immersions $f:\Delta\to M$ with $f(\Delta)\ni x$, and ρ_Δ is the hyperbolic metric on the unit disc $\Delta=\{z\in\mathbf{C};|z|<1\}$. We will call the pseudometric t_M the Thurston pseudometric on M. If t_M is non-degenerate it will be called the Thurston metric.

Recall that the Kobayashi metric on a Riemann surface, which coincides with the hyperbolic metric if non-degenerate, is defined by taking the infimum over all holomorphic immersions. (See [9].) The following properties are immediate consequences from the definitions of Thurston metric and Kobayashi hyperbolic metric.

Proposition 2.2. *For a $\mathbf{C}P^1$-manifold M, let k_M denote the Kobayashi pseudo-metric on M. Then the following hold:*

(1) $t_M \geq k_M$.

(2) If these metrics are non-degenerate on M and coincide at a non-zero tangent vector v, then these two metrics coincide on the entire tangent space TM.

(3) For the projective universal covering space $\tilde{M}$ of M, $t_{\tilde{M}}$ descends to t_M via the projective universal covering map $\tilde{M}\to M$.

(4) If $t_M(v)\neq 0$ for a vector $v\in T_zM$ at a point $z\in M$, then there is a projective mapping $f:\Delta\to M$ that attains the minimum in the definition of $t_M(v)$, and the mapping f is determined by z uniquely up to precomposition of automorphisms of Δ.

In the following, we assume that the underlying complex structure of the $\mathbf{C}P^1$-manifold M is hyperbolic, hence t_M does not degenerate.

For convenience, we consider the Thurston metric on the universal projective covering space $\tilde{M}$ rather than on M, as any extremal mapping $f:\Delta\to\tilde{M}$ which realizes the Thurston metric at $z\in\tilde{M}$ is an embedding.

For each point $z\in\tilde{M}$ the image $f(\Delta)$ by an extremal mapping f is a disc determined uniquely by z. (Note that the terminology “discs” makes sense in $\mathbf{C}P^1$-manifolds.) This disc is called the *maximal disc* for z. Let D_z denote the maximal disc for $z \in \tilde{M}$. Take a projective mapping f on the upper half plane to $\tilde{M}$ realizing the Thurston metric at z, and identify D_z with the upper half plane model of the hyperbolic 2-space $\mathbf{H}^2$ via f. Then we can compactify D_z with the circle at infinity $\mathbf{R} \cup \{\infty\}$ of $\mathbf{H}^2$. Let $\zeta \in \partial D_z$ be a boundary point. If the mapping $f : \mathbf{H}^2 \to \tilde{M}$ can be extended as a projective map beyond ζ, then ζ is identified with a point in the frontier of D_z in $\tilde{M}$, otherwise, we call ζ an *ideal boundary point*. Denote the set of all ideal boundary points of D_z by $\partial_\infty D_z$. Take the convex hull of $\partial_\infty D_z$ with respect to the hyperbolic metric of $D_z (= \mathbf{H}^2)$, and denote it by $C(\partial_\infty D_z)$. It is easy to see that $\partial_\infty D_z$ consists of at least 2 points, z can not be disjoint from $C(\partial_\infty D_z)$ the definition of the maximal disc, and there are three cases as follows (see Figure 1) ;

(i) $\partial_\infty D_z$ contains at least three points, and z is in the interior of $C(\partial_\infty D_z)$.

(ii) $\partial_\infty D_z$ contains at least three points, and z is in the frontier of $C(\partial_\infty D_z)$ in D_z.

(iii) $\partial_\infty D_z$ consists of two points, and $z \in C(\partial_\infty D_z)$.

FIGURE 1. z is in the convex hull of $C(\partial_\infty D_z)$

We may assume that 0 and ∞ are ideal boundary points, and z is on the imaginary axis. In the first case, the Thurston metric coincides with the hyperbolic metric $|dz|/\mathrm{Im}\, z$ (on the upper half plane model of D_z) near z.

In the third case, Thurston metric at z is equal to the flat metric $|dz|/|z|$. In the second case, the hyperbolic metric and the flat metric coincide on the imaginary axis.

It is easy to see that $\tilde{M}$ is decomposed into the union of hyperbolic pieces and flat lines by the convex hulls of ideal boundary points set $C(\partial_\infty D_z)$ of maximal discs D_z.

Example 2.3. Let $M = \tilde{M}$ be the union of two discs D and D' intersecting with angle $\theta \in [0, \pi)$ (Figure 2). For convenience, we employ the model such that the two intersecting points are 0 and ∞. Let S be the sector bounded by the ray perpendicular to ∂D and the ray perpendicular to $\partial D'$. It is easy to see that for $z \in S$ the maximal disc for z is the half plane with boundary orthogonal to the ray through z starting at 0. In this case the Thurston metric is equal to $|dz|/|z|$ on the ray. If z is outside of S and contained in D (resp. D'), then the maximal disc for z is D (resp. D'), and the Thurston metric near z coincides with the hyperbolic metric on D (resp. D').

Therefore, Thurston metric is hyperbolic in $D - S$ and $D' - S$, and is flat in S.

FIGURE 2. A projective surface consists of hyperbolic pieces and flat pieces

Note that in fact θ can be any positive number; if $\theta \geq \pi$, we distinguish each sheet over the overwrapping region by regarding the surface as $\{(re^{i\rho}, \rho) \in \mathbf{C} \times \mathbf{R} : r \neq 0, 0 < \rho < \theta\}$.

2.2. Bending a hyperbolic surface in $\mathbf{H}^3$

Next, we shall see that projective structures are obtained by bending the hyperbolic 2-space $\mathbf{H}^2$ in a locally convex way in the hyperbolic 3-space $\mathbf{H}^3$. In what follows, we will denote by $CH(E)$ the convex hull of a subset E in $\mathbf{H}^3 \cup \mathbf{C}P^1$, where $\mathbf{C}P^1$ is considered as a sphere at infinity of $\mathbf{H}^3$, to avoid mixing up the convex hull in $\mathbf{H}^3$ with that in $\mathbf{H}^2$.

We begin with a simple example. We will consider the Riemann sphere as the sphere at infinity of the hyperbolic space $\mathbf{H}^3$. Let D be a disc in the Riemann sphere. The convex hull $CH(D)$ of D in $\mathbf{H}^3$ is the half space bounded by the hyperbolic plane $CH(\partial D)$. The nearest point projection $D \to CH(\partial D)$ sends the hyperbolic structure of D to the hyperbolic structure of $CH(\partial D)$. (Namely, for each point $z \in D$, there is a unique horosphere at z which is tangent to $CH(\partial D)$. Map each z to the tangent point.) On the other hand, the hyperbolic structure of D coincides with the projective structure as a domain of $\mathbf{C}\mathrm{P}^1$. Hence in this case the projective structure on D is given by the hyperbolic surface $CH(\partial D)$ in $\mathbf{H}^3$ with nearest point projection.

Now, take a geodesic line $l \in CH(\partial D)$, fix an orientation of l and denote the left (resp. right) part of $CH(\partial D) - l$ by Δ^0 (resp. Δ^1). Take a positive number θ (for simplicity, we temporarily assume that $\theta < \pi$) and rotate Δ^1 along l by angle θ. Then we have a pleated surface R as in Figure 3.

FIGURE 3. Bending a hyperbolic surface in the hyperbolic 3-space $\mathbf{H}^3$ by angle θ produces a sector S with angle θ on the sphere at infinity, which is a Euclidean piece of the projective surface $\Omega = D + D'$.

We will call this procedure *bending the hyperbolic surface* $CH(\partial D)$ *along* l.

Now let us see what happens in the sphere at infinity when we bend $CH(\partial D)$ along l. (Roughly speaking, we get a new projective surface by pushing the bent surface down to the sphere at infinity via the nearest point projection.)

As we bend $CH(\partial D)$ in $\mathbf{H}^3$ along l, $CH(\partial D)$ splits into two totally geodesic pieces, which are the images of Δ^0 and Δ^1. We denote the images by the same symbols Δ^0 and Δ^1. For each of them, there is a unique circle on the sphere at infinity whose convex hull in $\mathbf{H}^3$ contains the piece. For Δ^0, the circle is the boundary of D. For Δ^1, the circle bounds the disc D' intersecting with D at the endpoints of l with angle θ. Therefore, when we bend $CH(\partial D)$ in $\mathbf{H}^3$ along l with angle θ, the original projective surface D turns into the domain $\Omega = D \cup D'$. This domain Ω has a projective structure as a domain of the projective surface $\mathbf{C}P^1$, which we observed in Example 2.3.

We can reconstruct the pleated surface R from Ω in the following way ([7]). Remember that we saw in Section 2.1 that for each $z \in \Omega$ there is a unique maximal disc D_z. For each $z \in \Omega$, take the convex hull of the circle ∂D_z in $\mathbf{H}^3$. Then send each point in the convex hull $C(\partial_\infty D_z)$ (defined in Section 2.1) of $\partial_\infty D_z$ in the hyperbolic surface D_z by the nearest point projection to the convex hull of $\partial_\infty D_z$ in $\mathbf{H}^3$. Recall that we saw in Example 2.3 that Ω is decomposed into hyperbolic pieces $D - S$ and $D' - S$ and a flat piece S with respect to Thurston metric t_Ω. Then by the nearest point projection, $D - S$ (resp. $D' - S$) is mapped to Δ^0 (resp. Δ^1) isometrically. As for the sector S, each flat line connecting 0 and ∞ is mapped to l isometrically. Thus the image of Ω is the pleated surface R, and the above mapping $\Omega \to R$ is the inverse of the procedure getting the projective structure Ω from the pleated surface R.

Thus the procedure bending $CH(\partial D)$ in $\mathbf{H}^3$ along a geodesic is equivalent to 'grafting' a flat part S into the hyperbolic structure on D.

As before, note that we do not have to restrict θ to be smaller than π: if $\theta \geq \pi$, distinguish overwrapping sheets.

Now we proceed to the case with a group action. Let Γ be a co-compact Fuchsian group acting on $\mathbf{H}^2$. Embed $\mathbf{H}^2$ in $\mathbf{H}^3$ as a totally geodesic surface. Let X denote the hyperbolic surface $\mathbf{H}^2/\Gamma$. Take a simple closed geodesic curve γ on X. The lift of γ on $\mathbf{H}^2$ is a Γ-invariant set of geodesic lines. We can bend $\mathbf{H}^2$ along each of these geodesics with angle θ step by step (see Epstein-Marden [2]). In each step, on the sphere at infinity, we have a new projective surface with a grafted part to the preceding step, as we did in the preceding example. (In each step distinguish the overwrapping sheets, if any, as we did in Example 2.3.) Then we end up with a simply connected projective manifold $\tilde{M}$ spread over the sphere at infinity, which is partly hyperbolic and partly flat.

In view of the construction of $\tilde{M}$, it is easy to see that there is a projective automorphism group $\tilde{\Gamma}$ acting on $\tilde{M}$ isomorphic to Γ. Hence in particular, $\tilde{M}/\tilde{\Gamma}$ is homeomorphic to X. To consider $\tilde{M}$ as spread over the Riemann sphere as above is to map $\tilde{M}$ to $\mathbf{C}P^1$ via the developing map and we have the holonomy representation $\chi : \Gamma \to \mathrm{PSL}(2, \mathbf{C})$. In fact, it is easy to see $\tilde{\Gamma} = \chi(P)$. Then the above bending procedure is given by an equivariant map from $\mathbf{H}^2$ to $\mathbf{H}^3$ with respect to Γ and the holonomy representation which is bent along the bending locus and isometric elsewhere.

Indeed, it is known that we can write down the holonomy representation $\chi : \Gamma \to \mathrm{PSL}(2, \mathbf{C})$ in terms of bending. (We omit the formulae. See [2, Chapter 3] for details. There, the homomorphism is called the quakebend homomorphism.)

It is also known that when the weighted simple closed curves converge to a measured lamination in the space of measured laminations, the equivariant maps converge to the equivariant map, bent along the measured lamination, which defines the corresponding projective structure.

See Epstein-Marden [2] for details.

2.3. Grafting along a simple closed curve

Grafting is the way to see the above procedure directly on the quotient surfaces $X = \mathrm{H}^2/\Gamma$ and $M = \tilde{M}/\tilde{\Gamma}$ as in the following way.

We provide two types of $\mathbf{C}P^1$-manifolds which we will paste together. Let X and γ be as in Section 2.2. First, take the lower half plane model of $\mathbf{H}^2$ such that the geodesic line $\{iy; y < 0\}$ is one of the component of the lift of γ. Let $g(z) = e^{l(\gamma)}z$ be the generator of the stabilizer of $\{iy; y < 0\}$ in Γ, where $l(\gamma)$ denotes the hyperbolic length of γ on X. Next, take the sector $\{z = re^{i\rho}; 0 < r < \infty, 0 \leq \rho \leq \theta\}$ equipped with the projective structure as a domain of $\mathbf{C}P^1$ (with no restrictions on θ). The group $< g >$ generated by g acts on this sector as a projective automorphism. Taking the quotient we get a flat annulus A_θ with height θ and circumference $l(\gamma)$.

Now we cut X along γ, and paste each side of γ to one of the boundary component of A_θ as Figure 4, in such a way that the length parameters of pasting sides match, and that the pair of points which is identified in X are connected by segments in A_θ orthogonal to the boundary.

FIGURE 4. grafting a flat annulus of height θ to X along γ

Note that the hyperbolic structure of X and the projective structure of A_θ match on the pasting locus. Therefore, the above pasting process yields a new $\mathbf{C}\mathrm{P}^1$-structure on Σ_g preserving the original projective structures of X and A_θ. We call this surgery *grafting a flat annulus of height* θ *to* X *along* γ, or *grafting* $\theta\gamma$ *to* X, and denote the resulted $\mathbf{C}P^1$-structure by $Gr_{\theta\gamma}(X)$.

Note that the metrics on the hyperbolic surface X and the flat annulus A_θ also match on the pasting locus and the resulted surface $Gr_{\theta\gamma}(X)$ is equipped with a metric which is partly hyperbolic and partly flat. It is easy to see that this metric is equal to the Thurston metric on $Gr_{\theta\gamma}(X)$. It follows that the underlying complex structure of $Gr_{\theta\gamma}(X)$ differs from X by Proposition 2.2 (4), unless $\theta = 0$.

It is also easy to see that this projective structure $Gr_{\theta\gamma}(X)$ has a projective universal covering space $\tilde{M}$, which we obtained by bending in Section 2.2.

2.4. grafting a general measured lamination and the parametrization theorem

Let P_g denote the set of all projective structures on the oriented closed surface Σ_g of genus g. Then as we have seen above, the grafting operation gives a mapping

$$Gr : T_g \times \mathbf{R}_+ \times \mathcal{S} \to P_g,$$

which sends each $(X, \theta, \gamma) \in T_g \times \mathbf{R}_+ \times \mathcal{S}$ to the projective structure obtained by grafting a flat annulus of height θ along the hyperbolic geodesic in the homotopy class of γ to the hyperbolic surface X, where $\mathcal{S}$ denotes the set of homotopy classes of simple closed curves. Now we can state Thurston's parametrization theorem.

Theorem 2.4 (Thurston). *The map Gr extends to a homeomorphism of $T_g \times \mathcal{ML}$ onto P_g, where $\mathcal{M}L$ denotes the space of measured laminations on Σ_g.*

Sketch of the proof. We have already seen that for any measured lamination $\mu = \theta\gamma$ supported on a simple closed curve γ and for any hyperbolic structure $X \in T_g$, grafting an annulus with height θ yields a projective structure. Recall that $\mathbf{R}_+ \times \mathcal{S}$ is a dense subset of $\mathcal{M}L$. The mapping $Gr : T_g \times \mathbf{R}_+ \times \mathcal{S} \to P_g$ is continuously extended to $T_g \times \mathcal{M}L$, as bending is defined for any measured lamination and depends on the lamination continuously. (See [2] for details.)

We shall describe the inverse correspondence: $P_g \to T_g \times \mathcal{M}L$. By the arguments in the preceding sections, it suffices to show that any projective structure on Σ_g is obtained from the bending procedure defined with an equivariant map $\mathbf{H}^2 \to \mathbf{H}^3$, bent along a measured lamination and isometric elsewhere.

Given a projective structure on Σ_g, take its projective universal covering $\tilde{M}$ and fix its developing map. Begin with an open set U in $\tilde{M}$ small enough so that the developing map restricted to U is homeomorphic. For each point z in U, take the maximal disc D_z for z and embed it into the Riemann sphere via the developing map. We identify D_z with its image. Then take the convex hull of the circle ∂D_z in $\mathbf{H}^3$ and denote it by R_z. R_z is a totally geodesic disc isometric to D_z with respect to the hyperbolic metrics on them via the nearest point projection. Now, as in Section 2.1, take the convex hull $C(\partial_\infty D_z)$ of the ideal boundary points of D_z. Then as we did in Section 2.2 for the simple case without group action, send $C(\partial_\infty D_z)$ into R_z via the nearest point projection between R_z and D_z. Denote the image of $C(\partial_\infty D_z)$ by P_z. If $C(\partial_\infty D_z)$ is of the type (i) in Section 2.1, then P_z is a convex domain of R_z which is the convex hull of $\partial_\infty D_z$ in $\mathbf{H}^3$. If $\partial_\infty D_z$ is of the type (iii), then P_z is the hyperbolic line in R_z connecting the two points in $C(\partial_\infty D_z)$. In any case, $P_z = P_{z'}$ for every $z' \in C(\partial_\infty D_z)$. Now, $\cup_{z \in U} P_z$ is a piece of a pleated surface in $\mathbf{H}^3$: there are a subset V of $\mathbf{H}^2$ and a mapping from V to $\cup_{z \in U} P_z \subset \mathbf{H}^3$, such that for each point $w \in V$ there is a straight line in V which is mapped isometrically to a hyperbolic line in $\mathbf{H}^3$. This piece of pleated surface defines locally the bending which gives the projective structure of $\cup_{z\in U} D_z$. (Here, $\cup_{z\in U} D_z$ is equipped with the projective structure as a domain of $\tilde{M}$.)

Beginning with U and continuing this procedure, it is easy to get an equivariant mapping $\mathbf{H}^2$ to $\mathbf{H}^3$, defining the bending which produce the projective surface $\tilde{M}$. See [7] for details.

3. Grafted structures on surfaces

Now, we are ready to discuss projective structures in terms of the geometric parametrization. Given a measured lamination μ, let $gr_\mu(X)$ stand for the underlying complex structure of the projective structure $Gr_\mu(X)$ for $X \in T_g$. For any fixed μ, this assignment gives a mapping $gr_\mu : T_g \to T_g$. We shall call this mapping *the grafting map defined by* μ.

First, we recall some facts about projective structures with Fuchsian holonomy representations. On any complex structure $X \in T_g$ there is a unique projective structure whose projective universal covering space is projectively equivalent to the hyperbolic 2-space $\mathbf{H}^2$, namely, the hyperbolic structure. The holonomy representation of this projective structure is a Fuchsian group Γ acting on $\mathbf{H}^2$ with quotient manifold $X = \mathbf{H}^2/\Gamma$. An 'exotic' projective structure with Fuchsian holonomy representation whose developing map is not a covering map was first constructed by Maskit [10]. Hejhal [6] and Goldman [4] made more topological and geometric approach to such projective structures. The following characterization of projective structures with Fuchsian holonomy representations was given by Goldman.

Theorem 3.1 (Goldman [4]). *A projective structure given by* $(X, \mu) \in T_g \times \mathcal{M}L$ *has a Fuchsian holonomy representation if and only if* μ *is an integral point of* $\mathcal{M}L$. *Here, a measured lamination* μ *is called an integral point if it is of the form* $\mu = \sum 2\pi m_i \gamma_i$ *with a disjoint union of nontrivial simple closed geodesics* $\{\gamma_i\}$ *and a set of positive integers* $\{m_i\}$.

Note that given a projective structure determined by a pair $(X, \mu) \in T_g \times \mathcal{M}L$, the underlying complex structure of $Gr_\mu(X)$ is hardly expressed by X and μ, unless $\mu = 0$. So far, in particular, it is unclear on which complex structures there exist projective structures with Fuchsian holonomy representations other than the hyperbolic structures. Our main result shows that on *any* complex structure and any integral point

$\mu \in \mathcal{M}L$ there is a unique projective structure with Fuchsian holonomy representation which is obtained by grafting μ to some hyperbolic structure $X \in T_g$:

Theorem 3.2. *For any integral point $\mu \in \mathcal{M}L$, the grafting map $gr_\mu : T_g \to T_g$ is a real analytic homeomorphism.*

Before proving this theorem, let us interpret it in terms of the parametrization of P_g (the space of projective structures) by the bundle of holomorphic quadratic differentials on Riemann surfaces $\pi : Q_g \to T_g$. This parametrization is given in the following way: for each projective structure, take the Schwarzian derivative of the developing map, where the Schwarzian derivative of a locally univalent meromorphic function f is defined by $(f''/f')' - (1/2)(f''/f')^2$. Then the Schwarzian derivative is a quadratic differential on the surface which is holomorphic with respect to the complex structure under the projective structure. (See Hejhal [6], for example, for details.) The canonical projection $\pi : Q_g \to T_g$ sends each projective structure to its underlying complex structure. Let $K \subset Q_g$ be the set of projective structures with discrete holonomy representations. For $X \in T_g$, let $Q(X)$ and $K(X)$ denote the fibers over X of $\pi : Q_g \to T_g$ and $\pi|K : K \to T_g$ respectively. For every $X \in T_g$, the interior point set $\operatorname{int} K(X)$ has a component containing 0, which coincides with the Bers slice. Theorem 3.2 implies the existence of components of $\operatorname{int} K(X)$ other than the Bers slice:

Corollary 3.3. *On any complex structure $X \in T_g$, there are infinitely many components of* $\operatorname{int} K(X)$.

Proof of Corollary 3.3. Fix an integral point $\mu \in \mathcal{M}L$ and a hyperbolic structure X'. The projective structure $Gr_\mu(X')$ has a Fuchsian holonomy group Γ' such that X' is holomorphically equivalent to $\mathbf{H}^2/\Gamma'$ (see [10]). For any Beltrami differential τ for Γ' on the Riemann sphere $\hat{\mathbf{C}}$, we can take a quasiconformal deformation of the projective structure $Gr_\mu(X')$ by τ (cf. [13]): let f^τ denote the quasiconformal homeomorphism of $\hat{\mathbf{C}}$ with Beltrami differential τ fixing $0, 1$ and ∞. Then $\gamma^\tau = f^\tau \circ \gamma \circ (f^\tau)^{-1}$ is a Möbius transformation for every $\gamma \in \Gamma'$ and $\Gamma^\tau = f^\tau \Gamma' (f^\tau)^{-1}$ is a quasifuchsian group. As $\gamma^\tau \circ f^\tau = f^\tau \circ \gamma$, we have another projective structure by replacing the local coordinate system $\{(U, \phi)\}$ of $Gr_\mu(X')$ to $\{(U, f^\tau \circ \phi)\}$ with holonomy representation Γ^τ. It is easy to see that this new projective structure depends only on the equivalence class of τ (cf. [13]). Therefore, we have an open set $QF(\mu)$ of K consisting of all quasiconformal deformations of $Gr_\mu(X')$.

Note that if Γ^τ is a Fuchsian group, the projective structure defined by the quasiconformal deformation of $Gr_\mu(X')$ with τ is equal to the projective structure $Gr_\mu(X^\tau)$, where X^τ is the hyperbolic surface obtained by the quasiconformal deformation of X' with the Beltrami differential τ. Hence by Theorem 3.2 the restriction $\pi|QF(\mu) : QF(\mu) \to T_g$ is surjective for each integral point $\mu \in \mathcal{M}L$. Therefore, the corollary follows if we show $QF(\mu) \cap QF(\nu) = \emptyset$ for any two different integral points μ and ν. To see this, take the inverse image of the limit set $\mathbf{R} \cup \infty$ of Γ' via the developing map on the universal cover of the $\mathbf{C}P^1$-manifold $Gr_\mu(X')$. Then the inverse image descends to a disjoint union of curves on $Gr_\mu(X')$. If the integral point μ is of the form $\mu = \sum 2n_i\pi\gamma_i$ for integers $\{n_i\}$ and simple closed curves $\{\gamma_i\}$, then the inverse image of the limit set of Γ' descends to the union of $2n_i$ curves each of which is homotopic to γ_i. On the other hand, it is easy to see that any quasiconformal deformation of the projective structure $Gr_\mu(X')$ maps this system of curves quasiconformally (the image depends only on the equivalence class of the Beltrami differential). Therefore, the homotopy class of these system of curves characterizes the open set $QF(\mu)$. Hence $QF(\mu) \cap QF(\nu) = \emptyset$ for any two different integral points μ and ν. q.e.d.

Remark. It was shown by Maskit [10] that there exists *some* X such that $\operatorname{int} K(X)$ (the interior of $K(X)$ in $Q(X)$) has *some* components other than the Bers slice. In [13] we discussed on $\operatorname{int} K(X)$ for such X (i.e., assuming the existence of such components on X), where we showed that any component of $\operatorname{int} K(X)$ is a component of $QF(\mu) \cap Q(X)$ for an integral point $\mu \in \mathcal{M}L$. What we have shown in the above corollary is that $QF(\mu) \cap Q(X)$ is a non-empty open set for every complex structure X and every integral point μ.

Proof of Theorem 3.2. To prove Theorem 3.2, it suffices to show the following for an integral point μ:

(1) $gr_\mu : T_g \to T_g$ is a proper mapping,

(2) $gr_\mu : T_g \to T_g$ is a local diffeomorphism, and

(3) $gr_\mu : T_g \to T_g$ is real analytic.

Proof of (1). The following theorem enables us to show that for *any* measured lamination μ (not necessarily an integral point), the grafting map $gr_\mu : T_g \to T_g$ is a proper map.

Theorem 3.4. *Let X be a hyperbolic surface and μ be a measured lamination. Let $h : gr_\mu(X) \to X$ denote the harmonic map with respect to the hyperbolic metric on X and $\mathcal{E}(h)$ be its energy. (Remember that the harmonic map between surfaces depends on the metric on the target surface but only on the conformal structure on the source surface.) Then*

$$\frac{1}{2} l_X(\mu) \le \frac{1}{2}\frac{l_X(\mu)^2}{E_{gr_\mu(X)}(\mu)} \le \mathcal{E}(h) \le \frac{1}{2} l_X(\mu) + 4\pi(g-1),$$

where $l_X(\mu)$ is the hyperbolic length of μ on X, and $E_{gr_\mu(X)}(\mu)$ is the extremal length of μ on the grafted surface $gr_\mu(X)$.

Proof of Theorem 3.4. For simplicity, we abbreviate $Y = gr_\mu(X)$. First, assume that μ is supported on a simple closed curve, so that $\mu = \theta\gamma$. Then the projective structure $Gr_\mu(X)$ consists of hyperbolic piece(s) whose union is identified with X and a flat annulus A_θ. We will use this geometric structure on Y. We define a mapping $f : Y \to X$ by collapsing the annulus A_θ to the geodesic curve γ on the hyperbolic surface X along the flat structure (i.e., translating each point of A_θ to γ along the segment perpendicular to γ) and sending the hyperbolic pieces of Y isometrically on the corresponding domains on X. Then f is among the competitive mappings for the harmonic map $h : Y \to X$. As f is isometric on the hyperbolic pieces, the contribution of this part for the total energy is the hyperbolic area of X. On the flat annulus A_θ, the direction parallel to the geodesic γ and the direction of the segment orthogonal to γ form an orthogonal frame in A_θ. The length of the former direction is preserved by f, while the image of the latter direction degenerates. Therefore, the contribution to the total energy of the flat part is $(1/2)\theta l_X(\gamma) = (1/2)l_X(\mu)$. Hence we have

$$\mathcal{E}(h) \le \mathcal{E}(f) \le \frac{1}{2} l_X(\mu) + \pi(4g-4).$$

On the other hand, by the left part of Minsky's inequality [11, Theorem 7.2],

$$\frac{1}{2}\frac{l_X(\mu)^2}{E_Y(\mu)} \le \mathcal{E}(h).$$

Note that the extremal length of γ in Y is not greater than that in A_θ, that is, $l_X(\gamma)/\theta$. It follows that

$$\frac{1}{2} l_X(\mu) = \frac{1}{2}\theta l_X(\gamma) = \frac{1}{2}\frac{l_X(\gamma)^2}{l_X(\gamma)/\theta} \le \frac{1}{2}\frac{l_X(\gamma)^2}{E_Y(\gamma)} = \frac{1}{2}\frac{l_X(\mu)^2}{E_Y(\mu)}.$$

We have shown the inequality in the statement of Theorem 3.4 for the case μ is supported on a simple closed curve. For a general measured lamination, we approximate μ by a sequence of measured laminations each of which is supported on a simple closed curve. The inequality follows from the continuity of the hyperbolic length of measured laminations on X and the continuity of grafting with respect to measured laminations. q.e.d.

Now we prove the properness of $gr_\mu : T_g \to T_g$ from Theorem 3.4. When a sequence of points in T_g leaves any compact set eventually, we will say 'the sequence tends to infinity' for simplicity. We have to show for any sequence $\{X_n\}$ tending to infinity the image $\{gr_\mu(X_n)\}$ also tends to infinity. Denote $Y_n = gr_\mu(X_n)$ for simplicity. By taking a subsequence if necessary, we may assume that either

(i) $\sup_n l_{X_n}(\mu) < \infty$, or

(ii) $\lim_{n\to\infty} l_{X_n}(\mu) = \infty$.

In the case (i) we show that $\{Y_n\}$ tends to infinity by contradiction. Assume that $\{Y_n\}$ stays in a compact set of T_g. As X_n tends to infinity, the energy of the harmonic map $h_n : Y_n \to X_n$ tends to infinity by a result of M. Wolf [16, Proposition 3.3]. This contradicts the assumption that $l_{X_n}(\mu)$ is uniformly bounded, considering the rightmost inequality of Theorem 3.4.

In the case (ii), by Theorem 3.4,

$$\lim_{n\to\infty} E_{Y_n}(\mu) = \lim_{n\to\infty} (l_{X_n}(\mu) + O(1)) = \infty.$$

Therefore, Y_n tends to infinity. q.e.d.

Remark 1 (Collapsing the grafted part is close to the harmonic map). In the above proof of Theorem 3.4, we showed that the difference between the total energy of the annulus collapsing map $f : Y \to X$ and that of the harmonic map $h : Y \to X$ is bounded by a universal constant depending only on the genus g. Therefore, we can say that f is close to the harmonic map when $l_X(\mu)$ is large, as the harmonic map between a pair of hyperbolic surfaces is unique by a result of Hartman [5]. Here we exhibit an intuitive explanation for this phenomenon.

First, note that the grafted part occupies a large portion on the entire surface when $l_X(\mu)$ is very large, in view of the Thurston metric on

$Gr_\mu(X)$. To collapse this large part likely results in 'significant stretch in the direction along the curve'. In general, the direction of 'maximal stretch' of any kinds of extremal mappings (e.g. Teichmüller mappings, extremal Lipschitz maps, or harmonic maps) plays the key role in measuring the difference between two surfaces: Kerckhoff [8] showed that the Teichmüller distance between two Riemann surfaces is described by the ratio of the extremal lengths of the direction of maximal stretch of the Teichmüller mapping. Similar results are proved for Lipschitz maps by Thurston [15], and for harmonic maps, by Minsky [11] and [12]. Now, as for grafting, it is natural to pay attention to harmonic maps to compare the grafted surface with the original surface for the following reason.

Recall (see Section 2.2) that grafting a measured lamination μ to a hyperbolic surface X is equivalent to bending which is realized by the equivariant map $g : \mathbf{H}^2 \to \mathbf{H}^3$, with respect to the Fuchsian group Γ with $\mathbf{H}^2/\Gamma = X$ and the holonomy representation of $Gr_\mu(X)$, which is bent along the lift of μ and isometric elsewhere. This is a generalization of a pleated surface for $\mathrm{PSL}(2, \mathbf{C})$-representation which is not necessarily discrete.

Assume for a moment that the holonomy representation of the projective structure is discrete. Then this equivariant map actually determines the pleated surface realizing the measured lamination μ in the quotient 3-manifold for the holonomy representation. On the other hand, Thurston gave a remark in [14] that realizing a measured lamination μ in a hyperbolic 3-manifold is a "harmonic map" from $[\mu] \in \mathcal{PML}$, where $\mathcal{PML}$ is the Thurston boundary of T_g. (A rough explanation is given in the following way: $[\mu] \in \mathcal{PML}$ is the limit of a degenerating sequence $\{Y_n\}$ of hyperbolic structures which shrink in the direction μ as $n \to \infty$. Therefore, the harmonic mapping from Y_n to a fixed hyperbolic 3-manifold stretchs along this direction μ significantly. From the definition of the energy, the harmonic map from Y_n sends this direction μ close to the realization of μ in the 3-manifold, and the image is contained in its convex core of the 3-manifold. Hence for large n the image is close to a pleated surface with pleating locus μ.) This intuitive claim was justified by Minsky [12]: a pleated surface is the limit (in a very strong sense) of the images of the harmonic maps from surfaces whose 'maximal stretch direction' is the pleating locus, when the pleating locus is complete.

Since collapsing the grafted part can be seen as a mapping from the grafted surface to the pleated surface in the quotient 3-manifold, it is natural to expect that collapsing the grafted part is close to being harmonic, when the grafted part is very large.

When the holonomy representation is not discrete, we can still think of harmonic maps in the following way: as in Donaldson [1], form a flat $\mathbf{H}^3$- bundle defined by

$$\mathcal{H} = \mathbf{H}^2 \times_\chi \mathbf{H}^3 \to Y,$$

where χ is the holonomy representation. For a section $s : Y \to \mathcal{H}$ take the vertical part of its derivative: $(Ds)_x : T_xY \to T_{s(x)}\mathcal{H}_x$ where x is a point on Y and $\mathcal{H}_x$ is the fiber over x. Define the energy by $\mathcal{E}(s) = \int_Y ||Ds||^2 dV$ where dV is the volume form. A *twisted harmonic map* is a critical point for the energy functional. Donaldson [1] showed the existence of the twisted harmonic map. In the same way, we can also define "pleated surfaces in the vertical direction of $\mathcal{H}$", which is equivalent to considering the equivariant map realizing bending. From the intuitive explanation of the relation between harmonic maps and pleated surfaces for 3-manifolds, it is reasonable to expect similar things to be true when the representation is not discrete.

Remark 2 (an alternative proof). When μ is supported on a simple closed curve, we can show the properness of the grafting map without using harmonic maps. In fact, when μ is supported on a simple closed curve, it is easy to see that in the case (i) the Teichmüller distance between Y_n and X_n is bounded by a constant independent of n. In the case (ii), we can prove that $E_{Y_n}(\mu) \geq l_{X_n}(\mu) + O(1)$ applying the Thurston metric on Y_n to the definition of the extremal length, for any μ. However, the author exhibited the proof using harmonic maps because it gives a better geometric perspective and also because it seems (to the author) that for the case (i) arguments by approximation would not work to give a uniform constant to bound the Teichmüller distance between Y_n and X_n for general measured laminations.

Remark 3 (properness with respect to $\mathcal{M}L$). Theorem 3.4 implies also that for a fixed hyperbolic surface X, the mapping $gr.(X) : \mathcal{M}L \to T_g$ is proper.

We continue the proof of Theorem 3.2. Although the properness of grafting map was proved for any measured lamination, we will assume that μ is an integral point of $\mathcal{M}L$ for the proofs of (2) and (3).

Proof of (2). Here we use the parametrization of projective structures by Q_g, i.e., the space of quadratic differentials. We will observe how the fiber of Q_g over each point $Y \in T_g$, i.e., the space of projective structures on a fixed complex structure, is mapped by the holonomy map. Let $Rep = Hom(\pi_1\Sigma_g, \mathrm{PSL}(2, \mathbf{C}))/\mathrm{PSL}(2, \mathbf{C})$ denote the space of $\mathrm{PSL}(2, \mathbf{C})$-representations of $\pi_1\Sigma_g$, and $hol : Q_g \to Rep$ denote the holonomy map, namely the mapping which sends each projective structure to its holonomy representation.

Let $X \in T_g$ be a hyperbolic surface. By the assumption that μ is an integral point, the holonomy representation of $Gr_\mu(X)$ is a Fuchsian group Γ_X with quotient surface X, hence the holonomy representation is in the space of real representations (namely, the equivalence class in Rep with a representative in $Hom(\pi_1\Sigma_g, \mathrm{PSL}(2, \mathbf{R}))$ which is denoted by $Rep_{\mathbf{R}}$. Now, let Y denote the complex structure under the projective structure $Gr_\mu(X)$, and $Q(Y)$ be the space of projective structures on Y, i.e., the fiber of Q_g over $Y \in T_g$. There is an element $\varphi \in Q(Y)$ corresponding to $Gr_\mu(X)$. Then $hol(Q(Y))$ intersects $Rep_{\mathbf{R}}$ at $hol(\varphi) = \Gamma_X$. By Faltings' theorem, (Faltings [3, Theorem 12]), this intersection is transversal. Therefore, at $hol(Gr_\mu(X)) = \Gamma_X$, we can take a basis $\{u_1, ..., u_{6g-6}\}$ of the (real) tangent space $T_{\Gamma_X}(hol(Q(Y)))$ and a basis $\{v_1, ..., v_{6g-6}\}$ of the (real) tangent space $T_{\Gamma_X}(Rep_{\mathbf{R}})$ such that $\{u_1, ..., u_{6g-6}, v_1, ..., v_{6g-6}\}$ forms the basis of the tangent space $T_{\Gamma_X}(Rep)$. Remember that hol is a local C^1-diffeomorphism (Hejhal). Therefore there is a neighborhood U of φ in Q_g and a neighborhood V of Γ_X in Rep such that $hol|U : U \to V$ is a C^1-diffeomorphism, and the inverse map $g : V \to U$ of $hol|U$ is well-defined. By the bundle structure of Q_g, we can take a neighborhood U' of Y in T_g such that the restriction $\pi|\pi^{-1}(U') : \pi^{-1}(U') \to U'$ is identified with the product of U' with $\mathbf{R}^{6g-6}$, where $\pi : Q_g \to T_g$ is the projection. Thus we may assume that U is the product of U' and an open set of $\mathbf{R}^{6g-6}$. Denote by ϕ the point of $\mathbf{R}^{6g-6}$ such that φ corresponds to $(Y, \phi) \in T_g \times \mathbf{R}^{6g-6}$. Then the tangent space $T_\varphi U$ is spanned by the 'direction of the base space' $T_Y T_g$ and the 'direction of the fiber' $T_\phi \mathbf{R}^{6g-6}$, and the derivative dg maps $T_\Gamma Rep$ onto $T_\varphi T_g$. Now, $\{dg(u_i)\}_{i=1,...,6g-6}$ is contained in the direction of fiber. Therefore, none of the non-zero vectors in $dg(T_{\Gamma_X} Rep_{\mathbf{R}})$ is contained in the direction of fiber. It follows that $d(\pi \circ g) : T_{\Gamma_X} Rep_{\mathbf{R}} \to T_Y T_g$ is surjective. As we can identify the component of the space of real representations containing Γ_X with the Teichmüller space, the composition of the restriction $g|Rep_{\mathbf{R}}$ with π is equal to gr_μ. Therefore, gr_μ is locally diffeomorphic at X. q.e.d.

Proof of (3). Let $QF(\mu)$ be the set of projective structures obtained by quasiconformal deformations of a grafted projective structure $Gr_\mu(X)$. Then $QF(\mu)$ is identified with the space of quasiconformal deformations of the holonomy representation of $Gr_\mu(X)$, which is a Fuchsian group (cf. [13]). Recall that T_g has a natural complex structure and the space of quasiconformal deformations of the Fuchsian group is identified with the complex manifold $T_g \times T_g$. With respect to this identification, the mapping $\Pi : QF(\mu) \to T_g$ which sends each projective structure in $QF(\mu)$ to the underlying complex structure is holomorphic (cf. [13]). Now, in the space of quasiconformal deformations of a Fuchsian group, the set of Fuchsian groups , which is identified with T_g, forms a real analytic submanifold. Therefore, the restriction of $\Pi : QF(\mu) \to T_g$ to this set of Fuchsian groups is real analytic. This restriction is the same mapping as $gr_\mu : T_g \to T_g$. q.e.d.

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