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THE GEOMETRY OF THE SPACE OF HOLOMORPHIC MAPS FROM A RIEMANN SURFACE TO A COMPLEX PROJECTIVE SPACE

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0. Introduction

In recent years there have been a number of papers on the homology and geometry of spaces of holomorphic maps of the Riemann sphere into complex varieties, [21], [5], [17], [18], [3], [9]. However, the very classic question of the structure of the spaces of holomorphic maps from complex curves M_g of genus $g \geq 1$ to complex varieties has proved to be very difficult. There is Segal's stability theorem, [21], which shows that the natural inclusion of the space of based holomorphic maps of degree k into the space of all based maps $Hol_k^*(M_g, V) \hookrightarrow Map_k^*(M_g, V)$ is a homotopy equivalence through a range of dimensions which increases with k when $V = \mathbf{P}^n$, the complex projective space. Also, there is the extension of this result by J. Hurtubise to further V; [11]. But that is about all.

In this paper we begin the detailed study of the topology of the $Hol_k^*(M_g, \mathbf{P}^n)$. We are able to completely determine these spaces and their homology when M_g is an elliptic curve, and we give an essentially complete determination in the case where M_g is hyperelliptic. In particular we determine the rational homology of these spaces when $k \geq 2g-1$ in the elliptic and hyperelliptic cases.

Let M_g be a genus g complex curve. The key analytic result on the structure of $Hol_k^*(M_g, \mathbf{P}^1)$ is Abel's Theorem which identifies the

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disjoint pairs of k-tuples of unordered points $\langle r_1, \ldots, r_k \rangle$ and $\langle p_1, \ldots, p_k \rangle$ on M_g that are the roots and poles of a meromorphic function on M_g in terms of the Abel-Jacobi embedding of M_g into its Jacobi variety, $\mu \colon M_g \to J(M_g)$. This extends to give necessary and sufficient conditions for an $(n+1)$ tuple $\{V_i \mid V_i = \langle x_{i,1}, \ldots, x_{i,k} \rangle, i = 0, \ldots, n\}$ of points in M_g with $\cap_0^n \{V_i\} = \emptyset$ to be the root data for a holomorphic map of M_g into $\mathbf{P}^n$, $n \geq 2$, and thus defines an embedding

$$Hol_k^*(M_g, \mathbf{P}^n) \subset (SP^k(M_g))^{n+1},$$

where $SP^k(X)$ is the k-fold symmetric product of X.

We begin by studying a compactification of the space $Hol_k^*(M_g, \mathbf{P}^n)$, which we denote $E_k^n(M_g)$ obtained by taking the closure of the embedding

$$Hol_k^*(M_g, \mathbf{P}^n) \subset (SP^k(M_g))^{n+1}$$

above. We show in (2.2) that $E_k^n(M_g)$ is given as the total space of a fibering,

$$(\mathbf{P}^{k-g})^{n+1} \longrightarrow E_k^n(M_g) \longrightarrow J(M_g)$$

for $k \geq 2g-1$, with $H^*(E_k^n(M_g); \mathbf{A}) = H^*(\mathbf{P}^{k-g}, \mathbf{A})^{n+1} \otimes H^*(J(M_g); \mathbf{A})$ for $\mathbf{A}$ any commutative ring of coefficients. Moreover, for $k \leq 2g-2$, $E_k^n(M_g)$ is stratified by strata which are fiberings

$$(\mathbf{P}^s)^{n+1} \to S_k^s \to A_k^s$$

where the A_k^s are subspaces of $J(M_g)$ determined by the curve. Specifically, $W_k^j \subset J(M_g)$ is the subspace of points in the image of the k^{th} Abel-Jacobi map

$$\mu_k \colon SP^k(M_g) \to J(M_g)$$

with $(\mu_k)^{-1}(x) = \mathbf{P}^s$ where $s \geq \max(k-g, 0) + j$, and $A_k^j = W_k^j - W_k^{j+1}$. Both spaces W_k^j and A_k^j are extensively studied in [1], [10] (cf. §5).

This compactification has the property that $Hol_k^*(M_g, \mathbf{P}^n)$ is open in $E_k^n(M_g)$, and we write $V_k^n(M_g)$ for the (closed) complement $E_k^n(M_g) - Hol_k^*(M_g, \mathbf{P}^n)$. The space $E_k^n(M_g)$ being a closed, compact manifold, Alexander-Poincaré duality now gives

$$\tilde{H}^{2k(n+1)-2ng-*}(Hol_k^*(M_g, \mathbf{P}^n); \mathbf{F}) \cong H_*(E_k^n(M_g), V_k^n(M_g); \mathbf{F})$$

for $k \geq 2g-1$. This then indicates that one can understand $Hol_k^*(M_g, \mathbf{P}^n)$ by first studying $V_k^n(M_g)$.

The space $V_k^n(M_g) \subset E_k^n(M_g)$ is the union of two pieces. The first, for $k > 2g-1$, is a subfibration $Z_k^n(M_g)$ of the form

$$\left\{ \bigcup_{i=0}^{n+1} (\mathbf{P}^{k-g})^i \times \mathbf{P}^{k-g-1} \times (\mathbf{P}^{k-g})^{n-i} \right\} \longrightarrow Z_k^n(M_g) \longrightarrow J(M_g)$$

with analogous definitions of $Z_k^n(M_g)$ for $k \leq 2g-1$ (see Definition 2.6). Moreover, it is easy to determine the relative cohomology groups, $H^*(E_k^n(M_g), Z_k^n(M_g); \mathbf{F})$ for $k > 2g-1$, and not too difficult for $k \leq 2g-2$, provided we know enough about the W_k^j.

In the case of hyperelliptic and elliptic curves the Riemann-Roch Theorem determines the W_k^j explicitly. Moreover, a complete description of the W_k^j for all curves of genus ≤ 6 is given in [1, pp.206 - 211].

In §5 we study the way in which the W_k^i determine

$$H^*(E_k^n(M_g), Z_k^n(M_g); \mathbf{A})$$

in detail and obtain a spectral sequence converging to these groups with explicit E^1-term.

Proposition 5.9. *Suppose that $k \leq g$. Then there is a spectral sequence converging to $H^*(E_k^n(M_g), Z_k^n(M_g); \mathbf{A})$ with E^1-term*

$$\begin{aligned} E^1 = & H_*(SP^k(M_g), SP^{k-1}(M_g); \mathbf{A}) \\ & \oplus \coprod_i H_*(W_k^i, W_{k-1}^i; \mathbf{A}) \otimes \tilde{H}^*(S^{2i(n+1)}; \mathbf{A}), \end{aligned}$$

and in case $k \geq g$ then

$$\begin{aligned} E^1 = & \Sigma^{2(k-g)(n+1)} H_*(J(M_g), W_{k-1}^{k-g}; \mathbf{A}) \\ & \oplus \coprod_{i > k-g} H_*(W_k^i, W_{k-1}^i; \mathbf{A}) \otimes \tilde{H}_*(S^{2i(n+1)}; \mathbf{A}). \end{aligned}$$

The second part of $V_k^n(M_g)$ consists of the image of an action map

$$\nu\colon\ M_g \times E_{k-1}^n(M_g) \rightarrow E_k^n(M_g)$$

introduced in (2.4) that puts in redundant roots, and is the main source of difficulty in recovering the homology of $Hol_k^*(M_g, \mathbf{P}^n)$ from the homology groups $H_*(E_k^n(M_g); \mathbf{A})$ or $H_*(E_k^n(M_g), Z_k^n(M_g); \mathbf{A})$. In order to handle this part we construct a spectral sequence which converges to the homology of the pair $(E^n_k(M_g), V^n_k(M_g))$ starting with the relative groups

$$H_*(E^n_k(M_g), Z^n_k(M_g); \mathbf{F}).$$

Theorem 4.3. *There is a spectral sequence converging to*

$$H_*(E^n_k(M_g), V^n_k(M_g); \mathbf{F})$$

for any field $\mathbf{F}$ *with* E^1*-term*

$$\begin{aligned} E^1 \;=\; \coprod_{\substack{i+j=k \\ i\geq 1}} & H_*(E^n_i(M_g), Z^n_i(M_g); \mathbf{F}) \\ & \otimes H_*(SP^j(\Sigma M_g), SP^{j-1}(\Sigma M_g); \mathbf{F}) \\ & \oplus H_*((SP^k(\Sigma M_g), SP^{k-1}(\Sigma M_g); \mathbf{F}). \end{aligned}$$

The rest of §4 discusses the structure of d_1 and various properties such as the multiplicative pairing of the spectral sequences discussed immediately after the proof of (4.3). Of course the structure of the groups $H_*(SP^j(\Sigma M_g), SP^{j-1}(\Sigma M_g); \mathbf{F})$ is well known from e.g. [7], [6], and [19]. It is also reviewed in §6.

Next we use these results to clarify the structure of the natural inclusions

$$Hol^*_k(M_g, \mathbf{P}^n) \hookrightarrow Map^*_k(M_g, \mathbf{P}^n).$$

The spectral sequence of (4.3) has a natural break at $i = 2g-1$ in the sense that $i \geq 2g-1$ implies that the relative homology groups

$$H_*(E^n_i(M_g), Z^n_i(M_g); \mathbf{F}) = \begin{cases} 0 * < 2(i-g)(n+1), \\ H_{*-2(i-g)(n+1)}(J(M_g); \mathbf{F}) & \text{otherwise}, \end{cases}$$

and, when $i < 2g-1$, the groups depend on the structure of μ_i, the W^r_i, and have to be determined case by case. We call the case $i \geq 2g-1$ the stable range for the spectral sequence and completely determine the differentials in this range. For $* > (2n-1)(k-2g+1)$ only the stable homology contributes to $H_*(E^n_k(M_g), V^n_k(M_g); \mathbf{F})$. Thus, by Poincaré duality, the unstable range only contributes homology above this range. On the other hand it is easily seen that the duals of the stable range classes inject into $H_*(Map^*_k(M_g, \mathbf{P}^n); \mathbf{F})$, and many of these classes live considerably above the stable range above. Thus, as was the case with

$Hol_k^*(\mathbf{P}^1, \mathbf{P}^n)$ ([5]) one has considerably more information about the map than was given by Segal's stability theorem.

In the final sections we study the cases of elliptic and hyperelliptic curves. Here, as indicated, the Riemann-Roch Theorem gives complete control of the W_k^r and consequently the E^1-term of the spectral sequence is within range of calculation. In particular, for elliptic curves the situation is completely understood.

Typical of the results in the elliptic case is

Lemma 9.4. *Let $I = (b_1, b_2)$ be the augmentation ideal in the polynomial ring $\mathbf{Q}[b_1, b_2]$ where $dim(b_i) = 2$, $i = 1, 2$. Then*

$$H_*(Hol_k^*(M_1, \mathbf{P}^1); \mathbf{Q}) \simeq \left\{ \mathbf{Q}[b_1, b_2]/I^{k+1} \right\} (1, e_1, e_2, h_1, h_2, v) \oplus \mathbf{Q}(w_1, w_2, \ldots, w_{2k-1}),$$

where $dim(e_i) = 1$, $dim(h_i) = 2$, $dim(w_i) = 2k - 3$ and $dim(v) = 3$.

Here M_1 is an arbitrary elliptic curve. Additionally, it turns out that the map

$$H^*(Map_k^*(M_1, \mathbf{P}^1); \mathbf{Q}) \longrightarrow H^*(Hol_k^*(M_1, \mathbf{P}^1); \mathbf{Q})$$

is surjective for all $k \geq 1$ in this case.

For hyperelliptic curves there are some technical questions which seem difficult to handle for finite field coefficients, but with some effort a complete determination of the E^1-term in (4.3) with $\mathbf{Q}$-coefficients is given in §10 –15, together with sufficient differentials to completely determine E^∞ and conclude that $H_*(Hol_k^*(M_g, \mathbf{P}^n); \mathbf{Q})$ injects into $H_*(Map_k^*(M_g, \mathbf{P}^n); \mathbf{Q})$ for $k \geq 2g - 1$ in the hyperelliptic case as well.

As the arguments are pretty involved we summarize the salient points here.

To begin, the Riemann-Roch Theorem gives a complete determination of the W_i^r as quotients of $SP^i(M_g)$ via an action discussed in the proof of Lemma 10.2,

$$SP^r(\mathbf{P}^1) \times SP^l(M_g) \longrightarrow SP^{l+2r}(M_g)$$

induced from the Abel-Jacobi map $\mu_2 : SP^2(M_g) \rightarrow J(M_g)$ which fails to be an embedding at only one point where it has inverse image a copy of $\mathbf{P}^1$, the $\mathbf{P}^1$ in the action map above.

These observations give the following result for hyperelliptic curves:

Lemma 10.2. *Suppose that M_g is hyperelliptic and $\tau \in J(M_g)$ is the hyperelliptic point. Let $k \geq 1$ and $t \leq \left[\frac{k}{2}\right]$. Then we have the following:*

(a) for $k \leq g$, the space W_k^t is $\mu_{k-2t}(SP^{k-2t}(M_g)) + t\tau$,

(b) for $2g-1 > k > g$, $t > k-g$, we also have

$$W_k^t = \mu_{k-2t}(SP^{k-2t}(M_g)) + t\tau.$$

Of course, this gives the W_k^t as quotients, so, in order to obtain information about the W_k^t we introduce some spectral sequences which take care of the details of the quotienting process in §11 through §13. Using them we are able to determine the rational homology of the W_k^t as follows.

Lemma 13.7.

(a) The inclusion $W_j \subset J(M_g)$ induces an injection in rational homology $H_*(W_j; \mathbf{Q}) \hookrightarrow H_*(J(M_g); \mathbf{Q}) = \Gamma(e_1, \ldots, e_{2g})$ with image the subvector space spanned by the subspaces

$$\left\{\Gamma_s(e_1, \ldots, e_{2g})[M_g]^t \mid s+t \leq j\right\}$$

where $[M_g] = \sum_1^g e_{2i-1}e_{2i}$ is the image of the fundamental class of M_g under the Abel-Jacobi map μ_*.

(b) $H_*(W_{j-1}; \mathbf{Q})$ injects into $H_*(W_j; \mathbf{Q})$ under the inclusion so the relative groups are given as

$$H_*(W_j, W_{j-1}; \mathbf{Q}) \cong H_*(W_j; \mathbf{Q})/H_*(W_{j-1}; \mathbf{Q}).$$

Next we turn to the homology of the spaces $Hol_k^*(M_g, \mathbf{P}^n)$ in the hyperelliptic cases. This involves using the spectral sequences and calculations above. Of course, the E^1-term in the spectral sequence (4.3) is very complex, but one is able to recognize in it the direct sum of a family of chain complexes, each of which calculates a part of a certain Tor or Ext group of the exterior algebra $\Gamma(e_1, \ldots, e_{2g})$ modulo the two sided ideal generated by $[M_g]$. This explains §14 which is devoted to the calculation of the relevant Tor-groups.

Finally, in §15 we are able to put these results together to obtain our main calculational result,

Theorem 15.1. *The natural map*

$$H_*(Hol_k^*(M_g, \mathbf{P}^n); \mathbf{Q}) \to H_*(Map_k^*(M_g, \mathbf{P}^n); \mathbf{Q})$$

is injective for $k \geq 2g$ and $n > 2$ if M_g is hyperelliptic.

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1. Preliminaries on the Abel-Jacobi map

We review some classical definitions and theorems about the algebraic geometry of curves. We begin by defining the Abel-Jacobi map together with the Jacobi variety $J(M_g)$ associated to any positive genus Riemann surface. Good references are [1] and [10].

Any Riemann surface M_g of genus $g \geq 1$ has g independent holomorphic sections of the cotangent bundle $\tau^*(M_g)$, (holomorphic 1-forms), $w_1, w_2, \ldots, w_g$, the *abelian differentials* on M_g.

Fix a basepoint $p_0 \in M_g$. Then we can associate to each point $p \in M_g$ and each path γ between p_0 and p the vector of integrals

$$\mu_\gamma(p) = \left(\int_{p_0}^{p} w_1, \ldots, \int_{p_0}^{p} w_g\right) \in \mathbf{C}^g.$$

Since any two paths γ and γ' between p_0 and p together determine a closed loop based at p_0, (which we will denote as L), we have that $\mu_\gamma(p)$ is well defined up to vectors of the form

$$\left(\int_L w_1, \ldots, \int_L w_g\right).$$

If we choose a set of loops $L_1, \ldots, L_{2g}$ which, in homology, form a basis for $H_1(M_g;\mathbf{Z})$, they give rise to the following $g\times 2g$ *period matrix*

$$\Omega=\begin{pmatrix}\int_{L_1}w_1 & \cdots & \int_{L_{2g}}w_1\\ \vdots & \ddots & \vdots\\ \int_{L_1}w_g & \cdots & \int_{L_{2g}}w_g\end{pmatrix}.$$

Thus, since the w_i are *closed*, the values $\mu_\gamma(p)$ depend only on p and not γ in the quotient torus

$$J(M_g)\ =\ \mathbf{C}^g/\Omega\cong(S^1)^{2g}.$$

Consequently, they give rise to a well defined map $\mu\colon M_g\to J(M_g)$ which is called the *Abel-Jacobi map* for M_g.

At this point we need to introduce symmetric products.

Definition. The *n-fold symmetric product $SP^n(X)$ of the space* X is defined to be the set of unordered n-tuples of points of X; i.e.,

$$SP^n(X)=X^n/\mathcal{S}_n,$$

where $\mathcal{S}_n$ is the symmetric group on n letters. A point in $SP^n(X)$ will be written in the form $\sum m_ix_i$ with $m_i>0$ and $\sum m_i=n$, or in the form

$$\langle x_1,x_2,\dots,x_n\rangle.$$

Remarks. Let X be any CW complex with base point $*$. Then there are inclusions $SP^n(X)\hookrightarrow SP^{n+1}(X)$ which identify $\sum_i n_iP_i$ with $\sum_i n_iP_i+*$, and we get the increasing sequence of spaces

$$*=SP^0(X)\subset SP^1(X)\subset\cdots\subset SP^{n-1}(X)\subset SP^n(X)\subset\cdots.$$

The union of this sequence is the infinite symmetric product $SP^\infty(X)$, based at $*$. Moreover, if X is path connected, then the homotopy type of $SP^\infty(X)$ is independent of the choice of $*$.

We can extend the Abel-Jacobi map to the symmetric products of M_g by the rule $\langle m_1,\dots,m_k\rangle\mapsto\mu(m_1)+\cdots+\mu(m_k)$, obtaining the family of maps

$$\mu_k\colon SP^k(M_g)\longrightarrow J(M_g).\tag{1.1}$$

The map μ is called the Abel-Jacobi map and it, together with the extensions μ_k, is a critical piece of the structure data for M_g.

It is a remarkable result due to Andreotti that for any M_g the symmetric products $SP^k(M_g)$ are all complex manifolds, indeed, complex algebraic varieties of complex dimension k. To see this note that the symmetric product $SP^k(\mathbf{C})$ is diffeomorphic to $\mathbf{C}^k$ via the map that takes the unordered collection $\langle z_1, \ldots, z_k \rangle$ of points in $\mathbf{C}$ to the coefficients of the monic polynomial of degree k with the z_i, $1 \leq i \leq k$ as roots. Since locally $SP^k(M_g)$ is modeled on $SP^k(\mathbf{C})$, the result follows.

Remark 1.2. In the special case that $M_g = \mathbf{P}^1$ a slight extension of the above argument identifies $SP^k(\mathbf{P}^1)$ with $\mathbf{P}^k$ which is now regarded as the space of all homogeneous polynomials of degree k in 2 variables.

Remark 1.3. In (1.14) we point out an extension of this result due to Mattuck which for $k \geq 2g-1$ identifies $SP^k(M_g)$ with the total space of a fibration over $(S^1)^{2g}$ with fiber $\mathbf{P}^{k-g}$.

Remark 1.4. Another way of thinking about $J(M_g)$ is as the Picard group of M_g, the space of isomorphism classes of holomorphic line bundles on M_g. From this point of view the addition in $J(M_g)$ corresponds to the tensor product of line bundles. Under this correspondence, the map μ takes $m \in M_g$ to the line bundle obtained from the trivial bundle over M_g by gluing in a copy of the negative Hopf bundle over m (see [10]).

We can associate to every holomorphic function $f \in Hol(M_g, \mathbf{P}^1)$ a divisor (f) defined by $(f) = \sum n_i Z_i - \sum m_j P_j$ where $\{Z_i\}$, $\{P_i\}$ are respectively the zeros and poles of f with multiplicities n_i and m_i respectively. By standard residue calculations, it is easy to see that $\sum n_i = \sum m_j$. The space of pairs of disjoint divisors on a Riemann surface (ζ, η), subject to the condition $\deg(\zeta) = \deg(\eta)$, constitutes then the first step in the description of the space of holomorphic maps from the surface to the Riemann sphere $\mathbf{P}^1$.

For maps of $\mathbf{P}^1$ to itself this description is enough for there do exist meromorphic maps having prescribed roots and poles (of equidegree); i.e., any divisor $D = \zeta - \eta$, $\deg D = 0$ is the divisor of a meromorphic function on $\mathbf{P}^1$. For the general case $g \geq 1$ it turns out that a pair (ζ, η) as above need not necessarily give rise to a meromorphic function and one needs a further condition.

Theorem 1.5 (Abel). *Given positive divisors D and D' on M_g, $deg D = deg D'$, then there exists an $f \in Hol(M_g, \mathbf{P}^1)$ so that $(f) = D - D'$ if and only if $\mu(D) = \mu(D')$.*

The f associated to the difference $D - D'$ is unique provided we specify in advance the image of p_0 (based maps) and neither D nor D' contains the basepoint, p_0. Additionally, given $D - D'$, there are unique *disjoint* positive divisors D_1, D_1' so that $D - D' = D_1 - D_1'$ and any f with $(f) = D - D'$ will have roots precisely the terms in D_1 and poles in D_1'. We make the following definition.

Definition 1.6. The *divisor* space $\mathrm{Div}_k(X)$ for a given space X is the set of pairs of *disjoint* positive divisors on X, i.e.,

$$\mathrm{Div}_k(X) = \left\{(D, D') \in SP^k(X) \times SP^k(X) \mid D \cap D' = \emptyset\right\}, \tag{1.7}$$

and more generally

$$\begin{aligned}\mathrm{Div}_k^n(X) = \Big\{ &(D_1, \ldots, D_{n+1} \mid D_i \in SP^k(X),\ 1 \leq i \leq n+1, \\ &D_1 \cap D_2 \cap \cdots \cap D_{n+1} = \emptyset\Big\}.\end{aligned} \tag{1.8}$$

Corollary 1.9. *The space of based holomorphic maps of degree k, $Hol_k^*(M_g, \mathbf{P}^1)$, is the inverse image of 0 under the subtraction map*

$$s\colon Div_k(M_g - p_0) \longrightarrow J(M_g),$$

where the subtraction map s is given by $s((D, D')) = \mu(D) - \mu(D')$.

More generally, and perhaps more usefully, we have

Corollary 1.10. *The space of based holomorphic maps of degree k, $Hol_k^*(M_g, \mathbf{P}^n)$ is the subspace of $Div_k^n(M_g - p_0)$ consisting of $(n+1)$-tuples of degree k positive divisors, subject to the following constraint:*

$$\mu_k(D_1) = \mu_k(D_2) = \cdots = \mu_k(D_{n+1}).$$

It was using this formulation of the space $Hol_k^*(M_g, \mathbf{P}^n)$ that Segal [21] proceeded to prove his stability result. We will shortly give and use a yet more explicit version of this corollary. But before we do so, here are some of the standard results on the Abel-Jacobi map that we will be using:

(1.11) The orginal map $\mu\colon M_g \to J(M_g)$ is an embedding and the (complex) dimension of the image of $\mu_d\colon SP^d(M_g) \to J(M_g)$ is d for $d \leq g$. In particular μ is onto for $d \geq g$ *(Jacobi inversion theorem).*

(1.12) The preimage of any point $\mu(p) \in J(M_g)$, $\mu_d^{-1}(\mu(p)) \in SP^d(M_g)$ is always a complex projective plane $\mathbf{P}^m$ for some $m \geq 0$.

(1.13) For $d \leq 2g-2$ the dimension m is less than or equal to $\frac{d}{2}$ *(Clifford)*.

(1.14) For $d \geq 2g-1$ the map μ_d makes $SP^d(M_g)$ into an analytic fiber bundle over $J(M_g)$ with fiber $\mathbf{P}^{d-g}$ *(Mattuck [16])*.

In §9 we will give more details on the structure of these maps for $g \leq 5$, but now we turn to the construction of an explicit model for the space $\mathrm{Hol}_k^*(M_g, \mathbf{P}^n)$ from our considerations thus far.

2. A compactified version of $\mathbf{Hol}_k^*(M_g, \mathbf{P}^n)$

It should be apparent from §1 that we are interested in the inverse image of 0 under $s\colon \mathrm{Div}_k(M_g - x_0) \to J(M_g)$ for holomorphic maps to $\mathbf{P}^1$, and generally in the inverse image of the iterated diagonal

$$\Delta^{n+1}(J(M_g) \subset \underbrace{J(M_g) \times \cdots \times J(M_g)}_{n+1-times}$$

under $\mu_k \times \cdots \times \mu_k$ for maps into $\mathbf{P}^n$.

Let the space $E_{i_0,i_1,\dots,i_n}$ be defined as the fiber product of $\mu \times \cdots \times \mu$ and Δ in the diagram below

$$(2.1)\qquad \begin{array}{ccc} E_{i_0,i_1,\dots,i_n} & \longrightarrow & SP^{i_0}(M_g) \times \cdots \times SP^{i_n}(M_g) \\ \Big\downarrow & & \Big\downarrow{\scriptstyle \mu\times\cdots\times\mu} \\ J & \xrightarrow{\Delta} & J(M_g) \times \cdots \times J(M_g). \end{array}$$

More explicitly

$$E_{i_0,i_1,\dots,i_n} = \{(D_0, \dots , D_n) \in SP^{i_0}(M_g) \times \cdots \times SP^{i_n}(M_g) \mid \mu(D_i) = \mu(D_j)\}.$$

Clearly $\mathrm{Hol}_k^*(M_g, \mathbf{P}^n) \subset E_{k,k,\dots,k}$ as the subset consisting of all the $\{(D_0, \dots, D_n)\}$ where no D_i contains $*$ and $\bigcap_0^n D_i = \emptyset$. Of course, for k sufficiently small $\mathrm{Hol}_k^*(M_g, \mathbf{P}^n)$ is empty and this is consistent with the fact that Riemann surfaces of positive genus do not admit holomorphic functions with a single pole. However, once $\mathrm{Hol}_k^*(M_g, \mathbf{P}^n)$ is non-empty, and k is sufficiently large, $\mathrm{Hol}_k^*(M_g, \mathbf{P}^n)$ will be open in

$E_{k,\dots,k}$ with $E_{k,\dots,k}$ as its closure. Then $E_{k,k,\dots,k}$ is the compactification of $\mathrm{Hol}_k^*(M_g, \mathbf{P}^n)$ mentioned in the title of this section.

Lemma 2.2. *For $i_j \geq 2g-1$, $0 \leq j \leq n$, we have a fibering*

$$\prod_{j=0}^{n} \mathbf{P}^{i_j - g} \longrightarrow E_{i_0,i_1,\dots,i_n} \longrightarrow J(M_g),$$

and the fiber is totally non-homologous to zero so

$$H_*(E_{i_0,i_1,\dots,i_n}) \cong H_*(\mathbf{P}^{i_0-g}) \otimes \cdots \otimes H_*(\mathbf{P}^{i_n-g}) \otimes H_*(J(M_g)).$$

Proof. In this range of dimensions $i_j \geq 2g-1$, one can see by virtue of Mattuck's Theorem that the space $E_{i_0,i_1,\dots,i_n}$ becomes the total space of a fibration

$$\mathbf{P}^{i_0-g} \times \cdots \times \mathbf{P}^{i_n-g} \longrightarrow E_{i_0,i_1,\dots,i_n} \longrightarrow J(M_g)$$

obtained from the pull-back of a product of Mattuck's fibrations

$$\mathbf{P}^{i_0-g} \times \cdots \times \mathbf{P}^{i_n-g} \longrightarrow SP^{i_0}(M_g) \times \ldots \times SP^{i_n}(M_g) \longrightarrow J(M_g)^{n+1}$$

by the diagonal inclusion Δ. Since Δ is injective and since the Serre spectral sequence for Mattuck's fibration collapses at E_2, the lemma follows from standard spectral sequence comparison arguments. q.e.d.

Further Properties of the $E_{i_0,\dots,i_n}$

First of all, we notice that for any $j \geq 0$, we have natural inclusions

$$E_{i_0,i_1,\dots,i_n} \hookrightarrow E_{i_0,\dots,i_j+1,\dots,i_n}$$

given by adding the basepoint in the $(j+1)^{st}$ position.

Secondly we observe that the map

$$\mu \times \cdots \times \mu \colon SP^{i_0}(M_g) \times \cdots \times SP^{i_n}(M_g) \to J(M_g)^{n+1}$$

is multiplicative by construction, and so it induces a multiplicative pairing on the $E_{i_0,\dots,i_n}$ which is commutative and associative:

$$\nu^{I,J} \colon E_{i_0,\dots,i_n} \times E_{j_0,\dots,j_n} \longrightarrow E_{i_0+j_0,\dots,i_n+j_n}. \tag{2.3}$$

Also, $E_{1,1,\dots,1} = M_g$ and the pairing above thus yields an action

$$\nu \colon \left(\coprod_{k=1}^{\infty} SP^k(M_g) \right) \times E_{i_0,\dots,i_n} \longrightarrow \coprod_{k=1}^{\infty} E_{i_0+k,\dots,i_n+k} \tag{2.4}$$

explicitly defined via the diagonal multiplication

$$SP^k(M_g)\times\big(SP^{i_0}(M_g)\times\cdots\times SP^{i_n}(M_g)\big)$$
$$\longrightarrow SP^{i_0+k}(M_g)\times\cdots\times SP^{i_n+k}(M_g),$$

which acts on points as follows

$$\Big(\sum m_s,(\langle m_{11},\ldots,m_{1,i_0}\rangle,\ldots,\langle m_{n1},\ldots,m_{ni_n}\rangle)\Big)$$
$$\mapsto\Big(\langle\sum m_s,m_{11},\ldots,m_{1,i_0}\rangle,\ldots,\langle\sum m_s,m_{n1},\ldots,m_{ni_n}\rangle\Big).$$

We can then give an explicit reformulation of the description of $\mathrm{Hol}_i^*(M_g,\mathbf{P}^n)$ in these terms.

Lemma 2.5. *Let $\tilde{E}_{i_0,\ldots,i_n}\subset E_{i_0,\ldots,i_n}$ be the subspace in which no m_{rs} is equal to the base point $*$. Then,*

$$Hol_i^*(M_g,\mathbf{P}^n)\cong\tilde{E}_{i,i,\ldots,i}-\mathrm{Image}(\nu)\cap\tilde{E}_{i,i,\ldots,i}.$$

The following constructions are now needed for the remainder of our discussion.

Definition 2.6. For each n, the space LE_i is the quotient

$$LE_i\;=\;E_{i,i,\ldots,i}\Big/\Big\{\bigcup E_{i,i,\ldots,i-1,i,\ldots,i}\Big\}.$$

The space QE_i is the quotient

$$LE_i\big/\left\{\mathrm{Image}(\nu)\right\}.$$

Remark 2.7. When $i\geq 2g-1$, the space $E_{i,\ldots,i}$ is a manifold of dimension $2(n+1)i-2ng=2(i-g)(n+1)+2g$. However when $i<2g-1$ there is no garantee that $E_{i,\ldots,i}$ is actually a manifold.

Using Alexander-Poincaré duality, we can now deduce

Lemma 2.8. *Assume $i\geq 2g-1$. Then for untwisted coefficients* $\mathbf{A}$ *we have*

$$H_j(Hol_i^*(M_g,\mathbf{P}^n);\mathbf{A})\cong H^{2(i-g)(n+1)+2g-j}(QE_i;\mathbf{A}).$$

Note at this point that the multiplication μ of (2.3) induces an associative, commutative multiplication on the QE_i as well:

$$\mu\colon QE_i \times QE_j \longrightarrow QE_{i+j}.$$

From Lemma (2.8), it is clear that it is the space QE_i that we wish to study in the remainder of this paper. Unfortunately, it is generally very hard to obtain the cohomology of such a space without a careful analysis of the piece we collapse out. So, in order to do this we follow the procedure of [5] and replace the cone on the union above by a more complex but much more structured space.

3. A model for QE_i

Consider the twisted product space

$$DE(M_g) = \Big(\bigcup E_{i_0,i_1,\dots,i_n}\Big) \times_t SP^\infty(cM_g), \tag{3.1}$$

where cT denotes the reduced cone on T, and the twisting t is given by the action above. Precisely, points of DE are of the form

$$\{(D_0,\dots,D_n),(t_1,z_1)\dots(t_l,z_l)\},\ D_i \in SP^{k_i}(M_g) \text{ and } \mu(D_i)=\mu(D_j)\}$$

with the identification that when $t_i = 0$ the point above is identified with

$$\{(D_0+z_i,\dots,D_n+z_i),(t_1,z_1)\dots\widehat{(t_i,z_i)}\dots(t_l,z_l)\},$$

where the entry (t_i,z_i) is deleted from the last set of coordinates. Clearly $\mu(D_r+z_i)=\mu(D_s+z_i)$ and the construction makes sense.

The space DE is filtered by the subspaces

$$DE_{k_0,k_1,\dots,k_n}(M_g) = \bigcup_{\substack{i_0+l\le k_0\\ \vdots \\ i_n+l\le k_n}} E_{i_0,i_1,\dots,i_n} \times_t SP^l(cM_g).$$

Observe that there are projection maps

$$p_{k_0,k_1,\dots,k_n}\colon DE_{k_0,k_1,\dots,k_n} \longrightarrow E_{k_0,k_1,\dots,k_n}/\{\mathrm{Image}(\nu)\}$$

where

$$(v_1,\dots,v_s,(t_1,w_1),\dots,(t_r,w_r)) \mapsto \{(v_1,\dots,v_s)\}$$

and the inverse images of points consist of contractible sets. This implies that the maps $p_{k_0,k_1,\dots,k_n}$ are acyclic and induce isomorphisms

$$H_*(DE_{k_0,k_1,\dots,k_n};\mathbf{F}) \longrightarrow H_*(E_{k_0,k_1,\dots,k_n}/\left\{\mathrm{Image}(\nu)\right\};\mathbf{F}).$$

We combine this with the isomorphism in Lemma 2.8 to get

Corollary 3.2. *Let $k \geq 2g-1$, then*

$$\begin{aligned}&\tilde{H}^{2k(n+1)-2ng-*}(Hol_k^*(M_g,\mathbf{P}^n);\mathbf{F})\\ &\quad\cong H_*(DE_{\underbrace{k,\dots,k}_{n+1}}/\bigcup_i DE_{k,\dots,\underbrace{k-1}_{ith-entry},\dots,k};\mathbf{F})\end{aligned}$$

This result is very useful because it is possible, using the filtration of the space $DE_{k,\dots,k}$ described above, to construct spectral sequences with known E_1-terms which converge to the cohomology of the relative spaces above for all $k \geq 1$.

4. The spectral sequence

The diagonal action. The diagonal multiplication introduced in (2.3)

$$SP^r(M_g) \times E_{i_0,i_2,\dots,i_n} \xrightarrow{\nu} E_{i_0+r,\dots,i_n+r}$$

induces an action in homology, ν_*,

$$\nu_*\colon \left(\coprod_{r=0}^{\infty} H_*(SP^r(M_g);\mathbf{F})\right) \otimes \left(\coprod_{j=1}^{\infty} H_*(LE_i;\mathbf{F})\right) \longrightarrow \coprod_{j=1}^{\infty} H_*(LE_{i+r};\mathbf{F}) \tag{4.1}$$

for any field coefficients $\mathbf{F}$. Of course, this quotient action fits together with the original action on the $E_{i,\dots,i}$ via the following commutative diagram

(4.2)

$$\begin{array}{ccc} H_*(SP^r(M_g);\mathbf{F}) \otimes H_*(E_{i,\dots,i};\mathbf{F}) & \xrightarrow{id\otimes p_*} & H_*(SP^r(M_g);\mathbf{F}) \otimes H_*(LE_i;\mathbf{F}) \\ \downarrow{\scriptstyle \nu_*} & & \downarrow{\scriptstyle \nu_*} \\ H_*(E_{i+r,\dots,i+r};\mathbf{F}) & \xrightarrow{p_*} & H_*(LE_{i+r};\mathbf{F}). \end{array}$$

Theorem 4.3. *There is a spectral sequence converging to* $H_*(QE_k;\mathbf{F})$ *with* E^1 *term*

$$\begin{aligned} E^1 \;=\; & \coprod_{\substack{i+j=k\\ i\geq 1}} \tilde{H}_*(LE_i;\mathbf{F})\otimes H_*(SP^j(\Sigma M_g),SP^{j-1}(\Sigma M_g);\mathbf{F}) \\ & \oplus H_*(SP^k(\Sigma M_g),SP^{k-1}(\Sigma M_g);\mathbf{F}) \end{aligned}$$

and $d_1(\Theta\otimes\{|a_1|\cdots|a_r|\}) = (-1)^{||\Theta||||a_1||}\nu_*(a_1\otimes\Theta)\otimes\{|a_2|\cdots|a_r|\}$.

Proof. From the model for QE constructed in §3 we have that

$$QE_k \cong DE_{k,\dots,k}/\bigcup DE_{k,\dots,k-1,\dots,k}$$

where

$$DE_{k,\dots,k}(M_g) = \bigcup_{i_r+j\leq k} E_{i_0,i_1,\dots,i_n}\times_t SP^j(cM_g).$$

Also, $LE_i = E_{i,\dots,i}/\left\{\bigcup E_{i,i,\dots,i-1,i,\dots,i}\right\}$ so we can write

$$QE_k = \bigcup_{i+j=k} LE_i\times_t\left(SP^j(cM_g)/SP^{j-1}(cM_g)\right),$$

where the twisting in the description above is given as before by

$$(h,\{(0,w_1),(t_2,w_2),\dots,(t_r,w_r)\}) \sim (\nu(w_1,h),\{(t_2,w_2),\dots,(t_r,w_r)\}).$$

To obtain the desired spectral sequence, introduce the filtration by

$$\mathcal{F}_r(QE_k) \;=\; \bigcup_{\substack{i\leq r\\ i+j=k}} LE_i\times_t\left(SP^j(cM_g)/SP^{j-1}(cM_g)\right),$$

and the remainder of the proof of the theorem is direct. q.e.d.

A multiplicative structure for the spectral sequence. The induced multiplication on the QE_i's,

$$\bar{\nu}\colon QE_i\times QE_j \longrightarrow QE_{i+j}$$

passes to the spectral sequences above and defines a pairing of E^1-terms:

$$\begin{aligned} & \left(H_*(LE_i;\mathbf{F})\otimes H_*(SP^j(\Sigma M_g),SP^{j-1}(\Sigma M_g);\mathbf{F})\right) \\ & \quad\otimes\left(H_*(LE_v;\mathbf{F})\otimes H_*(SP^w(\Sigma M_g),SP^{w-1}(\Sigma M_g);\mathbf{F})\right) \\ & \qquad\longrightarrow H_*(LE_{i+v};\mathbf{F})\otimes H_*(SP^{w+j}(\Sigma M_g),SP^{w+j-1}(\Sigma M_g);\mathbf{F}) \end{aligned}$$

for which the d_i's act as derivations. For this reason it is often convenient to consider all the spectral sequences above at once. In particular we can describe the direct sum of all the E_1-terms as a trigraded ring where an element $x \in E_1$ has tridegree $(i, j, *)$ if and only if

$$x \in H_s(LE_i) \otimes H_{*-s}(SP^j(\Sigma M), SP^{j-1}(\Sigma M); \mathbf{F}).$$

Remark 4.4. There is a related spectral sequence for $H_*(QE_k; \mathbf{F})$ obtained by filtering

$$QE_k = \bigcup_{i+j=k} LE_i \times_t \left(SP^j(cM_g)/SP^{j-1}(cM_g)\right)$$

in a somewhat different way. Instead of filtering by i in the expression above, filter by the number of *distinct* t's in the point

$$(h, \{(t_1, w_1), (t_2, w_2), \ldots, (t_r, w_r)\}).$$

In the space $SP^\infty(\Sigma M_g)$ this filtration results in the Eilenberg-Moore spectral sequence with E_2-term $Ext^{*,*}_{H_*(SP^\infty(M_g);\mathbf{F})}(\mathbf{F}, \mathbf{F})$. To describe the resulting spectral sequence most efficiently it is best to include all the QE_k's at once, and what results in our case is a trigraded E_2-term,

$$Ext^{r,s,*}_{H_{*,*}(SP^\infty(M_g);\mathbf{F})}(\mathbf{F}, \bigoplus_{i=1}^{\infty} H_*(LE_i; \mathbf{F})),$$

where the summand such that $r + s = k$ corresponds to the E_2-term of the spectral sequence converging to $H^*(QE_k; \mathbf{F})$.

Some remarks on differentials. There are similar spectral sequences for the Div-spaces (cf. 1.6) starting (as in 3.1) with the model

$$\left(\coprod_{j=0}^{\infty} \underbrace{SP^j(M_g) \times \cdots \times SP^j(M_g)}_{n+1-times}\right) \times_T SP^\infty(cM_g), \tag{4.5}$$

where T is now the diagonal twisting which identifies the point $(0, z)$ in the cone cM with the diagonal element $\Delta^{n+1}(z)$ in $(M_g)^{n+1}$ and then extends this multiplicatively. The associated spectral sequences have E_1-term

$$\begin{aligned}\coprod_{i+j=k} &\left[H^*(SP^i(M_g), SP^{i-1}(M_g); \mathbf{F})\right]^{n+1} \\ &\otimes H^*(SP^j(\Sigma M_g), SP^{j-1}(\Sigma M_g); \mathbf{F})\end{aligned} \tag{4.6}$$

and converge to $H^*(Div_{k,\dots,k}(M_g);\mathbf{F})$. (Here the superscript $n+1$ means the $(n+1)$-fold tensor product.) Moreover, as is clear, the inclusions

$$E_{i,\dots,i} \subset SP^i(M_g) \times \cdots \times SP^i(M_g)$$

induce maps of spectral sequences here (in cohomology) to the spectral sequences above for the Hol_k^* spaces. (Or in homology from the spectral sequences for the Hol_k^* spaces to these for the Div-spaces.)

But for the spectral sequences for the Div-spaces, and using H. Cartan's little constructions [4] to embed the homology into the chain complex one is able to construct an explicit (small) filtered chain complex with associated spectral sequence equal to that in Theorem 4.3, [12]. In particular one knows that $E_\infty = E_1$ for $n \geq 2$ for the Div-spaces spectral sequence 4.6.

In the next section we will identify a region of the spectral sequence for the Hol_k^* spaces where the induced map of homology spectral sequences is an injection. Hence, in this range for $n > 1$ the spectral sequence for $\mathrm{Hol}_k^*(M_g, \mathbf{P}^n)$ collapses at E_1. When $n = 1$ there are differentials, however our knowledge of the differentials in the Div-spectral sequence here implies considerable information about the differentials for the Hol_k^* spaces here as well.

The d^1-differential for the highest filtration terms of the spectral sequence. One region where the two sequences don't compare very well is the tail end of Theorem 4.3, the terms

$$\tilde{H}_*(M_g;\mathbf{F}) \otimes H_*(SP^{k-1}(\Sigma M_g), SP^{k-2}(\Sigma M_g);\mathbf{F})$$

and

$$H_*(SP^k(\Sigma M_g), SP^{k-1}(\Sigma M_g);\mathbf{F}).$$

More generally it can happen that $\mu_k\colon SP^k(M_g) \to J(M_g)$ is an embedding for $1 \leq k \leq m(M_g)$ in which case we have

Lemma 4.7. *If the Abel-Jacobi map $\mu_k\colon SP^k(M_g)\to J(M_g)$ is an embedding for $1 \leq k \leq m(M_g)$, then in this range we have $E_{k,\dots,k} \cong SP^k(M_g)$, $E_{k,\dots,k-1,k,\dots,k} \cong SP^{k-1}(M_g)$ included in $E_{k,\dots,k}$ via the usual base-point embedding*

$$SP^{k-1}(X) \subset SP^k(X), \qquad \langle x_1,\dots,x_{k-1}\rangle \mapsto \langle x_1,\dots,x_{k-1},*\rangle$$

and the action ν corresponds to the usual multiplication

$$SP^r(M_g) \times SP^{k-r}(M_g) \longrightarrow SP^k(M_g).$$

Consequently, in this range we have $LE_{k,\dots,k} \cong SP^k(M_g)/SP^{k-1}(M_g)$, $1 \le k \le m(M_g)$, *and the spectral sequence in this region is the corresponding spectral sequence for the quasi-fibration*

$$SP^\infty(M_g) \to SP^\infty(cM_g) \to SP^\infty(\Sigma M_g).$$

(The only statement above which might need clarification is the last. But recall that the spectral sequence with field coefficients has E_2-term

$$\bigoplus_{k,t} H^*(SP^k(M_g), SP^{k-1}(M_g); \mathbf{F}) \otimes H^*(SP^t(\Sigma M_g), SP^{t-1}(\Sigma M_g); \mathbf{F})$$

and all the differentials preserve the sum $k+t$. Moreover, in this region the chain embedding techniques of [12] are valid, so the two spectral sequences have the same (internal) differentials.)

For example, if M_g is not hyperelliptic then $\mu_2\colon SP^2(M_g) \to J(M_g)$ is an embedding. Also the map $\mu_3\colon SP^3(M_5) \to J(M_5)$ is an embedding for most curves of genus 5. (See the discussion in [1, chapter V].)

5. The Jacobi varieties W_i^j and a spectral sequence for the LE_i spaces

Our next step is to analyze the spectral sequence of Theorem 4.3. To do this we must understand the groups $H_*(LE_i)$. The LE_i are quotients of the spaces $E_i \equiv E_{i,\dots,i}$ defined in §2 and these latter spaces turn out to be built out of fibrations with projective spaces as fibers. Here there is a stratification of the image of $SP^n(M_g)$ in $J(M_g)$, and over each stratum we get such a fibration, though the dimensions of the fibers vary as we move from stratum to stratum.

Definition 5.1. The image of μ_d in $J(M_g)$ is written W_d. Also the set of points $y \in W_d$ so that $\mu_d^{-1}(y) = \mathbf{P}^m$ with $m \ge r$ is denoted W_d^r. Thus we have a decreasing filtration

$$W_d \supseteq W_d^1 \supseteq W_d^2 \supseteq \cdots \supseteq W_d^r \supseteq \cdots \supseteq W_d^{[d/2]+1} = \emptyset.$$

It is well known that $W_d = J(M_g)$ whenever $d \ge g$, and that the dimension m of a *generic* fiber $\mathbf{P}^m$ over W_d is 0 when $d \le g$ and $d-g$ when $d \ge g$.

Examples.

(5.2) The map μ_1 is always an embedding and so $W_1 \cong M_g$.

(5.3) In the genus 1 case the original map $\mu = \mu_1$ is the identity, and $J(M_1) = M_1$, while the μ_d are fiberings for $d \geq 2$.

Assume now that $g \geq 2$.

(5.4) The map $\mu_2 \colon SP^2(M_g) \to J(M_g)$ is an embedding unless M_g is hyperelliptic (cf §10). In the hyperelliptic case W_2^1 is a single point p and $\mu_2^{-1}(p) = \mathbf{P}^1$. It follows that W_2 can be identified with $SP^2(M_g)$ with a single $\mathbf{P}^1$ blown down. Indeed, we can check that the normal bundle to $\mathbf{P}^1 \subset SP^2(M_g)$ is ξ^{1-g}, the line bundle with self-interesection number $1-g$, and from this it follows that

(1) $SP^2(M_2)$ is the blowup of $J(M_2)$ at a single point (here any genus 2 surface is automatically hyperelliptic).

(2) For $g \geq 3$ we have that $W_2 = SP^2(M_g)/(\mathbf{P}^1 \sim *)$ is a manifold with a single isolated point singularity which looks like the cone on a Lens space L^3_{g-1}.

We can introduce the complementary subspace $A^i_k = W^i_k - W^{i+1}_k$. By definition, we have that $\forall x \in A^i_k$, $\mu_k^{-1}(x) = \mathbf{P}^i$. The result of Mattuck quoted in (1.14) takes actually the more general form.

Theorem 5.5 (Mattuck). $\mu_k^{-1}(A^i_k) \subset SP^k(M_g)$ *is the total space of a locally trivial analytic fibration* $\mathbf{P}^i \to \mu_k^{-1}(A^i_k) \to A^i_k$.

Remark 5.6. The following formula relating W^i_k and W^{i-1}_{k-1} is a special case of [10, (20), p. 53]:

$$(**) \qquad W^i_k = \bigcap_{m \in \mu(M_g)} (W^{i-1}_{k-1} + m),$$

that is, W^i_k is obtained as the intersection of all translates of elements of W^{i-1}_{k-1} by elements of $\mu(M_g) \subset J(M_g)$. In particular, this shows that

$$(* * *) \qquad W^i_k \subset W^{i-1}_{k-1} + * \subset W^{i-1}_k.$$

From now on we do not differentiate between $W^{i-1}_{k-1} \subset W_{k-1}$ and $W^{i-1}_{k-1} + * \subset W_k$; denoting both by W^{i-1}_{k-1}.

Furthermore, Gunning, [10, p. 54], gives the following dimension estimates for the associated containments.

Lemma 5.7. *If the subvariety* W^{i-1}_{k-1} *is non-empty then:*

- *dim* $W_k^i <$*dim* W_{k-1}^{i-1} *whenever* $2 \leq i \leq k \leq g$.
- *dim* $W_{k-2}^{i-1} <$ *dim* W_{k-1}^{i-1} *whenever* $2 \leq i \leq k \leq g$.

Remark 5.8. In chapter V of [1] techniques for determining the W_k^i's are extensively discussed, and results for $g \leq 6$ are completely given, (p. 206-211).

We now turn to the pull-back spaces $E_{k,\dots,k}$ and observe that the fibration of Lemma (5.5) induces in turn a fibration

$$(\mathbf{P}^i \times \cdots \times \mathbf{P}^i) \longrightarrow \mathcal{A}_k^i \xrightarrow{\pi} A_k^i,\ \mathcal{A}_k^i \subset E_{k,\dots,k}.$$

When collapsing $\mathcal{A}_{k-1}^i$, we get a quotient

$$X = \mathcal{A}_k^i / \mathcal{A}_{k-1}^i \hookrightarrow E_{k,\dots,k}/E_{k-1,\dots,k-1}$$

and a quotient map $X \longrightarrow LE_{k,\dots,k}$. The space X projects down via μ to

$$A_k^i / A_{k-1}^i = W_k^i / W_{k-1}^i;$$

this last equality being a consequence of the inclusion $W_k^{i+1} \subset W_{k-1}^i$ described above.

We then pass to the quotient LE_k and observe that since $W_k^i \subset W_{k-1}^{i-1}$, $i \geq 1$, we must further collape, along fibers this time, subsets of the form

$$\bigcup_{j \leq n+1} (\mathbf{P}^i)^{j-1} \times \mathbf{P}^{i-1} \times (\mathbf{P}^i)^{n+1-j} \hookrightarrow (\mathbf{P}^i)^{n+1}.$$

The fiber $(\mathbf{P}^i)^{n+1}$ has a top $2i(n+1)$ cell of the form

$$e^{2i(n+1)} = e_1^{2i} \times \cdots \times e_{n+1}^{2i},$$

where e_j^{2i} is the top $2i$ dimensional cell in the j-th copy $\mathbf{P}^i$ with boundary mapping to $\mathbf{P}^{i-1} \subset \mathbf{P}^i$. We then see that

$$(\mathbf{P}^i)^{n+1} / \cup_j (\mathbf{P}^i)^{j-1} \times \mathbf{P}^{i-1} \times (\mathbf{P}^i)^{n+1-j} \simeq e^{2i(n+1)} / \partial e^{2i(n+1)} \simeq S^{2i(n+1)}.$$

The space so obtained is denoted by T_k^i. In the case when $i = 0$ we have that $W_k^1 \subset W_{k-1} \subset W_k$, and hence $W_k / W_{k-1} \cong SP^k(M_g)/SP^{k-1}(M_g)$.

Now we filter the Jacobian according to the increasing sequence of W_i's

$$J(M_g) = W_g \supset W_{g-1} \supset \cdots \supset W_1 = M_g.$$

This induces a filtration on LE_k yielding a spectral sequence which by the preceeding discussion has E^1 term as follows.

Proposition 5.9. *Suppose that $k \leq g$. Then there is a spectral sequence converging to $\tilde{H}_*(LE_k)$ with E^1 term*

$$E^1 = H_*(SP^k(M_g), SP^{k-1}(M_g)) \oplus \coprod_i H_*(W_k^i, W_{k-1}^i) \otimes \tilde{H}_*(S^{2i(n+1)})$$

and in case $k \geq g$, then

$$\begin{aligned} E^1 = &\Sigma^{2(k-g)(n+1)} H_*(J(M_g), W_{k-1}^{k-g}) \\ &\oplus \coprod_{i>k-g} H_*(W_k^i, W_{k-1}^i) \otimes \tilde{H}_*(S^{2i(n+1)}). \end{aligned}$$

Remark 5.10. Note that

$$H_*(LE_k) = H_*(J(M_g)) \otimes \tilde{H}_*(S^{2(k-g)(n+1)}),$$

whenever $k > 2g-1$ for then $W_{k-1}^{k-g} = \emptyset$ and $W_k^i = W_{k-1}^i = J(M_g)$ otherwise.

Remark 5.11. The spectral sequence of (5.9) turns out to collapse at E^1 for all cases that we treat in this paper.

6. The structure of symmetric products

In this section, we describe the homology of the symmetric products $SP^n(M_g)$ for all n and g. Also, since it is required in the spectral sequences of Theorem 4.3 and Proposition 5.9, we give the structure of $H_*(SP^n(\Sigma M_g))$ again for all n, g. In the case of $SP^n(M_g)$ we follow the description given by I.G. Macdonald, [15], while the description for the suspension is taken from [19] and unpublished work of N. Steenrod.

There is an evident pairing $SP^n(X) \times SP^m(X) \xrightarrow{+} SP^{n+m}(X)$ given by addition of points, and this turns $SP^\infty(X)$ into an associative, abelian monoid with $*$ as a two sided identity. The Dold and Thom theorem states that

$$SP^\infty(X) \simeq \prod_1^\infty K(H_i(X;\mathbf{Z}), i) \tag{6.1}$$

is a product of Eilenberg-MacLane spaces if X is path connected [7]. Applying this to $X = M_g$ yields

$$SP^\infty(M_g) \simeq K(\mathbf{Z}^{2g}, 1) \times K(\mathbf{Z}, 2) \simeq (S^1)^{2g} \times \mathbf{P}^\infty, \tag{6.2}$$

where $K(\mathbf{Z}, 1) \simeq S^1$ while $K(\mathbf{Z}, 2) \simeq \mathbf{P}^\infty$, the infinite complex projective space. We then find that

$$H_*(SP^\infty(M_g); \mathbf{Z}) \cong \Lambda(e_1, \dots, e_{2g}) \otimes \Gamma(a), \tag{6.3}$$

where $\Lambda(,)$ denotes the exterior algebra on the stated generators while $\Gamma(a)$ denotes the *divided power algebra* on a: the ring with $\mathbf{Z}$-generators a_i, $i = 0, 1, \dots$ and multiplication $a_i a_j = \binom{i+j}{j} a_{i+j}$. This accords well with the multiplication of the $SP^n(M_g)$ described above. Often however it is convenient to work with the cohomology rings so we need the following description.

I.G. Macdonald's description of $H^*(SP^k(M_g); \mathbf{Z})$.

Consider the map $[M_g]^*\colon M_g \to \mathbf{P}^\infty = K(\mathbf{Z}, 2)$, taking the fundamental class to the dual of the orientation class, and the map

$$\vee e_i\colon M_g \to K(\mathbf{Z}^{2g}, 1) \simeq (S^1)^{2g}.$$

Both $\mathbf{P}^\infty$ and $(S^1)^{2g}$ are associative abelian H-spaces with the structure on $\mathbf{P}^\infty$ coming from the identification $SP^\infty(\mathbf{P}^1) = \mathbf{P}^\infty$ described earlier.

This allows us to extend $[M_g]^*$ and $\vee e_i$ to a multiplicative map

$$\theta(k)\colon (M_g)^k \to \mathbf{P}^\infty \times (S^1)^{2g}$$

which, by definition factors through the symmetric product $SP^k(M_g)$. In the limit, as $k \mapsto \infty$ this gives the Dold-Thom equivalence

$$SP^\infty(M_g) \to K\left((\mathbf{Z})^{2g}, 1\right) \times K(\mathbf{Z}, 2).$$

With respect to the maps $\theta(k)$ we can describe the cohomology ring $H^*(SP^k(M_g); \mathbf{Z})$ as follows:

Theorem 6.4 (I.G. Macdonald). *The cohomology ring of $SP^k(M_g)$ over the integers $\mathbf{Z}$ is generated by the elements*

$$f_1 = e_1^*, \dots, f_i = e_{2i-1}^*, \dots f_g = e_{2g-1}^*, f_1' = e_2^*, \dots f_g' = e_{2g}^*, \text{ and } b$$

subject to the following relations:

(i) The f_i's and the f_i''s anti-commute with each other and commute with b;

(ii) If $i_1, \ldots, i_a, j_1, \ldots, j_b, k_1, \ldots, k_c$ are distinct integers from 1 to g inclusive, then

$$f_{i_1} \cdots f_{i_a} f'_{j_1} \cdots f'_{j_b} (f_{k_1} f'_{k_1} - b) \cdots (f_{k_c} f'_{k_c} - b) b^q = 0$$

provided that

$$a + b + 2c + q = n + 1.$$

If $k < 2g$ all the relations above are consequences of those for which $q = 0$, and if $n > 2g - 2$ all the relations are consequences of the single relation

$$b^{k-2g+1} \prod_{i=1}^{g} (f_i f_i' - b) = 0.$$

(Actually Macdonald only stated this result in [15] for fields of characteristic zero as coefficients, however, since the relevant invariant maps that he used to prove the result are actually surjective over $\mathbf{Z}$, it is quite direct to extend the result to integers as well.)

The homology of $SP^n(X)$ for more general X

The homology of the spaces $SP^n(X)$ splits according to a result of Steenrod, [6], (see also [19]) which holds for X any CW-complex:

$$H_*(SP^\infty(X); \mathbf{Z}) = \bigoplus_j H_*(SP^j(X), SP^{j-1}(X); \mathbf{Z}). \tag{6.5}$$

Consequently, the ring $H_*(SP^\infty(M_g); \mathbf{Z})$ is bigraded, by defining $x \in H_*(SP^\infty(X); \mathbf{Z})$ to have bidegree (i, j) iff

$$x \in H_j(SP^i(M_g), SP^{i-1}(M_g); \mathbf{Z}) \subset H_*(SP^\infty(M_g); \mathbf{Z}).$$

The bigrading is multiplicative in the sense that the product map

$$SP^m(X) \times SP^n(X) \rightarrow SP^{n+m}(X)$$

in homology induces a bigraded ring map

$$H_{i,j}(SP^n(X); \mathbf{A}) \otimes H_{r,s}(SP^m(X); \mathbf{A}) \longrightarrow H_{i+r,j+s}(SP^{n+m}(X); \mathbf{A}),$$

where the first degree is the dimension and the second the bidegree.

The bigrading is preserved by the diagonal map, so the cohomology ring is bigraded as well, and the two structures together form a bigraded Hopf algebra. In the case of the $SP^k(M_g)$ we have that the coproduct on each of the 1-dimensional generators is primitive, while the 2-dimensional generator $[M_g]$ in $H_{1,2}(SP^\infty(M_g);\mathbf{Z})$ corresponding to the orientation class of M_g has coproduct

$$(6.6)\quad \Delta([M_g]) = [M_g]\otimes 1 + \sum_1^g (e_{2i-1}\otimes e_{2i} - e_{2i}\otimes e_{2i-1}) + 1\otimes [M_g].$$

The homology of $SP^n(\Sigma(M_g))$.

We will need the cohomology and homology of the spaces $SP^n(\Sigma(M_g))$ where ΣX denotes the suspension of X. From the Dold-Thom Theorem we have

$$(6.7)\qquad SP^\infty(\Sigma M_g) \simeq \left(\prod_1^{2g}\mathbf{P}^\infty\right)\times SP^\infty(S^3),$$

where we can assume that the generator for the homology of each $\mathbf{P}^\infty$ has bidegree $(1,2)$. Thus, this amounts to describing the homology and cohomology of $K(\mathbf{Z},3)$ and its associated bigrading.

With coefficients $\mathbf{F}_p$ for any odd prime p we have, [19],

$$(6.8)\qquad \begin{aligned} &H_*(K(\mathbf{Z},3);\mathbf{F}_p)\\ &\cong \Lambda(|a|,|\gamma_p|,\cdots|\gamma_{p^i}|\cdots)\otimes\Gamma(|a^{p-1}|a|)\\ &\quad\otimes\Gamma(|\gamma_p^{p-1}|\gamma_p|)\otimes\cdots\otimes\Gamma(|\gamma_{p^i}^{p-1}|\gamma_{p^i}|)\otimes\cdots\ . \end{aligned}$$

The generator γ_{p^i} corresponds to $[SP^{p^i}(M_g)]$ and $|\gamma_{p^i}|$ has filtration degree p^i and dimension $2p^i+1$ while $|\gamma_{p^{i-1}}^{p-1}|\gamma_{p^{i-1}}|$ has bidegree $(p^i, 2p^i+2)$.

In the case of the prime 2 we have

$$(6.9)\qquad H_*(K(\mathbf{Z},3);\mathbf{F}_2)\ \cong\ \Gamma[|a|,|a_2|,\ldots,|a_{2^i}|,...].$$

Remark 6.10. With $\mathbf{F}_p$ coefficients, the divided power algebra $\Gamma[a]$ splits as an algebra into a tensor product of truncated polynomial algebras [4]

$$\Gamma[a]\cong\mathbf{F}_p[a]/a^p\otimes\mathbf{F}_p[a_p]/(a_p)^p\otimes\cdots\ .$$

When $\mathbf{F} = \mathbf{Q}$, divided power algebras are isomorphic to polynomial algebras and the answer simplifies greatly; namely

$$H_*(SP^\infty(\Sigma M_g); \mathbf{Q}) = \mathbf{Q}[|e_1|, \dots, |e_{2g}|] \otimes \Lambda(a). \tag{6.11}$$

The answer is simpler to describe if we use cohomology. Here, $H^*(SP^\infty(S^3); \mathbf{F}_p)$ is always a tensor product of an exterior algebra on odd dimensional generators and a polynomial algebra on the even dimensional generators. In other words in the formula (6.8) above one replaces all the divided power algebras by polynomial algebras with generators of the same bidegrees to get the cohomology ring description.

We take advantage of the bigrading to write $H_*(SP^\infty(\Sigma M_g); \mathbf{F}_p)$ in the form

$$\coprod_1^\infty H_*(SP^n(\Sigma M_g), SP^{n-1}(\Sigma M_g); \mathbf{F}_p).$$

These summands can be written out more completely in terms of the bigrading of $SP^\infty(S^3)$ as follows. First, we write $H_*(SP^\infty(S^3); \mathbf{F}_p) = H_*(SP^\infty(\Sigma S^2); \mathbf{F}_p)$ in this way. The decomposition here has an alternate description, [5]:

$$H_*(SP^\infty(S^3); \mathbf{F}_p) = \bigoplus_1^\infty \Sigma^{4i} \mathcal{D}_i^*(p), \tag{6.12}$$

where the $\mathcal{D}_i$ are the Snaith splitting components of the loop space $\Omega^2 S^3$ for the prime p, and $\mathcal{D}_i^*$ means dual, where we index the dual by $\dim(\mathcal{D}_{i,j}) = -j$.

Then, going back to (6.7) we see that we can write the term

$$\begin{aligned} &H_*(SP^n(\Sigma M_g), SP^{n-1}(\Sigma M_g); \mathbf{F}_p) \\ &\qquad = \bigoplus_{j=0}^n \Sigma^{4j} \mathcal{D}_j^*(p) \otimes \mathbf{F}_p[b_1, \dots, b_{2g}]_{n-j}\ , \end{aligned} \tag{6.13}$$

and $\mathbf{F}_p[b_1, \dots, b_{2g}]_{n-j}$ is the free $\mathbf{Z}$-module on the degree $n-j$ monomials in the variables $b_1, \dots, b_{2g}$. There are $\binom{n-j+2g-1}{2g-1}$ such monomials.

Example. What follows is a list of generators for

$$H_*(SP^j(\Sigma M), SP^{j-1}(\Sigma M); \mathbf{Z}_p)$$

in the E^1 term mod(p) of the spectral sequence of Theorem 4.3 together with their tridegrees

$$\begin{array}{lcc} & \textit{Generator} & \textit{trigrading} \\ & |M| & (0,1,3) \\ (6.14) & |e_i| & (0,1,2) \\ & |\gamma_{p^i}| & (0,p^i,2p^i+1) \\ & |\gamma_{p^{i-1}}^{p-1}|\gamma_{p^{i-1}}| & (0,p^i,2p^i+2). \end{array}$$

Explicit d^1-differentials in the spectral sequence.

In Theorem 4.3 the differential d^1 is implicitly determined. Now that we have the explicit form of the homology groups $H_*(SP^n(M_g))$ and $H_*(SP^n(\Sigma M_g))$ we can make d^1 explicit. For example we have

$$\begin{aligned} & d^1(|f_1|^{s_1}\cdots|f_{2g}|^{k-s_1-\cdots-s_{2g-1}}) \\ (6.15)\qquad & = \sum_1^{2g} f_i \otimes |f_1|^{s_1}\cdots|f_i|^{s_i-1}\cdots|f_{2g}|^{k-s_1-\cdots-s_{2g-1}} \end{aligned}$$

and

$$\begin{aligned} & d^1(|M_g||f_1|^{s_1}\cdots|f_{2g}|^{k-s_1-\cdots-1} \\ (6.16)\qquad & = [M_g]\otimes|f_1|^{s_1}\cdots|f_{2g}|^{k-s_1-\cdots-1} \\ & - \left(d^1(|f_1|^{s_1}\cdots|f_{2g}|^{k-s_1-\cdots-1}\right)(1\otimes|M_g|)\,. \end{aligned}$$

It is easily checked that this part of the d^1-differential is injective to the term $\tilde{H}^*(M_g;\mathbf{F})\otimes\mathbf{F}[|f_1|,\ldots,|f_{2g}|]_{k-1}$ with cokernel spanned by the monomials of the following form:

$$(6.17)\qquad \{f_t|f_i|^{j_i}|f_{i+1}|^{j_{i+1}}\cdots|f_{2g}|^{k-1-j_1-\cdots-j_{2g-1}}\},$$

where $j_i > 0$, and $t > i$.

Remark 6.18. This is the only time the d^1 differential on the elements $|f_1|^{i_1}\cdots|f_{2g}|^{i_{2g}}$ is non-trivial, since in the remaining parts of the spectral sequence their images lie in the part which has been collapsed out.

Example 6.19. If $g = 1$ then the differentials in this region have the form $d^1(|f_1|^s|f_2|^{k-s}) = f_1\otimes|f_1|^{s-1}|f_2|^{k-s} + f_2\otimes|f_1|^s|f_2|^{k-s-1}$

provided s and $k-s$ are both greater than zero and

$$\begin{aligned} d^1(|T||f_1|^s|f_2|^{k-s-1}) &= f_1f_2 \otimes |f_1|^s|f_2|^{k-s-1} \\ &\quad - f_1 \otimes |T||f_1|^{s-1}|f_2|^{k-s-1} \\ &\quad - f_2 \otimes |T||f_1|^s|f_2|^{k-s-2}. \end{aligned}$$

The role of this last differential can be regarded as identifying terms of the form $f_1f_2 \otimes |f_1|^s|f_2|^{k-s-1}$ with terms involving $|T|^*$. Also, the first of these differentials is injective but not surjective. The quotient has as a basis the set of images of the elements $f_2 \otimes |f_1|^s|f_2|^{k-s-1}$ with $s > 0$. Consequently it has dimension $k-2$.

7. The stable range

In the case where $i \geq 2g$, we've seen already that the Abel-Jacobi map becomes an analytic fibration, and hence the projection

$$E_{i,\dots,i} \to J(M_g)$$

fibers as

$$\mathbf{P}^{i-g} \times \dots \times \mathbf{P}^{i-g} \longrightarrow E_{i,\dots,i} \longrightarrow J(M_g). \tag{7.1}$$

Because of this and other stabilization properties (§8), we refer to the range $i \geq 2g$ as *stable*. In this case, we see that the relative groups

$$\tilde{H}(LE_i, \mathbf{A}) \cong H_*(E_{i,i,\dots,i}, \bigcup E_{i,\dots,i-1,\dots,i}; \mathbf{A})$$

are given by

$$\begin{aligned} &\left[H_*(\mathbf{P}^{i-g}, \mathbf{P}^{i-g-1}; \mathbf{A})\right]^{n+1} \otimes H_*(J(M_g); \mathbf{A}) \\ &\qquad \cong \Sigma^{2(i-g)(n+1)} H_*(J(M_g); \mathbf{A}). \end{aligned} \tag{7.2}$$

Dually, in cohomology the relative group is given as

$$\begin{aligned} &b_0^{i-g} b_1^{i-g} \cdots b_n^{i-g} \otimes H^*(J(M_g); \mathbf{A}) \\ &\qquad = (b_0 \cdots b_n)^{i-g} \otimes H^*(J(M_g); \mathbf{A}), \end{aligned} \tag{7.3}$$

a form which is often of more use in calculations.

Remark 7.4. Of course, in both these forms, the inclusions $\mathbf{P}^{i-g} \subset SP^i(M_g)$ induce injections in homology onto $\mathbf{Z}$-direct summands, but even more, from Macdonald's Theorem 6.4 it follows that the inclusion of pairs

$$(\mathbf{P}^{i-g}, \mathbf{P}^{i-g-1}) \subset (SP^i(M_g), SP^{i-1}(M_g))$$

induces an inclusion in homology sending $H_{2(i-g)}(\mathbf{P}^{i-g}, \mathbf{P}^{i-g-1}; \mathbf{Z})$ isomorphically to the group $H_{2(i-g)}(SP^i(M_g), SP^{i-1}(M_g); \mathbf{Z}) = \mathbf{Z}$. (We will expore this fact further below.)

Thus in this stable range the E_1-term for $\mathrm{Hol}_k^*(M_g, \mathbf{P}^n)$ injects into the corresponding E_1-term for the Div-space as was asserted at the end of §4. In particular this gives as a corollary to the results of [12],

Lemma 7.5. *The spectral sequence of Theorem 4.3 collapses at E_1 in the stable range for $n \geq 2$.*

Now we turn to some fairly direct calculations which lead to the determination of the d^1-differential in the Hol_k^* spectral sequence for $n = 1$ in the stable range.

More exactly, in the stable range the relative group injects,

$$H^*(E_{i,i}, E_{i,i-1} \cup E_{i-1,i}; \mathbf{A}) \subset H^*(E_{i,i}, \mathbf{A})$$

as the principal ideal

$$((b_0 b_1)^{i-2g} W_0 W_1) \subset H^*(E_{i,i}; \mathbf{A}), \tag{7.6}$$

where W_j is the polynomial

(7.7)

$$\begin{aligned} W_j &= \prod_{t=1}^{g} (f_{2t-1} f_{2t} - b_j) \\ &= (-1)^g \left[b_j^g - \left(\sum_1^g f_{2t-1} f_{2t} \right) b_j^{g-1} + \cdots + (-1)^g \left(\prod_1^{2g} f_t \right) \right], \end{aligned}$$

which generates the kernel of the restriction map to $H^*(E_{i-1,i}; \mathbf{A})$ and $H^*(E_{i,i-1}; \mathbf{A})$ respectively according to Macdonald's Theorem 6.4.

Lemma 7.8. *In the stable range the differential d_1 in the spectral sequence of Theorem 4.3 vanishes for $n > 1$. For $n = 1$, it has the form*

$$d_1(W_0 W_1 b_0^{i-2g} b_1^{i-2g}) = 2 \left(\sum_1^g f_{2i-1} f_{2i} \right) (W_0 W_1 b_0^{i-2g-1} b_1^{i-2g-1}) |M_g^*|,$$

and it is multiplicative in the remaining generators. That is to say,

$$\Theta W_0 W_1 b_0^{i-2g} b_1^{i-2g} \xrightarrow{d_1} \pm\, \Theta d_1(W_0 W_1 b_0^{i-2g} b_1^{i-2g}).$$

Proof. We check the action map $M \times E_{i-1,\dots,i-1} \to E_{i,\dots,i}$ in cohomology. The term $f_{2r-1}f_{2r} - b_i$ maps to

$$f_{2r-1} \otimes f_{2r} - f_{2r} \otimes f_{2r-1} + 1 \otimes (f_{2r-1}f_{2r} - b_i),$$

since $f_{2r-1}f_{2r} - b = 0$ in $H^2(M_g; \mathbf{Z})$. Hence

$$W_i \mapsto \sum_{r=1}^{g} (f_{2r-1} \otimes f_{2r} - f_{2r} \otimes f_{2r-1}) 1 \otimes W_i(r) + 1 \otimes W_i,$$

where, as is evident $W_i(r) = \prod_{j \neq r}(f_{2j-1}f_{2j} - b_i)$. Thus, since $f_i f_j = 0$ in $H^*(M_g; \mathbf{Z})$ for $\langle i, j\rangle \neq \langle 2r-1, 2r\rangle$ for some r, it follows that

$$\begin{aligned} W_0 W_1 \;\mapsto\; & \sum_{r=1}^{g} 2b \otimes (f_{2r-1} f_{2r} W_0(r) W_1(r)) \\ & \quad + (f_{2r-1} \otimes f_{2r} - f_{2r} \otimes f_{2r-1}) 1 \otimes (W_0(r) W_1 + W_0 W_1(r)) \\ & + 1 \otimes W_0 W_1. \end{aligned}$$

Next note that $W_i b_i^{i-2g} = 0$ in $H^*(E_{i-1,\dots,i-1}; \mathbf{Z})$, so the image of $(b_0 b_1)^{i-2g} W_0 W_1$ is $2b \otimes \sum f_{2r-1} f_{2r} b_0^{i-2g-1} b_1^{i-2g-1} W_0 W_1$ since

$$b_i W_i(r) f_{2r-1} f_{2r} = W_i f_{2r-1} f_{2r}.$$

This proves the desired result. q.e.d.

Remark 7.9. Actually, what the argument above really determines is the coaction map in cohomology. Given the action map

$$SP^r(M_g) \times LE_i \longrightarrow LE_{i+r}$$

described earlier, and suppose $i \geq 2g$; then

$$U_{i+k} \mapsto b^k \otimes \frac{2^k}{k!} \left(\sum_{1}^{g} f_{2r-1} f_{2r} \right)^k U_i \ ,$$

where U_j is the generating class above for $H^{2(j-g)}(LE_j; \mathbf{Z})$ when $n = 1$.

8. Stabilization

For $d \geq g$, any $n \geq 1$ and any neighborhood $U_\epsilon(*)$, there are $n+1$ divisors $D_i \subset SP^d(U_\epsilon(*))$, $0 \leq i \leq n$ with $\mu(D_0) = \mu(D_1) = \cdots = \mu(D_n)$ and $\bigcap D_i = \emptyset$. By deforming $M_g - *$ to $M_g - U_\epsilon$ and then adding in the corresponding divisors, one obtains a stablization map τ which one can iterate

$$\text{(8.1)} \quad \mathrm{Hol}_k^*(M_g, \mathbf{P}^n) \xrightarrow{\tau} \mathrm{Hol}_{k+d}^*(M_g, \mathbf{P}^n) \longrightarrow \cdots \longrightarrow \mathrm{Hol}_{k+rd}^*(M_g, \mathbf{P}^n)$$

obtaining in the limit a space $\lim_{r \mapsto \infty} \mathrm{Hol}_{k+rd}^*(M_g, \mathbf{P}^n)$ which is homotopy equivalent to any component in the mapping space $\mathrm{Map}_*^*(M_g, \mathbf{P}^n)$. Note that this last statement is a consequence of Segal's stabilization theorem, [21].

The inclusion τ descends to a map of spectral sequences and by an argument similar to one in [5], we see the corresponding map on the E_1-term is given by

$$\text{(8.2)} \quad \begin{aligned} &\cup\, (b_0 b_1 \cdots b_n) \otimes id\colon \\ &\qquad H^*(LE_i; \mathbf{Z}) \otimes H^*(SP^j(\Sigma M_g), SP^{j-1}(\Sigma M_g); \mathbf{Z}) \\ &\qquad\qquad \longrightarrow H^*(LE_{i+1}; \mathbf{Z}) \otimes H^*(SP^j(\Sigma M_g), SP^{j-1}(\Sigma M_g); \mathbf{Z}). \end{aligned}$$

This relation between inclusion at the level of mapping spaces and cupping with $b_0 \cdots b_n$ in the Poincaré duals is quite important and follows basically by checking normal bundles (cf [12]). In any case the content of the preceeding remarks can be summarized in

Lemma 8.3. *In the stable range the cohomology of $Hol_k^*(M_g, \mathbf{P}^n)$ is isomorphic to the cohomology of $Map_k^*(M_g, \mathbf{P}^n)$ via the natural inclusion.*

This is again direct from the stabilization via cupping with $b_0 \cdots b_n$ and Segal's stability theorem.

The differentials which we know appear come from comparison with the Div-space spectral sequence. These are the differential d_1 already described and the further differentials which only hold $\mathrm{mod}(p)$ for odd p,

$$\text{(8.4)} \quad \begin{aligned} d(|\gamma_{p^i}|^*) &= \frac{1}{p^i}\left(\sum_1^g f_{2j-1} f_{2j}\right)^{p^i}, \\ d(|\gamma_{p^i}^{p-1}|\gamma_{p^i}|^*) &= \left[\frac{1}{p^i}\left(\sum_1^g f_{2j-1} f_{2j}\right)^{(p-1)p^i}\right] |\gamma_{p^i}|^*. \end{aligned}$$

The last differential in (8.4) is called the Kudo differential, [14]. There are no differentials for $p = 2$ (see next remark). Similarly, for $n > 2$ all the differentials are zero for all p and hence, even over the integers.

Remark 8.5. There is a sequence of Serre fibrations

$$\Omega^2(\mathbf{P}^n)\longrightarrow \mathrm{Map}^*(M_g,\mathbf{P}^n)\longrightarrow (\Omega(\mathbf{P}^n))^{2g} \xrightarrow{f^\dagger} \Omega(\mathbf{P}^n)$$

induced from the cofibration sequence associated to M_g;

$$S^1 \xrightarrow{f} \bigvee^{2g} S^1 \hookrightarrow M_g \longrightarrow S^2$$

which gives $\mathrm{Map}^*(M_g,\mathbf{P}^n)$ as the total space of a principal $\Omega^2(\mathbf{P}^n)$ fibration with classifying map $f^\dagger$. Then the discussion above shows by dualizing that in the Serre spectral sequence of the fibration

$$\Omega^2(\mathbf{P}^n)\longrightarrow \mathrm{Map}^*(M_g,\mathbf{P}^n)\longrightarrow (\Omega(\mathbf{P}^n))^{2g} \tag{8.6}$$

we have (for $n = 1$) the differentials $d^1(|b|^*) = 2\sum_1^g f_{2j-1}f_{2j}$, the transgressive mod$(p)$ differentials from $|b^{p^i}|^*$ to the divided power of $(\sum_1^g f_{2j-1}f_{2j})$, and the Kudo differentials for p odd. Of course, for $p = 2$ the differentials in the Serre spectral sequence are totally transgressive from the fiber to the base. Hence, there are no differentials and $E_2 = E_\infty$. Similarly, for $n > 2$ there are no differentials for any p and $E_2 = E_\infty$ in this spectral sequence even with integer coefficients though the only way we know to prove this is via the results of [13].

9. The genus one case

When $g = 1$, then $\mu\colon M_1 \to J(M_1)$ is a holomorphic homeomorphism identifying the Jacobi variety with T itself. Also, the stable range starts immediately in this case and hence, for all $n \geq 1$ the Abel-Jacobi map μ is a fibration

$$\mathbf{P}^{n-1} \longrightarrow SP^n(T) \xrightarrow{\mu} T.$$

In the spectral sequence for $QE_k(n)$ with $n \geq 2$ (dual to $Hol_k^*(T,\mathbf{P}^n)$) there are no differentials except for those at the tail end (6.15) and (6.16), but these are the only differentials and $E^1 = E^\infty$ away from this region, while $E^2 = E^\infty$ for the entire spectral sequence. In the case $n = 1$ the only differential is again d_1 which now also has a stable component given in (5.9), and again $E^2 = E^\infty$.

It also follows that duality gives the isomorphism

$$\tilde{H}^{2k(n+1)-2n-*}(\mathrm{Hol}_k^*(T,\mathbf{P}^n);\mathbf{F}) \cong H_*(QE_{\underbrace{k,\dots,k}_{n+1}};\mathbf{F}) \tag{9.1}$$

for all $k \geq 2$, with $Hol_1^*(T,\mathbf{P}^n)$ being empty for all n.

We now determine these homology groups explicitly for $n=1$. First, in the case where $\mathbf{F}$ has characteristic 2 there are no stable differentials and $E^1 = E^\infty$ in this range. Second, in case $\mathbf{F} = \mathbf{Q}$ we have

$$\begin{aligned} E^1(\mathrm{Hol}_k^*) &= \coprod_{i=2}^{k} \Sigma^{4(i-1)} H_*(T;\mathbf{Q}) \\ &\quad \otimes (\mathbf{Q}[|e_1|,|e_2|]_{k-i} \oplus |[T]| \otimes \mathbf{Q}[|e_1|,|e_2|]_{k-i-1}) \\ &\quad \oplus \tilde{H}_*(T;\mathbf{Q}) \otimes (\mathbf{Q}[|e_1|,|e_2|]_{k-1} \oplus |[T]| \otimes \mathbf{Q}[|e_1|,|e_2|]_{k-2}) \\ &\quad \oplus \mathbf{Q}[|e_1|,|e_2|]_k + |T|\mathbf{Q}[|e_1|,|e_2|]_{k-1}, \end{aligned} \tag{9.2}$$

where $\mathbf{Q}[|e_1|,|e_2|]_r$ is the $(r+1)$-dimensional subspace spanned by the monomials of degree r; that is: $|e_1|^r$, $|e_1|^{r-1}|e_2|,\dots,|e_2|^r$. The only differential is d_1, generated by

$$d_1(\Sigma^{4(k-j-1)}e_1e_2|T|) = \Sigma^{4(k-j)}1,$$

and the unstable differentials of (6.15) and (6.16). Thus one has directly (compare (6.19) for the last term) that

$$\begin{aligned} E^2 &= \coprod_{i=2}^{k} \Sigma^{4(i-1)} \tilde{H}_*(T;\mathbf{Q}) \otimes \mathbf{Q}[|e_1|,|e_2|]_{k-i} \\ &\quad \oplus \Sigma^{4(i-1)}|T|(1,e_1,e_2)\mathbf{Q}[|e_1|,|e_2|]_{k-i-1} \\ &\quad \oplus \mathbf{Q}\{e_2|e_1|^{k-1}, e_2|e_1|^{k-2}|e_2|,\dots,e_2|e_1||e_2|^{k-2}\} \\ &\quad \oplus |T|(1,e_1,e_2)\mathbf{Q}[|e_1|,|e_2|]_{k-2} \\ &= E^\infty. \end{aligned} \tag{9.3}$$

In the remaining case where $\mathbf{F} = \mathbf{F}_{p^n}$ has characteristic p an odd prime, as explained in (6.6), (6.7) we can write the E_1-term as

$$\coprod_{i+j=k} \Sigma^{4(i-1)} H^*(T;\mathbf{F}) \otimes \sum_{l=0}^{j} \Sigma^{4l}\mathcal{D}_j^*(p) \otimes \mathbf{F}[|b_1,b_2]_{j-l},$$

where $\mathcal{D}_j^* = 0$ if $j \neq pk$ or $pk+1$ and with the usual special considerations at the tail end of the filtration. Incidently, the case $j = pk+1$

implies that $\Sigma^{4j}\mathcal{D}_j^* = |M|\Sigma^{4(j-1)}\mathcal{D}_{j-1}^*$. The differential is as before, and we have that in the stable range

$$\begin{aligned} E^2 &= \coprod_{i=0}^{k} \Sigma^{4(i+j-1)}\tilde{H}_*(T;\mathbf{F}) \otimes \mathcal{D}_j^* \otimes \mathbf{F}[b_1,b_2]_{k-i-j} \\ &\qquad \oplus \Sigma^{4(i+j-2)}|T|\mathbf{F}[b_1,b_2]_{k-i-j-1} \\ &= E^\infty. \end{aligned}$$

In each case the homology injects into $H_*(\mathrm{Map}_k^*(T,\mathbf{P}^1);\mathbf{F})$. Thus the geometric interpretation of each of the homology and cohomology classes in $H_*(\mathrm{Hol}_k^*(T,\mathbf{P}^1);\mathbf{F})$ can be regarded as the same as that for the corresponding class in the mapping space.

For example with rational coefficients we have the following corollary to 9.3 and 9.1.

Lemma 9.4. *Let $I = (|e_1|,|e_2|)$ be the augmentation ideal in $\mathbf{Q}[|e_1|,|e_2|]$. Then*

$$\begin{aligned} H_*(Hol_k^*(T,\mathbf{P}^1);\mathbf{Q}) \;&\cong\; \mathbf{Q}(1,e_1,e_2,h_{2,1},h_{2,2},v_3) \otimes \mathbf{Q}[|e_1|,|e_2|]/I^{k-1} \\ &\qquad \oplus \mathbf{Q}\{e_2|e_1|^{k-1},\ldots,e_2|e_1||e_2|^{k-2}\}. \end{aligned}$$

In particular the Poincaré series for $H_*(\mathrm{Hol}_k^*(T,\mathbf{P}^1);\mathbf{Q})$ is

$$(1+2x+2x^2+x^3)(1+2x^2+3x^4+\cdots+(k-1)x^{2(k-2)})+(k-2)x^{2(k-1)-1}.$$

Note how this compares with the corresponding Poincaré series for the mapping space $H_*(\mathrm{Map}^*(T,\mathbf{P}^1);\mathbf{Q})$:

$$\frac{(1+2x+2x^2+x^3)}{(1-x^2)^2}.$$

The case $n \geq 2$. Any component in the mapping space is given as the total space of the Serre fibration

$$\Omega^2 S^{2n+1} \to E \to (S^1 \times \Omega S^{2n+1})^2$$

with

$$H_*(\mathrm{Map}^*(T,\mathbf{P}^n);\mathbf{F}) = H_*(\Omega^2 S^{2n+1};\mathbf{F}) \otimes H_*(T;\mathbf{F}) \otimes \mathbf{F}[g_{2n},g_{2n}'].$$

Consequently, with $\mathbf{Q}$ as coefficients the Poincaré series for the mapping space has the form

$$\frac{(1+x)^2(1+x^{2n-1})}{(1-x^{2n})^2},$$

while for $\mathrm{Hol}_k^*(T,\mathbf{P}^n)$ we have

$$(1+x)^2(1+x^{2n-1})(1+2x^{2n}+3x^{4n}+\cdots+(k-1)x^{(k-2)2n})+(k-2)x^{2(k-1)n-1}.$$

Remark 9.5. Notice that in this case all the calculations were independent of the particular elliptic curve T that we were studying, even though different T are not holomorphically isomorphic. In fact this is not unexpected since we have

Theorem 9.6. *Let T be any non-singular elliptic curve. Then the homeomorphism type of $Hol_k^*(T,\mathbf{P}^n)$ is independent of T.*

Proof. Let T and T' be two given tori with corresponding lattices Λ and $\Lambda' \subset \mathbf{C}$. Then there is a map $\Lambda \longrightarrow \Lambda'$ which induces a homeomorphism

$$T = \mathbf{C}/\Lambda \xrightarrow{f} T' = \mathbf{C}/\Lambda',$$

and such that the following commutes for all n;

$$\begin{array}{ccc} SP^n(T) & \xrightarrow{SP^n(f)} & SP^n(T') \\ \Big\downarrow \mu & & \Big\downarrow \mu \\ T & \xrightarrow{f} & T'. \end{array}$$

It follows therefore that $E_{i,i,\dots i} \cong E'_{i,i,\dots,i}$ where $E_{i,i,\dots i}$ (resp. $E'_{i,i,\dots,i}$) is the pushout as in §3

$$\begin{array}{ccc} E_{i,i,\dots,i} & \longrightarrow & SP^i(T)\times\cdots\times SP^i(T) \\ \downarrow & & \downarrow \\ T & \xrightarrow{\Delta} & T\times\cdots\times T \end{array}$$

(respectively, T is replaced by T'). It is now clear that

$$\mathrm{Hol}_n^*(T,\mathbf{P}^n) \cong \tilde{E}_{i,\dots i} - \mathrm{Image}(\nu) \cong \tilde{E}'_{i,\dots i} - \mathrm{Image}(\nu) \cong \mathrm{Hol}_n^*(T,\mathbf{P}^n),$$

and the proof is complete. q.e.d.

10. The decomposition for hyperelliptic curves

The surface M_g is hyperelliptic if and only if there is a degree-two (branched) holomorphic map $f\colon M_g \to \mathbf{P}^1$. This map f fits in the following diagram

$$\begin{tikzcd} M_g \arrow[rr, "s"] \arrow[dr] & & P^{2g-2} \\ & P^1 \arrow[ur] & \end{tikzcd}$$

where s is the canonical map, [8, p. 247].

From the point of view of the Abel-Jacobi maps, M_g is hyperelliptic if and only if we have a cofibering of the form

$$\mathbf{P}^1 \hookrightarrow SP^2(M_g) \longrightarrow SP^2(M_g)/\mathbf{P}^1 = W_2 \subset J(M_g).$$

The holomorphic embedding $\mathbf{P}^1 \hookrightarrow SP^2(M_g)$ is now constructed by associating to $z \in \mathbf{P}^1$ the pair $f^{-1}(z) \in SP^2(M_g)$ where the points are counted with multiplicity, i.e., the ramification points are counted twice. The image $\tau \in J(M_g)$ of $\mathbf{P}^1$ under μ_2 is called the hyperelliptic point of M_g.

Lemma 10.2. *Suppose that M_g is hyperelliptic and $\tau \in J(M_g)$ is the hyperelliptic point. Let $k \geq 1$ and $t \leq \left[\frac{k}{2}\right]$. Then:*

(a) for $k \leq g$, the space W_k^t is $\mu_{k-2t}(SP^{k-2t}(M_g)) + t\tau$,

(b) for $2g-1 > k > g$, $t > k-g$, we also have

$$W_k^t = \mu_{k-2t}(SP^{k-2t}(M_g)) + t\tau.$$

Proof. Note that $SP^r(\mathbf{P}^1) = \mathbf{P}^r$ so the pairing

$$SP^{2l}(M_g) \times SP^{k-2l}(M_g) \longrightarrow SP^k(M_g)$$

induces an action

$$\nu\colon\ \mathbf{P}^l \times SP^{k-2l}(M_g) \xrightarrow{=} SP^l(\mathbf{P}^1) \times SP^{k-2l}(M_g) \longrightarrow SP^k(M_g),$$

and the image of

$$\nu(\langle p_1, \ldots, p_l\rangle, \langle m_1, \ldots, m_{k-2l}\rangle)$$

under the Abel-Jacobi map is $l\tau + \mu_{k-2l}(\langle m_1, \ldots, m_{k-2l}\rangle$. From this follows directly the fact that W_k^t is at least as large as asserted in (a). The inclusion argument for (b) is a similar dimension check when we recall that generically $\mu_{g+s}^{-1}(j) = \mathbf{P}^s$, so we are checking for points with inverse image $\mathbf{P}^t$ where $t > s$ and $\nu(\mathbf{P}^t \times SP^{g+s-2t}(M_g)) \subset \mu_{g+s}^{-1}(W_{g+s}^t)$.

The converse is given on page 13 of [1] for $k \leq g$ where it is pointed out that as a consequence of the geometric version of the Riemann-Roch Theorem every $\mathbf{P}^t$ inverse image under the Abel-Jacobi map for $SP^k(M_g)$ has the form $\mu^{-1}(t\tau) + m_1 + \cdots + m_{k-2t}$ where no two of the m_i's are conjugate in M_g under the hyperelliptic involution. To obtain the remaining part of the converse one uses the residual series isomorphism

$$W_k^t \cong W_{2g-2-k}^{g-k+t-1}, \qquad D \leftrightarrow K - D,$$

which holds for all curves M_g, $k \geq g$. q.e.d.

For later use we also specify an appropriate base point for the construction. Let $* \in M_g$ be one of the ramification points of $f\colon M_g \to \mathbf{P}^1$. Then $2\mu(*) = \tau$, and we choose $*$ as the base point in M_g. Also the base point of $\mathbf{P}^1$ is $\langle *, * \rangle$, and the inclusion $f\colon \mathbf{P}^1 \to SP^2(M_g)$ is based. With respect to this choice, the composite inclusion

$$SP^k(M_g) \hookrightarrow SP^{k+1}(M_g) \hookrightarrow SP^{k+2}(M_g)$$

commutes with the $\mathbf{P}^1$-action. Moreover, in the actual construction of the Abel-Jacobi map μ given at the beginning of §1 we can take our base point in M_g to be $*$, so that $\mu(*) = 0 \in J(M_g)$, and the hyperelliptic point $\tau = 0$ as well. This has the advantage that the inclusion $W_k^t + \tau \subset W_{k+2}^{t+1}$ becomes simply $W_k^t \subset W_{k+1}^{t+1}$.

11. Some preliminary constructions

Let V be a space with a commutative, associative multiplication. For example

$$V = SP^\infty(X, *),\ V = \coprod_1^\infty SP^n(X),\ V = \coprod_1^\infty SP^i(X) \times SP^i(X),$$

etc. If $f\colon X \to V$ is any continuous map, then f defines an action

$$\mu_f\colon \coprod SP^k(X) \times V \longrightarrow V$$

by $\mu_f(\langle x_1,\dots,x_k\rangle, v) = f(x_1)f(x_2)\cdots f(x_k)v$.

Suppose now that V, $f_1\colon \mathbf{P}^1 \to V$ and V', $f_2\colon \mathbf{P}^1\to V'$ are given with V, V' as above. Then we can form the bicomplex

$$B(f_1,f_2) \;:=\; V\times_{f_1}\left(\coprod_k SP^k(I\times\mathbf{P}^1)\right)\times_{f_2} V', \tag{11.1}$$

defined by the equivalence relation built from the basic relations

$$\begin{aligned}&(v,\langle(t_1,p_1),\dots,(t_k,p_k)\rangle,v')\\ &\qquad\sim\begin{cases}(v_1\cdot f_1(p_i),\langle(t_1,p_1),\dots,\widehat{(t_i,p_i)},\dots,(t_k,p_k)\rangle,v') & \text{if } t_i=0,\\ (v_1,\langle(t_1,p_1),\dots,\widehat{(t_i,p_i)},\dots,(t_k,p_k)\rangle,f_2(p_i)\cdot v') & \text{if } t_i=1.\end{cases}\end{aligned}$$

Remark 11.2. This is just the 2-sided bar construction which is associated to the homological functor $Tor_A(M_{A},{}_A N)$. But here, left and right modules are equivalent since A is commutative.

Remark 11.3. This is an unbased construction. If f_1 and f_2 are based maps with $f_i(*) = id$, $i=1,2$, then there is an associated reduced construction, $\tilde{B}(f_1,f_2)$. Also, the fact that we are using $\mathbf{P}^1$ plays no real role in the definition. It is just there because that is all we use in the applications of this construction to our study of $Hol_k^*(M_g,\mathbf{P}^n)$ for M_g hyperelliptic.

Example 11.4. Suppose that $V=\coprod SP^k(X)$ and

$$f_1\colon \mathbf{P}^1\to SP^i(X).$$

Then write

$$V_j \;=\; \coprod_{k=0}^{\infty} SP^{j+ik}(X), \qquad 0\le j<i$$

so we have separate actions on each V_j.

Example 11.5. Suppose $f_2\colon \mathbf{P}^1\to *$ where $*$ has the obvious commutative and associative action. Then

$$B(f_1,f_2) = V\times_{f_1}\coprod SP^n(I\times\mathbf{P}^1)\times_{f_2} * \simeq V\times_{f_1}\coprod SP^n(c\mathbf{P}^1)$$

is a space that we have seen many times before.

Example 11.6. $\Delta_{n+1}\colon \mathbf{P}^1 \longrightarrow (\mathbf{P}^1)^{n+1}$ induces an action on $\left(\coprod SP^k(\mathbf{P}^1)\right)^{n+1}$ which restricts, in a manner similar to that in example (1) to an action on

$$\mathcal{E}_n(i_1,\ldots,i_{n+1}) \;=\; \coprod_{k=0}^{\infty} SP^{i_1+k}(\mathbf{P}^1)\times\cdots\times SP^{i_{n+1}+k}(\mathbf{P}^1)$$

for each $(n+1)$-tuple $(i_1,\ldots,i_{n+1})$ of non-negative integers with at least one of the $i_j = 0$.

12. Models for the spaces $E_{i,\ldots,i}$, $LE_{i,\ldots,i}$ and W_k^t when M_g is hyperelliptic

We specialize to

$$\begin{aligned} &BH(f,n)\\ &\quad= \left(\coprod_m SP^m(M_g)\right)\times_f\left(\coprod_m SP^m(I\times\mathbf{P}^1)\right)\\ &\qquad\times_{f_2}\mathcal{E}_n(0,0,\ldots,0), \end{aligned} \tag{12.1}$$

where $f\colon \mathbf{P}^1 \hookrightarrow SP^2(M_g)$ embeds $\mathbf{P}^1$ as the inverse image under the Abel-Jacobi map of the hyperelliptic point, 0, for some hyperelliptic structure on M_g, and f_2 is the structure map Δ_{n+1} of (11.6).

This construction specializes, as in (11.5), to give two disjoint subspaces,

$$BH_0(f,n) \;=\; V_0\times_f\left(\coprod_m SP^m(I\times\mathbf{P}^1)\right)\times_{f_2}\mathcal{E}_n(0,\ldots,0), \tag{12.2}$$

$$BH_1(f,n) \;=\; V_1\times_f\left(\coprod_m SP^m(I\times\mathbf{P}^1)\right)\times_{f_2}\mathcal{E}_n(0,\ldots,0), \tag{12.3}$$

where $V_0 = \coprod SP^{2k}(M_g)$ and $V_1 = \coprod SP^{2k+1}(M_g)$. Each $BH_\epsilon(f,n)$ breaks up into further components as follows. We define a grading by letting

$$\begin{aligned} &(\langle m_1,\ldots,m_l\rangle,\langle(t_1,p_1),\ldots,(t_r,p_r)\rangle,(w_{s,1},\ldots,w_{s,n+1}))\\ &\quad\in SP^l(M_g)\times SP^r(I\times\mathbf{P}^1)\times\prod_1^{n+1} SP^s(\mathbf{P}^1) \end{aligned} \tag{12.4}$$

have grading degree $l + 2r + 2s$. This grading is obviously preserved by the identifications and hence induces the desired decomposition of the quotient $BH_\epsilon(f)$. We write $\mathcal{G}_s(f,n)$ for the component of points of grading degree s.

This is the general case, however, there is a special case $BH_\epsilon(f,-1)$ which is also needed where $V_2 \sim *$ consists of a single point. In this case the grading degree is *not preserved* by the identifications, and the grading only gives a *filtration* of $BH_\epsilon(f,-1)$. We write $\mathcal{F}_k(BH_\epsilon(f,-1))$ for the filtering subspace consisting of all points of grading degree $\leq k$. Moreover, in this case the identification has the form $(v,(t,x),*) \sim (vf(x),*)$ when $t=0$ and $(v,(t,x),*) \sim (v,*)$ when $t=1$.

Lemma 12.5. *We have the following homotopy equivalences in the case where M_g is hyperelliptic:*

(a) $\mathcal{G}_k(BH_\epsilon(f,n)) \simeq E_{k,\ldots,k}$ *for* $k \equiv \epsilon \bmod (2)$ *provided* $k \leq g$.

(b) $\mathcal{F}_k(BH_\epsilon(f,-1)) \simeq W_k$, $k \equiv \epsilon \bmod (2)$ *and* $k < g$.

(These constructions are modeled on the analogous constructions in [20] and the proofs are the same.)

Remark 12.6. The point of these constructions is to handle the algebraic complexities of systematically collapsing out $\mathbf{P}^n$'s. As one sees from the lemma, the effect is to introduce $SP^n(\Sigma\mathbf{P}^1)$'s which provide a measure of the geometry of the collapsed spaces as the collapsing gets more and more complex.

13. Some spectral sequences for the $BH_\epsilon(f,n)$

We can bifilter $\mathcal{G}_k(BH_\epsilon(f,n))$ and $\mathcal{F}_k(BH_\epsilon(f,-1))$ by the number of $SP^t(I \times \mathbf{P}^1)$ terms appearing, obtaining spectral sequences converging to the various spaces above as well as certain useful quotients. The E^1-terms are as follows:

$$\begin{aligned} &\coprod_{k+2l+2t=v} H_*(SP^k(M_g)) \\ &\qquad \otimes H_*(SP^l(\Sigma\mathbf{P}^1_+), SP^{l-1}(\Sigma\mathbf{P}^1_+)) \otimes \left(H_*(\mathbf{P}^t)\right)^{n+1} \end{aligned} \tag{13.1}$$

converging to $H_*(E_{v,\ldots,v})$ with field coefficients and $v < g$,

$$\coprod_{k+2l\leq v} H_*(SP^k(M_g)) \otimes H_*(SP^l(\Sigma\mathbf{P}^1_+), SP^{l-1}(\Sigma\mathbf{P}^1_+)) \tag{13.2}$$

with $k \equiv v \bmod (2)$ converging to $H_*(W_v)$ for $v < g$, and

$$\coprod_{k+2l=v} H_*(SP^k(M_g)) \otimes H_*(SP^l(\Sigma\mathbf{P}^1_+), SP^{l-1}(\Sigma\mathbf{P}^1_+)) \tag{13.3}$$

converging to $H_*(W_v/W_v^1)$. (The notation $\Sigma\mathbf{P}^1_+$ means the reduced suspension on the union of $\mathbf{P}^1$ with a disjoint base point denoted $+$, it has the homotopy type of the wedge $\Sigma(\mathbf{P}^1) \vee S^1$.)

If we relativize we obtain a spectral sequence with E^1-term

$$\begin{aligned} &\coprod_{k+2(l+t)=v} H_*(SP^k(M_g), SP^{k-1}(M_g)) \\ &\qquad \otimes H_*(SP^l(\Sigma\mathbf{P}^1_+), SP^{l-1}(\mathbf{P}^1_+)) \otimes \tilde{H}_*(S^{2(n+1)t}) \\ &\qquad\qquad \Longrightarrow \tilde{H}_*(LE_{v,\ldots,v}). \end{aligned} \tag{13.4}$$

All these spectral sequences are algebraic in the sense of [12], and can be modeled by simply replacing the terms $H_*(SP^l(\Sigma\mathbf{P}^1))$ by the cobar construction on

$$H_*\left(\coprod SP^l(\mathbf{P}^1_+)\right) = \mathbf{Z}[t] \otimes \Gamma(b).$$

In particular, the relative spectral sequence splits as a direct sum of spectral sequences each of the form

$$\begin{aligned} &\left[\coprod_{k+2l=v-2t} H_*(SP^k(M_g), SP^{k-1}(M_g)) \otimes H_*(SP^l(\Sigma\mathbf{P}^1_+), SP^{l-1}(\Sigma\mathbf{P}^1_+))\right] \\ &\qquad \otimes \tilde{H}_*(S^{2t(n+1)}) \\ &\qquad\qquad \Longrightarrow \Sigma^{2t(n+1)}\left(H_*(W_{v-2t}/(W^1_{v-2t} \cup W_{v-2t-1}))\right) \end{aligned} \tag{13.5}$$

for any $n > 0$.

In the case where the coefficients are the rationals, these spectral sequences become quite simple. First $H_*(SP^\infty(\Sigma\mathbf{P}^1_+); \mathbf{Q}) = \Lambda(h_1, h_3)$, and

$$H_*(SP^i(\Sigma\mathbf{P}^1), SP^{i-1}(\Sigma\mathbf{P}^1); \mathbf{Q}) = 0$$

for $i \geq 3$. Second, when we write

$$H_*(SP^k(M_g), SP^{k-1}(M_g); \mathbf{Z}) = \bigoplus_{s=0}^{k} \Lambda_s(e_1, \ldots, e_{2g}) \otimes [M_g]^{k-s},$$

for $k < g$, we see that the action map

$$\begin{aligned} * [\mathbf{P}^1]\colon H_*(SP^k(M_g), SP^{k-1}(M_g); \mathbf{Z}) \\ \to H_{*+2}(SP^{k+2}(M_g), SP^{k-1}(M_g); \mathbf{Z}) \end{aligned}$$

is just multiplication by $-\sum_{j=1}^g e_{2j-1}e_{2j}$ since

$$[\mathbf{P}^1] = [M_g] - \sum_1^g e_{2j-1}e_{2j}$$

in $H_*(SP^2(M_g); \mathbf{Z})$. Third, from [2], this map is injective in homology for $k \leq g-2$. Thus we have

Lemma 13.6.

$$\begin{aligned} & H_*(LE_{v,\dots,v}; \mathbf{Q}) \\ & \quad = \sum_{t=0}^{[v/2]} \frac{\Sigma^{2t(n+1)} H_*(SP^{v-2t}(M_g), SP^{v-2t-1}(M_g); \mathbf{Q})}{im\Big(* \big(\sum_{j=1}^g e_{2j-1}e_{2j}\big)\Big)} \end{aligned}$$

for $v \leq g$, which has Poincaré series

$$\sum_{t=0}^{[v/2]} x^{2t(n+1)} S_{v-2t},$$

where $S_r = \sum_{l \leq r} \left[\binom{2g}{l} - \binom{2g}{l-1}\right] x^{2r-l}$.

Similarly, we can analyze $H_*(W_j; \mathbf{Q})$.

Lemma 13.7.

(a) The inclusion $W_j \subset J(M_g)$ induces an injection in rational homology $H_*(W_j; \mathbf{Q}) \hookrightarrow H_*(J(M_g); \mathbf{Q}) = \Lambda(e_1, \dots, e_{2g})$ with image the subvector space spanned by the subspaces

$$\left\{\Lambda_s(e_1, \dots, e_{2g})[M_g]^t \mid s + t \leq j\right\},$$

where $[M_g] = \sum_1^g e_{2i-1}e_{2i}$ is the image of the fundamental class of M_g under the Abel-Jacobi map μ_*,

(b) $H_*(W_{j-1}; \mathbf{Q})$ injects into $H_*(W_j; \mathbf{Q})$ under the inclusion, so the relative groups are given as

$$H_*(W_j, W_{j-1}; \mathbf{Q}) \cong H_*(W_j; \mathbf{Q}) / H_*(W_{j-1}; \mathbf{Q}).$$

Proof. We have that the E^1-term of the spectral sequence above is given as

$$H_*(SP^j(M_g);\mathbf{Q}) \oplus H_*(SP^{j-2}(M_g);\mathbf{Q})(1,h_1,h_3) \oplus \sum_{2\le 2l\le j} H_*(SP^{j-2l}(M_g);\mathbf{Q})(1,h_1,h_3,h_1h_3).$$

$d_1(\Theta h_1) = i_*(\Theta) - \Theta$ where $\Theta \in H_*(SP^{j-2l}(M_g))$ and

$$i\colon SP^{j-2l}(M_g) \to SP^{j-2l+2}(M_g)$$

is a base point inclusion. Similarly

$$d_1(\Theta h_1 h_3) = d_1(\Theta h_1)h_3 \pm (\Theta h_1)d_1(h_3)$$

where, of course, this last term is just multiplication by $[M_g]-\sum e_{2i-1}e_{2i}$. Ignoring the last piece one sees that the d_1-differential on h_1 reduces the calculation of E^2 to the following complex:

$$H_*(SP^{j-2}(M_g);\mathbf{Q})h_3 \longrightarrow H_*(SP^j(M_g);\mathbf{Q}), \qquad \Theta h_3 \mapsto \Theta([M_g] - \sum_1^g e_{2i-1}e_{2i})$$

from which the first statement of the lemma follows. The second statement is then immediate. q.e.d.

The remaining groups $H_*(LE_{v,\dots,v};\mathbf{Q})$ in the hyperelliptic case.

It is convenient to write $v = g+s$, $1 \le s < g-1$ when $v > g$. In this case we have that $J(M_g)$ is filtered by $W_0 \subset W_1 \subset \cdots W_{g-1} \subset J(M_g)$, and over points in $W_{g-s-2l} - W_{g-s-2l-2}$ the inverse image in $E_{v,\dots,v}$ is $(\mathbf{P}^{s+l})^{n+1}$, while the inverse image over $J(M_g) - W_{g-s-2}$ is $(\mathbf{P}^s)^{n+1}$. Thus, we obtain a spectral sequence with E^1-term:

$$\begin{aligned} &H_*(J(M_g), W_{g-s-2}) \otimes (H_*(\mathbf{P}^s))^{n+1} \\ &\oplus \sum_{l=1}^{[g-s/2]} H_*(W_{g-s-2l}, W_{g-s-2l-2}) \otimes \left(H_*(\mathbf{P}^{s+l})\right)^{n+1} \\ &\Longrightarrow H_*(E_{g+s,\dots,g+s}) \end{aligned} \tag{13.8}$$

with field coefficients.

Similarly, the corresponding E^1-term for $H_*(LE_{g+s,\ldots,g+s})$ is given by

$$\begin{aligned} & H_*(J(M_g), W_{g-s-1}) \otimes \big(H_*(\mathbf{P}^s, \mathbf{P}^{s-1})\big)^{n+1} \\ (13.9) \qquad & \oplus \sum_{l=1}^{[g-s/2]} H_*(W_{g-s-2l}, W_{g-s-2l-1}) \otimes \Big(H_*(\mathbf{P}^{s+l}, \mathbf{P}^{s+l-1})\Big)^{n+1} \\ & \qquad \Longrightarrow H_*(LE_{g+s,\ldots,g+s}). \end{aligned}$$

Note here that in the formula above the term involving the relative groups

$$H_*(W_{g-s-2l-1}, W_{g-s-2l-2})$$

is zero. That is because, in this region of $J(M_g)$ and this range for v the inverse image of a point is $\mathbf{P}^{s+l}$ in both $E_{g+s,\ldots,g+s}$ and $E_{g+s-1,\ldots,g+s-1}$.

Again, because of this gap the spectral sequence collapses and $E^1 = E^\infty$. Of course the calculations become considerably more complex with $\mathbf{F}_p$ coefficients.

14. The Ext-groups for Λ_g/S_g

By inspection, in the spectral sequence of (5.9) we find the that d_1-differential gives rise to a series of complexes that occur in the calculation of Ext-groups for the ring V_g defined below. In this section we calculate these Ext-groups and apply the results in §15 to determine the structure of the rational homology of the space $Hol_k^*(M_g, \mathbf{P}^n)$ with M_g hyperelliptic and $n \geq 2$.

Notation 14.1.

(1) $\Lambda_g = \Lambda(e_1, \ldots, e_{2g}) = H_*(J(M_g), \mathbf{Q})$,

(2) $f_g = \sum_{i=1}^{g} e_{2i-1}e_{2i} \in \Lambda_g$,

(3) $S_g \subset \Lambda_g$ is the ideal (f_g), and

(4) $\Lambda_g/S_g = V_g$ is the quotient.

V_g occurs quite often in both our study of $\mathrm{Hol}_d^*(M_g, \mathbf{P}^n)$ and $\mathrm{Map}_d^*(M_g, \mathbf{P}^n)$. In particular we need the groups $Ext_{\Lambda_g}^{*,*}(V_g, \mathbf{Q})$ which we will describe as modules over $Ext_{\Lambda_g}^{*,*}(\mathbf{Q}, \mathbf{Q}) = \mathbf{Q}[h_1, \ldots, h_{2g}]$ where $h_i = |e_i|^*$, $1 \leq i \leq 2g$.

In [2] it is shown that

$$*f_g\colon (\Lambda_g)_i \longrightarrow (\Lambda_g)_{i+2}$$

is injective for $i \leq g-1$ and surjective for $i \geq g-1$. Consequently, $(V_g)_i = 0$ for $i > g$ while $(V_g)_i = \mathbf{Q}^{N(g,i)}$ for $0 \leq i < g$ where

$$N(g,i) \;=\; \begin{cases} \binom{2g}{i} - \binom{2g}{i-2} & \text{for } 2 \leq i \leq g, \\ 2g & \text{for } i = 1, \\ 1 & \text{for } i = 0. \end{cases}$$

As a special case note that

$$V_1 \;=\; \Lambda_1/\{e_1e_2\},$$

so there is a short exact sequence of Λ_1-modules

$$0 \longrightarrow \mathbf{Q} \overset{e_1e_2}{\hookrightarrow} \Lambda_1 \longrightarrow V_1 \longrightarrow 0,$$

where $\mathbf{Q}$ denotes the "trivial" module given by the augmentation map $\epsilon\colon \Lambda_g \to \mathbf{Q}$, $\epsilon(e_i) = 0$, $\epsilon(1) = 1$. This exact sequence gives rise to the long exact sequence of Ext-groups:

$$\begin{aligned} \cdots \longrightarrow Ext^{i-1,j-2}_{\Lambda_1}(\mathbf{Q},\mathbf{Q}) &\xrightarrow{\delta} Ext^{i,j}_{\Lambda_1}(V_1,\mathbf{Q}) \\ \longrightarrow Ext^{i,j}_{\Lambda_1}(\Lambda_1,\mathbf{Q}) \longrightarrow Ext^{i,j}_{\Lambda_1}(\mathbf{Q},\mathbf{Q}) &\xrightarrow{\delta} \cdots \ , \end{aligned}$$

and since $Ext^{i,j}_{\Lambda_1}(\Lambda_1,\mathbf{Q}) = \begin{cases} 0 & (i,j) \neq (0,0), \\ \mathbf{Q} & (i,j) = (0,0), \end{cases}$ we have

$$\begin{aligned} Ext^{*,*}_{\Lambda_1}(V_1,\mathbf{Q}) \;&=\; \mathbf{Q}e_{0,0} \oplus Ext^{*,*}_{\Lambda_1}(\mathbf{Q},\mathbf{Q})P_{1,2} \\ &=\; \mathbf{Q}e_{0,0} \oplus \mathbf{Q}[h_1,h_2]P_{1,2}\ , \end{aligned}$$

where $P_{1,2}$ is a generating element in $Ext^{1,2}_{\Lambda_1}(V_1,\mathbf{Q})$.

Before we can describe the structure of the modules $Ext^{*,*}_{\Lambda_g}(V_g,\mathbf{Q})$ for $g > 1$, we need some preliminary constructions.

First, we will often be dealing with a situation where we have a ring $R_{i+1} = R_i \otimes_{\mathbf{Q}} R_1$ given as the tensor product of augmented rings, together with a module, M, over R_i. M becomes a module over R_{i+1} via the following obvious composition

$$\begin{aligned} (R_i \otimes R_1) \otimes M &\xrightarrow{id\otimes id} (R_i \otimes R_1) \otimes (M \otimes_{\mathbf{Q}} \mathbf{Q}) \xrightarrow{1\otimes T\otimes 1} (R_i \otimes M) \otimes (R_1 \otimes \mathbf{Q}) \\ &\xrightarrow{\mu_i\otimes\epsilon} M \otimes_{\mathbf{Q}} \mathbf{Q} = M. \end{aligned}$$

Then, by change of rings,

$$Ext_{R_{i+1}}(M, \mathbf{Q}) \;=\; Ext_{R_i}(M, \mathbf{Q}) \otimes Ext_{R_1}(\mathbf{Q}, \mathbf{Q}). \tag{14.2}$$

Second, we will need to consider modules of the following kind: given a module M over the ring R and an exact sequence of R-modules

$$0 \longrightarrow K \hookrightarrow (R)^N \longrightarrow M \longrightarrow 0,$$

then K is written ΩM and is unique up to direct sum with a free R-module. The following result is well known.

Lemma 14.3. *$Ext_R^{i,j}(\Omega M, N) \cong Ext_R^{i+1,j}(M, N)$ for all $i \geq 1$, and*

$$Ext_R^{0,*}(\Omega M, N) \rightarrow Ext_R^{1,*}(M, N)$$

is surjective for all R-modules N.

We say ΩM is *minimal* if the natural map

$$Ext_R^{0,*}(\Omega M, \mathbf{Q}) \rightarrow Ext_R^{1,*}(M, \mathbf{Q})$$

is an isomorphism as well. Of course, ΩM is minimal if and only if the map $R^N \rightarrow M$ induces isomorphisms of $Ext^{0,*}(\ , \mathbf{Q})$-groups.

Consider the following module:

$$M_1 = I \subset \mathbf{Q}[h_1, h_2]; \tag{14.4}$$

it is the augmentation ideal, $I = (h_1, h_2)$. Moreover, $M_1 = \Omega\mathbf{Q}$ and is even minimal for $R = \mathbf{Q}[h_1, h_2]$; thus

$$Ext^{i,j}_{\mathbf{Q}[h_1,h_2]}(M_1, \mathbf{Q}) = Ext^{i+1,j}_{\mathbf{Q}[h_1,h_2]}(\mathbf{Q}, \mathbf{Q})$$

for all j and $i \geq 0$. Moreover, $Ext_{\mathbf{Q}[h_1,h_2]}(\mathbf{Q}, \mathbf{Q}) = \Lambda(|h_1|^*, |h_2|^*)$; thus

$$Ext^{i,j}_{\mathbf{Q}[h_1,h_2]}(M_1, \mathbf{Q}) \;=\; \begin{cases} \mathbf{Q}\{|h_1|^*, |h_2|^*\} & \text{if } (i,j) \;=\; (0,2), \\ \mathbf{Q}\{|h_1|^*|h_2|^*\} & \text{if } (i,j) \;=\; (1,4). \end{cases}$$

It follows that a minimal resolution for M_1 over $\mathbf{Q}[h_1, h_2]$ has the form

$$\mathbf{Q}[h_1, h_2]h_{1,2} \longrightarrow \mathbf{Q}[h_1, h_2]\{[h_1], [h_2]\} \longrightarrow M_1 \longrightarrow 0,$$

where $[h_i] \mapsto h_i$ and $h_{1,2} \mapsto h_1[h_2] - h_2[h_1]$.

Next we consider M_1 as a module over

$$\mathbf{Q}[h_1,h_2,h_3,h_4] = \mathbf{Q}[h_1,h_2] \otimes \mathbf{Q}[h_3,h_4]$$

as above, and define M_2 as the kernel of the surjective homomorphism

$$\pi_2 \colon \mathbf{Q}[h_1,h_2,h_3,h_4]\{b_1,b_2\} \longrightarrow M_1 \longrightarrow 0,$$

$\pi_2(b_i) = h_i$. Clearly $M_2 = \Omega M_1$ as a module over $\mathbf{Q}[h_1,h_2,h_3,h_4]$ and is minimal as well. Consequently we have

$$\begin{aligned}
Ext^{0,*}_{\mathbf{Q}[h_1,\ldots,h_4]}(M_2,\mathbf{Q}) &= Ext^{1,*}_{\mathbf{Q}[h_1,\ldots,h_4]}(M_1,\mathbf{Q}) \\
&= Ext^{2,*}_{\mathbf{Q}[h_1,h_2]}(\mathbf{Q},\mathbf{Q}) \oplus Ext^{1,*}_{\mathbf{Q}[h_1,h_2]}(\mathbf{Q},\mathbf{Q}) \\
&\quad \otimes Ext^{1,*}_{\mathbf{Q}[h_3,h_4]}(\mathbf{Q},\mathbf{Q}) \\
&= \mathbf{Q}^5
\end{aligned}$$

with generators $|h_1|^*|h_2|^*$, $|h_1|^*|h_3|^*$, $|h_1|^*|h_4|^*$, $|h_2|^*|h_3|^*$, $|h_2|^*|h_4|^*$. Likewise,

$$\begin{aligned}
Ext^1(M_2,\mathbf{Q}) &= Ext^2(M_1,\mathbf{Q}) \\
&\cong Ext^2_{\mathbf{Q}[h_1,h_2]}(\mathbf{Q},\mathbf{Q}) \otimes Ext^1_{\mathbf{Q}[h_3,h_4]}(\mathbf{Q},\mathbf{Q}) \\
&\quad \oplus Ext^1_{\mathbf{Q}[h_1,h_2]}(\mathbf{Q},\mathbf{Q}) \otimes Ext^1_{\mathbf{Q}[h_3,h_4]}(\mathbf{Q},\mathbf{Q}) \\
&= \mathbf{Q}^4.
\end{aligned}$$

To obtain M_3 regard M_2 as a module over $\mathbf{Q}[h_1,\ldots,h_6] = R_3$, and define $M_3 = \Omega M_2$, the kernel in the exact sequence

$$0 \longrightarrow M_3 \hookrightarrow R_3^5 \longrightarrow M_2 \longrightarrow 0,$$

where the map to M_2 takes the basis for R_3^5 to the five generators of M_2 corresponding to the description above. M_3 is again minimal.

In general M_n is given as a module over $R_n = \mathbf{Q}[h_1,\ldots,h_{2n}]$, indeed is a submodule of a free R_n-module, and M_{n+1} is ΩM_n where M_n is now regarded as a module over $R_{n+1} = R_n \otimes \mathbf{Q}[h_{2n+1},h_{2n+2}]$.

Note that for M_1 we have $Ext^0(M_1,\mathbf{Q}) = Ext^{0,2}(M_1,\mathbf{Q})$, and for M_2, $Ext^0(M_2,\mathbf{Q}) = Ext^{0,4}(M_2,\mathbf{Q})$. In other words, for these modules Ext^0 is concentrated in a single bidegree. Also, note that $Ext_{R_n}(\mathbf{Q},\mathbf{Q}) = \Lambda(|h_1|^*,\ldots,|h_{2n}|^*)$ has each generator $|h_i|^*$ occurring in bidegree $(1,2)$, so all the elements in Ext^i occur in a single bidegree $(i, 2i)$ here as well. In fact we have by a direct induction

Lemma 14.5. *The action map*

$$Ext^{*,*}_{R_n}(\mathbf{Q},\mathbf{Q}) \otimes Ext^{0,*}_{R_n}(M_n,\mathbf{Q}) \longrightarrow Ext^{i,j}_{R_n}(M_n,\mathbf{Q})$$

is surjective in all degrees, and $Ext^{0,j}_{R_n}(M_n,\mathbf{Q}) = 0$ *unless* $j = 2n$. *Consequently,*

$$Ext^{i,j}_{R_n}(M_n,\mathbf{Q}) = 0$$

unless $j = 2n + 2i$.

As an immediate corollary we see that the M_i, each being minimal, are all unique, since, in as much as all the generators of M_i occur in the same degree, the map of the minimal free module onto M_i is well defined up to an isomorphism, hence the same is true of the kernel, M_{i+1}.

With this backround discussion complete we are able to state our main result.

Theorem 14.6. *Suppose* $g \geq 2$. *Then*

$$Ext^{i,j}_{\Lambda_g}(V_g,\mathbf{Q}) = \begin{cases} \mathbf{Q} & \text{if } (i,j) = (0,0), \\ \mathbf{Q} & \text{if } (i,j) = (1,2), \\ (M_g)_{i-2,j+i-4} & \text{if } i \geq 2. \end{cases}$$

In particular, as a module over $Ext_{\Lambda_g}(\mathbf{Q},\mathbf{Q})$

$$Ext_{\Lambda_g}(V_g,\mathbf{Q}) = \mathbf{Q} \oplus \mathbf{Q} \oplus M_g,$$

where the first generator gives $Ext^{0,0}$, *the second gives* $Ext^{1,2}$ *and the rest is shifted by* $(2, -i+4)$.

Example 14.7. We have

$$Ext_{\Lambda_2}(V_2,\mathbf{Q}) = \mathbf{Q}e_{0,0} \oplus \mathbf{Q}P_{1,2} \oplus \mathbf{Q}[h_1,\ldots,h_4]\{b_1,b_2,\ldots,b_5\}/\mathcal{R},$$

where $b_i \in Ext^{2,4}_{\Lambda_2}(V_2,\mathbf{Q})$, and $\mathcal{R}$ is the set of four relations

$$\mathcal{R} = \{h_5b_1,\ h_6b_1,\ h_5b_3 - h_6b_2,\ h_5b_5 - h_6b_4\}.$$

Proof. A resolution of V_g over Λ_g starts in the following way

$$0 \longrightarrow K_g \hookrightarrow \Lambda_g P_{1,2} \xrightarrow{*f_g} \Lambda_g e_{0,0} \longrightarrow V_g \longrightarrow 0,$$

and we have already seen that $(K_g)_i = 0$ for $i \leq g+1$. Thus it follows that

$$Ext^{r,s}_{\Lambda_g}(V_g, \mathbf{Q}) = 0 \text{ whenever } r \geq 2 \text{ and } s \leq g+1.$$

On the other hand, since K_g has $N(g,g)$ generators in $g+2$, we have that

$$Ext^{2,g+2}_{\Lambda_g}(V_g, \mathbf{Q}) \; = \; \mathbf{Q}^{N(g,g)}.$$

We now give an inductive analysis of the V_g. In particular we assume the theorem is true for V_{g-1} and begin by establishing a relation between V_g and V_{g-1}. Actually, we start with $g = 1$ where the theorem is not quite true as stated, but $Ext_{\Lambda_1}(V_1, \mathbf{Q})$ is given as an extension

$$0 \longrightarrow M_1 \longrightarrow Ext_{\Lambda_1}(V_1, \mathbf{Q}) \longrightarrow \mathbf{Q}e_{0,0} \oplus \mathbf{Q}P_{1,2} \longrightarrow 0,$$

which turns out to be sufficient to start the induction. There is a surjection

$$\pi_g \colon V_g \longrightarrow V_{g-1}$$

induced by the projection $p_g \colon \Lambda_g \rightarrow \Lambda_{g-1}$,

$$p_g(e_i) \;=\; \begin{cases} e_i & \text{for } i \leq i \leq 2g-2, \\ 0 & for \text{i = 2g-1or } i = 2g, \end{cases}$$

which takes f_g to f_{g-1}. Let N_g be the kernel of π_g, and $\mathcal{K}_g$ be the kernel of p_g. Then we can write

$$\mathcal{K}_g \cong \Lambda_{g-1}\{e_{2g-1},\ e_{2g},\ e_{2g-1}e_{2g}\}$$

and

$$\mathcal{K}_g / f_g \mathcal{K}_g \;=\; V_{g-1} \otimes \mathbf{Q}\{e_{2g-1},\ e_{2g},\ e_{2g-1}e_{2g}\}$$

as a module over Λ_g. However, the surjection

$$\bar{p}_g \colon \mathcal{K}_g / f_g \mathcal{K}_g \longrightarrow N_g \longrightarrow 0$$

is *not an isomorphism.* In fact we have

Lemma 14.8. *The kernel of $\bar{p}_g$ is a direct sum of trivial modules over Λ_g concentrated in degree $g+1$,*

$$Ker(\bar{p}_g) = Ker(\bar{p}_g)_{g+1} \cong \mathbf{Q}^{N(g-1,g-1)}$$

as a module over Λ_g.

Proof. For $2 \le i \le g-1$, using the expansion

$$\binom{r}{i} = \binom{r-2}{i} + 2\binom{r-2}{i-1} + \binom{r-2}{i-2}$$

valid for $r \ge i \ge 2$, we have

$$\begin{aligned} Dim(N_g)_i &= \left[\binom{2g}{i} - \binom{2g}{i-2}\right] - \left[\binom{2g-2}{i} - \binom{2g-2}{i-2}\right] \\ &= \left\{2\binom{2g-2}{i-1} + \binom{2g-2}{i-2}\right\} - \left\{2\binom{2g-2}{i-3} + \binom{2g-2}{i-4}\right\}. \end{aligned}$$

This shows that $\bar{p}_g$ is an isomorphism in this range. Likewise,

$$\begin{aligned} Dim(N_g)_g &= \binom{2g}{g} - \binom{2g}{g-2} \\ &= \binom{2g-2}{g} + 2\binom{2g-2}{g-1} - 2\binom{2g-2}{g-3} - \binom{2g-2}{g-4} \\ &= \left[\binom{2g-2}{g-2} - \binom{2g-2}{g-4}\right] - 2\left[\binom{2g-2}{g-1} - \binom{2g-2}{g-3}\right], \end{aligned}$$

using the equality $\binom{2g-2}{g} = \binom{2g-2}{g-2}$, which gives the isomorphism in dimension g. However, $(N_g)_{g+1} = 0$ while

$$Dim((\mathcal{K}_g/f_g\mathcal{K}_g)_{g+1}) = Dim((V_{g-1})_{g-1}) = N(g-1,g-1),$$

and the lemma follows since $(\mathcal{K}_g/f_g\mathcal{K}_g)_t = 0$ for $t \ge g+2$. q.e.d.

Consequently we have a short exact sequence of Λ_g-modules

$$0 \longrightarrow \mathbf{Q}^{N(g-1,g-1)} \longrightarrow \mathcal{K}_g/f_g\mathcal{K}_g \longrightarrow N_g \longrightarrow 0$$

and a long exact sequence of Ext-modules

$$(*) \qquad \begin{aligned} &\cdots \xrightarrow{\delta} Ext_{\Lambda_g}(N_g, \mathbf{Q}) \longrightarrow Ext_{\Lambda_g}(\mathcal{K}_g, \mathbf{Q}) \\ &\xrightarrow{i^*} \coprod_{i=1}^{N(g-1,g-1)} Ext_{\Lambda_g}(\mathbf{Q},\mathbf{Q})\chi_i \xrightarrow{\delta} \cdots \end{aligned}$$

where the generator, χ_i for each of the $Ext_{\Lambda_g}(\mathbf{Q},\mathbf{Q})$ terms occurs in bidegree $(0, g+1)$.

By change of rings we have

$$\begin{aligned} Ext_{\Lambda_g}(\mathcal{K}_g,\mathbf{Q}) &\cong Ext_{\Lambda_{g-1}}(V_{g-1},\mathbf{Q}) \\ &\qquad \oplus Ext_{\Lambda(e_{2g-1},e_{2g})}(\{e_{2g-1},e_{2g},e_{2g-1}e_{2g}\},\mathbf{Q}) \\ &= Ext_{\Lambda_{g-1}}(V_{g-1},\mathbf{Q})\otimes \mathbf{Q}[h_{2g-1},h_{2g}](b_1,b_2)/\mathcal{R}, \end{aligned}$$

where $\mathcal{R}$ is the relation $h_{2g}b_1 - h_{2g-1}b_2 = 0$. By our inductive assumption this implies that the generators of $Ext_{\Lambda_g}(\mathcal{K}_g,\mathbf{Q})$ as a module over $Ext_{\Lambda_g}(\mathbf{Q},\mathbf{Q})$ occur in bidegrees $(0,1)$, $(1,3)$, and $(2,g+2)$. But in these bidegrees $Ext_{\Lambda_g}(\mathbf{Q},\mathbf{Q})e_{0,g+1}$ is identically zero. It follows that the map i^* in $(*)$ is zero so

$$\delta\colon \coprod_{i=1}^{N(g-1,g-1)} Ext_{\Lambda_g}(\mathbf{Q},\mathbf{Q})\chi_i \longrightarrow Ext_{\Lambda_g}(N_g,\mathbf{Q})$$

is an injection, and therefore

$$Ext_{\Lambda_g}(N_g,\mathbf{Q}) = \coprod_{i=1}^{N(g-1,g-1)} Ext_{\Lambda_g}(\mathbf{Q},\mathbf{Q})e^i_{1,g+1} \oplus Ext_{\Lambda_g}(\mathcal{K}_g,\mathbf{Q})$$

as modules over $Ext_{\Lambda_g}(\mathbf{Q},\mathbf{Q})$ up to a possible extension problem.

The remainder of the proof is direct. The exact sequence

$$0\longrightarrow N_g \longrightarrow V_g \xrightarrow{\pi_g} V_{g-1} \longrightarrow 0$$

gives us the long exact sequence of Ext-groups

$$\begin{aligned} \cdots Ext^i_{\Lambda_g}(N_g,\mathbf{Q}) &\xrightarrow{\delta} Ext^{i+1}_{\Lambda_{g-1}}(V_{g-1},\mathbf{Q})\otimes \mathbf{Q}[h_{2g-1},h_{2g}] \\ &\longrightarrow Ext_{\Lambda_g}(V_g,\mathbf{Q}) \longrightarrow \cdots . \end{aligned}$$

The map δ on the piece $Ext_{\Lambda_g}(\mathcal{K}_g,\mathbf{Q})$ in $Ext_{\Lambda_g}(N_g,\mathbf{Q})$ injects to

$$Ext_{\Lambda_{g-1}}(V_{g-1},\mathbf{Q})\otimes I,$$

where $I \subset \mathbf{Q}[h_{2g-1},h_{2g}]$ is the augmentation ideal (h_{2h-1},h_{2g}). The resulting quotient is $Ext_{\Lambda_{g-1}}(V_{g-1},\mathbf{Q})$ and so the calculation reduces to the determination of the map

$$\delta\colon \coprod_{i=1}^{N(g-1,g-1)} Ext_{\Lambda_g}(\mathbf{Q},\mathbf{Q})\chi_i \longrightarrow M_{g-1}$$

from the inductive assumption. However, we know $Ext^{*,j}_{\Lambda_g}(V_g, \mathbf{Q}) = 0$ for $* \geq 2$ and $j \leq g+1$, and also

$$Ext^{2,g+1}_{\Lambda_{g-1}}(V_{g-1}, \mathbf{Q}) = \mathbf{Q}^{N(g-1,g-1)},$$

so it follows that δ must give an isomorphism from $(1, g+1)$ to $(2, g+1)$. Since M_{g-1} is generated over $Ext_{\Lambda_{g-1}}(\mathbf{Q}, \mathbf{Q})$ by the elements in this dimension, consequently δ is surjective and the kernel is minimal and hence M_g, with the generators all occuring in bidegree $(2, g+2)$. The induction is complete. q.e.d.

15. The rational homology of $\mathbf{Hol}^*_k(M_g, \mathbf{P}^n)$ for M_g hyperelliptic

We now use the results of §10-§14 to prove the following theorem.

Theorem 15.1. *The map*

$$H_*(Hol^*_k(M_g, \mathbf{P}^n); \mathbf{Q}) \rightarrow H_*(Map^*_k(M_g, \mathbf{P}^n); \mathbf{Q})$$

is injective for $k \geq 2g$ and $n > 2$.

Proof. Consider the subquotient of the E^1 term in the spectral sequence of (5.9) converging to $H_*(QE_{v,\ldots,v}; \mathbf{Q})$ defined as the direct sum

$$\sum_{v=2s}^{g+s-1} H_*(W_{v-2s}, W_{v-2s-1}) \otimes \tilde{H}_*(S^{2s(n+1)}) \\ \otimes H_*(SP^{k-v}(\Sigma M_g), SP^{k-v-1}(\Sigma M_g); \mathbf{Q})$$

for a given $s = 0, 1, 2, \ldots$ with the term

$$H_*(J(M_g), W_{g-s+1}) \otimes \tilde{H}_*(S^{2s(n+1)}) \\ \otimes H_*(SP^{k-g-s}(\Sigma M_g), SP^{k-v-s-1}(\Sigma M_g); \mathbf{Q})$$

added but playing a special role. We have determined that

$$H_*(W_j, W_{j-1}) = \sum_{l=0}^{[j/2]} (V_g)_{j-2l}[M]^l,$$

where $[M] \sim \sum_1^g e_{2i-1}e_{2i}$ in $H_*(J(M_g))$. Also,

$$H_*(SP^k(\Sigma M_g), SP^{k-1}(\Sigma M_g)) \\ = \mathbf{Q}[|e_1|, \ldots, |e_{2g}|]_k \oplus \mathbf{Q}[|e_1|, \ldots, |e_{2g}|]_{k-1}|[M_g]|$$

and the (sub-quotient) d_1 differentials are given by $d_1(|[M_g]|) = [M_g]$, $d_1(|e_i|) = e_i$, including the map from the term $H_*(W_{g-s+1}, W_{g-s}) \otimes \cdots$ to the extreme term

$$H_*(J(M_g), W_{g-s}) \otimes \cdots .$$

If we first apply the differential to $|[M_g]|$, we obtain the complex

$$\left\{ \sum_{v=2s}^{g+s-1} (V_g)_{v-2s} \otimes \mathbf{Q}[|e_1|, \ldots, |e_{2g}|]_{k-v} \right\} \otimes \tilde{H}_*(S^{2s(n+1)}),$$

with differentials as specified as well as an appropriate quotient of the extreme term with differentials mapping to it from the $v = g+s-1$-term in the sum above.

Note that the complex in brackets above is a piece of a direct summand,

$$\sum_{w=0}^{g-2} (V_g)_w \otimes \mathbf{Q}[|e_1|, \ldots, |e_{2g}|]_{k-2s-w}$$

of the complex $(V_g) \otimes \mathbf{Q}[|e_1|, \ldots, |e_{2g}|]$ which has, as its homology the groups

$$Tor_*^{\Lambda_g}(\mathbf{Q}, V_g).$$

Here, the extreme term contains the next term in the resolution as a direct summand, but contains $H_*(J(M_g), W_{g-s-1}) \otimes \cdots$ as well as the complementary summand. On the other hand we have already determined these Tor-groups in §14 and have shown, in particular that $Tor_{l,m}^{\Lambda_g}(\mathbf{Q}, V_g) = 0$ unless $m = m(g, l)$ is

$$m(g,l) = \begin{cases} l+g & \text{if } l \geq 2, \\ 2 & \text{if } l = 1, \\ 0 & \text{if } l = 0. \end{cases}$$

But, in the main part of the complex the m's which appear are always less than $m(g, l)$ so they contribute nothing by E^2, while at E^2 the term $H_*(J(M_g), W_{g-s}) \otimes \cdots$, which is a surjective image of the stable term $\tilde{H}(J(M_g)) \otimes \cdots$, contributes a quotient, consequently still a surjective image of the stable term, and the result follows for $n > 2$ by the collapsing of the stable spectral sequence at E^2. q.e.d.

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