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TORUS BUNDLES AND THE GROUP COHOMOLOGY OF $GL(N, \mathbb{Z})$

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Abstract

We prove the vanishing of a certain characteristic class of flat vector bundles when the structure groups of the bundles are contained in $GL(N, \mathbb{Z})$. We do so by explicitly writing the characteristic class as an exact form on the base of the bundle.

In this paper we consider certain characteristic classes of flat complex vector bundles, which are known in algebraic K-theory as the Borel regulator classes. We prove that if the structure group of a rank-N vector bundle is contained in $GL(N, \mathbb{Z})$ with N odd, then the Borel class of degree $2N-1$ vanishes. Our proof is analytic in nature and is a special, but interesting, case of a more general theorem concerning the direct images of flat vector bundles under smooth submersions [4].

The background to our result is the following. First, let N be a positive even integer and let E be a real oriented rank-N vector bundle over a connected manifold B. Then the rational Euler class $\chi_{\mathbb{Q}}(E)$ is an element of $\mathrm{H}^N(B; \mathbb{Q})$. Sullivan showed that if E is a flat vector bundle whose structure group is contained in $SL(N, \mathbb{Z})$, then $\chi_{\mathbb{Q}}(E) = 0$ [12]. Let Λ be the integer lattice in E. Then $M = E/\Lambda$ is the total space of a torus bundle over B. Sullivan's proof was by a simple topological argument involving this torus bundle.

Bismut and Cheeger observed that Sullivan's result follows from the Atiyah-Singer families index theorem, applied to the vertical signature operators on the torus bundle [2]. They also showed that one can write

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a certain differential-form representative of $\chi_{\mathbb{Q}}(E)$ explicitly as an exact form, by means of the so-called eta-form of [1].

In [4] we proved a real analog of the Riemann-Roch-Grothendieck (RRG) theorem. The geometric setup of our theorem involved a smooth fiber bundle $\pi : M \to B$ with closed connected fibers Z. If F is a flat complex vector bundle on M, then it has a direct image $H\left(Z; F\big|_Z\right)$ which is an alternating sum of flat complex vector bundles on B, given by the cohomologies of the fibers Z (with value in $F\big|_Z$). We defined certain characteristic classes of flat complex vector bundles and showed a relationship between the characteristic classes of F and $H\left(Z; F\big|_Z\right)$. We actually proved a more refined statement at the level of differential forms on B, which involved a so-called analytic torsion form on B.

The motivation of the present paper was to see what the RRG-type theorem of [4] says in the case of the torus bundle described above. We now state the precise results.

Let B be a connected smooth manifold. Let E be a complex rank-N vector bundle over B with a flat connection ∇^E. We define certain characteristic classes $\{n^{\mathrm{z}}_j(\nabla^E)\}_{j=1}^N$, with $n^{\mathrm{z}}_j(\nabla^E) \in \mathrm{H}^{2j-1}(B;\mathbb{R})$. These classes are pulled back from universal classes

$$n^{\mathrm{z}}_{j,\mathbb{C}^N} \in \mathrm{H}^{2j-1}(BGL(N,\mathbb{C})_\delta;\mathbb{R}),$$

where δ denotes the discrete topology on $GL(N,\mathbb{C})$. The classes $\{n^{\mathrm{z}}_{j,\mathbb{C}^N}\}_{j=1}^N$ can be characterized by the fact that the continuous real-valued group cohomology of $GL(N,\mathbb{C})$ is an exterior algebra

$$\Lambda\left(n^{\mathrm{z}}_{1,\mathbb{C}^N}, n^{\mathrm{z}}_{2,\mathbb{C}^N}, \ldots, n^{\mathrm{z}}_{N,\mathbb{C}^N}\right)$$

[5]. They are stable classes in the sense that they come from classes in $\mathrm{H}^{2j-1}(BGL(\infty,\mathbb{C})_\delta;\mathbb{R})$, and give rise to the Borel regulators on the algebraic K-theory of number fields.

Our main result is the following :

Theorem 0.1. *Let N be a positive odd integer. Let E be a flat complex rank-N vector bundle over B whose structure group is contained in $GL(N,\mathbb{Z})$. Then $n^{\mathrm{z}}_N(\nabla^E)$ vanishes in $\mathrm{H}^{2N-1}(B;\mathbb{R})$.*

An equivalent formulation of the theorem is:

Theorem 0.2. *Let N be a positive odd integer. Let $i : GL(N,\mathbb{Z}) \to GL(N,\mathbb{C})$ be the natural inclusion. Then $i^*\left(n^{\mathrm{z}}_{N,\mathbb{C}^N}\right)$ vanishes in the group cohomology $\mathrm{H}^{2N-1}(GL(N,\mathbb{Z});\mathbb{R})$.*

Borel showed that for $j > 1$, the classes $i^*\left(n^{\mathrm{z}}_{j,\mathbb{C}^N}\right)$ are nonzero if N is sufficiently large compared to j [5]. Ronnie Lee informs us that he showed many years ago in unpublished work that $i^*\left(n^{\mathrm{z}}_{j,\mathbb{C}^N}\right)$ is nonzero if $1 < j < N$. Furthermore, Jens Franke informs us that Theorem 0.2 is a special case of his general results on the group cohomology of arithmetic groups [8]; his method of proof seems to be completely different from ours.

As shown in Section 1, Theorem 0.1 follows fairly directly from the RRG-type theorem of [4], when applied to a torus bundle. However, in this special case of a torus bundle one can greatly simplify the arguments in [4]. Thus in this paper we give an alternative self-contained proof of Theorem 0.1. This parallels the corresponding proof of Sullivan's result in [2]. Given $t > 0$, in Section 2 we define a form δ_t on the total space of E^* with the property that if λ is a flat section of E^*, then $\lambda^*\delta_t$ is closed on B. Letting Λ^* denote the dual lattice to Λ, we consider the closed form $\sum_{\mu\in\Lambda^*}\mu^*\delta_t$ on B. We show that its de Rham cohomology class $\left[\sum_{\mu\in\Lambda^*}\mu^*\delta_t\right]$ is independent of t. Taking $t \to 0$, one can see that $\left[\sum_{\mu\in\Lambda^*}\mu^*\delta_t\right]$ is a nonzero constant times $n^{\mathrm{z}}_N(\nabla^E)$. We then define forms ρ_t on the total space of E, which are again closed after being pulled back by flat sections, and which have the property that $\sum_{\mu\in\Lambda^*}\mu^*\delta_t$ is proportionate to $\sum_{m\in\Lambda}m^*\rho_t$. Taking $t \to \infty$, one sees that $\sum_{m\in\Lambda}m^*\rho_t \to 0$. This proves Theorem 0.1. By keeping track of the t-dependence, we find an explicit form on B, defined as a Dirichlet-type series, whose differential represents $n^{\mathrm{z}}_N(\nabla^E)$.

The elementary proof of Theorem 0.1 given in Section 2 is in fact a transcription of the proof of the RRG-type theorem of [4], in the special case of a torus bundle. The transcription proceeds by Fourier analysis along the fibers. In Section 3 we assume a knowledge of [4] and make the relationship explicit.

An open question is whether there is a simple topological proof of Theorem 0.1 along the lines of Sullivan's proof of his result.

The paper is organized as follows. In Section 1 we define the relevant characteristic classes of flat vector bundles by means of differential-form representatives. Given a complex vector bundle F with flat connection ∇^F, a Hermitian metric h^F on F and a real invariant power series P on the space of $(N \times N)$-complex matrices, we define a closed differential form $P^z(\nabla^F, h^F) \in \Omega^{\mathrm{odd}}(B)$, whose de Rham cohomology class $P^z(\nabla^F) \in \mathrm{H}^{\mathrm{odd}}(B;\mathbb{R})$ is independent of h^F. In the case of $P(A) = \mathrm{Tr}\left[A^j\right]$, we obtain the above-mentioned cohomology class $n_j^z(\nabla^F) \in \mathrm{H}^{2j-1}(B;\mathbb{R})$. We study the relationships between the classes and describe them in terms of the cohomology of the classifying space $BGL(N,\mathbb{C})_\delta$. We then recall the RRG-type theorem of [4] and show how it implies Theorems 0.1 and 0.2. We also prove that Theorem 0.2 is a special case of a general vanishing theorem for the group cohomology of a discrete group which acts smoothly on a compact manifold.

In Section 2 we start with a real oriented rank-N vector bundle $\mathcal{V}$. We review Berezin integrals, the Thom form $\alpha_t \in \Omega^N(\mathcal{V})$ of Mathai-Quillen and its transgressing form $\beta_t \in \Omega^{N-1}(\mathcal{V})$. If $\mathcal{V}$ is flat, we define the form $\delta_t \in \Omega^{2N-1}(\mathcal{V})$ by a slight modification of the definition of α_t, and also construct its transgressing form $\epsilon_t \in \Omega^{2N-2}(\mathcal{V})$. We then define an auxiliary form $\rho_t \in \Omega^{2N-1}(\mathcal{V})$ and its transgressing form $\sigma_t \in \Omega^{2N-2}(\mathcal{V})$. We give an elementary proof of Theorem 0.1 along the lines sketched above.

In Section 3 we first consider the total space of a flat vector bundle $\mathcal{V}$. Using the superconnection formalism of [4], we define a form $\widetilde{\delta}_t \in \Omega^{2N-1}(\mathcal{V})$ and its transgressing form $\widetilde{\epsilon}_t \in \Omega^{2N-2}(\mathcal{V})$. We then show that $\widetilde{\delta}_t$ and $\widetilde{\epsilon}_t$ are essentially the same as the forms δ_t and ϵ_t of Section 2. Using Fourier analysis on the fibers of the torus bundle M, we prove that the closed form $f\left(C'_t, h^W\right)$ of [4] reduces to $\sum_{\mu\in\Lambda^*}\mu^*\delta_t$, and that the transgressing form $\frac{1}{t}\, f^\wedge\left(C'_t, h^W\right)$ of [4] reduces to $\sum_{\mu\in\Lambda^*}\mu^*\epsilon_t$. This establishes the link between the results of Section 2 and those of [4].

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I. Characteristic classes of flat vector bundles

In this section we describe certain characteristic classes of flat vector bundles.

The section is organized as follows. In a) we establish our conventions. Given a rank-N complex vector bundle F on a base B, a flat connection ∇^F on F, a Hermitian metric h^F on F and an invariant power series P on the space of complex $(N \times N)$-matrices, in b) we define a closed form $P^{\mathrm{z}}(\nabla^F, h^F) \in \Omega^{\mathrm{odd}}(B)$. Its de Rham cohomology class $P^{\mathrm{z}}(\nabla^F)$ is independent of h^F. We describe relationships between the forms $P^{\mathrm{z}}(\nabla^F, h^F)$ for different choices of P and F. In c) we relate $P^{\mathrm{z}}(\nabla^F)$ to the Borel regulator classes. In d) we recall the RRG-type theorem of [4] and prove Theorems 0.1 and 0.2.

a) Conventions

Except where otherwise indicated, we will take all vector spaces in this paper to be over $\mathbb{C}$. The covariant functor Λ sends a vector space V to its exterior algebra $\Lambda(V)$, and a linear map $T : V \to W$ to an algebra homomorphism $\Lambda(T) : \Lambda(V) \to \Lambda(W)$.

If A is an $N \times N$ complex matrix, put

$$
\begin{aligned}
c(A) &= \det(I + A), \\
\mathrm{ch}(A) &= \mathrm{Tr}\left[e^A\right], \\
\mathrm{Td}(A) &= \det\left(\frac{A}{I - e^{-A}}\right), \\
n_j(A) &= \mathrm{Tr}\left[A^j\right], \qquad j \in \mathbb{N}.
\end{aligned}
\tag{1.1}
$$

Let $\{c_j(A)\}_{j=1}^N$ be the symmetric functions of A, satisfying

$$
c(\lambda A) = 1 + \lambda c_1(A) + \ldots + \lambda^N c_N(A). \tag{1.2}
$$

Let B be a smooth connected manifold. If E is a smooth vector bundle over B, we let $C^\infty(B; E)$ denote the smooth sections of E, and $L^2(B; E)$ the L^2 measurable sections of E. We let $\Lambda(T^*B)$ denote the complexified exterior bundle of B, and $\Omega(B)$ the space of smooth sections of $\Lambda(T^*B)$. We put $\Omega(B; E) = C^\infty(B; \Lambda(T^*B) \otimes E)$. We will say that a differential form is real if it can be written with real coefficients.

b) Characteristic classes of flat vector bundles

Let $P(A)$ be an ad-invariant power series on the space of $N \times N$ complex matrices A. Then P can be expressed as a power series in the variables $\{n_j(A)\}_{j=1}^N$. We will say that P is real if P has real coefficients in these variables.

Let F be a complex rank-N vector bundle on B, endowed with a flat connection ∇^F. The antidual bundle $\overline{F}^*$ inherits a flat connection $\nabla^{\overline{F}^*}$. Let h^F be a Hermitian metric on F. We do not require that ∇^F be compatible with h^F. The metric h^F induces a Hermitian metric $h^{\overline{F}^*}$ on $\overline{F}^*$ and a $C^\infty(B)$-linear isometry

$$\widehat{h}^F : \Omega\left(B;F\right) \to \Omega\left(B;\overline{F}^*\right). \tag{1.3}$$

Then the adjoint flat connection $\left(\nabla^F\right)^*$ on F is given by

$$\left(\nabla^F\right)^* = \left(\widehat{h}^F\right)^{-1} \nabla^{\overline{F}^*} \widehat{h}^F. \tag{1.4}$$

Define $\omega\left(\nabla^F, h^F\right) \in \Omega^1(B; \mathrm{End}(F))$ by

$$\omega\left(\nabla^F, h^F\right) = \left(\nabla^F\right)^* - \nabla^F = \left(h^F\right)^{-1}\left(\nabla^F h^F\right). \tag{1.5}$$

In the rest of this section, except where otherwise indicated, we will abbreviate $\omega\left(\nabla^F, h^F\right)$ by ω. With respect to a locally-defined covariantly-constant basis of F, h^F is locally a Hermitian matrix-valued function on B, and we can write ω more simply as

$$\omega = \left(h^F\right)^{-1} dh^F. \tag{1.6}$$

Definition 1.1. The connection $\nabla^{F,u}$ on F is given by

$$\nabla^{F,u} = \nabla^F + \frac{\omega}{2}. \tag{1.7}$$

It is easy to see that $\nabla^{F,u}$ is compatible with h^F, and

$$\nabla^{F,u}\omega = 0. \tag{1.8}$$

The curvature of $\nabla^{F,u}$ is given by

$$\left(\nabla^{F,u}\right)^2 = -\frac{\omega^2}{4}. \tag{1.9}$$

Definition 1.2. Define $P(\nabla^F, h^F) \in \Omega^{\mathrm{even}}(B)$ by

$$P(\nabla^F, h^F) = P\left(\frac{\omega^2}{8i\pi}\right). \tag{1.10}$$

Lemma 1.3. *We have*

$$P(\nabla^F, h^F) = P(0). \tag{1.11}$$

Proof. We can write $P(A)$ as a power series in the variables $\{n_j(A)\}_{j=1}^N$. Clearly if $j = 0$ then $n_j(\nabla^F, h^F) = 1$. If $j > 0$ then

$$\operatorname{Tr}\left[\omega^{2j}\right] = \operatorname{Tr}\left[\omega \cdot \omega^{2j-1}\right] = -\operatorname{Tr}\left[\omega^{2j-1} \cdot \omega\right] = -\operatorname{Tr}\left[\omega^{2j}\right] = 0, \tag{1.12}$$

and so $n_j(\nabla^F, h^F) = 0$. The lemma follows. q.e.d.

Let z be an odd Grassmann variable, so that $z^2 = 0$. Given $\alpha \in \Omega(B) \widehat{\otimes} \mathbb{C}[z]$, we can write α in the form

$$\alpha = \alpha_0 + z\alpha_1 \tag{1.13}$$

with $\alpha_0, \alpha_1 \in \Omega(B)$. Put

$$\operatorname{Tr}_z[\alpha] = \alpha_1. \tag{1.14}$$

Definition 1.4. Define $P^z(\nabla^F, h^F) \in \Omega^{\mathrm{odd}}(B)$ by

$$P^z(\nabla^F, h^F) = \operatorname{Tr}_z\left[P(\frac{\omega^2}{8i\pi} + z\frac{\omega}{2})\right]. \tag{1.15}$$

In particular,

$$n_j^z(\nabla^F, h^F) = j \cdot 2^{-(2j-1)}(2i\pi)^{-(j-1)} \operatorname{Tr}\left[\omega^{2j-1}\right]. \tag{1.16}$$

Lemma 1.5. *If P and Q are two ad-invariant power series, then*

$$(PQ)^z(\nabla^F, h^F) = P(0) \cdot Q^z(\nabla^F, h^F) + P^z(\nabla^F, h^F) \cdot Q(0). \tag{1.17}$$

Proof. This follows from Lemma 1.3. q.e.d.

Lemma 1.6. *The odd form $P^z(\nabla^F, h^F)$ is closed, and its de Rham cohomology class is independent of h^F. If P is real, then $P^z(\nabla^F, h^F)$ is also real.*

Proof. In the case $P(A) = n_j(A)$, $j > 0$, a simple proof of the statement of the lemma was given in [4, Theorems 1.8 and 1.11]. The general case follows from expressing $P(A)$ as a power series in the variables $\{n_j(A)\}_{j=1}^N$ and using Lemma 1.5. q.e.d.

Lemma 1.7. *We have*

$$P^{\mathrm{z}}(\nabla^{\overline{F}^*}, h^{\overline{F}^*}) = -P^{\mathrm{z}}(\nabla^F, h^F). \tag{1.18}$$

Proof. One has

$$\omega\left(\nabla^F, h^F\right) = -\left(\widehat{h}^F\right)^{-1} \omega\left(\nabla^{\overline{F}^*}, h^{\overline{F}^*}\right) \widehat{h}^F. \tag{1.19}$$

The lemma follows from (1.15). q.e.d.

Definition 1.8. Let $P^{\mathrm{z}}(\nabla^F) \in \mathrm{H}^{\mathrm{odd}}(B;\mathbb{R})$ denote the de Rham cohomology class of $P^{\mathrm{z}}(\nabla^F, h^F)$.

Remark 1.9. Given the polynomial P, there is a corresponding Cheeger-Chern-Simons class $\widetilde{P}(\nabla^F) \in \mathrm{H}^{\mathrm{odd}}(B;\mathbb{C}/\mathbb{Z})$ of the flat bundle F [7]. If $P(A) = n_j(A)$ then, up to a multiplicative constant, $P^{\mathrm{z}}(\nabla^F)$ is the same as the imaginary part of $\widetilde{P}(\nabla^F)$ [4, Proposition 1.14].

Theorem 1.10. *We have*

$$\omega^{2N} = 0. \tag{1.20}$$

In particular, for $j > N$,

$$n_j^{\mathrm{z}}(\nabla^F, h^F) = 0. \tag{1.21}$$

For $j \leq N$,

$$c_j^{\mathrm{z}}(\nabla^F, h^F) = \frac{(-1)^{j-1}}{j}\, n_j^{\mathrm{z}}(\nabla^F, h^F). \tag{1.22}$$

Proof. Identity (1.20) appears in [11, Theorem 4.1]. We give a direct proof, which is essentially the same as that of [11]. By the Cayley-Hamilton theorem,

$$(\frac{\omega^2}{8i\pi})^N + \sum_{j=1}^N (-1)^j c_j(\nabla^F, h^F)\cdot(\frac{\omega^2}{8i\pi})^{N-j} = 0. \tag{1.23}$$

By Lemma 1.3, for $j > 0$,

$$c_j(\nabla^F, h^F) = 0, \tag{1.24}$$

which together with (1.23) implies (1.20) immediately.

Equation (1.21) follows from (1.16) and (1.20). Finally, if $j \leq N$ then Newton's formula gives an identity of polynomials:

$$n_j - c_1 n_{j-1} + \ldots + (-1)^{j-1} c_{j-1} n_1 + (-1)^j j c_j = 0. \tag{1.25}$$

Using Lemma 1.5 we thus obtain equation (1.22). q.e.d.

Let $E = E_+ \oplus E_-$ be a $\mathbb{Z}_2$-graded complex vector bundle on B. Let $\nabla^E = \nabla^{E_+} \oplus \nabla^{E_-}$ be a flat connection on E which preserves the splitting $E = E_+ \oplus E_-$. Let $h^E = h^{E_+} \oplus h^{E_-}$ be a Hermitian metric on E such that E_+ and E_- are orthogonal. Put

$$P^{\mathrm{z}}(\nabla^E, h^E) = P^{\mathrm{z}}(\nabla^{E_+}, h^{E_+}) - P^{\mathrm{z}}(\nabla^{E_-}, h^{E_-}). \tag{1.26}$$

Given the flat vector bundle F, the associated flat vector bundle $\Lambda(\overline{F}^*)$ has a natural $\mathbb{Z}_2$-grading. If h^F is a Hermitian metric on F, then there is an induced Hermitian metric $h^{\Lambda(\overline{F}^*)}$ on $\Lambda(\overline{F}^*)$.

Theorem 1.11. *We have*

$$\mathrm{ch}^{\mathrm{z}}\left(\nabla^{\Lambda(\overline{F}^*)}, h^{\Lambda(\overline{F}^*)}\right) = \frac{1}{N}\, n_N^{\mathrm{z}}(\nabla^F, h^F). \tag{1.27}$$

Proof. In general,

$$\mathrm{ch}(\Lambda(A)) = \det(I - e^A) = (-1)^N\ \mathrm{Td}(-A)^{-1} \cdot \det(A). \tag{1.28}$$

Using (1.17), we get

$$\begin{aligned} \mathrm{ch}^{\mathrm{z}}\left(\nabla^{\Lambda(\overline{F}^*)}, h^{\Lambda(\overline{F}^*)}\right) &= (-1)^N c_N^{\mathrm{z}}\left(\nabla^{\overline{F}^*}, h^{\overline{F}^*}\right) \\ &= -\ \frac{1}{N}\, n_N^{\mathrm{z}}\left(\nabla^{\overline{F}^*}, h^{\overline{F}^*}\right). \end{aligned} \tag{1.29}$$

The theorem now follows from Lemma 1.7. q.e.d.

Corollary 1.12. *In* $\mathrm{H}^{\mathrm{odd}}(B; \mathbb{R})$, *one has the equality*

$$\mathrm{ch}^{\mathrm{z}}\left(\nabla^{\Lambda(\overline{F}^*)}\right) = \frac{1}{N}\, n_N^{\mathrm{z}}(\nabla^F). \tag{1.30}$$

In particular, $\mathrm{ch}^{\mathrm{z}}\left(\nabla^{\Lambda(\overline{F}^*)}\right)$ *is concentrated in degree* $2N-1$.

Proof. This is an immediate consequence of Theorem 1.11. q.e.d.

c) Topological description of the characteristic classes

The classes $n_j^{\mathrm{z}}(\nabla^F)$ are the characteristic classes (of flat vector bundles) which are of interest to us. A more topological description of them can be given as follows. Let V be a finite-dimensional complex vector space. Let $\mathrm{H}_c^*(GL(V);\mathbb{R})$ denote the continuous group cohomology of $GL(V)$, meaning the cohomology of the complex of Eilenberg-Maclane cochains on $GL(V)$ which are continous in their arguments. Let $GL(V)_\delta$ denote $GL(V)$ with the discrete topology and let $BGL(V)_\delta$ denote its classifying space. The cohomology group $\mathrm{H}^*(BGL(V)_\delta;\mathbb{R})$ is isomorphic to the (discrete) group cohomology $\mathrm{H}^*(GL(V);\mathbb{R})$. There is a forgetful map

$$\mu_V : \mathrm{H}_c^*(GL(V);\mathbb{R}) \to \mathrm{H}^*(BGL(V)_\delta;\mathbb{R}). \tag{1.31}$$

Fix a basepoint $* \in B$. Put $\Gamma = \pi_1(B,*)$ and let $h : B \to B\Gamma$ be the classifying map for the universal cover of B, defined up to homotopy. Let V be the fiber of F above $*$. The holonomy of F is a homomorphism $r : \Gamma \to GL(V)$, and induces a map $Br : B\Gamma \to BGL(V)_\delta$. Then the flat bundle F is classified by the homotopy class of maps $\nu = Br \circ h : B \to BGL(V)_\delta$. One can show that there is a class $n_{j,V}^{\mathrm{z}} \in \mathrm{H}^{2j-1}(GL(V);\mathbb{R})$ such that $n_j^{\mathrm{z}}(\nabla^F) = \nu^*(n_{j,V}^{\mathrm{z}})$, and a class $N_{j,V}^{\mathrm{z}} \in \mathrm{H}_c^{2j-1}(GL(V);\mathbb{R})$ such that $n_{j,V}^{\mathrm{z}} = \mu_V(N_{j,V}^{\mathrm{z}})$. For example, $N_{1,V}^{\mathrm{z}}$ is given by the homomorphism $g \to \ln|\det(g)|$ from $GL(V)$ to $(\mathbb{R},+)$.

Put $G = GL(V)$ and $K = U(V)$. Denote the Lie algebras of G and K by $\gamma = gl(V)$ and $\kappa = u(V)$, respectively. The quotient space γ/κ is isomorphic to the space of Hermitian endomorphisms of V, and carries an adjoint representation of K. One has that $\mathrm{H}_c^*(GL(V);\mathbb{R})$ is isomorphic to $\mathrm{H}^*(\gamma,K;\mathbb{R})$, the cohomology of the complex $\mathrm{C}^*(\gamma,K;\mathbb{R}) = \mathrm{Hom}_K(\Lambda^*(\gamma/\kappa),\mathbb{R})$ [6, Chapter IX, §5]. In fact, the differential of this complex vanishes, and so $\mathrm{H}_c^*(GL(V);\mathbb{R}) = \mathrm{C}^*(\gamma,K;\mathbb{R})$ [6, Chapter II, Corollary 3.2]. Thus the classes $\{n_j^{\mathrm{z}}(\nabla^F)\}_{j=1}^\infty$ arise indirectly from K-invariant forms on γ/κ. It is possible to see the relationship between $n_j^{\mathrm{z}}(\nabla^F)$ and $\mathrm{C}^{2j-1}(\gamma,K;\mathbb{R})$ more directly [4, §1g]. In particular, define a $(2j-1)$-form Φ_j on γ/κ by sending Hermitian endomorphisms

$M_1, \ldots, M_{2j-1}$ to

$$\Phi_j(M_1, \ldots, M_{2j-1}) = \sum_{\sigma \in S_{2j-1}} (-1)^{\mathrm{sign}(\sigma)} \operatorname{Tr}\left[M_{\sigma(1)} \ldots M_{\sigma(2j-1)}\right]. \tag{1.32}$$

Then Φ_j is an element of $\mathrm{C}^{2j-1}(\gamma, K; \mathbb{R})$ which, up to an overall multiplicative constant, corresponds to $N^{\mathrm{z}}_{j,V}$.

The compact dual of the symmetric space G/K is G^d/K, where $G^d = U(V) \times U(V)$. Let $\gamma^d = u(V) \oplus u(V)$ be the Lie algebra of G^d. Duality gives an isomorphism between $\mathrm{H}^*(\gamma, K; \mathbb{R})$ and $\mathrm{H}^*(\gamma^d, K; \mathbb{R}) = \mathrm{H}^*(U(V); \mathbb{R}) = \Lambda(x_1, x_3, ..., x_{2\dim(V)-1})$. It follows that the classes $\{N^{\mathrm{z}}_{j,V}\}_{j=1}^{\dim(V)}$ are algebraically independent.

If V is the complexification of a real vector space $V_{\mathbb{R}}$, then one can apply the same arguments with $G = GL(V_{\mathbb{R}})$, $K = O(V_{\mathbb{R}})$ and $G^d = U(V)$. One obtains that if the flat complex vector bundle F is the complexification of a flat real vector bundle $F_{\mathbb{R}}$ and j is even then $n^{\mathrm{z}}_j(\nabla^F)$ vanishes.

d) Proof of the main theorem

We first review the RRG-type theorem of [4]. Let $Z \to M \xrightarrow{\pi} B$ be a smooth fiber bundle with connected base B and connected closed fibers $Z_b = \pi^{-1}(b)$. Let F be a flat complex vector bundle on M. Let $H\left(Z; F\big|_Z\right)$ denote the $\mathbb{Z}$-graded complex vector bundle on B whose fiber over $b \in B$ is isomorphic to the cohomology group $\mathrm{H}^*(Z_b, F\big|_{Z_b})$. It has a canonical flat connection $\nabla^{H(Z;F|_Z)}$ which preserves the $\mathbb{Z}$-grading. Let TZ be the vertical tangent bundle of the fiber bundle and let $o(TZ)$ be its orientation bundle, a flat real line bundle on M. Let $e(TZ) \in \mathrm{H}^{\dim(Z)}(M; o(TZ))$ be the Euler class of TZ.

Theorem 1.13 [4]. *For any positive integer j, one has an equality in $\mathrm{H}^{2j-1}(B; \mathbb{R})$:*

$$n^{\mathrm{z}}_j\left(\nabla^{H(Z;F|Z)}\right) = \int_Z e(TZ) \cdot n^{\mathrm{z}}_j(\nabla^F). \tag{1.33}$$

We now prove Theorem 0.1 of the introduction.

Theorem 0.1. *Let B be a connected smooth manifold. Let N be a positive odd integer. Let E be a flat complex rank-N vector bundle* *over B whose structure group is contained in $GL(N, \mathbb{Z})$. Then $n^{\mathrm{z}}_N(\nabla^E)$ vanishes in $\mathrm{H}^{2N-1}(B; \mathbb{R})$.*

Proof. We may assume without loss of generality that E is real. Let $*$ be a basepoint in B and put $\Gamma = \pi_1(B, *)$. Let $\widetilde{B}$ denote the universal cover of B. The holonomy of E is a homomorphism $\rho : \Gamma \to GL(N, \mathbb{Z})$ such that

$$E = \widetilde{B} \times_\rho \mathbb{R}^N. \tag{1.34}$$

Put

$$\Lambda = \widetilde{B} \times_\rho \mathbb{Z}^N \tag{1.35}$$

and $M = E/\Lambda$. Then M is the total space of a fiber bundle over B with fiber $Z = \mathbb{R}^N/\mathbb{Z}^N$.

Let F be the trivial flat complex line bundle on M. Then $n^{\mathrm{z}}_j(\nabla^F) = 0$. As the fiber Z is an N-torus, one can easily say what the cohomology bundle $H(Z; F|_Z)$ is. Namely, as $\mathrm{H}^*(T^N; \mathbb{C}) \cong \Lambda\left(\mathbb{R}^{N*}\right) \otimes \mathbb{C}$, we have

$$H(Z; F|_Z) = \Lambda\left(E^*\right) \otimes \mathbb{C}. \tag{1.36}$$

Theorem 1.13 now implies that

$$n^{\mathrm{z}}_j\left(\nabla^{\Lambda(E^*)}\right) = 0. \tag{1.37}$$

By Corollary 1.12, it follows that

$$\mathrm{ch}^{\mathrm{z}}\left(\nabla^{\Lambda(E^*)}\right) = \frac{1}{N}\, n^{\mathrm{z}}_N(\nabla^E). \tag{1.38}$$

By the definition of ch^{z}, the term of degree $2N-1$ of $\mathrm{ch}^{\mathrm{z}}\left(\nabla^{\Lambda(E^*)}\right)$ is proportional to $n^{\mathrm{z}}_N\left(\nabla^{\Lambda(E^*)}\right)$. The theorem now follows from combining (1.37), in the case $j = N$, and (1.38). q.e.d.

Recall from Section 1c that $n^{\mathrm{z}}_{N,\mathbb{C}^N}$ denotes both an element of the (discrete) group cohomology $\mathrm{H}^{2N-1}(GL(N, \mathbb{C}); \mathbb{R})$ and the corresponding element of $\mathrm{H}^{2N-1}(BGL(N, \mathbb{C})_\delta; \mathbb{R})$. We now prove Theorem 0.2 of the introduction.

Theorem 0.2. *Let N be a positive odd integer. Let $i : GL(N, \mathbb{Z}) \to GL(N, \mathbb{C})$ be the natural inclusion. Then $i^*\left(n^{\mathrm{z}}_{N,\mathbb{C}^N}\right)$ vanishes in the group cohomology $\mathrm{H}^{2N-1}(GL(N, \mathbb{Z}); \mathbb{R})$.*

Proof. It is known that there is a model for $BGL(N,\mathbb{Z})$ which is a CW-complex with a finite number of cells in each degree. For $K >> 2N-1$, let $BGL(N,\mathbb{Z})^K$ be the K-skeleton of $BGL(N,\mathbb{Z})$. Let B be a smooth connected manifold (possibly with boundary) which is homotopy equivalent to $BGL(N,\mathbb{Z})^K$. Let $h : B \to BGL(N,\mathbb{Z})$ be the classifying map for $\widetilde{B}$, defined up to homotopy. Let $Bi : BGL(N,\mathbb{Z}) \to BGL(N,\mathbb{C})$ be the map induced by i. Let $\mathcal{E}$ be the canonical $\mathbb{C}^N$-bundle on $BGL(N,\mathbb{C})$ and put $E = (Bi \circ h)^*\mathcal{E}$. Then the discussion in Section 1c yields that $n^{\mathrm{z}}_N\left(\nabla^E\right) = h^*(Bi)^*\left(n^{\mathrm{z}}_{N,\mathbb{C}^N}\right)$. By Theorem 0.1, $n^{\mathrm{z}}_N\left(\nabla^E\right) = 0$. As h is highly connected, it follows that $(Bi)^*\left(n^{\mathrm{z}}_{N,\mathbb{C}^N}\right) = 0$, which is equivalent to the theorem. q.e.d.

Theorem 0.2 is in fact a special case of the following theorem.

Theorem 1.14. *Let Γ be a discrete group such that there is a CW-complex $B\Gamma$ which is a $K(\Gamma,1)$-space with a finite number of cells in each degree. Let Z be a connected closed smooth manifold and let Γ act on Z by a homomorphism $\rho : \Gamma \to \mathrm{Diff}(Z)$. For each integer $p \in [0, \dim(Z)]$, there is an induced representation $r_p : \Gamma \to GL(\mathrm{H}^p(Z;\mathbb{C}))$. If j is a positive integer, we can pullback the group cohomology class $n^{\mathrm{z}}_{j,\mathrm{H}^p(Z;\mathbb{C})}$ under r_p to obtain $r_p^*\left(n^{\mathrm{z}}_{j,\mathrm{H}^p(Z;\mathbb{C})}\right) \in \mathrm{H}^{2j-1}(\Gamma;\mathbb{R})$. Then we have an equality in $\mathrm{H}^{2j-1}(\Gamma;\mathbb{R})$:*

$$\sum_{p=0}^{\dim(Z)} (-1)^p \; r_p^*\left(n^{\mathrm{z}}_{j,\mathrm{H}^p(Z;\mathbb{C})}\right) = 0. \tag{1.39}$$

The proof of Theorem 1.14 is similar to that of Theorem 0.2. Theorem 0.2 is the special case of Theorem 1.14 when $\Gamma = GL(N,\mathbb{Z})$ and $Z = T^N$.

Remark 1.15. One can give a more direct proof of Theorem 1.14 in the case $j=1$. If $j=1$, equation (1.39) says that for all $\gamma \in \Gamma$,

$$\sum_{p=0}^{\dim(Z)} (-1)^p \;\ln\left|\det r_p(\gamma)\big|_{\mathrm{H}^p(Z;\mathbb{C})}\right| = 0. \tag{1.40}$$

Let T^p denote the torsion subgroup of $\mathrm{H}^p(Z;\mathbb{Z})$. Then $\mathrm{H}^p(Z;\mathbb{C}) \cong (\mathrm{H}^p(Z;\mathbb{Z})/T^p)\otimes\mathbb{C}$. As $\rho(\gamma)$ is a diffeomorphism of Z, it acts as an automorphism of the lattice $\mathrm{H}^p(Z;\mathbb{Z})/T^p \subset \mathrm{H}^p(Z;\mathbb{C})$. Thus $\det r_p(\gamma)\big|_{\mathrm{H}^p(Z;\mathbb{C})} = \pm 1$, and equation (1.39) follows.

II. Elementary proof of the main theorem

In this section we use Berezin integrals to define certain forms δ_t and ρ_t on the total space of a flat vector bundle, which are closed when pulled back to the base by a flat section. We also define transgressing forms ϵ_t and σ_t. We use these forms to prove Theorem 0.1.

The section is organized as follows. In a) we briefly review the Berezin integral. In b) we review the construction of Mathai-Quillen of the Thom form α_t of a real rank-N vector bundle $\mathcal{V}$. If $\mathcal{V}$ is flat, we then define the forms $\delta_t \in \Omega^{2N-1}(\mathcal{V})$ and $\epsilon_t \in \Omega^{2N-2}(\mathcal{V})$, and establish their basic properties. In c) we construct useful auxiliary forms $\rho_t \in \Omega^{2N-1}(\mathcal{V})$ and $\sigma_t \in \Omega^{2N-2}(\mathcal{V})$. In d) we use these forms, along with the Poisson summation formula, to prove Theorem 0.1.

a) Berezin integrals

Let V now be a real oriented inner-product space of dimension N. Let $\{\psi_k\}_{k=1}^N$ be an oriented orthonormal basis of V. We can identify $\Lambda(V)$ with $\Lambda(\psi_1,\dots,\psi_N)$. The Berezin integral $\int^B : \Lambda(V) \to \mathbb{R}$ is defined to be the linear functional which vanishes on $\Lambda^k(V)$ unless $k = N$, in which case it is given by

$$\int^B \psi_1\psi_2\dots\psi_N = 1. \tag{2.1}$$

If A is a graded-commutative superalgebra over $\mathbb{R}$, there is an extension of the Berezin integral to a linear map

$$\int^B : A \,\widehat{\otimes}\, \Lambda(V) \to A \tag{2.2}$$

such that for $a \in A$ and $\alpha \in \Lambda(V)$,

$$\int^B a\cdot\alpha = a\int^B \alpha. \tag{2.3}$$

Let $\widehat{V}$ be another copy of V. The exterior algebra

$$\Lambda\left(V \oplus \widehat{V}\right) = \Lambda\left(\psi_1,\dots,\psi_N,\widehat{\psi}_1,\dots,\widehat{\psi}_N\right) \tag{2.4}$$

has a bigrading as

$$\Lambda\left(V \oplus \widehat{V}\right) = \bigoplus_{k,l=1}^{N} \Lambda^k(V)\,\widehat{\otimes}\,\Lambda^l(\widehat{V}). \tag{2.5}$$

Then the Berezin integral $\int^B : \Lambda\left(V \oplus \widehat{V}\right) \to \mathbb{R}$ vanishes on

$$\Lambda^k(V) \mathbin{\widehat{\otimes}} \Lambda^l(\widehat{V})$$

unless $k = l = N$, in which case it is given by

$$\int^B \psi_1 \dots \psi_N \widehat{\psi}_1 \dots \widehat{\psi}_N = 1. \tag{2.6}$$

The Berezin integral on $\Lambda\left(V \oplus \widehat{V}\right)$ is independent of the orientation of V.

If W is an $(N \times N)$-matrix, we write

$$\left\langle \psi, W\widehat{\psi} \right\rangle = \sum_{i,j} \psi_i W_{ij} \widehat{\psi}_j. \tag{2.7}$$

Lemma 2.1. *If V is an antisymmetric $(N \times N)$-matrix with even entries, and W is a symmetric $(N \times N)$-matrix with odd entries, then*

$$\begin{aligned} &\int^B \left\langle \psi, W\widehat{\psi} \right\rangle e^{\frac{1}{2}\langle \psi, V\psi\rangle - \frac{1}{2}\left\langle \widehat{\psi}, V\widehat{\psi}\right\rangle} \\ &\qquad = -\pi^{N-1} \operatorname{Tr}_z \left[\det\left(\frac{V}{i\pi} + zW\right)\right]. \end{aligned} \tag{2.8}$$

Proof. Equation (2.8) can be checked by putting V into normal form. q.e.d.

Let $A \in \operatorname{End}(V)$ be antisymmetric. Then we can identify A with an element of $\Lambda^2(V)$ by

$$A \to \frac{1}{2} \ \langle \psi, A\psi \rangle \, . \tag{2.9}$$

b) Thom-like forms

Let $\mathcal{V}$ be a real rank-N oriented vector bundle over a connected smooth manifold B, with projection map $\pi : \mathcal{V} \to B$. Let $h^{\mathcal{V}}$ be a metric on $\mathcal{V}$ and let $\nabla^{\mathcal{V}}$ be a compatible connection. The connection $\nabla^{\mathcal{V}}$ gives a splitting

$$T\mathcal{V} = \pi^* TB \oplus \pi^* \mathcal{V}. \tag{2.10}$$

There is an induced connection $\nabla^{\Lambda(\mathcal{V})}$ on $\Lambda(\mathcal{V})$. Put $\mathcal{E} = \Lambda(\pi^*\mathcal{V})$, a $\mathbb{Z}$-graded vector bundle on $\mathcal{V}$ with connection $\nabla^{\mathcal{E}} = \pi^*\nabla^{\Lambda(\mathcal{V})}$. We first define the Thom-form of Mathai-Quillen [10], following the notation of [3, Section 1.6].

The Berezin integral gives a map $\int^B : \Omega(\mathcal{V};\mathcal{E}) \to \Omega(\mathcal{V})$. There are certain elements of $\Omega(\mathcal{V};\mathcal{E})$ of interest, namely:

1. the tautological section $\mathbf{x} \in \Omega^0(\mathcal{V};\Lambda^1(\pi^*\mathcal{V}))$;
2. the element $|\mathbf{x}|^2 \in \Omega^0(\mathcal{V};\Lambda^0(\pi^*\mathcal{V}))$;
3. the element $\nabla^{\mathcal{E}}\mathbf{x} \in \Omega^1(\mathcal{V};\Lambda^1(\pi^*\mathcal{V}))$;
4. the curvature $R^{\mathcal{V}} = \pi^*\left(\nabla^{\mathcal{V}}\right)^2$. Using (2.9), we can think of $R^{\mathcal{V}}$ as an element of $\Omega^2(\mathcal{V};\pi^*\Lambda^2(\mathcal{V}))$.

Let $i_{\mathbf{x}} : \Omega^p(\mathcal{V};\Lambda^q(\pi^*\mathcal{V})) \to \Omega^p(\mathcal{V};\Lambda^{q-1}(\pi^*\mathcal{V}))$ be interior multiplication by the tautological section. For any $t > 0$ and $\alpha \in \Omega(\mathcal{V};\mathcal{E})$, one has

$$d\int^B \alpha = \int^B \left(\nabla^{\mathcal{E}} + 2\sqrt{t}\, i_{\mathbf{x}}\right)\alpha. \tag{2.11}$$

Definition 2.2. For $t > 0$, define $A_t \in \Omega(\mathcal{V};\mathcal{E})$ by

$$A_t = \frac{R^{\mathcal{V}}}{2} + \sqrt{t}\ \nabla^{\mathcal{E}}\mathbf{x} + t\ |\mathbf{x}|^2. \tag{2.12}$$

Definition 2.3. For $t > 0$, the Mathai-Quillen form $\alpha_t \in \Omega^N(\mathcal{V})$ is given by

$$\alpha_t = (-1)^{\frac{N(N+1)}{2}}\pi^{-\frac{N}{2}}\int^B e^{-A_t}. \tag{2.13}$$

The form $\beta_t \in \Omega^{N-1}(\mathcal{V})$ is given by

$$\beta_t = -\ (-1)^{\frac{N(N+1)}{2}}\pi^{-\frac{N}{2}}\int^B \frac{\mathbf{x}}{2\sqrt{t}}\ e^{-A_t}. \tag{2.14}$$

Theorem 2.4 ([10, Thm. 7.6], [3, Prop. 1.53]).
a. The form α_t is closed.
b. In addition,

$$\frac{\partial \alpha_t}{\partial t} = d\beta_t. \tag{2.15}$$

Proof. a. One can check that

$$\left(\nabla^{\mathcal{E}} + 2\sqrt{t}\, i_{\mathbf{x}}\right) A_t = 0. \tag{2.16}$$

Using (2.11) and (2.13), part a. of the theorem follows.

Part b. of the theorem can be easily checked directly, but we will give a more general construction which will be of use later. Put $B' = B \times \mathbb{R}^+$ and $\mathcal{V}' = \mathcal{V} \times \mathbb{R}^+$. Define $\pi' : \mathcal{V}' \to B'$ by $\pi'(f,s) = (\pi(f), s)$. Let $\rho_B : B' \to B$ and $\rho_{\mathcal{V}} : \mathcal{V}' \to \mathcal{V}$ be the projection maps. Then $\mathcal{V}' = \rho_B^* \mathcal{V}$. Using the product structure on $\mathcal{V}'$, we can write exterior differentiation on $\Omega(\mathcal{V}')$ as

$$d' = d + ds\ \partial_s. \tag{2.17}$$

Let $h^{\mathcal{V}'}$ be the metric on $\mathcal{V}'$ which restricts to $s \cdot h^{\mathcal{V}}$ on $\mathcal{V} \times \{s\}$. Then

$$\nabla^{\mathcal{V}'} = \rho_B^* \nabla^{\mathcal{V}} + \frac{ds}{2s} \tag{2.18}$$

is a connection on $\mathcal{V}'$ which is compatible with $h^{\mathcal{V}'}$. Furthermore, $\left(\nabla^{\mathcal{V}'}\right)^2 = \rho_B^* \left(\nabla^{\mathcal{V}}\right)^2$.

We now apply the preceding formalism to the vector bundle $\mathcal{V}'$ with metric $h^{\mathcal{V}'}$ and connection $\nabla^{\mathcal{V}'}$. Using an obvious notation, $A'_1 \in \Omega(\mathcal{V}'; \mathcal{E}')$ is given by

$$A'_1 = \frac{R^{\mathcal{V}}}{2} + \sqrt{s}\ \left(\nabla^{\mathcal{E}} \mathbf{x} + ds \wedge \frac{\mathbf{x}}{2s}\right) + s|\mathbf{x}|^2 = A_s + ds \wedge \frac{\mathbf{x}}{2\sqrt{s}}. \tag{2.19}$$

Then

$$\alpha'_1 = \int^B e^{-A'_1} = \left(\int^B e^{-A_s}\right) - ds \wedge \int^B \frac{\mathbf{x}}{2\sqrt{s}}\ e^{-A_s}. \tag{2.20}$$

From part a. of the theorem,

$$d'\alpha'_1 = 0. \tag{2.21}$$

Part b. of the theorem now follows from (2.17), (2.20) and (2.21).

q.e.d.

Remark 2.5. The closed form α_t is a Thom form in the sense that it is rapidly decreasing at infinity and the fiberwise integral $\pi_*\alpha_t$ is identically 1 on B.

We now assume that $\mathcal{V}$ has a flat connection $\nabla^{\mathcal{V}}$. Let $h^{\mathcal{V}}$ be a metric on $\mathcal{V}$. Define $\omega \in \Omega^1(B;\mathrm{End}(\mathcal{V}))$ as in (1.5) and $\nabla^{\mathcal{V},u}$ as in (1.7).

Lemma 2.6. *The j-form ω^j takes value in the symmetric endomorphisms of $\mathcal{V}$ if $j \equiv 0,1 \pmod 4$, and in the antisymmetric endomorphisms of $\mathcal{V}$ if $j \equiv 2,3 \pmod 4$.*

Proof. One sees from (1.5) that ω takes value in the symmetric endomorphisms of $\mathcal{V}$. Then $\left(\omega^j\right)^T = (-1)^{\frac{j(j-1)}{2}}\omega^j$. q.e.d.

Let $\widehat{\mathcal{V}}$ be another copy of $\mathcal{V}$. Put $\widehat{\mathcal{E}} = \Lambda\left(\pi^*\widehat{\mathcal{V}}\right)$. Following (2.6), the Berezin integral gives a map $\int^B : \Omega\left(\mathcal{V};\mathcal{E}\widehat{\otimes}\widehat{\mathcal{E}}\right) \to \Omega\left(\mathcal{V}\right)$.

There are certain elements of $\Omega\left(\mathcal{V};\mathcal{E}\ \widehat{\otimes}\ \widehat{\mathcal{E}}\right)$ of interest, namely:

1. the tautological section $\mathbf{x} \in \Omega^0(\mathcal{V};\Lambda^1(\pi^*\mathcal{V})\ \widehat{\otimes}\ \Lambda^0(\pi^*\widehat{\mathcal{V}}))$;
2. the tautological section $\widehat{\mathbf{x}} \in \Omega^0(\mathcal{V};\Lambda^0(\pi^*\mathcal{V})\ \widehat{\otimes}\ \Lambda^1(\pi^*\widehat{\mathcal{V}}))$;
3. the element $|\mathbf{x}|^2 = |\widehat{\mathbf{x}}|^2 \in \Omega^0(\mathcal{V};\Lambda^0(\pi^*\mathcal{V})\ \widehat{\otimes}\ \Lambda^0(\pi^*\widehat{\mathcal{V}}))$;
4. the element $\left\langle\psi,\widehat{\psi}\right\rangle \in \Omega^0(\mathcal{V};\Lambda^1(\pi^*\mathcal{V})\ \widehat{\otimes}\ \Lambda^1(\pi^*\widehat{\mathcal{V}}))$;
5. the element $\nabla^{\mathcal{E}\widehat{\otimes}\widehat{\mathcal{E}},u}\mathbf{x} \in \Omega^1(\mathcal{V};\Lambda^1(\pi^*\mathcal{V})\ \widehat{\otimes}\ \Lambda^0(\pi^*\widehat{\mathcal{V}}))$;
6. the element $\nabla^{\mathcal{E}\widehat{\otimes}\widehat{\mathcal{E}},u}\widehat{\mathbf{x}} \in \Omega^1(\mathcal{V};\Lambda^0(\pi^*\mathcal{V})\ \widehat{\otimes}\ \Lambda^1(\pi^*\widehat{\mathcal{V}}))$;
7. the curvature $R^{\mathcal{V},u} = \left(\pi^*\nabla^{\mathcal{V},u}\right)^2 = -\frac{\pi^*\omega^2}{4}$. We will think of $R^{\mathcal{V},u}$ as an element of $\Omega^2(\mathcal{V};\Lambda^2(\pi^*\mathcal{V})\ \widehat{\otimes}\ \Lambda^0(\pi^*\widehat{\mathcal{V}}))$, namely $R^{\mathcal{V},u} = -\frac{1}{8}\left\langle\psi,(\pi^*\omega^2)\psi\right\rangle$.
8. the element $\widehat{\omega} \in \Omega^1(\mathcal{V};\Lambda^1(\pi^*\mathcal{V})\ \widehat{\otimes}\ \Lambda^1(\pi^*\widehat{\mathcal{V}}))$ given by $\widehat{\omega} = \left\langle\psi,(\pi^*\omega)\widehat{\psi}\right\rangle$;
9. the element $\widehat{\omega^2} \in \Omega^2(\mathcal{V};\Lambda^0(\pi^*\mathcal{V})\ \widehat{\otimes}\ \Lambda^2(\pi^*\widehat{\mathcal{V}}))$ given by

$$\widehat{\omega^2} = \frac{1}{2}\left\langle\widehat{\psi},(\pi^*\omega^2)\widehat{\psi}\right\rangle.$$

As in (2.11), for any $t > 0$ and $\alpha \in \Omega\left(\mathcal{V}; \mathcal{E} \widehat{\otimes} \widehat{\mathcal{E}}\right)$, one has

$$d \int^B \alpha = \int^B \left(\nabla^{\mathcal{E}\widehat{\otimes}\widehat{\mathcal{E}},u} + 2\sqrt{t}\; i_{\mathbf{x}}\right) \alpha. \tag{2.22}$$

Definition 2.7. For $t > 0$, define $B_t \in \Omega\left(\mathcal{V}; \mathcal{E} \widehat{\otimes} \widehat{\mathcal{E}}\right)$ by

$$B_t = \frac{R^{\mathcal{V},u}}{2} + \sqrt{t}\; \nabla^{\mathcal{E}\widehat{\otimes}\widehat{\mathcal{E}},u}\mathbf{x} + t\; |\mathbf{x}|^2 + \frac{1}{8}\; \widehat{\omega^2}. \tag{2.23}$$

Definition 2.8. For $t > 0$, define $\delta_t \in \Omega^{2N-1}(\mathcal{V})$ by

$$\delta_t = \int^B \left(\frac{1}{4}\; \widehat{\omega} - \sqrt{t}\; \widehat{\mathbf{x}}\right) e^{-B_t} \tag{2.24}$$

and $\epsilon_t \in \Omega^{2N-2}(\mathcal{V})$ by

$$\epsilon_t = -\int^B \left(\frac{1}{4t} \left\langle \psi, \widehat{\psi} \right\rangle + \frac{\mathbf{x}}{2\sqrt{t}} \left(\frac{1}{4}\; \widehat{\omega} - \sqrt{t}\; \widehat{\mathbf{x}}\right)\right) e^{-B_t}. \tag{2.25}$$

Theorem 2.9.

a. Let U be an open subset of B and let $\lambda : U \to \mathcal{V}$ be a flat section of $\mathcal{V}$ over U. Then for $t > 0$,

$$d\left(\lambda^* \delta_t\right) = 0. \tag{2.26}$$

b. In addition,

$$\frac{\partial\left(\lambda^* \delta_t\right)}{\partial t} = d\left(\lambda^* \epsilon_t\right). \tag{2.27}$$

Proof. a. As in (2.16), we have

$$\left(\nabla^{\mathcal{E}\widehat{\otimes}\widehat{\mathcal{E}},u} + 2\sqrt{t}\; i_{\mathbf{x}}\right) \left(\frac{R^{\mathcal{V},u}}{2} + \sqrt{t}\; \nabla^{\mathcal{E}\widehat{\otimes}\widehat{\mathcal{E}},u}\mathbf{x} + t\; |\mathbf{x}|^2\right) = 0. \tag{2.28}$$

From (1.8) it follows that

$$\left(\nabla^{\mathcal{E}\widehat{\otimes}\widehat{\mathcal{E}},u} + 2\sqrt{t}\; i_{\mathbf{x}}\right) \widehat{\omega^2} = 0. \tag{2.29}$$

Furthermore,

$$(2.30)\quad \left(\nabla^{\mathcal{E}\widehat{\otimes}\widehat{\mathcal{E}},u}+2\sqrt{t}\;i_{\mathbf{x}}\right)\left(\frac{1}{4}\;\widehat{\omega}-\sqrt{t}\;\widehat{\mathbf{x}}\right)=-\sqrt{t}\;\nabla^{\mathcal{E}\widehat{\otimes}\widehat{\mathcal{E}},u}\widehat{\mathbf{x}}+\frac{\sqrt{t}}{2}\;i_{\mathbf{x}}\widehat{\omega}.$$

Using (2.28), (2.29) and (2.30), we obtain

$$(2.31)\qquad d\delta_t=\int^B\left(-\sqrt{t}\;\nabla^{\mathcal{E}\widehat{\otimes}\widehat{\mathcal{E}},u}\widehat{\mathbf{x}}+\frac{\sqrt{t}}{2}\;i_{\mathbf{x}}\widehat{\omega}\right)e^{-B_t}.$$

As elements of $\Omega^1(U;\Lambda^0(\mathcal{V})\;\widehat{\otimes}\;\Lambda^1(\widehat{\mathcal{V}}))$, there is an equality

$$(2.32)\qquad \lambda^*\left(\nabla^{\mathcal{E}\widehat{\otimes}\widehat{\mathcal{E}},u}\widehat{\mathbf{x}}\right)=\frac{1}{2}\;\lambda^*\left(i_{\mathbf{x}}\widehat{\omega}\right).$$

Equation (2.26) follows.

To prove b., we continue with the setup of the proof of Theorem 2.4.b. Let $\nabla^{\mathcal{V}'}$ now be the flat connection $\rho_B^*\nabla^{\mathcal{V}}$. Then $\lambda'=\rho_B^*\lambda$ is a flat section. Since

$$(2.33)\qquad \left(\nabla^{\mathcal{V}'}\right)^*=\rho_B^*\left(\nabla^{\mathcal{V}}\right)^*+\frac{ds}{s}$$

and

$$(2.34)\qquad \omega'=\rho_B^*\omega+\frac{ds}{s},$$

we have

$$(2.35)\qquad \left(\omega'\right)^2=\rho_B^*\omega^2$$

and

$$(2.36)\qquad \widehat{\omega}'=\rho_{\mathcal{V}}^*\widehat{\omega}+\left\langle\psi,\frac{ds}{s}\widehat{\psi}\right\rangle=\rho_{\mathcal{V}}^*\widehat{\omega}-\frac{ds}{s}\left\langle\psi,\widehat{\psi}\right\rangle.$$

The unitary connection on $\mathcal{V}'$ is given by

$$(2.37)\qquad \nabla^{\mathcal{V}',u}=\rho_B^*\nabla^{\mathcal{V},u}+\frac{ds}{2s}.$$

Using an obvious notation, we can then write $B_1'\in\Omega\left(\mathcal{V}';\mathcal{E}'\;\widehat{\otimes}\;\widehat{\mathcal{E}}'\right)$ as

$$(2.38)\qquad B_1'=B_s+ds\wedge\frac{\mathbf{x}}{2\sqrt{s}}.$$

Thus

$$
\begin{aligned}
\delta_1' &= \int^B \left(\frac{1}{4}\,\widehat{\omega}' - \sqrt{s}\,\widehat{\mathbf{x}}\right) e^{-B_1'} \\
&= \int^B \left(\frac{1}{4}\,\widehat{\omega} - \sqrt{s}\,\widehat{\mathbf{x}}\right) e^{-B_s} \\
&\quad - ds \wedge \int^B \left(\frac{1}{4s}\left\langle \psi, \widehat{\psi}\right\rangle + \frac{\mathbf{x}}{2\sqrt{s}}\left(\frac{1}{4}\,\widehat{\omega} - \sqrt{s}\,\widehat{\mathbf{x}}\right)\right) e^{-B_s}.
\end{aligned}
\tag{2.39}
$$

From part a. of the theorem,

$$
d'\left(\lambda'^{*}\delta_1'\right) = 0. \tag{2.40}
$$

Part b. of the theorem now follows from (2.17), (2.39) and (2.40).

q.e.d.

Remark 2.10. If N is odd then $\alpha_t = \beta_t = 0$. If N is even then $\delta_t = \epsilon_t = 0$.

Remark 2.11. There is an evident analogy between the properties of (α_t, β_t) and (δ_t, ϵ_t). However, the important difference is that α_t can be pulled-back by an arbitrary section of $\mathcal{V}$ to get a closed form on B, whereas δ_t can only be pulled-back by a flat section of $\mathcal{V}$ if one wants to get a closed form on B.

c) Some auxiliary forms

We use the notation of Section 2b. Suppose that the holonomy of the flat connection $\nabla^{\mathcal{V}}$ preserves a volume form η on the fibers. We will assume that the metric $h^{\mathcal{V}}$ is such that the volume form on the fibers induced by $h^{\mathcal{V}}$ equals η. The holonomy of the adjoint flat connection $(\nabla^{\mathcal{V}})^*$ also preserves η.

Definition 2.12. Define $\mathrm{Vol} \in \Omega^N(\mathcal{V})$ to be the extension of η to $\mathcal{V}$, using $(\nabla^{\mathcal{V}})^*$.

More precisely, if U is a contractible open set in B and $U \times \mathbb{R}^N$ is a trivialization of $\mathcal{V}$ which is covariantly-constant with respect to $(\nabla^{\mathcal{V}})^*$, let $\phi : U \times \mathbb{R}^N \to \mathbb{R}^N$ be projection onto the fiber. Then on $U \times \mathbb{R}^N$, one has $\mathrm{Vol} = \phi^*\eta$. Clearly Vol is closed. Consider the Berezin integral $\int^B : \Omega\left(\mathcal{V}; \mathcal{E}\right) \to \Omega\left(\mathcal{V}\right)$.

Theorem 2.13. *We have*

$$
\mathrm{Vol} = (-1)^{\frac{N(N-1)}{2}} \int^B e^{(\nabla^{\mathcal{E}})^* \mathbf{x}}. \tag{2.41}
$$

If $\lambda : U \to \mathcal{V}$ is a section of $\mathcal{V}$ which is flat with respect to $\nabla^{\mathcal{V}}$ then

$$\lambda^* \mathrm{Vol} = (\omega\lambda)_1 \wedge \ldots \wedge (\omega\lambda)_N. \tag{2.42}$$

Proof. In terms of the trivialization $U \times \mathbb{R}^N$ above, we can write $(\nabla^{\mathcal{E}})^* \mathbf{x} = \langle d\phi, \psi \rangle$. Then

$$\begin{aligned}(-1)^{\frac{N(N-1)}{2}} \int^B e^{\langle d\phi,\psi\rangle} &= (-1)^{\frac{N(N-1)}{2}} \int^B \prod_{j=1}^N d\phi_j \cdot \psi_j \\ &= d\phi_1 \wedge \ldots \wedge d\phi_N.\end{aligned} \tag{2.43}$$

Equation (2.41) follows. As elements of $\Omega^1(U;\mathcal{V})$, we have

$$\lambda^*((\nabla^{\mathcal{E}})^*\mathbf{x}) = \left(\nabla^{\mathcal{E}} + \omega\right)\lambda = \omega\lambda. \tag{2.44}$$

Equation (2.42) follows. q.e.d.

Now consider the Berezin integral $\int^B : \Omega\left(\mathcal{V}; \widehat{\mathcal{E}}\right) \to \Omega\left(\mathcal{V}\right)$.

Definition 2.14. For $t > 0$, define $\rho_t \in \Omega^{2N-1}(\mathcal{V})$ by

$$\rho_t = e^{-\frac{|\mathbf{x}|^2}{4t}}\, t^{-N}\, \mathrm{Vol} \cdot \int^B \frac{\widehat{\mathbf{x}}}{\sqrt{t}}\, e^{-\frac{1}{8}\widehat{\omega^2}} \tag{2.45}$$

and $\sigma_t \in \Omega^{2N-2}(\mathcal{V})$ by

$$\sigma_t = -e^{-\frac{|\mathbf{x}|^2}{4t}}\, t^{-N-1}\, i_{\mathbf{x}} \mathrm{Vol} \cdot \int^B \frac{\widehat{\mathbf{x}}}{\sqrt{t}}\, e^{-\frac{1}{8}\widehat{\omega^2}}. \tag{2.46}$$

Theorem 2.15.

a. Let U be an open subset of B and let $\lambda : U \to \mathcal{V}$ be a flat section of $\mathcal{V}$ over U. Then for $t > 0$,

$$d\left(\lambda^* \rho_t\right) = 0. \tag{2.47}$$

b. In addition,

$$\frac{\partial\left(\lambda^*\rho_t\right)}{\partial t} = d\left(\lambda^*\sigma_t\right). \tag{2.48}$$

Proof. If N is even, then $\rho_t = \sigma_t = 0$ and the theorem is trivially true. Thus we may assume that N is odd.

a. We have

$$d\rho_t = - \, e^{-\frac{|\mathbf{x}|^2}{4t}} \, t^{-N} \, \mathrm{Vol} \cdot \left(\frac{i_{\widehat{\mathbf{x}}}(\nabla^{\widehat{\mathcal{E}},u}\widehat{\mathbf{x}})}{2t} \int^B \frac{\widehat{\mathbf{x}}}{\sqrt{t}} \, e^{-\frac{1}{8}\widehat{\omega^2}} + \int^B \frac{\nabla^{\widehat{\mathcal{E}},u}\widehat{\mathbf{x}}}{\sqrt{t}} \, e^{-\frac{1}{8}\widehat{\omega^2}} \right). \tag{2.49}$$

Furthermore,

$$\lambda^* \left(\nabla^{\widehat{\mathcal{E}},u}\widehat{\mathbf{x}}\right) = \nabla^{\mathcal{V},u}\lambda = \left(\nabla^{\mathcal{V}} + \frac{\omega}{2}\right)\lambda = \frac{\omega}{2}\,\lambda \tag{2.50}$$

and

$$\lambda^* \left(\, i_{\widehat{\mathbf{x}}}(\nabla^{\widehat{\mathcal{E}},u}\widehat{\mathbf{x}})\right) = - \, \frac{1}{2} \, \langle \lambda, \omega\lambda \rangle \, . \tag{2.51}$$

Equation (2.47) now follows from the explicit representation of $\lambda^*\mathrm{Vol}$ as an N-form in (2.42).

b. We continue with the notation of the proof of Theorem 2.4.b, except that we now take $h^{\mathcal{V}'}$ to be the metric on $\mathcal{V}'$ which restricts to $\frac{1}{s} \cdot h^{\mathcal{V}}$ on $\mathcal{V} \times \{s\}$. Let $\nabla^{\mathcal{V}'}$ be the flat connection $\rho_B^* \nabla^{\mathcal{V}}$. Then

$$\left(\nabla^{\mathcal{V}'}\right)^* = \rho_B^* \left(\nabla^{\mathcal{V}}\right)^* - \frac{ds}{s}. \tag{2.52}$$

Using (2.41) and (2.52), we have

$$\mathrm{Vol}' = (-1)^{\frac{N(N-1)}{2}} \int^B e^{(\nabla^{\mathcal{V}})^* \mathbf{x} - \frac{ds}{s}\mathbf{x}} = \mathrm{Vol} - \frac{ds}{s} \wedge i_{\mathbf{x}}\mathrm{Vol}. \tag{2.53}$$

Thus

$$\rho_1' = e^{-\frac{|\mathbf{x}|^2}{4s}} \, s^{-N} \, \mathrm{Vol} \cdot \int^B \frac{\widehat{\mathbf{x}}}{\sqrt{s}} \, e^{-\frac{1}{8}\widehat{\omega^2}} - ds \wedge e^{-\frac{|\mathbf{x}|^2}{4s}} \, s^{-N-1} \, i_{\mathbf{x}}\mathrm{Vol} \cdot \int^B \frac{\widehat{\mathbf{x}}}{\sqrt{s}} \, e^{-\frac{1}{8}\widehat{\omega^2}}. \tag{2.54}$$

From part a. of the theorem,

$$d' \left(\lambda'^* \rho_1'\right) = 0. \tag{2.55}$$

Part b. of the theorem now follows from (2.17), (2.54) and (2.55).

q.e.d.

d) Proof of the main theorem

Let N be a positive odd integer, and B be a connected smooth manifold. Let $\widetilde{B}$ be the universal cover of B and put $\Gamma = \pi_1(B)$. Let $\rho : \Gamma \to SL(N,\mathbb{Z})$ be a homomorphism. When convenient, we will consider $SL(N,\mathbb{Z})$ to be a subgroup of $SL(N,\mathbb{R})$ or $SL(N,\mathbb{C})$. Define a flat real rank-N vector bundle on B:

$$E = \widetilde{B} \times_\rho \mathbb{R}^N. \tag{2.56}$$

Let $\Lambda \subset E$ be the lattice

$$\Lambda = \widetilde{B} \times_\rho \mathbb{Z}^N. \tag{2.57}$$

Let E^* be the dual vector bundle to E and let $\Lambda^* \subset E^*$ be the dual lattice:

$$\Lambda^* = \{\mu \in E^* : \forall \lambda \in \Lambda, \langle \mu, \lambda \rangle \in 2\pi\mathbb{Z}\}. \tag{2.58}$$

Let ∇^E be the canonical flat connection on E. Then ∇^E preserves the lattice Λ, and the dual flat connection ∇^{E^*} preserves Λ^*.

From (2.56), there are volume forms on the fibers of E coming from the standard volume form on $\mathbb{R}^N$. Let $\mathrm{Vol}(E/\Lambda)$ denote the common volume of the quotients of the fibers by Λ. Choose an inner product h^E on E which is compatible with these volume forms. We write ω_E for $\omega\left(\nabla^E, h^E\right) \in \Omega^1(B;\mathrm{End}(E))$ and ω_{E^*} for $\omega\left(\nabla^{E^*}, h^{E^*}\right) \in \Omega^1(B;\mathrm{End}(E^*))$.

We now apply the formalism of Section 2b to the case $\mathcal{V} = E^*$. Define $\delta_t \in \Omega^{2N-1}(E^*)$ as in (2.24) and $\epsilon_t \in \Omega^{2N-2}(E^*)$ as in (2.25). If U is a contractible open subset of B, then over U the lattice Λ consists of a countable number of disjoint copies of U. Thus the forms $\sum_{\mu\in\Lambda^*} \mu^*\delta_t$ and $\sum_{\mu\in\Lambda^*} \mu^*\epsilon_t$ are well-defined on U, and we obtain global forms $\sum_{\mu\in\Lambda^*} \mu^*\delta_t \in \Omega^{2N-1}(B)$ and $\sum_{\mu\in\Lambda^*} \mu^*\epsilon_t \in \Omega^{2N-2}(B)$.

Theorem 2.16. *We have*

$$d \sum_{\mu\in\Lambda^*} \mu^*\delta_t = 0 \tag{2.59}$$

and

$$\frac{\partial}{\partial t} \sum_{\mu \in \Lambda^*} \mu^* \delta_t = d \sum_{\mu \in \Lambda^*} \mu^* \epsilon_t. \tag{2.60}$$

Proof. This is a consequence of Theorem 2.9. q.e.d.

Theorem 2.17. *If K is a compact subset of B, then there is a constant $c > 0$ such that on K, as $t \to \infty$,*

$$\sum_{\mu \in \Lambda^*} \mu^* \delta_t = \frac{1}{2}\, \pi^{N-1}\, c_N^{\mathrm{z}} \left(\nabla^E, h^E\right) + \mathcal{O}\left(e^{-ct}\right). \tag{2.61}$$

If $N > 1$ then

$$\sum_{\mu \in \Lambda^*} \mu^* \epsilon_t = \mathcal{O}\left(e^{-ct}\right), \tag{2.62}$$

and if $N = 1$ then

$$\sum_{\mu \in \Lambda^*} \mu^* \epsilon_t = -\frac{1}{4t} + \mathcal{O}\left(e^{-ct}\right). \tag{2.63}$$

Proof. It is enough to consider only the contribution of $\mu = 0$, as the other terms will be exponentially damped in t. From (2.24) it follows that

$$0^* \delta_t = \frac{1}{4} \int^B \left\langle \psi, \omega_{E^*} \widehat{\psi} \right\rangle e^{\frac{1}{16}\langle \psi, \omega_{E^*}^2 \psi \rangle - \frac{1}{16}\langle \widehat{\psi}, \omega_{E^*}^2 \widehat{\psi} \rangle}. \tag{2.64}$$

Using Lemma 2.1 with $V = \frac{\omega_{E^*}^2}{8}$ and $W = \frac{\omega_{E^*}}{2}$, we obtain

$$0^* \delta_t = -\frac{1}{2}\, \pi^{N-1}\, c_N^{\mathrm{z}} \left(\nabla^{E^*}, h^{E^*}\right). \tag{2.65}$$

Equation (2.61) now follows from Lemma 1.7.

By (2.25) we get

$$0^* \epsilon_t = -\frac{1}{4t} \int^B \left\langle \psi, \widehat{\psi} \right\rangle e^{\frac{1}{16}\langle \psi, \omega_{E^*}^2 \psi \rangle - \frac{1}{16}\langle \widehat{\psi}, \omega_{E^*}^2 \widehat{\psi} \rangle}. \tag{2.66}$$

Lemma 1.3 now gives

$$0^* \epsilon_t = -\frac{1}{4t} \int^B \left\langle \psi, \widehat{\psi} \right\rangle, \tag{2.67}$$

from which (2.62) and (2.63) follow. q.e.d.

Applying the results of Section 2c to the case $\mathcal{V} = E$, we define $\rho_t \in \Omega^{2N-1}(E)$ as in (2.45) and $\sigma_t \in \Omega^{2N-2}(E)$ as in (2.46).

Theorem 2.18. *The forms* $\sum_{m \in \Lambda} m^* \rho_t \in \Omega^{2N-1}(B)$ *and* $\sum_{m \in \Lambda} m^* \sigma_t \in \Omega^{2N-2}(B)$ *satisfy*

$$d \sum_{m \in \Lambda} m^* \rho_t = 0 \tag{2.68}$$

and

$$\frac{\partial}{\partial t} \sum_{m \in \Lambda} m^* \rho_t = d \sum_{m \in \Lambda} m^* \sigma_t. \tag{2.69}$$

Proof. This is a consequence of (2.47) and (2.48). q.e.d.

Theorem 2.19. *We have*

$$\sum_{\mu \in \Lambda^*} \mu^* \delta_t = 2^{-3N-1} \, \pi^{-\frac{N}{2}} \, \mathrm{Vol}(E/\Lambda) \sum_{m \in \Lambda} m^* \rho_t \tag{2.70}$$

and

$$\sum_{\mu \in \Lambda^*} \mu^* \epsilon_t = 2^{-3N-1} \, \pi^{-\frac{N}{2}} \, \mathrm{Vol}(E/\Lambda) \sum_{m \in \Lambda} m^* \sigma_t. \tag{2.71}$$

Proof. From (2.23) it follows that

$$\begin{aligned} \mu^* B_t = & -\frac{1}{16} \langle \psi, \omega_{E^*}^2 \psi \rangle + \sqrt{t} \, \left\langle \frac{1}{2} \, \omega_{E^*} \mu, \psi \right\rangle \\ & + t \, |\mu|^2 + \frac{1}{16} \left\langle \widehat{\psi}, \omega_{E^*}^2 \widehat{\psi} \right\rangle \\ = & \frac{1}{16} \, |\omega_{E^*} \psi|^2 + \sqrt{t} \, \left\langle \frac{1}{2} \, \mu, \omega_{E^*} \psi \right\rangle \\ & + t \, |\mu|^2 + \frac{1}{16} \left\langle \widehat{\psi}, \omega_{E^*}^2 \widehat{\psi} \right\rangle \\ = & t \left| \mu + \frac{1}{4\sqrt{t}} \, \omega_{E^*} \psi \right|^2 + \frac{1}{16} \left\langle \widehat{\psi}, \omega_{E^*}^2 \widehat{\psi} \right\rangle. \end{aligned} \tag{2.72}$$

Furthermore,

$$\mu^* \left(\frac{1}{4}\, \widehat{\omega} - \sqrt{t}\; \widehat{\mathbf{x}} \right) = -\sqrt{t}\; \left\langle \left(\mu + \frac{1}{4\sqrt{t}}\; \omega_{E^*} \psi \right), \widehat{\psi} \right\rangle . \tag{2.73}$$

Let z be an auxiliary odd variable, satisfying $z^2 = 0$. Define Tr_z as in (1.14). Then by equation (2.72), in terms of the Berezin integral $\int^B : \Omega \left(B; \Lambda(E^*) \mathbin{\widehat{\otimes}} \Lambda(\widehat{E^*}) \right) \to \Omega\,(B)$, we have

$$\mu^* \delta_t = \mathrm{Tr}_z \left[\int^B e^{-t\, \left| \mu + \frac{1}{4\sqrt{t}} \omega_{E^*} \psi + \frac{1}{2\sqrt{t}} z \widehat{\psi} \right|^2 - \frac{1}{16} \left\langle \widehat{\psi}, \omega^2_{E^*} \widehat{\psi} \right\rangle} \right] . \tag{2.74}$$

Using the Poisson summation formula, in general one has that if $b \in E^*$ then

$$\sum_{\mu \in \Lambda^*} e^{-t\; |\mu + b|^2} = (4\pi t)^{-\frac{N}{2}}\; \mathrm{Vol}(E/\Lambda) \sum_{m \in \Lambda} e^{-\frac{|m|^2}{4t}} e^{i \langle m, b \rangle}. \tag{2.75}$$

Applying this to (2.74) yields

$$\begin{aligned} \sum_{\mu \in \Lambda^*} \mu^* \delta_t &= 2^{-N}\; \pi^{-\frac{N}{2}}\; t^{-\frac{N}{2}}\; \mathrm{Vol}(E/\Lambda) \\ &\qquad \cdot \mathrm{Tr}_z \left[\int^B \sum_{m \in \Lambda} e^{-\frac{|m|^2}{4t}} e^{i \left\langle m, \frac{1}{4\sqrt{t}} \omega_{E^*} \psi + \frac{1}{2\sqrt{t}} z \widehat{\psi} \right\rangle - \frac{1}{16} \left\langle \widehat{\psi}, \omega^2_{E^*} \widehat{\psi} \right\rangle} \right] \\ &= 2^{-N-1}\; \pi^{-\frac{N}{2}}\; t^{-\frac{N}{2}}\; i\; \mathrm{Vol}(E/\Lambda) \\ &\qquad \cdot \sum_{m \in \Lambda} e^{-\frac{|m|^2}{4t}} \int^B e^{i \left\langle m, \frac{1}{4\sqrt{t}} \omega_{E^*} \psi \right\rangle}\; \frac{\widehat{m}}{\sqrt{t}}\; e^{-\frac{1}{16} \left\langle \widehat{\psi}, \omega^2_{E^*} \widehat{\psi} \right\rangle} \end{aligned} \tag{2.76}$$

Define new Grassmann variables by $\eta = \left(\widehat{h}^E \right)^{-1} \psi$ and $\widehat{\eta} = \left(\widehat{h}^E \right)^{-1} \widehat{\psi}$. By (1.19) we obtain

$$\langle m, \omega_{E^*} \psi \rangle = - \langle m, \omega_E \eta \rangle \tag{2.77}$$

and

$$\left\langle \widehat{\psi}, \omega^2_{E^*} \widehat{\psi} \right\rangle = \left\langle \widehat{\eta}, \omega^2_E \widehat{\eta} \right\rangle . \tag{2.78}$$

From (2.76), (2.77) and (2.78), in terms of the Berezin integral

$$\int^B : \Omega\left(B; \Lambda(E) \widehat{\otimes} \Lambda(\widehat{E})\right) \to \Omega(B),$$

it follows that

$$\begin{aligned}
&\sum_{\mu\in\Lambda^*} \mu^* \delta_t \\
&\qquad = 2^{-3N-1} \pi^{-\frac{N}{2}} \operatorname{Vol}(E/\Lambda) \sum_{m\in\Lambda} e^{-\frac{|m|^2}{4t}} t^{-N} \\
&\qquad\quad \cdot \int^B (-1)^{\frac{N(N-1)}{2}} e^{\langle m, \omega_E \eta\rangle} \frac{\widehat{m}}{\sqrt{t}} e^{-\frac{1}{16}\langle \widehat{\eta}, \omega_E^2 \widehat{\eta}\rangle} \\
&\qquad = 2^{-3N-1} \pi^{-\frac{N}{2}} \operatorname{Vol}(E/\Lambda) \sum_{m\in\Lambda} m^* \rho_t.
\end{aligned} \tag{2.79}$$

This proves (2.70). In view of the constructions of the proofs of Theorems 2.9.b and 2.15.b, equation (2.71) follows automatically from (2.70). q.e.d.

Corollary 2.20. *If K is a compact subset of B, then there is a constant $c > 0$ such that on K, as $t \to 0$,*

$$\sum_{\mu\in\Lambda^*} \mu^* \delta_t = \mathcal{O}\left(e^{-\frac{c}{t}}\right) \tag{2.80}$$

and

$$\sum_{\mu\in\Lambda^*} \mu^* \epsilon_t = \mathcal{O}\left(e^{-\frac{c}{t}}\right). \tag{2.81}$$

Proof. As the sums in (2.70) and (2.71) can be taken over nonzero m, the theorem follows. q.e.d.

We now reprove Theorem 0.1 of the introduction.

Theorem 0.1. *Let B be a smooth connected manifold. Let N be a positive odd integer and let E be a flat complex rank-N vector bundle over B with structure group contained in $GL(N,\mathbb{Z})$. Then $n_N^{\mathrm{z}}(\nabla^E)$ vanishes in $\mathrm{H}^{2N-1}(B;\mathbb{R})$.*

Proof. Without loss of generality, we can assume that E is real. Suppose first that the structure group is contained in $SL(N,\mathbb{Z})$. By (2.60), the de Rham cohomology class of $\sum_{\mu\in\Lambda^*}\mu^*\delta_t$ is independent of t. The theorem follows from combining (2.61), (2.70) and (2.80). One can check that the arguments still go through if E is not orientable. q.e.d.

We now write $c_N^{\mathrm{z}}(\nabla^E, h^E)$ explicitly as an exact form on B.

Definition 2.21. If $N > 1$ and $s \in \mathbb{C}$, define $\phi(s) \in \Omega^{2N-2}(B)$ by

$$\phi(s) = -\int_0^\infty t^s \sum_{\mu\in\Lambda^*} \mu^*\epsilon_t \, dt. \tag{2.82}$$

Using (2.62) and (2.81), we see that $\phi(s)$ is well-defined and is a holomorphic function on $\mathbb{C}$.

Theorem 2.22. *For $\Re(s) << 0$,*

$$\begin{aligned}\phi(s) &= 2^{-N-2s}\,\pi^{-\frac{N}{2}}\,\Gamma(N+\frac{1}{2}-s)\ \mathrm{Vol}(E/\Lambda)\\ &\cdot \sum_{\substack{m\in\Lambda\\ m\neq 0}} (|m|)^{2s-2N-1}\, m^*\left(i_{\mathbf{x}}\mathrm{Vol}\cdot\int^B \widehat{\mathbf{x}}e^{-\frac{1}{8}\widehat{\omega^2}}\right).\end{aligned} \tag{2.83}$$

Proof. This follows from (2.46) and (2.71). q.e.d.

Hence the right-hand side of (2.83) must also have a holomorphic continuation to $\mathbb{C}$.

Theorem 2.23. *If $N > 1$ then*

$$d\phi(0) = \frac{1}{2}\,\pi^{N-1}\,c_N^{\mathrm{z}}\left(\nabla^E, h^E\right). \tag{2.84}$$

Proof. This follows from (2.60), (2.61), (2.80) and (2.82). q.e.d.

III. Fourier analysis on torus bundles

In this section we describe how in the special case of a flat torus bundle, the results of [4] become equivalent to the results of Section 2. We assume a familiarity with [4].

The section is organized as follows. In a) we review the relationship between supertraces and the Berezin integral. Given a flat rank-N vector bundle $\mathcal{V}$, in b) we define its leafwise topology $\mathcal{V}_{\mathcal{F}}$ and use the formalism of [4] to define a closed form $\widetilde{\delta}_t \in \Omega^{2N-1}(\mathcal{V}_{\mathcal{F}})$. We also construct the transgressing form $\widetilde{\epsilon}_t \in \Omega^{2N-2}(\mathcal{V}_{\mathcal{F}})$, and give the relationship between $(\widetilde{\delta}_t, \widetilde{\epsilon}_t)$ and the forms (δ_t, ϵ_t) of Section 2. In c) we use Fourier analysis on the fibers of a torus bundle to show how the results of [4] become equivalent to those of Section 2.

a) Supertraces and the Berezin integral

Let V be a Hermitian vector space of dimension N with orthonormal basis $\{e_k\}_{k=1}^N$. Given $X \in V$, define operators on $\Lambda(V)$ by

$$\begin{aligned} c(X) &= (X\,\wedge) - i(X), \\ \widehat{c}(X) &= (X\,\wedge) + i(X). \end{aligned} \tag{3.1}$$

Then for $X, Y \in V$,

$$\begin{aligned} c(X)c(Y) + c(Y)c(X) &= -2\langle X, Y\rangle, \\ \widehat{c}(X)\widehat{c}(Y) + \widehat{c}(Y)\widehat{c}(X) &= 2\langle X, Y\rangle, \\ c(X)\widehat{c}(Y) + \widehat{c}(Y)c(X) &= 0. \end{aligned} \tag{3.2}$$

Thus c and $\widehat{c}$ generate two graded-commuting Clifford algebras. Put $c_i = c(e_i)$ and $\widehat{c}_i = \widehat{c}(e_i)$. Among the monomials in the c_i's and $\widehat{c}_i$'s with less than or equal to N factors of each, the only nonzero supertrace occurs as

$$\mathrm{Tr}_s\left[c_1 \dots c_N \widehat{c}_1 \dots \widehat{c}_N\right] = (-1)^{\frac{N(N+1)}{2}} 2^N. \tag{3.3}$$

We now relate the above supertrace to the Berezin integral of (2.6). Let $\widehat{V}$ be another copy of V. Let $M \in \mathrm{End}(V \oplus \widehat{V})$ be a skew-symmetric endomorphism. Let $\det^{1/2}\left(\frac{\sin(M)}{M}\right) \in \mathbb{C}$ be the square-root of $\det\left(\frac{\sin(M)}{M}\right)$ which extends to a holomorphic function on the space of skew-symmetric endomorphisms, with value 1 at $M = 0$. As notation, we write C and Ψ for the $(2n)$-vectors $\begin{pmatrix} c \\ \widehat{c} \end{pmatrix}$ and $\begin{pmatrix} \psi \\ \widehat{\psi} \end{pmatrix}$, respectively. Let A be a graded-commutative superalgebra. Let $J = \begin{pmatrix} j \\ \widehat{j} \end{pmatrix}$ be a $(2N)$-vector of odd elements of A.

The next theorem is a consequence of [10, (2.13)].

Theorem 3.1. *One has an identity in A :*

$$\begin{aligned}&\operatorname{Tr}_s\left[e^{\frac{1}{2}\langle C,MC\rangle+\langle J,C\rangle}\right]\\ &\quad=(-1)^{\frac{N(N+1)}{2}}\,2^N\det{}^{1/2}\left(\frac{\sin(M)}{M}\right)\int^B e^{\frac{1}{2}\langle\Psi,M\Psi\rangle+\langle J,\Psi\rangle}.\end{aligned}\tag{3.4}$$

One can extend (3.4) to allow M to have entries which are even elements of A, provided that the entries are nilpotent or that A is a Banach algebra.

b) Thom-like forms II

Let $\mathcal{V}$ be a complex rank-N vector bundle over a smooth connected manifold B. Suppose that $\mathcal{V}$ has a flat connection $\nabla^{\mathcal{V}}$. Let $\nabla^{\Lambda(\mathcal{V})}$ be the induced flat connection on the $\mathbb{Z}$-graded vector bundle $\Lambda(\mathcal{V})$. The tangent vectors of $\mathcal{V}$ which are horizontal for $\nabla^{\mathcal{V}}$ define an integrable distribution on $\mathcal{V}$, and hence a foliation $\mathcal{F}$ which is transverse to the fibers of $\mathcal{V}$. Let $\mathcal{V}_{\mathcal{F}}$ denote the total space of $\mathcal{V}$ with the leaf topology, a basis of which is given by the connected components of intersections $S \cap L$ of open sets S in V with leaves L of $\mathcal{F}$ [9, p. 2]. The connected components of $\mathcal{V}_{\mathcal{F}}$ are exactly the leaves of $\mathcal{F}$. For example, if B is a point then $\mathcal{V}_{\mathcal{F}}$ is $\mathbb{C}^N$ with the discrete topology. In particular, if U is a contractible open set in B and $\lambda : U \to \mathcal{V}_{\mathcal{F}}$ is a continuous section of $\mathcal{V}_{\mathcal{F}}$ over U, then λ is automatically a flat section. Note that $\mathcal{V}_{\mathcal{F}}$ is a non-second-countable manifold whose dimension is that of B.

Let $\pi : \mathcal{V}_{\mathcal{F}} \to B$ be the projection map. Put

$$\mathcal{E} = \Lambda\left(\pi^*\mathcal{V}\right),$$

a $\mathbb{Z}$-graded vector bundle on $\mathcal{V}_{\mathcal{F}}$. It is equipped with the flat connection $\nabla^{\mathcal{E}} = \pi^*\nabla^{\Lambda(\mathcal{V})}$.

Definition 3.2. The operator $(\mathbf{x}\,\wedge) \in C^\infty(\mathcal{V}_{\mathcal{F}}; \operatorname{Hom}(\mathcal{E}^\bullet, \mathcal{E}^{\bullet+1}))$ acts at $x \in \mathcal{V}_{\mathcal{F}}$ as exterior multiplication by x on $\mathcal{E}_x = \Lambda\left(\pi^*\mathcal{V}\right)_x = \Lambda\left(\mathcal{V}_{\pi(x)}\right)$.

Definition 3.3. The superconnection A' on $\mathcal{E}$ is given by

$$A' = \sqrt{-1}\,(\mathbf{x}\,\wedge) + \nabla^{\mathcal{E}}.\tag{3.5}$$

Clearly A' has total degree 1.

Theorem 3.4. *The superconnection A' is flat.*

Proof. Clearly $(\mathbf{x}\,\wedge)^2 = \left(\nabla^{\mathcal{E}}\right)^2 = 0$. Given $b \in B$, let U be a contractible neighborhood of b. We can trivialize the vector bundle $\mathcal{V}$ over U so that if $\sigma : U \to \mathbb{C}^N$ is a section of $\mathcal{V}\big|_U$, then $\nabla^{\mathcal{V}}(\sigma) = d\sigma$. Now $\pi^{-1}(U) \cong U \times \mathbb{C}^N$, and the topology on $U \times \mathbb{C}^N$ which is induced from $\mathcal{V}_{\mathcal{F}}$ is the product of the topology of U with the discrete topology on $\mathbb{C}^N$. The operator d on $\Omega\left(\pi^{-1}(U)\right)$ is effectively exterior differentiation in the U-direction on $U \times \mathbb{C}^N$, and we can write it as d_U. Thus

$$\Lambda\left(\pi^*\mathcal{V}\right)\big|_{\pi^{-1}(U)} \cong (U \times \mathbb{C}^N) \times \Lambda\left(\mathbb{C}^N\right). \tag{3.6}$$

Writing a local section $s : U \times \mathbb{C}^N \to \Lambda\left(\mathbb{C}^N\right)$ of $\mathcal{E}$ as $s(u,x)$, we have

$$\begin{aligned}(\mathbf{x}\ \wedge s)(u,x) &= x \wedge s(u,x),\\ (\nabla^{\mathcal{E}} s)(u,x) &= d_U s(u,x).\end{aligned} \tag{3.7}$$

It is now clear that

$$\left[\nabla^{\mathcal{E}}, (\mathbf{x}\,\wedge)\right] = 0, \tag{3.8}$$

so that the theorem follows. q.e.d.

Remark 3.5. Theorem 3.4 would not be true if we used the ordinary topology on $\mathcal{V}$. This can be seen in the case where B is a point.

Let $h^{\mathcal{V}}$ be a Hermitian metric on $\mathcal{V}$. Then there is an induced Hermitian metric $h^{\mathcal{E}}$ on $\mathcal{E}$.

Definition 3.6. Let $i_{\mathbf{x}} \in C^\infty(\mathcal{V}_{\mathcal{F}}; \mathrm{Hom}(\mathcal{E}^\bullet, \mathcal{E}^{\bullet-1}))$ be the operator which acts at $x \in \mathcal{V}_{\mathcal{F}}$ as interior multiplication by x on $\mathcal{E}_x = \Lambda\left(\pi^*\mathcal{V}\right)_x = \Lambda\left(\mathcal{V}_{\pi(x)}\right)$.

Define $\omega\left(\nabla^{\mathcal{V}}, h^{\mathcal{V}}\right) \in \Omega^1(B; \mathrm{End}(\mathcal{V}))$ as in (1.5). In the rest of this section we will abbreviate $\omega\left(\nabla^{\mathcal{V}}, h^{\mathcal{V}}\right)$ by ω. We can lift ω to an operator $\pi^*\omega \in \Omega^1(\mathcal{V}_{\mathcal{F}}; \mathrm{End}(\pi^*\mathcal{V}))$, and extend it to an operator $\Lambda\left(\pi^*\omega\right) \in \Omega^1(\mathcal{V}_{\mathcal{F}}; \mathrm{End}(\Lambda(\pi^*\mathcal{V})))$.

Let A'^* be the adjoint superconnection to A' with respect to $h^{\mathcal{E}}$.

Theorem 3.7. *One has*

$$A'^* = -\sqrt{-1}\ i_{\mathbf{x}} + \nabla^{\mathcal{E}} + \Lambda\left(\pi^*\omega\right). \tag{3.9}$$

Proof. Clearly $(\mathbf{x}\wedge)^* = i_{\mathbf{x}}$. In terms of the local trivializations of the proof of Theorem 3.4, on U we have

$$\omega = \left(h^{\mathcal{V}}\right)^{-1}\left(dh^{\mathcal{V}}\right). \tag{3.10}$$

Since on $U \times \mathbb{C}^N$

$$\left(\nabla^{\mathcal{E}}\right)^* = d_U + \left(h^{\mathcal{E}}\right)^{-1}\left(d_U h^{\mathcal{E}}\right) = \nabla^{\mathcal{E}} + \Lambda\left(\pi^*\omega\right), \tag{3.11}$$

the theorem is proved. q.e.d.

Following [4, Definition 2.10], for $t > 0$ put

$$\begin{aligned} C'_t &= \sqrt{-1}\,\sqrt{t}\,(\mathbf{x}\wedge) + \nabla^{\mathcal{E}}, \\ C''_t &= -\sqrt{-1}\,\sqrt{t}\,i_{\mathbf{x}} + \nabla^{\mathcal{E}} + \Lambda\left(\pi^*\omega\right). \end{aligned} \tag{3.12}$$

Then the D_t of [4, (2.29)] is given by

$$D_t = -\frac{1}{2}\,\sqrt{-1}\,\sqrt{t}\,\widehat{c}(\mathbf{x}) + \frac{1}{2}\Lambda\left(\pi^*\omega\right). \tag{3.13}$$

Let $f(z) = z\ \exp(z^2)$.

Definition 3.8. Using the notation of [4, Section 2c], for $t > 0$, define $\widetilde{\delta}_t \in \Omega^{\mathrm{odd}}(\mathcal{V}_{\mathcal{F}})$ and $\widetilde{\epsilon}_t \in \Omega^{\mathrm{even}}(\mathcal{V}_{\mathcal{F}})$ by

$$\widetilde{\delta}_t = f\left(C'_t, h^{\mathcal{E}}\right), \quad \widetilde{\epsilon}_t = \frac{1}{t}\ f^{\wedge}\left(C'_t, h^{\mathcal{E}}\right). \tag{3.14}$$

As a consequence of [4, Theorems 1.8 and 2.11], $\widetilde{\delta}_t$ is closed and

$$\frac{\partial \widetilde{\delta}_t}{\partial t} = d\widetilde{\epsilon}_t. \tag{3.15}$$

Definition 3.9. Define $\psi \in \Omega^{\mathrm{even}}(\mathcal{V}_{\mathcal{F}})$ to be the right-hand-side of [4, (2.74)].

Let $s : B \to \mathcal{V}_{\mathcal{F}}$ be the zero-section.

Theorem 3.10. *On $\mathcal{V}_{\mathcal{F}} \backslash \operatorname{Image}(s)$, the differential form $f\left(\nabla^{\mathcal{E}}, h^{\mathcal{E}}\right)$ is exact and*

$$f\left(\nabla^{\mathcal{E}}, h^{\mathcal{E}}\right) = d\psi. \tag{3.16}$$

In addition,

$$s^* f\left(\nabla^{\mathcal{E}}, h^{\mathcal{E}}\right) = f\left(\nabla^{\Lambda(\mathcal{V})}, h^{\Lambda(\mathcal{V})}\right). \tag{3.17}$$

Proof. In our case, the complex [4, (2.61)] becomes

$$\begin{aligned}(\mathcal{E}, \mathbf{x}\wedge) : 0 \longrightarrow \Lambda^0(\pi^*\mathcal{V}) &\xrightarrow{\mathbf{x}\wedge} \Lambda^1(\pi^*\mathcal{V}) \xrightarrow{\mathbf{x}\wedge} \cdots \\ &\xrightarrow{\mathbf{x}\wedge} \Lambda^n(\pi^*\mathcal{V}) \longrightarrow 0.\end{aligned} \tag{3.18}$$

It is acyclic on $\mathcal{V}_{\mathcal{F}}\backslash \operatorname{Image}(s)$, and so we obtain (3.16) by [4, Theorem 2.22]. Equation (3.17) follows from the naturality of the constructions.

q.e.d.

We now relate $\widetilde{\delta}_t$ and $\widetilde{\epsilon}_t$ to the forms δ_t and ϵ_t of Definition 2.8.

Theorem 3.11. *Let $I : \mathcal{V}_{\mathcal{F}} \to \mathcal{V}$ be the identity map. Then*

$$\widetilde{\delta}_t = (-1)^{\frac{N-1}{2}}\ 2\ \pi^{-(N-1)}\ I^*\delta_{\frac{t}{4}} \tag{3.19}$$

and

$$\widetilde{\epsilon}_t = (-1)^{\frac{N-1}{2}}\ \frac{1}{2}\ \pi^{-(N-1)}\ I^*\epsilon_{\frac{t}{4}}. \tag{3.20}$$

Proof. Let z be an auxiliary odd variable, satisfying $z^2 = 0$. Define Tr_z as in (1.14). From [4, Definition 1.7], it follows that

$$\begin{aligned} f\left(C_t', h^{\mathcal{E}}\right) &= (2i\pi)^{1/2}\varphi \operatorname{Tr}_s\left[D_t e^{D_t^2}\right] \\ &= (2i\pi)^{1/2}\varphi \operatorname{Tr}_z\left[\operatorname{Tr}_s\left[e^{D_t^2 + zD_t}\right]\right]. \end{aligned} \tag{3.21}$$

As ω is a symmetric one-form, we have

$$\begin{aligned} \Lambda(\omega) &= \sum_{kl} \omega_{kl}\ (e_k\ \wedge)\ i(e_l) \\ &= \frac{1}{4}\sum_{kl} \omega_{kl}\ (\widehat{c}_k + c_k)\ (\widehat{c}_l - c_l) \\ &= -\frac{1}{2}\sum_{k,l} \omega_{kl}\widehat{c}_k c_l = \frac{1}{2}\sum_{k,l}\widehat{c}_k \omega_{kl} c_l = \frac{1}{2}\left\langle \widehat{c}, \omega c\right\rangle. \end{aligned} \tag{3.22}$$

For simplicity, in the rest of the proof we write ω in place of $\pi^*\omega$. Thus we can write the D_t of (3.13) as

$$D_t = -\frac{i\sqrt{t}}{2}\left\langle \widehat{c}, \mathbf{x} + \frac{i}{2\sqrt{t}}\,\omega c\right\rangle. \tag{3.23}$$

One can check that

$$D_t^2 + zD_t = -\frac{t}{4}\left|\mathbf{x} + \frac{i}{2\sqrt{t}}\,\omega c + \frac{i}{\sqrt{t}}\,z\widehat{c}\right|^2 + \frac{1}{16}\left\langle \widehat{c}, \omega^2\widehat{c}\right\rangle. \tag{3.24}$$

By (3.4) we obtain

$$\begin{aligned}\mathrm{Tr}_{\mathrm{z}}\left[\mathrm{Tr}_s\left[e^{D_t^2+zD_t}\right]\right] &= (-1)^{\frac{N(N+1)}{2}}\, 2^N \\ &\times \mathrm{Tr}_{\mathrm{z}}\left[\det{}^{1/2}\left(\frac{\sin(M)}{M}\right)\int^B e^{-\frac{t}{4}\left|\mathbf{x}+\frac{i}{2\sqrt{t}}\omega\psi+\frac{i}{\sqrt{t}}z\widehat{\psi}\right|^2+\frac{1}{16}\left\langle\widehat{\psi},\omega^2\widehat{\psi}\right\rangle}\right],\end{aligned} \tag{3.25}$$

where the matrix M is given by

$$M = \begin{pmatrix} -\frac{\omega^2}{8} & \frac{z\omega}{4} \\ -\frac{z\omega}{4} & \frac{\omega^2}{8}\end{pmatrix}. \tag{3.26}$$

Due to the nature of the operator Tr_{z}, we can consider it to be acting on either the first or second factor of the term in brackets in the right-hand-side of (3.25). As the z in M occurs in the off-diagonal entries,

$$\mathrm{Tr}_{\mathrm{z}}\left[\det{}^{1/2}\left(\frac{\sin(M)}{M}\right)\right] = 0. \tag{3.27}$$

Thus

$$\begin{aligned}\mathrm{Tr}_{\mathrm{z}}\left[\mathrm{Tr}_s\left[e^{D_t^2+zD_t}\right]\right] &= (-1)^{\frac{N(N+1)}{2}}\, 2^N \det{}^{1/2}\left(\frac{\sin(M_0)}{M_0}\right) \\ &\times \mathrm{Tr}_{\mathrm{z}}\left[\int^B e^{-\frac{t}{4}\left|\mathbf{x}+\frac{i}{2\sqrt{t}}\omega\psi+\frac{i}{\sqrt{t}}z\widehat{\psi}\right|^2+\frac{1}{16}\left\langle\widehat{\psi},\omega^2\widehat{\psi}\right\rangle}\right],\end{aligned} \tag{3.28}$$

where the matrix M_0 is given by

$$M_0 = \begin{pmatrix} -\frac{\omega^2}{8} & 0 \\ 0 & \frac{\omega^2}{8}\end{pmatrix}. \tag{3.29}$$

By Lemma 1.3, $\det^{1/2}\left(\frac{\sin(M_0)}{M_0}\right) = 1$. Hence

$$
\begin{aligned}
&\mathrm{Tr}_{\mathrm{z}}\left[\mathrm{Tr}_s\left[e^{D_t^2+zD_t}\right]\right] \\
&\quad = (-1)^{\frac{N(N+1)}{2}}\, 2^N\ \mathrm{Tr}_{\mathrm{z}}\left[\int^B e^{-\frac{t}{4}\left|\mathbf{x}+\frac{i}{2\sqrt{t}}\omega\psi+\frac{i}{\sqrt{t}}z\widehat{\psi}\right|^2+\frac{1}{16}\left\langle\widehat{\psi},\omega^2\widehat{\psi}\right\rangle}\right] \\
&\quad = (-1)^{\frac{N(N-1)}{2}}\, 2^N\ \mathrm{Tr}_{\mathrm{z}}\left[\int^B e^{-\frac{t}{4}\left|\mathbf{x}+\frac{1}{2\sqrt{t}}\omega\psi+\frac{1}{\sqrt{t}}z\widehat{\psi}\right|^2-\frac{1}{16}\left\langle\widehat{\psi},\omega^2\widehat{\psi}\right\rangle}\right].
\end{aligned}
\tag{3.30}
$$

On the other hand, (2.74) implies that

$$
I^*\delta_{\frac{t}{4}} = \mathrm{Tr}_{\mathrm{z}}\left[\int^B e^{-\frac{t}{4}\left|\mathbf{x}+\frac{1}{2\sqrt{t}}\omega\psi+\frac{1}{\sqrt{t}}z\widehat{\psi}\right|^2-\frac{1}{16}\left\langle\widehat{\psi},\omega^2\widehat{\psi}\right\rangle}\right].
\tag{3.31}
$$

Equation (3.19) follows from combining (3.21), (3.30) and (3.31). Equation (3.20) follows similarly. q.e.d.

Remark 3.12. It would have been more natural to express the results of Section 2b in terms of the forms $I^*\delta_t$ and $I^*\epsilon_t$. This is because it is only the flat structure of $\mathcal{V}$ which counts, not its topological structure.

c) Fourier decomposition

We follow the geometric setup of Section 2d. Put $M = E/\Lambda$. Then M is the total space of a fiber bundle over B with fiber $Z = T^N$. There is a horizontal distribution T^HM on M given by pushing forward the horizontal vectors for ∇^E under the quotient map $E \to M$. Let F be the trivial complex line bundle on M, with the standard Hermitian metric h^F. Let h^E be an inner-product on E. We assume that the induced volume forms on the fibers are preserved by ∇^E. Hereafter, we write ω_{E^*} for $\omega\left(\nabla^{E^*}, h^{E^*}\right)$. The metric h^E also induces a Riemannian metric g^{TZ} on the vertical tangent bundle TZ.

Let $\overline{W}$ be the $\mathbb{Z}$-graded Hilbert bundle on B, whose fiber over $b \in B$ is isomorphic to $L^2(Z_b; \Lambda(T^*Z_b))$. As the torus Z_b has trivial tangent bundle, we have isomorphisms

$$
\begin{aligned}
C^\infty(Z_b;\Lambda(T^*Z_b)) &\cong \Lambda(E_b^*) \otimes C^\infty(Z_b), \\
L^2(Z_b;\Lambda(T^*Z_b)) &\cong \Lambda(E_b^*) \otimes L^2(Z_b).
\end{aligned}
\tag{3.32}
$$

Given μ_b in the lattice Λ_b^*, there is a well-defined function $e^{\sqrt{-1}\langle\mu_b,\cdot\rangle} \in C^\infty(Z_b)$. By Fourier analysis, for each $b \in B$ there is an orthogonal decomposition

$$\overline{W}_b = \bigoplus_{\mu_b \in \Lambda_b^*} \Lambda\left(E_b^*\right) \otimes \mathbb{C}e^{\sqrt{-1}\langle\mu_b,\cdot\rangle}. \tag{3.33}$$

If U is a contractible open subset of B, then the orthogonal decompositions of $\{\overline{W}_b\}_{b\in U}$ piece together to give an orthogonal decomposition

$$L^2(U;\overline{W}\big|_U) = \bigoplus_{\mu\in C^\infty(U;\Lambda^*)} L^2(U;\Lambda(E^*\big|_U)) \otimes \mathbb{C}e^{\sqrt{-1}\langle\mu,\cdot\rangle}. \tag{3.34}$$

Theorem 3.13. *With respect to the orthogonal decomposition* (3.33), *the superconnection d^M splits as*

$$d^M = \bigoplus_{\mu\in C^\infty(U;\Lambda^*)} \left(\sqrt{-1}\,(\mu\ \wedge) + \nabla^{\Lambda(E^*|_U)}\right) \otimes I. \tag{3.35}$$

Proof. This follows from [4, Proposition 3.4], but we will give a direct proof. We can trivialize E over U as $E = U \times \mathbb{C}^N$ so that if $\sigma : U \to \mathbb{C}^N$ is a section of $E\big|_U$, then $\nabla^E(\sigma) = d\sigma$. With respect to this trivialization, $M\big|_U = U \times T^N$. There is an isomorphism

$$\alpha : \Omega\,(U)\ \widehat{\otimes}\ \Lambda\left((\mathbb{C}^N)^*\right) \otimes C^\infty(T^N) \to \Omega\,(U)\ \widehat{\otimes}\ \Omega\left(T^N\right). \tag{3.36}$$

Acting on $\Omega\,(U)\ \widehat{\otimes}\ \Omega\left(T^N\right)$, we have $d^M = (I\ \widehat{\otimes}\ d_{T^N}) + (d_U\ \widehat{\otimes}\ I)$. Then in terms of our trivializations, the superconnection d^M can be written as $\alpha^{-1}\left((I\ \widehat{\otimes}\ d_{T^N}) + (d_U\ \widehat{\otimes}\ I)\right)\alpha$.

Given $f \in C^\infty(U)$, $s \in \Lambda\left((\mathbb{C}^N)^*\right)$ and $\mu \in C^\infty(U;\Lambda^*)$, we have

$$\alpha\left(f \otimes s \otimes e^{\sqrt{-1}\langle\mu,\cdot\rangle}\right) = f \otimes s e^{\sqrt{-1}\langle\mu,\cdot\rangle}, \tag{3.37}$$

$$\left(I\ \widehat{\otimes}\ d_{T^N}\right)\left(f \otimes s e^{\sqrt{-1}\langle\mu,\cdot\rangle}\right) = f \otimes \left(\sqrt{-1}\ \mu \wedge s\ e^{\sqrt{-1}\langle\mu,\cdot\rangle}\right) \tag{3.38}$$

and

$$\left(d_U\ \widehat{\otimes}\ I\right)\left(f \otimes s e^{\sqrt{-1}\langle\mu,\cdot\rangle}\right) = d_U f \otimes s e^{\sqrt{-1}\langle\mu,\cdot\rangle}. \tag{3.39}$$

Thus, acting on $C^\infty(U) \mathbin{\widehat{\otimes}} \Lambda\left((\mathbb{C}^N)^*\right) \otimes \mathbb{C}e^{\sqrt{-1}\langle\mu,\cdot\rangle}$ gives

$$d^M = \left(I \mathbin{\widehat{\otimes}} \sqrt{-1}\,(\mu\,\wedge) \otimes I\right) + \left(d_U \mathbin{\widehat{\otimes}} I \otimes I\right). \tag{3.40}$$

Using the isomorphism

$$C^\infty(U;\Lambda(E^*\big|_U)) \cong C^\infty(U) \otimes \Lambda\left((\mathbb{C}^N)^*\right), \tag{3.41}$$

we see that acting on $C^\infty(U;\Lambda(E^*\big|_U)) \otimes \mathbb{C}e^{\sqrt{-1}\langle\mu,\cdot\rangle}$, one has

$$d^M = \left(\sqrt{-1}\,(\mu\,\wedge) + d_U\right) \otimes I. \tag{3.42}$$

However, in terms of our trivializations, $\nabla^{\Lambda(E^*|_U)} = d_U$. The theorem follows. q.e.d.

Theorem 3.14. *With respect to the orthogonal decomposition* (3.33), *the superconnection* $\left(d^M\right)^*$ *splits as*

(3.43)
$$\left(d^M\right)^* = \bigoplus_{\mu\in C^\infty(U;\Lambda^*)} \left(-\sqrt{-1}\; i(\mu) + \nabla^{\Lambda(E^*|_U)} + \Lambda\left(\omega_{E^*|_U}\right)\right) \otimes I.$$

Proof. Let $h^{\mathbb{C}}$ be the standard Hermitian metric on $\mathbb{C}$. The restriction of h^W to $L^2(U;\Lambda(E^*\big|_U)) \otimes \mathbb{C}e^{\sqrt{-1}\langle\mu,\cdot\rangle}$ is $h^{\Lambda(E^*|_U)} \otimes h^{\mathbb{C}}$. Thus it is enough to compute the adjoint of $\sqrt{-1}\,(\mu\,\wedge) + \nabla^{\Lambda(E^*|_U)}$, acting on $L^2(U;\Lambda(E^*\big|_U))$, with respect to the Hermitian metric $h^{\Lambda(E^*|_U)}$. Clearly the adjoint of $(\mu\,\wedge)$ is $i(\mu)$. The adjoint of $\nabla^{\Lambda(E^*|_U)}$ is

$$\begin{aligned} &\nabla^{\Lambda(E^*|_U)} + \left(h^{\Lambda(E^*|_U)}\right)^{-1}\left(\nabla^{\Lambda(E^*|_U)} h^{\Lambda(E^*|_U)}\right)\\ &\quad = \nabla^{\Lambda(E^*|_U)} + \omega_{\Lambda(E^*|_U)}\\ &\quad = \nabla^{\Lambda(E^*|_U)} + \Lambda\left(\omega_{E^*|_U}\right). \end{aligned} \tag{3.44}$$

The theorem follows. q.e.d.

We can now remove the restriction to the open set U, and work globally on B. Put

$$\widehat{c}(\mu) = (\mu\,\wedge) + i(\mu). \tag{3.45}$$

For $t > 0$ and $\mu \in \Lambda^*$, let

$$D_{t,\mu} = -\sqrt{-1}\,\frac{\sqrt{t}}{2}\,\widehat{c}(\mu) + \frac{1}{2}\,\Lambda\left(\omega_{E^*}\right). \tag{3.46}$$

By defining D_t as in [4, (3.50)], Theorems 3.13 and 3.14 give that

$$D_t = \bigoplus_{\mu \in \Lambda^*} D_{t,\mu} \otimes I. \tag{3.47}$$

Then defining $f\left(C'_t, h^W\right)$ as in [4, (3.83)] and $f^\wedge\left(C'_t, h^W\right)$ as in [4, (3.103)], we have

$$\begin{aligned} f\left(C'_t, h^W\right) &= \sum_{\mu \in \Lambda^*} (2i\pi)^{1/2} \varphi \operatorname{Tr}_s\left[f\left(D_{t,\mu}\right)\right], \\ f^\wedge\left(C'_t, h^W\right) &= \sum_{\mu \in \Lambda^*} \varphi \operatorname{Tr}_s\left[\frac{N}{2} f'\left(D_{t,\mu}\right)\right], \end{aligned} \tag{3.48}$$

where the supertraces on the right-hand side of (3.48) are finite-dimensional supertraces on $\Lambda\left(E^*\right)$. Applying the results of the previous section with $\mathcal{V} = E^*$, from (3.48) and Definition 3.8 we see that

$$\begin{aligned} f\left(C'_t, h^W\right) &= \sum_{\mu \in \Lambda^*} \mu^* \widetilde{\delta}_t, \\ \frac{1}{t}\, f^\wedge\left(C'_t, h^W\right) &= \sum_{\mu \in \Lambda^*} \mu^* \widetilde{\epsilon}_t. \end{aligned} \tag{3.49}$$

Theorem 3.15. *One has*

$$\frac{\partial}{\partial t} f\left(C'_t, h^W\right) = \frac{1}{t} d f^\wedge\left(C'_t, h^W\right). \tag{3.50}$$

Proof. This follows from [4, Theorem 3.20], or more directly from equation (3.15). q.e.d.

Theorem 3.16. *If K is a compact subset of B, then there is a constant $c > 0$ such that on K, as $t \to \infty$,*

$$\begin{aligned} f\left(C'_t, h^W\right) &= f\left(\nabla^{\Lambda(E^*)}, h^{\Lambda(E^*)}\right) + \mathcal{O}(e^{-ct}), \\ f^\wedge\left(C'_t, h^W\right) &= -\frac{1}{2} + \mathcal{O}(e^{-ct}) \quad \textit{if} \ \dim(E) = 1, \\ &= \mathcal{O}(e^{-ct}) \qquad\qquad \textit{if} \ \dim(E) > 1. \end{aligned} \tag{3.51}$$

Proof. The contribution of the terms in (3.49) with $\mu \neq 0$ is exponentially small in t. Thus it suffices to look at the $\mu = 0$ term. As

$$(2i\pi)^{1/2}\varphi \operatorname{Tr}_s \left[f\left(D_{t,0}\right)\right] = f\left(\nabla^{\Lambda(E^*)}, h^{\Lambda(E^*)}\right), \tag{3.52}$$

the first line of (3.51) follows. Now

$$\varphi \operatorname{Tr}_s \left[\frac{N}{2} f'\left(D_{t,0}\right)\right] = \varphi \operatorname{Tr}_s \left[\frac{N}{2} f'\left(\frac{1}{2}\ \Lambda\left(\omega_{E^*}\right)\right)\right]. \tag{3.53}$$

By Lemma 1.3, this is the same as

$$\operatorname{Tr}_s \left[\frac{N}{2} f'(0)\right] = \frac{1}{2} \sum_{p=0}^{N} (-1)^p\ p \binom{N}{p}, \tag{3.54}$$

from which the rest of Theorem 3.16 follows. q.e.d.

Remark 3.17. Theorem 3.16 is the same as Theorem 2.17, which was proved in terms of Berezin integrals.

Combining Theorem 3.11 and (3.49), we see that the form $\sum_{\mu\in\Lambda} \mu^* \delta_t$ of Section 2d is essentially the same as the form $f\left(C_t', h^W\right)$, and the form $\sum_{\mu\in\Lambda} \mu^* \epsilon_t$ of Section 2d is essentially the same as the form $\frac{1}{t}\ f^\wedge\left(C_t', h^W\right)$. Thus the analysis of Section 2d simply consists of explicitly verifying the general properties of the forms $f\left(C_t', h^W\right)$ and $f^\wedge\left(C_t', h^W\right)$ in the special case of a torus fibration.

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