J. DIFFERENTIAL GEOMETRY
45 (1997) 170-195

MEASURED SOLENOIDAL RIEMANN SURFACES AND HOLOMORPHIC DYNAMICS

MEIYU SU

Abstract

We study, from a measure theoretic point of view, the lamination structure on the inverse limit space $\varprojlim(\bar{\mathbb{C}}, f)$ for an arbitrary rational map f on the sphere $\bar{\mathbb{C}}$. It turns out that there is an ergodic holomorphic foliated dynamical object $\mathcal{L}$, namely a self mapping of a measured solenoidal Riemann surface, which continuously injects into the inverse limit space, with full image and with leaves conformally isomorphic to the complex plane $\mathbb{C}$.

1. Introduction

The purpose of this note is to state and prove the following theorem which associates a holomorphic foliated dynamical object to an arbitrary rational map $f : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$. The definitions are given below.

Theorem. *Given an arbitrary rational map $f : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ of degree $d \geq 2$ on the sphere $\bar{\mathbb{C}}$, there is an ergodic measured solenoidal Riemann surface $\mathcal{L}$ whose leaves are isomorphic to the complex plane $\mathbb{C}$, a holomorphic bijection $F:\mathcal{L} \to \mathcal{L}$, and a holomorphic map $\pi:\mathcal{L} \to \bar{\mathbb{C}}$ so that $f^n \circ \pi = \pi \circ F^n$ for every positive integer n. Moreover, the induced map of $\mathcal{L}$ into the reduced inverse limit space $\varprojlim(\mathbb{C}', f)$ is a continuous injection whose image intersects every fiber in full measure for the natural multiplicity fiber measures class (see the proof in Section 4).*

Remarks and Definitions.

1. *A topological solenoid*, simply *a solenoid*, $\mathcal{S}$ is a topological space with *local box charts* homeomorphic to a product of the form (a

Received July 10, 1996.

totally disconnected space T) cross (a k-ball D) so that overlap maps preserve the k-ball factor. A solenoid $\mathcal{S}$ is naturally foliated or laminated by its path connected components which are called global leaves. Holonomy homeomorphisms can be partially defined from one transversal to another by transporting points along leaves. (See [14] for the definition and examples.)

A measured solenoid $\mathcal{S}$ is a topological solenoid equipped with a finite holonomy invariant transverse measure class μ so that any open set in each box transversal has a positive μ-measure.

A measured solenoid is a *solenoidal Riemann surface* if, in addition, the overlap maps on box charts which preserve the two-disk factor are also biholomorphic in the two-disk direction. (See Section 4.2 for the general definition.)

2. *The reduced inverse limit space* of $f : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ is $\widetilde{\mathbb{C}'} = \varprojlim(\mathbb{C}', f)$ consisting of backward strings $\tilde{z} = (\ldots, z_n, \ldots, z_1, z_0)$ such that $f(z_n) = z_{n-1}$, for all $z_n \in \mathbb{C}'$, where

$$\mathbb{C}' = \begin{cases} \mathbb{C}^* & \text{if } f \sim z \mapsto z^d \text{ or } z \mapsto z^{-d}, \\ \mathbb{C} & \text{if } f \sim \text{a polynomial}, \\ \bar{\mathbb{C}} & \text{otherwise}, \end{cases}$$

and $\sim$ means conformally conjugate. The induced map $F : \widetilde{\mathbb{C}'} \to \widetilde{\mathbb{C}'}$ by f is a homeomorphism relative to the product topology when $\widetilde{\mathbb{C}'}$ is viewed as a subspace of the countable Cartesian product $\Pi\mathbb{C}'$.

3. Note that the projections $\pi_n : \widetilde{\mathbb{C}'} \to \mathbb{C}'$ defined by $\pi_n(\tilde{z}) = z_n, z_n \in \mathbb{C}'$, are continuous and satisfy $\pi_n \circ F = f \circ \pi_n$ and $\pi_{n+k} = f^k \circ \pi_n$, for all $n, k \geq 0$.

4. The fiber of π_n over every point $z \in \mathbb{C}'$ is a Cantor set (ref. Section 2.1) provided with a natural normalized multiplicity measure μ_z (see Section 2.2 for the definition of μ_z).

5. $\mathcal{L}$ is ergodic in the sense that for any two positive measure sets of box transversals, almost all the points in one are connected by leaves to points of the other (see Section 3.3.4 for the proof).

6. The triple $(\mathcal{L}, F, \pi)$ is well defined up to measure theoretic isomorphisms, namely, if there is another triple $(\mathcal{L}', \pi', F')$ which satisfies

the required properties, then there exists a one-to-one map Φ between them such that

$$\Phi \circ F = F' \circ \Phi \text{ and } \pi = \pi' \circ \Phi$$

on full measure subsets of leaves for the multiplicity fiber measures. Φ is a measure preserving Borel ismorphism in the transverse direction and a bi-holomorphism in the leaf direction (see Section 4.1 for the proof).

7. The topology on the solenoid $\mathcal{L}$ is not intended to be canonical. It is used to determine the measure theory structure (compare to Lyubich and Minsky [8]).

Here is a sketch of the idea of the proof of the theorem. The construction of a measured solenoid $\mathcal{L}$ starts by introducing the definition of the regular set $\mathcal{R}$ of the inverse limit space to be the union of all regular boxes. Given an arbitrary $\epsilon > 0$, we find, over every sufficiently small two-disk D on the sphere, a *maximal box* B which covers at least $1 - \epsilon$ amount of the μ_z-measure of the fiber $\pi^{-1}(z)$, for $z \in D$. Then $\mathcal{R}$ has full measure of every fiber in $\tilde{\mathbb{C}}'$. It is also shown that $\mathcal{R}$ can be covered by countably many maximal boxes so that all but finitely many of them have transversals of positive measures for the multiplicity fiber measures. Thus we can work out the construction in a countable fashion. We then show that relative to the multiplicity fiber measures, almost all global leaves of the regular set $\mathcal{R}$ are conformally equvalent to the complex plane $\mathbb{C}$ (these leaves are called affine leaves of $\mathcal{R}$, and the affine part of $\mathcal{R}$ consisting of all affine leaves is denoted by $\mathcal{A}$). A finer topology, called *the fine-topology*, on $\mathcal{R}$ is introduced using the *density-topology* on fibers. By deleting global leaves of $\mathcal{A}$ that pass through the density-boundaries of the countably many maximal boxes in $\mathcal{A}$, one constructs a fine-open subset $\mathcal{L}$ which is the measured solenoid satisfying the required properties stated in the theorem.

We point out that the definition of the regular set $\mathcal{R}$ here coincides with the one given by Lyubich and Minsky in [8]. They have a parallel construction of *an affine orbifold lamination* in the regular set $\mathcal{R}$ of the inverse limit space by refining the topology in a completely different manner (for more details of their construction see [8]).

The structure of this note is as follows.

Section 2 gives the definition of the natural normalized multiplicity measures supported on the fibers of π. Such fiber measures are non-atomic, holonomy invariant and F quasi-invariant in the reduced inverse limit space $\widetilde{\mathbb{C}}'$.

Section 3 is devoted to the construction of the maximal boxes in the reduced inverse limit space over two-disks in $\mathbb{C}'$ via the projection $\pi = \pi_0$. The F-invariant regular set $\mathcal{R}$ is defined as the union of all boxes in $\widetilde{\mathbb{C}}'$. Leaves of $\mathcal{R}$ are all Riemann surfaces, and almost all of them are isomorphic to the complex plane. The map F restricted to this affine part becomes affine in the leaf direction.

The existence of the solenoidal Riemann surface $\mathcal{L}$ is presented in Section 4 by introducing a new topology on the regular set $\mathcal{R}$, the so called fine-topology for which the complex structure on leaves of $\mathcal{L}$ is continuous. Also a general definition of a measured solenoid is given in this section. Our construction provides non-trivial examples of these objects.

Acknowledgments

I wish to express my deepest gratitude to Professor Dennis Sullivan for introducing me to this wonderful field, for his constant support and expert guidance encouraging me to pursue this study. Professors A. Baider and F.P. Gardiner have given many useful comments to this work, for which I would like to thank them very much. I also express my thanks to Professors J. Milnor, M.Y. Lyubich, and Dr. Y. Minsky for their comments and conversations with me.

2. The multiplicity fiber measures

In this section, we construct the normalized multiplicity fiber measures on the fibers in the reduced inverse limit space. Let us begin with the basic notions and some known results which will be used in the sequel.

2.1. Basic notions and some known results.

Let $f : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ be an arbitrary rational map of degree $d \geq 2$ on the Riemann sphere $\bar{\mathbb{C}}$. A point z is said to be exceptional if the grand orbit

$$\{\xi \in \bar{\mathbb{C}} : f^n(z) = f^m(\xi) \ for\ some\ n, m \geq 0\}$$

is finite. One knows that the exceptional set of f contains at most two points and that f is conformally conjugate, denoted by $\sim$, to $z \mapsto z^n$ or $z \mapsto z^{-n}$ if f has two exceptional points, and to a polynomial if f has one exceptional point (see [1] for a reference). Recall,

$$\mathbb{C}' = \begin{cases} \mathbb{C}^* & \text{for } f \sim z \mapsto z^n \text{ or } z^{-n}, \\ \mathbb{C} & \text{for } f \sim \text{a polynomial}, \\ \bar{\mathbb{C}} & \text{otherwise.} \end{cases}$$

Without lost of generality, we only work with f restricted to $\mathbb{C}'$ in what follows:

- Let $C = \{z \in \mathbb{C}' : f'(z) = 0\}$ denote the critical set of f, and let $PC = \bigcup_{n=1}^{\infty} f^n(C)$ denote the post critical set of f in $\mathbb{C}'$.
- The reduced inverse limit space
$$\begin{aligned} \widetilde{\mathbb{C}'} &= \varprojlim(\mathbb{C}', f) \\ &= \{\tilde{z} = (\ldots, z_n, \ldots, z_1, z_0) : f(z_n) = z_{n-1}, z_n \in \mathbb{C}', n \geq 0\}, \end{aligned}$$
consisting of backward orbit strings of f, is a subspace of the countable Cartesian product space $\Pi\mathbb{C}'$ with the usual topology.
- The induced map
$$F : \widetilde{\mathbb{C}'} \to \widetilde{\mathbb{C}'} \text{ defined by } F(\tilde{z}) = (\ldots, z_n, \ldots, z_1, z_0, f(z_0))$$
becomes a homeomorphism with the inverse map erasing the first coordinate of $\tilde{z}$.
- To simplify the notation, we sometimes denote
$$\tilde{z} = (\ldots, z_n, \ldots, z_1, z_0)$$
by (z_n).
- Consider the projection $\pi_n : \widetilde{\mathbb{C}'} \to \mathbb{C}'$ defined by $\pi_n(\tilde{z}) = z_n$. For every $n \geq 0$, the fiber $\pi_n^{-1}(z)$ over each point $z \in \mathbb{C}'$ is *a Cantor set* because z is not exceptional and $2 < |f^{-n}(z)| \leq d^n$, where $|\cdot|$ denotes the cardinality of the set. In particular, if $z \notin PC$, the fiber $\pi_n^{-1}(z)$ is a standard d-adic Cantor set.

- The following identities hold for all $n, k \geq 0$:

$$\pi_{n-k} = f^k \circ \pi_n, \ \pi_n = \pi_{n+k} \circ F^k, \ \textit{and} \ f^n \circ \pi = \pi \circ F^n,$$

where $\pi = \pi_0$. From now on, we work with the projection $\pi = \pi_0$. Similar ideas work for all other π_n.

- There is a one-to-one correspondence between the set of periodic points of F in $\widetilde{\mathbb{C}'}$ to that of f in $\mathbb{C}'$. We denote by $\tilde{S}$ the set of periodic points of F in $\widetilde{\mathbb{C}'}$ over PC. Then $\tilde{S}$ is a finite set.

- A rotation domain of f is a Siegel disk or a Herman ring of f in the sphere. For a rotation domain D of f in the sphere, there is a unique F-invariant lift $\tilde{D}$ contained in the $\widetilde{\mathbb{C}}$. We call such a corresponding $\tilde{D}$ a *rotation domain* of F.

Lemma 2.1 (Arc Lifting Property). *Given any continuous curve*

$$\gamma : [0,1] \to \mathbb{C}' \ \textit{with} \ z = \gamma(0), w = \gamma(1),$$

there exists a continuous curve $\tilde{\gamma} : [0,1] \to \widetilde{\mathbb{C}'}$ for each $\tilde{z}$ in $\pi^{-1}(z)$, so that $\tilde{\gamma}(0) = \tilde{z}$ and $\pi \circ \tilde{\gamma}(t) = \gamma(t)$ for all $t \in [0,1]$. If γ is contained in $\mathbb{C}' \setminus PC$, then the lift is unique for each $\tilde{z} \in \pi^{-1}(z)$, i.e. there is a Cantor bundle of curves in $\widetilde{\mathbb{C}'}$ over γ.

Proof. In fact, let $z_n = \pi_n(\tilde{z})$, for every $n \geq 0$, there is a lifted arc γ_n over γ starting from z_n and ending at some $w_n \in f^{-n}(w)$ via the map f^n in $\mathbb{C}'$. Then $\tilde{\gamma} = (\ldots, \gamma_n, \ldots, \gamma_1, \gamma)$ is the curve over γ starting at $\tilde{z}$ and ending at a point $\tilde{w} \in \pi^{-1}(w)$. When $\gamma \subset \mathbb{C}' \setminus PC$, the lifted curve $\tilde{\gamma}$ is uniquely determined by the initial point $\tilde{z} \in \pi^{-1}(z)$, but the fiber over z is a Cantor set in $\widetilde{\mathbb{C}'}$. q.e.d.

Take $a, b \in \mathbb{C}' \setminus PC$. Denote by $\Pi_1(\mathbb{C}' \setminus PC, a, b)$ the set of homotopy classes of curves in $\mathbb{C}' \setminus PC$ starting at a and ending at b. To every $[\gamma] \in \Pi_1(\mathbb{C}' \setminus PC, a, b)$, we can associate with a homeomorphism $h_\gamma : \pi^{-1}(a) \to \pi^{-1}(b)$ defined by $h_\gamma(\tilde{z}) = \tilde{w}$ iff the lifted curve $\tilde{\gamma}$ over γ initiated at $\tilde{z}$ ends at $\tilde{w} \in \pi^{-1}(w)$. We call h_γ the *holonomy map* from $\pi^{-1}(a)$ to $\pi^{-1}(b)$ representing $[\gamma]$. When $a = b$, all such holonomy maps from $\pi^{-1}(a)$ to itself form *the holonomy group* acting on the fiber representing the fundamental group $\Pi_1(\mathbb{C}' \setminus PC, a)$ of $\mathbb{C}' \setminus PC$ with base point a.

Lemma 2.2. *Holonomy maps are isometries from $\pi^{-1}(a)$ to $\pi^{-1}(b)$ relative to the metric*

$$\rho(\tilde{z}, \tilde{z'}) = \frac{1}{n}(or \ \frac{1}{d^n}),$$

where n is the first index for which $z_n \neq z'_n$.

Proof. The proof is left to the reader. q.e.d.

2.2. The fiber measures.

Now we define the natural normalized multiplicity measures on fibers in $\widetilde{\mathbb{C}'}$.

Start with a point $z \in \mathbb{C}' \setminus PC$, the fiber $\pi^{-1}(z)$ over z of the projection π is a d-adic Cantor set. Let $E_n = f^{-n}(z)$. Then E_n contains d^n distinct points in $\mathbb{C}' \setminus PC$. Define $\mu_n(z_n) = \frac{1}{d^n}$, for each $z_n \in E_n$, which gives the natural cardinality measure on E_n, $n \geq 1$. It is clear that the family of all cylinder sets of the form

$$\tilde{A} = \{\tilde{z} \in \pi^{-1}(z) : z_n \in A_n, A_n \subset E_n \text{ for some } n \geq 0\}$$

forms a ring which generates the Borel algebra $\mathcal{B}_z$ on the fiber $\pi^{-1}(z)$. Define

$$\mu(\tilde{A}) = \frac{|A_n|}{d^n},$$

for a cylinder $\tilde{A}$, where $|A_n|$ is the cardinality of $A_n \subset E_n$. By the Kolmogrov Thoerem in standard measure theory (see [3]), μ can be naturally extended to $\mathcal{B}_z$ as a probability measure μ_z .

When $z \in PC$ but z is not in a periodic critical orbit of f in the sphere $\mathbb{C}'$, we can define the multiplicity measure on $\pi^{-1}(z)$ using a similar idea. Each point $\tilde{z} \in \pi^{-1}(z)$ has a finite multiplicity in this case, so the cardinality counted with multiplicity makes sense.

If a point $z \in \mathbb{C}'$ is a periodic critical point of local degree m, then $m < d$ since z is not a exceptional point of f. For simplicity we assume that z is a fixed point. There are at least two preimages of z under f. Then each of the preimage has multiplicity at most $m \leq d-1$. At the n-th level z, as a preimage of z under f^n, has multiplicity at most m^n. Since $\frac{m^n}{d^n} \to 0$, as $n \to \infty$ the multiplicity measure of the fixed point $\tilde{z} = (\dots, z, \dots, z, z)$ in the fiber $\pi^{-1}(z)$ is zero. For all other points on the fiber, multiplicity through all steps are uniformly bounded by the multiple of local degrees of critical points hit by the backward string.

Thus the multiplicity measure on such a fiber is also well defined and has no atoms.

Proposition 2.1. *For any $z \in \mathbb{C}'$, there exists a non-atomic probability measure μ_z supported on the fiber $\pi^{-1}(z)$ which measures the multiplicity of points, call it the multiplicity measure. All such multiplicity fiber measures satisfy the following additional properties:*

1. *If h is a holonomy map from the fiber $\pi^{-1}(u)$ to $\pi^{-1}(w)$, then for $E \subset \pi^{-1}(u)$ measurable, $\mu_w(h(E)) = \mu_u(E)$. Thus the multiplicity measures are holonomy invariant.*

2. *The fiber measures are also F quasi-invariant in the sense that F restricted to the fiber $\pi^{-1}(z)$ is measurable and $\mu_{f(z)}(F(E)) > 0$ if and only if $\mu_z(E) > 0$.*

Proof. Call a special cylinder set of form

$$\tilde{A} = \{\tilde{u} = (u_n) : u_m = a_m \ for\ some\ a_m \in f^{-m}(a)\ and\ some\ n \geq 0\}$$

a block in the fiber $\pi^{-1}(a)$. Then all block sets in the fiber generates the Borel algebra $\mathcal{B}_a$.

1. When $a, b \in \mathbb{C}' \setminus PC$, h is a holonomy representing $[\gamma], \gamma \subset \mathbb{C}' \setminus PC$ initiated at a and ending at b. By the definition, h maps a block $\tilde{A}$ in the fiber $\pi^{-1}(a)$ to a block set $h(\tilde{A}) = \{\tilde{w} : w_n = b_n = h_n(a_n)\}$ of the fiber $\pi^{-1}(b)$ with the base point b_n, where $b_n = \gamma_n(1)$ is uniquely determined by the starting point a_n of $\gamma_n \in f^{-n}(a), n \geq 1$. Hence $\mu_w(h(\tilde{A})) = 1\,/\,d^n = \mu_u(\tilde{A})$.

2. F restricted to the fiber $\pi^{-1}(z)$ maps a block $\tilde{A} = \{\tilde{z} \in \pi^{-1}(z) : z_n = a_n\}$ to a block $F(\tilde{A}) = \{\tilde{w} = F(\tilde{z}) : z_n = a_n\} = \{\tilde{w} : w_{n+1} = a_n\}$ in the fiber $\pi^{-1}(f(z))$. So $\mu_{f(z)}(F(\tilde{A})) = 1\,/\,d^{n+1} = \frac{1}{d}\,\mu_z(\tilde{A})$.

q.e.d.

3. Boxes and the regular set

Intuitively, the reduced inverse limit space $\widetilde{\mathbb{C}'}$ is laminated by its path connected components with singularities related to the post critical set PC of f. It is hard to see if path connected components of $\widetilde{\mathbb{C}'}$ are Riemann surfaces, in general, because of the complexity of the set of singularities. We present, in this section, the study of the complex structure on the regular set in the inverse limit space $\widetilde{\mathbb{C}}'$.

3.1. Definition of the regular set $\mathcal{R}$.

Let $\tilde{z} = (z_n)$ be an arbitrary point in $\widetilde{\mathbb{C}}'$. Given any $n \geq 0$, pick a two-disk neighborhood D_n of $\pi_n(\tilde{z}) = z_n$ in $\mathbb{C}'$ and denote by D_{n+k} the component of $f^{-(k)}(D_n)$ containing $\pi_{n+k}(\tilde{z}) = z_{n+k}$. We call the subset

$$\tilde{D}(\tilde{z}, n) = \{\tilde{w} \in \widetilde{\mathbb{C}}' : w_j \in D_j \text{ for all } j \geq n\}$$

a *leafwise neighborhood* of $\tilde{z}$ in the inverse limit space $\widetilde{\mathbb{C}}'$ which is the path connected component of $\pi_n^{-1}(D_n)$ containing $\tilde{z}$.

Definition 3.1. A subset B of $\widetilde{\mathbb{C}}'$ is called a box if it is a globally trivial fibration over an open connected subdomain D_n of the sphere $\mathbb{C}'$ via some projection π_n. Path connected components in B are called local leaves and a fiber in B is called a transversal of B.

Define $\mathcal{R} =$ the union of all boxes in $\widetilde{\mathbb{C}}'$ and call it the regular set.

Observe the following useful facts:

- Given a point $\tilde{z}$ in $\mathcal{R}$, the local leaf $\tilde{D}(\tilde{z})$ through $\tilde{z}$ in a box B containing $\tilde{z}$ is also a box whose transversal contains a single point. $F(\tilde{D})$ is a box containing $F(\tilde{z})$, and $F^{-1}(\tilde{D})$ is a box containing $F^{-1}(\tilde{z})$. Thus the regular set $\mathcal{R}$ is totally F-invariant.

- If a point $\tilde{z} \in \mathcal{R}$, there is *a regular leafwise neighborhood* $\tilde{D}(\tilde{z}, n)$ (call it *a regular disk* in the inverse limit) of $\tilde{z}$ which is a univalent lift of a two-disk neighborhood D_n of $\pi_n(\tilde{z})$ via the projection π_n. Thus the regular set is the union of all regular disks. Note that the definition of the regular set $\mathcal{R}$ here is equivalent to that given in [8].

- Path connected components of $\mathcal{R}$ are called global leaves. One can check that global leaves of $\mathcal{R}$ are Riemann surfaces for which local leaves play the role of charts.

- Holonomy maps can be partially defined naturally on fibers of the regular set $\mathcal{R}$ by the restrictions of holonomy maps defined in Subsection 2.1. Choose a complete transversal T which is a union of box transversals and intersects every global leaf of $\mathcal{R}$. All partially defined holonomy maps from T to itself generate the *μ-invariant holonomy groupoid* on T.

- A box B is said to have a *positive transverse measure* if its transversal T has a positive multiplicity measure in the fiber containing T. All transversals of B have the same multiplicity measure by holonomy invariance.

Boxes in the regular set $\mathcal{R}$ play the role of charts. One expects that there are enough open boxes in $\mathcal{R}$. In the critically finite case, where PC is a finite set in the sphere, all but finitely many points in $\mathcal{R}$ can be covered by open boxes. We find, however, there are no open boxes in the inverse limit space if the post critical set PC is dense in the sphere! This is why we take a measure theoretic approach to the study below.

3.2. Maximal boxes.

Let $D \subset \mathbb{C}'$ be a two-disk. A component D_n of $f^{-n}(D)$ is said to be *regular* if $f^n : D_n \to D$ is univalent. If D_n is not regular, we say it is *critical.*

Proposition 3.1. *Let z be an arbitrary point of $\mathbb{C}' \setminus PC$. Then given any $\epsilon > 0$, there exists a box B in $\widetilde{\mathbb{C}'}$ so that the transversal $T = B \cap \pi^{-1}(z)$ in $\mathcal{B}_z$ has the μ_z-measure $\mu_z(T) > 1 - \epsilon$ in the fiber $\pi^{-1}(z)$.*

Proof. Given $\epsilon > 0$, we choose $N > 0$ large enough so that $\frac{1}{d^N} < \frac{\epsilon}{2}$. Pick a two-disk neighborhood D of z whose topological closure in $\mathbb{C}'$ is contained in $\mathbb{C}' \setminus \bigcup_{j=1}^{N} f^j(C)$. Since there is no critical value of f in any of $f^{-k}(D)$ at each level $k < N$, at most $M = 2d - 2$ components of $f^{-N}(D)$, among the d^N, intersect the set of critical values $f(C)$. Hence there are at most $Md^{n-N-1} + \cdots + Md = 2d(d^{n-N} - 1)$ critical components of $f^{-n}(D)$, among the d^n (counted with multiplicity of local degrees of critical points) when $n > N$. Let

$$A_n = \{z_n \in E_n = f^{-n}(z_0)\text{: the component } D_n \text{ of } f^{-n}(D) \text{ containing } z_n \text{ is regular } \}$$

and let $T = \{\tilde{z} \in \pi^{-1}(z_0) : z_n \in A_n, \text{ for all } n \geq 0\}$. Then T is μ-measurable because it is the complement of the union of critical blocks in the fiber. The multiplicity measure of T is at least $\frac{d^n - 2d(d^{n-N}-1)}{d^n} > 1-\epsilon$. We have also attached to every point $\tilde{z} \in T$ a regular disk neighborhood $\tilde{D}(\tilde{z}) = \{\tilde{w} \in \mathcal{L}\text{: } w_n \in D_n(z_n)\}$ which is the univalent lift at $\tilde{z}$ over the two-disk neighborhood D of z via the projection $\pi = \pi_0$.

Set $B \equiv \bigcup_{\tilde{z} \in T} \tilde{D}(\tilde{z})$. Then B is a box over the two-disk D via the projection $\pi = \pi_0$ which has the positive transverse measure $\geq 1 - \epsilon$.

q.e.d.

Such a box B constructed in the proof of the above proposition is called *the maximal box* over D via the projection π.

Corollary 3.1. *For every point $\tilde{p}$ of $\widetilde{\mathbb{C}}'$ and every $\epsilon > 0$, there is a box B so that $B \cap$ (the fiber containing $\tilde{p}$) has transverse measure $\geq 1 - \epsilon$.*

Proof. It suffices to consider every point $\tilde{p} = (p_n) \in \widetilde{\mathbb{C}}$, not in the periodic cycle of F over the grand orbit of a critical point of f. There is an $n_0 \geq 0$ so that p_m is not in PC, for all $m \geq n_0$. Thus we have a box B' over a disk neighborhood D_{n_0} of z_{n_0} via the projection π so that $B' \cap$ the fiber containing $\tilde{p}^{(n_0)}$ has transverse measure $\geq 1 - \epsilon$. Think of $B = F^{n_0}(B')$ as a box over D_{n_0} via the projection π_{n_0}. Then B is the box whose transversal has the multiplicity fiber measure $\geq 1 - \epsilon$ in the fiber $\pi_{n_0}^{-1}(p_{n_0})$.

Notice that B has multiplicity fiber measure $\geq (1 - \epsilon) / d^{n_0}$ in the fiber $\pi^{-1}(p)$, where $p = \pi(\tilde{p})$. q.e.d.

Corollary 3.2. *The regular set $\mathcal{R}$ has full measure in every fiber.*

Maximal boxes have the following properties.

Proposition 3.2. *The regular set $\mathcal{R}$ can be covered by countably many maximal boxes. Moreover, such maximal boxes can be chosen so that all but finitely many of them have positive transverse measures.*

Proof. Choose a countable basis $\mathcal{T} = \{D_m\}$ of two-disks on the sphere $\mathbb{C}'$. Over every element D_m in the basis, take the maximal box B_{mn} in the inverse limit space $\widetilde{\mathbb{C}}'$ via the projection π_n for each n. We claim that the collection $\mathcal{F} = \{B_{mn}\}$ of all such maximal boxes forms a cover of the regular set $\mathcal{R}$. In fact, for $\tilde{z} = (z_n) \in \mathcal{R} \setminus \tilde{S}$, there exists an $N \geq 0$ so that $z_n = \pi_n(\tilde{z}) \notin PC$ for all $n \geq N$. Pick D_m in the basis $\mathcal{T}$ of the sphere containing $z_N \in D_m$ and take the maximal box B_{m0} over D_m via the projection π_0. Then B_{m0} contains point $\tilde{z}^N = (..., z_{N+1}, z_N)$ and, by Proposition 3.1, it has positive μ-measure when D_m has small radius in the sphere. The transversal of $F^N(B_{m0})$ is contained in the transversal of the maximal box B_{mN} over D_m via the projection π_N. Therefore B_{mN} has positive μ-measure in the fiber $\pi_n^{-1}(z_N)$ and $\tilde{z} \in B_{mN}$.

There are only finitely many points in $\mathcal{R} \cap \tilde{S}$. For each point $\tilde{z}$ in the intersection, there is a box $B \ni \tilde{z}$. We can choose the base disk D of B to be an element of the basis $\mathcal{T}$. Notice that the box B may have zero μ-measure or the transversal of B may contain only a single point.
q.e.d.

Remark. The regular set $\mathcal{R}$ is independent of the choice of the countable basis of the sphere $\mathbb{C}'$.

Definition 3.2. Let $B = T \times D$ be a box in the regular set $\mathcal{R}$. We call

$$\partial_T B = \{\tilde{w} = (w_n) : w_n \in \partial D_n(z_n) \text{ for all } n \geq 0 \text{ and all } \tilde{z} = (z_n) \in T\}$$

the Vertical Boundary of B in $\widetilde{\mathbb{C}'}$, where $\partial D_n(z_n)$ is the topological boundary of the component $D_n(z_n)$ of $f^{-n}(D)$ containing z_n in the sphere $\mathbb{C}'$ (we may assume that B is a box over D via the projection π).

Corollary 3.3 (The Vertical Boundary Covering Property.) *Let $B = T \times D$ be the maximal box over a two-disk D in the basis chosen as in the proof of Proposition 3.2 via the projection π. Then the vertical boundary $\partial_T B$ can be covered to within a measure zero (with respect to the multiplicity fiber measures) by countably many other maximal boxes.*

Corollary 3.3 enables us to attach finitely many pieces of regular local leaves around the leafwise boundary of a local leaf $\tilde{D}(\tilde{z})$ in B. In doing so we may need to discard a null subset of the local leaves in B relative to the multiplicity fiber measures. Applying this procedure, we can prolong almost all local leaves in B further and further to obtain the global leaves of $\mathcal{R}$, by deleting a countable union of null set of leaves.

3.3. Complex structures on leaves of $\mathcal{R}$.

This subsection is distributed to the study of complex structures on leaves of the regular set $\mathcal{R}$ of the inverse limit space $\widetilde{\mathbb{C}'}$.

3.3.1. The point of density.

The concept of a point of density of a subset in a fiber is given in this paragraph.

Definition 3.3. Let $\tilde{E}$ be a Borel subset of the fiber $\pi^{-1}(z)$ in the inverse limit space over a point $z \in \mathbb{C}'$ with $\mu_z(\tilde{E}) > 0$. A $\tilde{z} \in \pi^{-1}(z)$ is called a density point of $\tilde{E}$ in the fiber if the percentage of $\tilde{E}$-points in the block set $\tilde{A}^{(m)} = \{\tilde{w} = (w_n) : w_m = z_m\}$ tends to one as $m \to \infty$.

Lemma 3.1. *The set of density points of $\tilde{E}$ intersects $\tilde{E}$ in full measure.*

Proof. The family $\mathcal{V}$ of all block sets in the Cantor set fiber $\pi^{-1}(z)$ containing $\tilde{E}$ forms a sequence of generating partition in the sense of Mañé in [10]. By Theorem 5.4 (on p.12 in [10]), the sequence of functions defined by

$$\phi_m(\tilde{p}) := \frac{\mu_z(\tilde{E} \cap \tilde{A}^{(m)})}{\mu_z(\tilde{A}^{(m)})}$$

converges in measure to the characteristic function $\phi_{\tilde{E}}$, in particular there exists a subsequence $\{m_n\}_{n=1}^{\infty}$ such that ϕ_{m_n} converges to $\phi_{\tilde{E}}$ almost everywhere. q.e.d.

Remark. The family $\mathcal{V}$ of all block set in the Cantor set fiber $\pi^{-1}(z)$ can also be viewed as a Vitali System in the sense of Shilov and Gurevich [12]. Let $\phi_{\tilde{E}}(\tilde{A}) = \mu_z(\tilde{E} \cap \tilde{A})$ for $\tilde{A}$ measurable in the fiber. Then the derivative

$$D_V \phi_{\tilde{E}}(\tilde{p}) = \lim_{m \to \infty} \frac{\phi_{\tilde{E}}(A^{(m)}(\tilde{p}))}{\mu_z(A^{(m)}(\tilde{p}))}$$

of function $\phi_{\tilde{E}}$ with respect to the Vitali system $\mathcal{V}$ exists on a set of full measure and coincide with the characteristic function of $\tilde{E}$ on $\tilde{E}$ (See The Lebesgue-Vitali Theorem in [12]).

Lemma 3.2 (The Intersection Property). *Let B, B' be two boxes in $\mathcal{L}$. Suppose that $\tilde{z} \in B \cap B'$ is a density point of both the transversal T of B and the transversal T' of B' containing $\tilde{z}$. Then $\tilde{z}$ is a density point of the intersection $T \cap T'$.*

3.3.2. The natural extension of Lyubich measure.

In this paragraph, we discuss *the natural extension* of *the Lyubich measure* on the sphere to the inverse limit space. In [7], Lyubich showed that for an arbitrary rational map f of degree $d \geq 2$ on the sphere, there exists a unique f-invariant ergodic probability measure ν of maximal entropy supported on the Julia set J_f for which periodic points of f are asymptotically uniformly distributed. Note that in the case of polynomials, the measure of meximal enropy was first constructed by Brolin. We call such a measure ν *the Lyubich measure* for an arbitrary rational map for convenience.

Lemma 3.3. *There exists a measure $\tilde{\nu}$ on $\tilde{\mathbb{C}}'$, called the natural extension of the Lyubich Measure ν, which projects to ν via π and satisfies the following:*

(a) $\tilde{\nu}$ is F-invariant and ergodic on $\widetilde{\mathbb{C}}'$,

(b) fiber measures defined by the disintegration of $\tilde{\nu}$ and indexed by almost all z in $(J_f \backslash PC)$ are extended to the fibers over $z \in \mathbb{C}' \backslash PC$ by the natural measures μ_z that we defined in Section 2.2, and

(c) the regular set $\mathcal{R}$ is a $\tilde{\nu}$-full measure subset of the inverse limit space.

Proof. Suppose that ν is defined on the Borel algebra $\mathcal{B}$. By Kolmogorov theorem, the F-invariant natural extension $\tilde{\nu}$ of Lyubich measure ν to the inverse limit space, defined on the Borel algebra $\mathcal{B}_\infty$ which contains all sets of the form

$$\tilde{A} = \tilde{A}_{m,A} = \{\tilde{z} = (z_n) \in \tilde{\mathbb{C}}' : z_m \in A\},$$

where $m \geq 0$ and $A \in \mathcal{B}$, exists and projects to ν via π.

(a) can be derived from [2] by J. R. Brown (also see [4]).

(b) is true in every finite step of the disintegration procedure and the construction of μ_n for every $n \geq 0$; it can also pass to the limit, as $n \to \infty$, in the weak sense.

(c) Observe that Lyubich measure ν on $\mathbb{C}'$ has support on the Julia set J_f. So the $J_f \setminus PC$ has the full ν-measure of the sphere $\mathbb{C}'$. For every point $z \in J_f \setminus PC$, boxes in $\mathcal{R}$ cover μ_z almost all points of the fiber $\pi^{-1}(z)$. Therefore the union of boxes in $\mathcal{R}$ covers $\widetilde{\mathbb{C}}'$ $\tilde{\nu}$-a.e. by the disintegration theory. q.e.d.

3.3.3. Complex structures on leaves of $\mathcal{R}$.

We now discuss the nice Riemann surface structures on leaves of the regular set $\mathcal{R}$ of the inverse limit space. The following statements are from Lyubich and Minsky (see the proof in [8]).

Lemma 3.4 [8]. *(a) All leaves of $\mathcal{R}$ are simply connected except Herman rings.*

(b) If $d = deg(f) > 1$, then there are no compact leaves in $\mathcal{R}$. Therefore, leaves are parabolic or hyperbolic conformally isomorphic to the complex plane $\mathbb{C}$ or the disk D.

(c) If a leaf ℓ is isomorphic to the complex plane $\mathbb{C}$, then it is dense in the reduced inverse limit space $\widetilde{\mathbb{C}}'$.

3.3.4. Affine leaves of $\mathcal{R}$.

We show that, in this paragraph, almost all leaves of the regular set are conformally isomorphic to the complex plane. The following statement given in [8] is a very useful argument in the study of the dynamics of a rational map on the sphere (see also [7] and [1]).

Lemma 3.5 (The Shrinking Lemma). *Let $U \subset \mathbb{C}'$ be a simply connected domain which is not contained in a rotation domain of f, let k be a natural number, and let $\{U_{jn}\}$ be a family of all connected components of $f^{-n}(U)$ with the property that the degree of $f^n : U_{jn} \to U$ is at most k for all j and n. Then given any simply connected domain W compactly contained in U, $diam(W_{jn}) \to 0$, as $n \to \infty$ independently of j, where $W_{jn} = f^{-n}(W) \cap U_{jn}$ and diam denotes the spherical diameter.*

Proposition 3.3. *Relative to the natural extension $\tilde{\nu}$ of Lyubich measure, hence to the multiplicity fiber measures μ, almost all leaves of $\mathcal{R}$ are isomorphic to the complex plane $\mathbb{C}$.*

Proof. Given $z \in J_f \setminus PC$ in the sphere, the Lyubich measure $\nu(D(z)) > 0$ for any disk neighborhood $D(z)$ of z in $\mathbb{C}'$. Let $R_T = \mathcal{R} \cap \pi^{-1}(z) \setminus \tilde{S}$. Choose a density point $\tilde{z} = (z_n) \in R_T$ in the fiber so that $\tilde{z}$ is not in the rotation domains of F. By Proposition 3.2, we can find a maximal box $B(\tilde{z}, \epsilon) = T(\tilde{z}) \times D(\epsilon) \ni \tilde{z}$ over a two-disk neighborhood $D(\epsilon)$ of $z_n = \pi_n(\tilde{z})$ with $diam(D(\epsilon)) = \epsilon > 0$ in $\mathbb{C}'$ and $\mu(T(\tilde{z}) > 0$. We may assume that $n = 0$ and $z = \pi(\tilde{z})$. Then $\nu(D(z, \epsilon)) > 0$. By the disintegration theorem, we have $\tilde{\nu}(B(\tilde{z}, \epsilon)) > 0$. Denote the "half"-box of $B(\tilde{z}, \epsilon) = T(\tilde{z}) \times D(\frac{\epsilon}{2})$ over the base disk $D(\frac{\epsilon}{2})$ of diameter $\frac{\epsilon}{2}$. Then $B(\tilde{z}, \frac{\epsilon}{2})$ still has a positive $\tilde{\nu}$-measure. Notice that F is $\tilde{\nu}$-ergodic, so is its inverse F^{-1}. Hence the forward and backward orbit of $\tilde{w}$ visit the half box $B(\tilde{z}, \frac{\epsilon}{2})$ infinitely often for $\tilde{\nu}$-a.e. $\tilde{w} \in B(\tilde{z}, \epsilon)$. We can choose a density point $\tilde{w}$ in T, $\tilde{w}$ is not in a rotation domain of F, so that $F^{-n_j}(\tilde{w}) \in B(\tilde{z}, \frac{\epsilon}{2}), n_j \to \infty$, as $j \to \infty$. The regular leafwise neighborhood $\tilde{D}(F^{-n_j}(\tilde{w}), \frac{\epsilon}{2}) \subset B(\tilde{z}, \epsilon)$ can be viewed as the univalent lift of the two-disk neighborhood $D(w_{n_j}, \frac{\epsilon}{2})$ of w_{n_j} through the projection π. On the other hand, the Shrinking lemma tells us that the diameter of $\pi(F^{-n_j}(\tilde{D}(\tilde{w},\epsilon)))$ tends to zero as $n_j \to \infty$. Therefore $F^{-n_j}(\tilde{D}(\tilde{w},\epsilon)) \subset \tilde{D}(F^{-n_j}(\tilde{w}),\frac{\epsilon}{2})$ for sufficiently large n_j. The annulus $\tilde{D}(F^{-n_j}(\tilde{w}),\frac{\epsilon}{2}) \setminus F^{-n_j}(\tilde{D}(\tilde{w},\epsilon))$ is univalently mapped to an annulus on the sphere with a large modulus for sufficiently large n_j. In the leaf $\ell(\tilde{w})$ we obtain a sequence of increasing disks around the disk $\tilde{D}(\tilde{w},\frac{\epsilon}{2}) \ni \tilde{w}$, by mapping $\tilde{D}(F^{-n_j}(\tilde{w},\frac{\epsilon}{2}))$ into $\ell(\tilde{w})$ via F^{n_j}, so that the sum of modulii is ∞. Thus the global leaf $\ell(\tilde{w})$ through $\tilde{w}$ is conformally isomorphic to the complex plane $\mathbb{C}$ by a standard argument in complex analysis theory (see [5]). We have showed that for $\tilde{\nu}$-a.e. $\tilde{w}$ in $B(\tilde{z},\epsilon)$, the global leaf $\ell(\tilde{w})$ is isomorphic to $\mathbb{C}$. F is $\tilde{\nu}$-ergodic and maps leaf to leaf isomorphically, therefore $\tilde{\nu}$ almost all leaves of $\mathcal{R}$ are isomorphic to the complex plane $\mathbb{C}$. Since μ_z is the fiber measure indexed by $\pi(\tilde{z}) = z$ via disintegration of $\tilde{\nu}$, almost all leaves are isomorphic to $\mathbb{C}$ relative to μ_z. The result then follows from the holonomy invariance of the multiplicity fiber measures. q.e.d.

Remark. If a leaf ℓ of $\mathcal{R}$ is isomorphic to the complex plane $\mathbb{C}$, then it possesses an affine structure induced by the isomorphism of $\ell \to \mathbb{C}$. The homeomorphism F maps a leaf $\ell(\tilde{z})$ through point $\tilde{z} \in \mathcal{R}$ to the leaf $\ell(F(\tilde{z}))$ through $F(\tilde{z})$ bi-holomorphically, thus if $\ell(\tilde{z})$ is isomorphic to $\mathbb{C}$, so is $\ell(F(\tilde{z}))$. Therefore, F becomes affine between them.

Let $\mathcal{A}$ be the union of all affine leaves of $\mathcal{R}$ and call it the *affine part* of $\mathcal{R}$. The above remark implies that $\mathcal{A}$ is F-invariant.

Corollary 3.4. *Let $f : \mathbb{C}' \to \mathbb{C}'$ be a rational map of degree $d \geq 2$. Then the affine part $\mathcal{A}$ of the regular set in the inverse limit space possesses the following properties:*

1. *$\mathcal{A}$ has full multiplicity measure in every fiber of π in the inverse limit space $\tilde{\mathbb{C}}'$.*

2. *The selfmapping F on $\mathcal{A}$ is affine from leaf to leaf.*

$\mathcal{A}$ also has a nice ergodic property relative to the multiplicity fiber measures.

Proposition 3.4. *The affine part $\mathcal{A}$ of $\mathcal{R}$ is μ-ergodic in the sense that for any two positive measure subsets T and T' of box transversals of the affine part, almost all points in one are connected by leaves of $\mathcal{A}$ to the other.*

Proof. It suffices to show that for a point $z \in \mathbb{C}' \setminus PC$ and any Borel set T , $T' \subset \pi^{-1}(z) \cap \mathcal{A}$ with positive multiplicity measure, leaves through T intersect T'.

Take a density point $\tilde{z} \in T$ and a density point $\tilde{z'} \in T'$. Let

$$N(\tilde{z}, k) = \{\tilde{u} = (u_n) : u_k = z_k \text{ for some } k > 0\}$$

be a block set containing $\tilde{z}$ in the fiber so that

$$\mu_z(N(\tilde{z}, k) \cap T) \geq 0.99\mu_z(N(\tilde{z}, k)),$$

and this inequality holds for all large k. Let

$$N'(\tilde{z'}, j) = \{\tilde{v} = (v_n) : v_j = z'_j \text{ for some } j > 0\}$$

be a block set so that

$$\mu_z(N(\tilde{z'}, j) \cap T') \geq 0.99\mu_z(N(\tilde{z'}, j)),$$

and this inequality holds for all large j.

Choose smallest k and j which make these two inequalities true and let $m = max(k, j)$. (That is, the above two inequalities true for all $j, k \geq m$). Notice that the leaf $\ell(\tilde{z})$ through $\tilde{z}$ is dense in $\tilde{\mathbb{C}}'$ for the usual topology, it should go through a point $\tilde{w}$ in the block neighborhood $N(\tilde{z'}, 3m)$ of $\tilde{z'}$. Any holonomy map h which takes $\tilde{z}$ to $\tilde{w}$ is an isometry with respect to the metric $\rho(\tilde{p}, \tilde{q}) = \frac{1}{d^n}$ in the fiber by Lemma 2.2. Thus $h(N(\tilde{z}, 2m))$ is a block neighborhood of $\tilde{z'}$ contained in $N(\tilde{z'}, m)$, which implies that

$$\mu_z(h(N(\tilde{z}, 2m)) \cap T') \geq 0.99\mu_z(h(N(\tilde{z}, 2m))).$$

The multiplicity fiber measure μ_z is h-invariant, so

$$\begin{aligned}\mu_z(h(N(\tilde{z}, 2m)) \cap h(T)) &= \mu_z(N(\tilde{z}, 2m) \cap T) \\ &\geq 0.99\mu_z(N(\tilde{z}, 2m)) = 0.99\mu_z(h(N(\tilde{z}, 2m))).\end{aligned}$$

The last two inequalities implies that $h(T) \cap T' \cap h(N(\tilde{z}, 2m)) \neq \emptyset$.

q.e.d.

4. The solenoidal Riemann surface $\mathcal{L}$

In this section we show that for an arbitrary rational map f from the sphere $\bar{\mathbb{C}}$ on to itself, there exists a solenoidal Riemann surface $\mathcal{L}$ continuously injected into the affine part of the inverse limit space $\widetilde{\mathbb{C}'} = \varprojlim(\mathbb{C}', f)$.

4.1. The main result.

Theorem 4.1. *For an arbitrary rational map f of degree $d \geq 2$ on the sphere $\bar{\mathbb{C}}$, there exists an ergodic solenoidal Riemann surface $\mathcal{L}$ whose leaves are isomorphic to the complex plane $\mathbb{C}$, a holomorphic bijection F on $\mathcal{L}$, and a holomorphic map $\pi : \mathcal{L} \to \bar{\mathbb{C}}$ so that $f^n \cdot \pi = \pi \cdot F^n$ for every positive integer n. Moreover, the induced map of $\mathcal{L}$ into the inverse limit space $\widetilde{\mathbb{C}'} = \varprojlim(\mathbb{C}', f)$ is a continuous injection with image intersecting every fiber in full measure for the multiplicity fiber measure class.*

To prove this theorem, we need to introduce a new topology on the regular set $\mathcal{R}$ of the inverse limit space $\widetilde{\mathbb{C}'}$.

Definition 4.1. Let $\pi^{-1}(z)$ be a fiber in $\widetilde{\mathbb{C}'}$ over a point $z \in \mathbb{C}'$. Let $\mathcal{B}_z$ be the Borel algebra generated by block subsets on the fiber. A Borel subset $\tilde{E}$ is said to be density open if the multiplicity measure of $\tilde{E}$ is positive and every point in $\tilde{E}$ is a density point of $\tilde{E}$. This defines a finer topology on the fiber than the usual one (the usual topology is generated by the family of all blocks as an open basis).

A box $B = T \times D$ in the regular set $\mathcal{R}$ is said to be fine-open if its transversal T is density open in the fiber containing T. The topology on $\mathcal{R}$ generated by all fine-open boxes as basis is finer than the usual topology induced from $\widetilde{\mathbb{C}'}$. Call this new topology the fine-topology on $\mathcal{R}$.

Remark. Suppose $B = T \times D$ is a box in $\mathcal{R}$ which has positive transversal measure. Let T^o be the subset of all density point in T. Then T^o is density open in the fiber containing T and $\mu(T^o) = \mu(T)$. Thus $B^o = T^o \times D$ is a fine-open sub-box of B and it has the same transverse measure.

Proof of Theorem 4.1. In Section 3, we have constructed a countable family $\Gamma = \{B_m = T_m \times D_m\}_{m=1}^{\infty}$ of maximal boxes whose union covers the regular set $\mathcal{R}$, and B_m are chosen so that all but finitely many of them have positive multiplicity fiber measures. For every such maximal box $B_m = T_m \times D_m$, the intersection of B with the affine part $\mathcal{A}$ is a box in $\mathcal{A}$ of the same μ-measure as B_m. Denote again by B_m such a box in $\mathcal{A}$ to save notation, that is, we have a countable family Γ of maximal boxes contained in the affine part $\mathcal{A}$.

For each maximal box $B_m = T \times D_m \in \Gamma$, take the fine-open interior

$B_m^o = T_m^o \times D_m$. Since every global leaf of the regular set intersects each fiber in a countable subset, the subset $T_m \setminus T_m^o$ of density-boundary points has a zero μ-measure in T_m. By deleting all global leaves that pass through points in $T_m \setminus T_m^o$, we obtain a full μ-measure fine-open sub-box of B_m. Performing the same process on every box in the family Γ. Let $B'_m = T'_m \times D_m$ be the fine-open box by deleting all global leaves that pass through the density-boundary points in $T_k \setminus T_k^o$ of $B_k \in \Gamma$, for all $k = 1, 2, ...$ Then B'_m is fine-open in $\mathcal{A}$, and has the same transversal measure as B_m in the fiber containing T_m. Define

$$\mathcal{L} = \bigcup_{m=1}^{\infty} B'_m.$$

Then $\mathcal{L}$ carries the fine-topology generated by fine-open boxes of $\mathcal{R}$ contained in $\mathcal{L}$. $\mathcal{L}$ also possesses the following properties:

- The inclusion $I : \mathcal{L} \to \widetilde{\mathbb{C}}'$ is continuous.
- $\mathcal{L}$ intersects every fiber in a full μ-measure subset since every B'_m has the same μ-measure as B_m in the transversal and the regular set $\mathcal{R}$ intersects every typical fiber in full μ-measure subset in the reduced inverse limit space.
- All global leaves in $\mathcal{L}$ are affine.
- From the construction of $\mathcal{L}$, the complex and the affine structures in leaf direction are continuous relative to the fine-topology.
- $\mathcal{L}$ is F-invariant and F becomes affine in the leaf direction, since the multiplicity fiber measures μ are F quasi-invariant. F maps a density boundary point $\tilde{z}$ in the transversal T of a maximal box B to the density-boundary point in the transversal $F(T)$ of the box $F(B)$ and also maps an interior point to an interior point in transversals.
- $\mathcal{L}$ is ergodic relative to the multiplicity fiber measures.

We complete the proof of the theorem. q.e.d.

Remark. Up to Borel isomorphisms (mod 0) for the multiplicity fiber measure class, the solenoid $\mathcal{L}$ constructed in the proof of Theorem 4.1 is independent of the choice of the countable basis of two-disks of the sphere.

Proposition 4.1. *The triple $(\mathcal{L}, F, \pi)$ is well-defined by the properties of Theorem 4.1 up to measure theoretic isomorphisms on full measure subsets of leaves which are biholomorphic in the leaf direction and measure preserving in the transversal direction.*

Proof. Suppose that there exists another triple $(\mathcal{L}', \pi', F')$ satisfying the properties stated in the main Theorem 4.1. Then there is a continuos injection $J : \mathcal{L}' \to \tilde{\mathbb{C}}'$ so that the F-invariant ergodic image $L'' = J(\mathcal{L}')$ has full μ-measure in any given fiber $\pi^{-1}(z)$ in $\tilde{\mathbb{C}}'$. Take the intersection $Y = L'' \cap \pi^{-1}(z) \cap \mathcal{L}$. By ergodicity, $\mu_z(Y) = 1$. Delete all global leaves of $\mathcal{L}$ and L'' that pass through the null set of points outside Y as well as their forward and backward F-images. We are left with an F-invariant affine solenoid $\mathcal{S}$ which still has full multiplicity measure on every fiber in $\tilde{\mathbb{C}}'$. Thus $I^{-1}(\mathcal{S}) \subset \mathcal{L}$ is F-invariant and $J^{-1}(\mathcal{S}) \subset \mathcal{L}'$ is F'-invariant. Moreover, the composite map $\Phi = J^{-1} \circ I : I^{-1}(\mathcal{S}) \to J^{-1}(\mathcal{S})$ is a homeomorphism which is bi-holomorphic in the leaf direction and measure preserving Borel isomorphic in the transversal direction for the multiplicity measure class. q.e.d.

Remark. The object $\mathcal{L}$ may not be canonical as a topological lamination. We may lose the compactness, even the local compactness in the fiber for the fine-topology. On the other hand we do have, from the point of view of measure theory, the ergodicity of the solenoid.

4.2. Measured solenoids.

We recall the definition of a topological solenoid given by Sullivan in [14]. *A topological solenoid*, or simply a solenoid, $\mathcal{S}$ is a topological space with local box charts homeomorphic to the product of the form (a totally disconnected space T) cross (a k-ball) so that overlap maps preserve the k-ball factor.

Subsets $T \times \{z\}$ of a box chart $B = T \times D$ are called *transversals* and those $\{t\} \times D$ of B are called *local leaves* in B. A solenoid $\mathcal{S}$ is naturally foliated or laminated by its path connected compoments which are called *global leaves* obtained by gluing local leaves together via overlap homeomorphisms. We say a subsets $T \subset \mathcal{S}$ a transversal if T is a union of transversals of local box charts and every global leaf intersects T locally at a unique point.

Let T and T' be two transversals of a solenoid $\mathcal{S}$. A point $x \in T$ can be connected by a path γ in global leaf to a point $x' \in T'$. Then there are finitely many local box charts whose union covers the compact path γ. One can check that there is a deck of global leaves, going along with the curve γ, connecting all points in an open neighborhood of x in T to an open neighborhood of x' in T'. This defines a homeomorphism h partially from the transversal T to another T' by transporting points along global leaves. Such an h is known as a holonomy map.

A family $\{\mu\}$ of finite measures supported on transversals of a solenoid $\mathcal{S}$ is said to be a tranverse measure class if one is a positive scalar multiple of the other restricted on subsets of the same transversal. A transverse measure class μ is said to be holonomy invariant if all holonomy maps preserve μ. See [14] for examples of solenoids.

In this subsection, we give a general definition of a measured solenoid. Let us begin by recalling some foundamentals of measure theory (see [11] and [9] for details). A Borel space is a set X together with a Borel algebra $\mathcal{B}$ of subsets of X such that $\mathcal{B}$ is closed under intersection, complement, and countable union. A Borel space $(X, \mathcal{B})$ is said to be *standard* if it is Borel isomorphic to a Borel subset of a complete separable metric space. A standard Borel space is countably generated, consequently, countably separated.

Definition 4.2. Let $L = (X, \mathcal{B}, \Gamma)$ be a standard Borel space with a countable separating and generating family Γ of Borel subsets of X. L is said to be a measured solenoid if it possesses the following properties:

(α) Every element B in Γ is a box Borel isomorphic to the product of

$$\text{(a Borel transversal } T\text{) cross (a } k\text{-ball } D \text{ in } \mathbf{R}^k\text{)},$$

where T is measurable in a Borel subspace $Y \subset X$ equipped with a σ-finite measure μ_Y and $\mu_Y(T) > 0$.

(β) [The Vertical Boundary Covering Property] For every box $B = T \times D$ in Γ, the vertical boundary $\partial_T B = T \times \partial D$ of B is a Borel subset of X and can be covered by other boxes in Γ.

(γ) [The Intersection Property] If $B_n = T_n \times D_n$, $B_m = T_m \times D_m \in \Gamma$, and $B_n \cap B_m \neq \emptyset$, then for every point $p \in B_n \cap B_m$, there exists a box $B = T \times D \in \mathcal{B}$ containing p such that the open ball $D \subset D_n \cap D_n$, $T \subset (T_n \times \{p\}) \cap (T_m \times \{p\})$, and $\mu_n(T) = \mu_m(T) > 0$, where μ_n and μ_m are the transverse measures on T_n and T_m respectively.

(δ) Overlap maps on intersections of boxes preserve the k-ball factor and they are homeomorphisms in the k-ball direction and quasi-invariant Borel isomorphisms in the transverse direction.

A measured solenoid is said to be a measured solenoidal Riemann surface if $k = 2$, and the transition maps are bi-holomorphic in the two-disk direction.

Here are some useful facts derived directly from this definition:

- The vertical boundary covering property enables us to extend local leaves, the isomorphic image of the k-ball in a box, to maximal manifolds, the so called global leaves.
- Holonomy maps can be partially defined from one transversal to another by applying the intersection property. It is clear that the transversal measures are holonomy invariant,. i.e., holonomy maps are Borel isomorphisms.

As an example of a measured solenoid, one can check the following statement.

Proposition 4.2. *A topological solenoid $\mathcal{S}$ with a standard Borel structure generated by its topology yields a measured solenoid if there is a holonomy invariant finite transverse measure class on the induced transverse Borel algebras such that open transversals of box charts have positive measures.*

The converse is also true in the following sense. Given a measured solenoid $L = (X, \mathcal{B}, \Gamma)$, there is a topological solenoid $\mathcal{S}$ continuously injected into L whose image has full measure of every transversal.

That is, a measured solenooid $L = (X, \mathcal{B}, \Gamma)$ is Borel isomorphic to a topological solenoid $\mathcal{S}$ mod zero for the transverse measure class and the Borel isomorphism is homeormorphic in the leaf direction and measure preserving in the transverse direction.

The proof of this proposition is left to the reader.

Remark. One can take the first part of the above proposition as a definition of a measured solenoid. The point of the general definition is the transverse measure theory plus a certain countability instead of the topology.

Corollary 4.1. *Up to Borel isomorphism mod zero for the multiplicity fiber measure class, $f \mapsto \mathcal{L}$ determines a well-defined measured* *solenoidal Riemann surface $\mathcal{L}$ with dynamics F for every rational map f.*

Proof. The regular set $\mathcal{R}$, hence the solenoid $\mathcal{L}$, is a standard Borel space with the Borel algebra generated by the fine-open boxes which coincides with the one generated by the usual topology. The countable generating family of boxes is the family of the countable maximal fine-open boxes partitioned by block sets of the Cantor set fibers. The transverse measure class is composed of all normalized multiplicity fiber measures. The rest follows from Proposition 4.1. q.e.d.

4.3. General discussion.

The discussion on the inverse limit space of a rational map of the sphere given in Sections 2.2, 3.3, and 4.1 can be generalized as follows:

Consider an inverse system

$$\cdots \xrightarrow{f_{n+1}} X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0$$

of branched coverings $f_n : X_n \to X_{n-1}$ of compact Riemann surfaces X_n to X_{n-1} of degree $d_n \geq 2$, for $n \geq 1$. Assume that X_n has the pull-back complex structure of X_{n-1} via f_n so that f_n is holomorphic. Let

$$\begin{aligned} C_n = \{x \in X_0 : x = (\Pi_{j=1}^n f_j)(b_n), b_n &\text{ is a branched point of the} \\ \text{composite map } \Pi_{j=1}^n f_j = f_1 \circ f_2 \circ \cdots \circ f_n\}, &\qquad n \geq 1, \end{aligned}$$

let $PC = \bigcup_{n=1}^{\infty} C_n$ denote *the post critical set* of the system

$$\{X_n \xrightarrow{f_n} X_{n-1}\}_{n=1}^{\infty},$$

and let $M_{n,k}$ denote the number of branched points, counted by multiplicities of local degrees at branched points, of the composite map

$$\Pi_{j=n}^{n+k} f_j = f_n \circ f_{n+1} \circ \cdots \circ f_{n+k} : X_{n+k} \to X_n, \quad \text{for } n \geq 1, k \geq 0.$$

Form the inverse limit space

$$X_\infty = \varprojlim(\cdots \xrightarrow{f_{n+1}} X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_3} X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0)$$

consisting of backward strings $x_\infty = (\ldots, x_n, x_{n-1}, \ldots, x_1, x_0)$ so that $f_n(x_n) = x_{n-1}, x_n \in X_n$. X_∞ is a compact Hausdorff subspace of the countable Cartesian product $\Pi_{n=0}^{\infty} X_n$ with respect to the usual induced topology. Define the projections $\pi_n : X_\infty \to X_n$ by $\pi_n(x_\infty) = x_n$.

We say that a point $x_\infty = (x_n) \in X_\infty$ is *eventually pinching* if $x_n = \pi_n(x_\infty) \in X_n$ are branched points of f_n for infinitely many n's. Denote by P_∞ the set of all eventually pinching points in X_∞. Assume that for every $x_\infty = (x_n) \in P_\infty$, $\Pi_{j=1}^{n} \frac{d_j(x_j)}{d_j} \to 0, as\, n \to \infty$, where $d_j(x_j)$ is the local degree of $f_j : X_j \to X_{j-1}$ at point $x_j \in X_j$. Then each fiber $\pi_n^{-1}(x_n)$ over every $x_n \in X_n$ is a Cantor set equipped with a normalized non-atomic multiplicity fiber measure μ_{x_n} which measures the multiplicities of points on the fiber. All such measures form the multiplicity fiber measure class $\mu = \{\nu_{x_n} : x \in X_n, n = 0, 1, \dots\}$ constructed as the one given in Section 2.2. Holonomy maps can be defined in the same way as in Section 2.1 representing the fundamental sets of homotopy classes of curves in $X_0 \setminus PC$. It is clear that the multiplicity fiber measure class μ is holonomy invariant.

The following result is a generalization of Theorem 4.1.

Theorem 4.2. *Suppose*

(i) $\Pi_{j=1}^{n} \frac{d_j(x_j)}{d_j} \to 0,$ *as* $n \to \infty$, *for* $x_\infty = (x_n) \in P_\infty$ *and*

(ii) the ratio $\frac{M_{n,k}}{\Pi_{j=1}^{n+k} d_j} \to 0,$ *as* $n \to \infty$ *and* $k \to \infty$.

Then there exists a measured solenoidal Riemann surface $\mathcal{S}$ *associated to the inverse system*

$$\cdots \xrightarrow{f_{n+1}} X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0$$

so that the inclusion of $\mathcal{S}$ *into the inverse limit space* X_∞ *is a continuous injection whose image has full multiplicity measure in every fiber in* X_∞ *and every projection* $\pi_n : \mathcal{S} \to X_n$ *becomes holomorphic relative to the complex structures on leaves.*

Proof. The process of construction of $\mathcal{S}$ is completely analogous to that of $\mathcal{L}$ for a rational map system on the sphere. Here we sketch the steps.

Step 1. Construction of the natural multiplicity fiber measure class $\mu = \{\mu_{x_n} : x_n \in X_n, n = 0, 1, \dots\}$ given in Section 2.2 valdis because of the assumption (i). μ is also holonomy invariant as stated in Proposition 2.1.

Step 2. Definitions of boxes and the regular set $\mathcal{R}$ are the same as given in Definition 3.1 by replacing $\mathbb{C}'$ by X_n.

(a) Construction of a maximal box around the fiber $\pi^{-1}(x_0)$ for $x_0 \in X_0 \setminus PC$ given in Proposition 3.1 is still valid by replacing $\mathbb{C}' \setminus \bigcup_{j=1}^{N} f^j C$ by $X_0 \setminus C_N$ and $\frac{d^N - 2d(d^{n-N}-1)}{d^n}$ by $\frac{M_{n,N}}{\Pi_{j=N}^{N+n} d_j}$. Consequently, $\mathcal{R}$ intersects every fiber in full μ-measure.

(b) The regular set $\mathcal{R}$ can be covered by a countable family $\mathcal{F}$ of maximal boxes. Choose a countable basis $\mathcal{T} = \{D_{mn}\}_{m=1}^{\infty}$ of two-disk charts for every X_n in the system. Over every D_{mn} take the maximal box B_{mn} in X_∞ via the projection π_n for every $n \geq 0$. Then the collection of all such maximal boxes forms a cover of $\mathcal{R}$. Notice that there may be infinitely many B_{mn}'s whose transversals have zero μ-mearsure.

(c) Global leaves of the regular set $\mathcal{R}$ are Riemann surfaces on which local regular leaves play a role of local charts. Overlap maps are compositions of finitely many f_n's which are biholomorphisms.

Step 3. Take the same definition of the fine-topology as given in Definition 4.1 on the regular set $\mathcal{R}$. By deleting the global leaves of $\mathcal{R}$ passing through μ-density boundary points in the transversal T_{mn} of B_{mn} for every B_{nm} in the countable family $\mathcal{F}$ of maximal boxes obtained in (b), we are left with the measured solenoidal Riemann surface $\mathcal{S}$ satisfying the properties stated in the theorem. q.e.d.

Remark. In this general setting, there is no induced dynamics on the measured solenoid $\mathcal{S}$.

References

[1] A. F. Beardon, *Iteration of rational functions,* Springer, Berlin, 1991.

[2] J. R. Brown, *Inverse limits, entropy, and weak isomorphisms for discrete dynamical systems,* Trans. Amer. Math. Soc. **164** (1972) 55-65.

[3] I. Cornfeld, S. Formin & Ya. G. Sinai, *Ergodic theory*, Spriger, Berlin, 1980.

[4] J. R. Choksi, *Inverse limits of measure spaces,* Proc. London Math. Soc. **(3) 8** (1958) 321-342.

[5] J. E. Fornæss & N. Sibony, *Complex dynamics in higher dimensions,* Lecture notes, Univ. of Paris at Orsay, France.

[6] J. H. Hubbard & R. Oberste-Vorth, *Hénon mappings in the complex domain I,* Inst. Hautes Études Sci. Publ. Math. (1992).

[7] M. Yu. Lyubich, *Entropy property of rational endmorphisms of the Riemann sphere,* Ergodic Theory Dynamical Systems **3** (1983) 351-385.

[8] M. Yu. Lyubich & Y. Minsky, *Laminations in holomorphic dynamics,* J. Differential Geom. **47** (1997) 17-94.

[9] G. W. Mackey, *Borel structure in groups and their duals,* Trans. Amer. Math. Soc. **85** (1957) 134-165.

[10] R. Mañé, *Ergodic theory and differentiable dynamics,* Spriger, Berlin, 1983.

[11] V. Rohlin, *On the Fundamental Ideas of Measure Theory,* Trans. Amer. Math. Soc. **10** (1962) 1-54.

[12] G. E. Shilov & B. L. Gurevich, *Integral, measure & derivative: A unified approach,* Translated by R. A. Silverman, Dover Publ., New York, 1977, 209-215.

[13] D. Sullivan, *Bounds, quadratic Differentials, and Renormalization Conjectures,* Math. into the Twenty-first Century, Vol. 2, Amer. Math. Soc., 1991.

[14] ______, *Linking the universalities of Milnor-Thurston, Feigenbaum and Ahlfors-Bers: topological methods in modern mathematics,* Publish or Perish, Boston, 1993.

THE GRADUATE SCHOOL & UNIVERSITY CENTER
CITY UNIVERSITY OF NEW YORK