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COMPLETE EMBEDDED MINIMAL SURFACES OF FINITE TOTAL CURVATURE

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Abstract

A general construction, for complete embedded minimal surfaces of finite total curvature in Euclidean three-space, is carried out. In particular, examples with an arbitrary number of ends are given for the first time. The construction amounts to desingularizing the circles of intersection of a collection of coaxial catenoids and planes. The desingularization process uses Scherk's singly periodic surfaces for an approximate construction which is subsequently corrected by singular perturbation methods.

1. Introduction

Historical background.

Among all minimal surfaces, those which are complete, embedded in E^3, and of finite total curvature, form a very restricted class which has fascinated many geometers. It is remarkable that besides the classical examples of the catenoid and the plane no other examples were known until the early eighties. At that time Costa [2], [3] discovered a new complete minimal surface of finite total curvature, which was proved to be embedded [8]. This sparked a great deal of activity in this subject, and some more new examples were first found by Hoffman and Meeks [9], [10], and later by others [30], [31] [18] [12] [7]. We refer the reader to the excellent survey article of Hoffman and Karcher [6] where a detailed account is given and many more references can be found.

This paper is motivated by a systematic study of a sequence of such surfaces of increasing genus by Hoffman and Meeks [11]. They proved that the sequence in consideration tends to the union of a catenoid and

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a plane, the plane intersecting the catenoid through its waist. They proved also that if the surfaces are appropriately scaled and positioned, then the areas of high curvature tend to a classical singly periodic minimal surface, Scherk's fifth surface [26]. This motivated the following question: Is it possible in general to desingularize the intersections of intersecting minimal surfaces by replacing neighborhoods of the intersections with Scherk's singly periodic surfaces?

Up to now all complete embedded minimal surfaces of finite total curvature have been found by using a global version [22], [23] of the Enneper-Weierstrass representation. The nature of this method requires that the Riemann surface structure of the minimal surface to be found has to be specified first, then the complex theoretic data be guessed, and finally analyzed to check period closing and embeddedness. It is only due to remarkable effort and persistence that this somewhat implicit method has given a satisfactory geometric understanding for some families of such surfaces. The class of examples found this way is also quite limited because each new example has to be studied anew. Even in simple cases, like examples desingularizing two intersecting catenoids, there have been no (existence) results up to now.

In this paper we present a construction which answers the question above in cases of high symmetry. This way we obtain a geometrically clear general construction for complete embedded minimal surfaces of finite total curvature. This is the only construction up to now which gives such surfaces with arbitrarily many—at least three—ends. Although the genus of the surfaces can not be arbitrarily prescribed because of the high symmetry, it takes arbitrarily high values, essentially all high values compatible with the symmetry. The main drawback of the method is that due to its nature it has nothing to say about examples of low genus.

We remark that with further work we have proved a more general desingularization theorem, where the intersecting minimal surfaces lie in a general three-dimensional Riemannian manifold, are required to satisfy certain very general hypotheses, and no symmetry is assumed [16]. Another general construction for complete minimal surfaces of finite total curvature, with proofs based on similar ideas to the ones presented here, allows one to take connected sums of complete minimal surfaces by using catenoids as bridges [32]. In this last construction though, embeddedness has to be sacrificed.

Before we proceed with a more detailed discussion we mention that Traizet working independently [29] has found new examples of singly periodic embedded minimal surfaces in E^3, by a construction similar in spirit, but different in the general strategy—it follows [13] rather than [14]—to ours. Also that properties of the spaces of embedded minimal surfaces of finite total curvature have been studied before in the abstract [25] [24].

The main results.

The main result is as follows: Consider the space $\mathcal{M}$ whose points are arrangements of $N_{\mathcal{K}}$ coaxial catenoids and $N_{\mathcal{P}}$ planes perpendicular to the common axis (figure 1); clearly $\mathcal{M}$ can be given the structure of a manifold of dimension $2N_{\mathcal{K}}+N_{\mathcal{P}}$. Then for each (large) $m \in \mathbb{N}$ there are open subsets $\mathcal{M}_m$ of $\mathcal{M}$ such that $\bigcup \mathcal{M}_m$ is dense in $\mathcal{M}$, and for each $\underline{x} \in \mathcal{M}_m$ we can desingularize the corresponding arrangement of catenoids and planes to obtain a complete, embedded if $N_{\mathcal{P}} \leq 1$, minimal surface M of finite total curvature, which relates to the catenoids and planes of $\underline{x}$ as follows:

Each circle of intersection $\mathcal{C}_k$ in the arrangement of the catenoids and planes has been replaced by a suitably bent, slightly deformed, and appropriately scaled down Scherk singly periodic surface where $2mm_k$ fundamental domains (that is mm_k handles) are used along the full length of the circle $\mathcal{C}_k$. The m_k's are small constants specified beforehand and for the simplest examples can all be taken $m_k = 1$. The removal of the $\mathcal{C}_k$'s decomposes the catenoids and planes to discs, annuli, and ends, which after being slightly perturbed, become parts of M, attached to the wings of the Scherk pieces replacing the $\mathcal{C}_k$'s. The whole surface M is symmetric under reflections with respect to a set of planes through the axis, and where subsequent planes form an angle π/m. The genus of M is clearly $m \sum m_k + C(\underline{x})$.

We discuss now to some more detail the change which the ends suffer. Notice that the slight perturbation of the ends, which we mentioned, makes them asymptotic at infinity to catenoids or planes, which may differ slightly from the original ones in size and position. In particular planar ends may become asymptotic to small catenoids. In order to have well defined parameters for the minimal surfaces we construct, we can and do arrange, that the top ends of each catenoid in $\underline{x}$ do stay asymptotic to the original end. For the planar ends we can not do the same, but we arrange at least that the catenoid to which they become asymptotic, has its waist on the original plane. Hence the number of free continuous parameters before we subtract homotheties and translations is $2N_{\mathcal{K}} + N_{\mathcal{P}}$, that is the total number of ends.

The strategy of the construction and proof.

The theorem proved in this paper is an application of the gluing techniques developed by Schoen [28] and Kapouleas [13] - [15], in the form they evolved to in [14]. Actually the general philosophy of our approach here is almost identical to the one in [14], and we proceed to outline it now in a way which points out the similarities and some differences. In both cases the surfaces we wish to construct can be decomposed into "standard" pieces and pieces joining them. In the Wente fusion case as standard pieces we consider the spherical and Enneper-like negatively curved regions of the Wente tori, while the joining pieces are rather complicated and in the appropriate intrinsic metric resemble long cylinders growing out at regular intervals from another cylinder. In the current construction the standard pieces are the Scherk surfaces, and the discs, annuli, and ends, of the planes and catenoids to be desingularized. The joining pieces are the wings of the Scherk surfaces, which modulo the symmetries become very long—compared to the length of the meridian—cylinders. In both cases we have two different scales for the standard pieces: The spheres in the Wente fusion and the discs, annuli, and ends, in the current construction have size and curvature of order 1, while the other standard pieces in both cases have small size of order (by definition) τ and curvature of order τ^{-2}.

The construction requires to construct first a surface which approximately satisfies the desired geometric condition, and then find a graph over it which satisfies the condition exactly, by solving the relevant partial differential equation for the function whose graph we are considering. This function is to be found by singular perturbation methods by solving the linearized equation and then correcting for the hopefully much smaller nonlinear terms. It turns out that the initial mistake is of order τ, and the method works when τ is small enough. The success of the method rests on our ability to solve the linearized equation and produce solutions of the same size as the inhomogeneous term.

Our approach requires that we solve the linearized equation separately on each region consisting of a standard piece joined with those joining pieces with which it is in contact, and then patch up the solutions. The patching up introduces some error which should be small so that it can be corrected by iteration. At this point we are faced with two difficulties: First, the linearized equation on the standard pieces has kernel, which persists in the regions mentioned, in the form of small eigenvalues. Second, for the patching up to give a small error, we need decay of the solutions along the joining pieces, so that the solutions are small close to the boundary of these regions. We can temporarily bypass both problems by solving modulo functions w and $\overline{w}$ which we use as follows: By adding w's we make the right-hand side orthogonal to the eigenfunctions of small eigenvalue. By adding $\overline{w}$'s we can appropriately change the solution so that we control the low harmonics on the cross section of the joining cylinders at the end close to the standard piece. This forces the required fast decay along the joining piece.

It remains to correct in the direction of the w's and $\overline{w}$'s. The philosophy for that, which we have called the "geometric principle" in [14] - [15], is that this can be achieved by introducing dislocations, that is repositioning the pieces of the surface relative to each other. In the current case the parameters which will control these dislocations are the $\theta_{i,j}$'s and the $\varphi_{i,j}$'s which change the angles between the wings of a Scherk surface and the angles between the wings and the Scherk surface respectively. In [14] we used relative translations and rotations of various suitable pieces of the surfaces.

We point out finally some differences between the two constructions. In the current construction we do not have ready to use joining pieces and this makes the construction of the initial surfaces subtler. On the other hand, the joining pieces in the current case are much simpler, and so are the decays we need along them. The last difference concerns the small eigenvalues on the standard pieces. While in [14] the relevant standard pieces were effectively disjoint (round) spheres, here we have Scherk surfaces, which (see Section 2) can still be thought of as made of spherical pieces, but which do not disconnect as $\tau \to 0$.

Outline of the paper.

The paper has three parts. The first part deals with the desingularizing surfaces used to replace the circles of intersection and consists of Sections 2, 3, and 4. In Section 2 we study carefully the Scherk surfaces and understand the approximate kernel well enough for our purposes in this paper. In Section 3 we carefully construct the desingularizing surfaces by appropriately deforming the Scherk surfaces. In Section 4 we define and study the functions w and $\overline{w}$ which we have discussed already, and we estimate the mean curvature of the desingularizing surfaces.

In the second part we construct the initial surfaces. It consists of

Section 5, where we study the configurations of the minimal surfaces to be desingularized, and Section 6, where the initial surfaces are actually constructed, and their aspects not known already, studied.

In the third part consisting of Sections 7 and 8 we solve the partial differential equation to produce the desired minimal surfaces. In Section 7 we study the linear theory and in Section 8 the rest.

Finally the paper contains two appendices. In appendix A we study the existence, uniqueness, and decay of solutions, to appropriate linear equations along long cylinders, under conditions which occur repeatedly in this paper. In appendix B we state some standard local facts for the mean curvature of surfaces and graphs in E^3 to facilitate reference to them.

Notation and conventions.

We discuss now some notation and conventions we use throughout this paper. First of all because we have many cut and paste situations it is useful to define functions $\psi[a,b]:\mathbb{R}\to[0,1]$ by

$$\psi[a,b](s)=\psi((s-a)/(b-a)), \tag{1.1}$$

where ψ is a fixed cutoff function which is smooth, increasing, vanishes on $(-\infty,1/3)$ and $\psi\equiv 1$ on $(2/3,\infty)$. Notice that $\psi[a,b]$ transits from 0 at a to 1 at b.

We often have a function s defined on the surfaces which we define with values in $\mathbb{R}\cup\{\infty\}$. If V is subset of such a surface we use the notation

$$V_{\leq a}:=\{p\in V: s(p)\leq a\},\qquad V_{\geq a}:=\{p\in V: s(p)\geq a\}. \tag{1.2}$$

E^3 denotes the 3-dimensional Euclidean space equipped with the usual Euclidean metric. $\mathbb{S}^n$ denotes the standard round n-sphere of radius 1. ν, g, A, H, K denote respectively the oriented unit normal, induced metric, second fundamental form, mean curvature, and induced Gauss curvature of an immersed surface in E^3. The invariants of a surface S are occasionally distinguished by using S as a subscript, for example ν_S denotes the Gauss map of a surface S. If $F:S_1\to S_2$ is a differentiable map between manifolds, then we use $F_*(f)$ and $F^*(f)$ to denote the push-forward and pull-back respectively of tensor fields and functions f by F.

We very often consider in this paper graphs over surfaces. To facilitate reference to them we fix now some terminology: Suppose we have an immersed surface S in E^3, which is immersed by $X : S \to E^3$ and has Gauss map $\nu : S \to \mathbb{S}^2$. If we have a C^1 function $f : S \to \mathbb{R}$, and $X + f\nu$ is an immersion, we denote by S_f the immersed surface defined by $X + f\nu$, and we call it the graph of f over S. Moreover we use $X + f\nu$ and its inverse to define projections $\Pi : S \to S_f$ and $\Pi' : S_f \to S$ associated to the graph. When we refer to projections of S to S_f or S_f to S we mean always these projections. Finally, we often use these projections to push-forward or pull-back functions and tensor fields from S to S_f or from S_f to S.

We use C (sometimes with subscripts) to denote positive constants throughout the paper, each time a possibly different one. These constants are always assumed to depend only on parameters explicitly mentioned except for certain parameters according to conventions discussed later. Because we have many small constants in this paper (mainly δ's and ϵ's), we have developed the habit of distinguishing between them by using subscripts which serve as mnemonic devises to help recalling in what context the constant was defined.

Weighted Hölder norms are defined by

$$\left\|\phi\colon C^{k,\alpha}(\Omega,g,f)\right\| := \sup_{x\in\Omega} f^{-1}(x)\|\phi\colon C^{k,\alpha}(\Omega\cap B(x),g)\|, \tag{1.3}$$

where Ω is a domain, g is the metric with respect to which we take the $C^{k,\alpha}$ norm, f is a weight function, and $B(x)$ is the geodesic ball centered at x of radius 1. We often omit the domain or the metric when implied by the context, or the function f when $f \equiv 1$.

$|\,.\,|$ denotes unless otherwise stated the maximum norm of elements of finite vector spaces.

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2. The Scherk surfaces

As we have already mentioned in the introduction the surfaces we use to desingularize the intersections in the initial approximate construction are constructed by using Scherk's singly periodic surfaces. These form a one-parameter family of singly periodic embedded minimal surfaces. The most symmetric of them was discovered by Scherk in 1835 [26], who also discovered the doubly-periodic conjugates of all of them. They are referred to usually as Scherk's fifth surfaces, Scherk's singly periodic surfaces, or Scherk-towers [21] [11] [4] [17], but we will refer to them as Scherk surfaces $\Sigma(\theta)$ for simplicity, where $\theta \in (0, \pi/2)$ is the nontrivial parameter of the family. We assume that E^3 is equipped with a Cartesian coordinate system $O.xyz$; $\Sigma(\theta)$ is defined then by the equation [21]

$$\cos^2\theta \cosh \frac{x}{\cos\theta} - \sin^2\theta \cosh \frac{y}{\sin\theta} = \cos z. \tag{2.1}$$

Because these surfaces degenerate as $\theta \to 0, \pi/2$, and we do need uniform bounds on their geometry, we assume from now on that θ is restricted by

$$\theta \in [10\,\delta_\theta, \frac{\pi}{2} - 10\,\delta_\theta] \tag{2.2}$$

for some small $\delta_\theta > 0$ which will be determined later. Before we discuss further the Scherk surfaces it is convenient to develop some new notation in order to facilitate their description:

Notation 2.3. We use $\vec{e}_x$, $\vec{e}_y$, and $\vec{e}_z$ to denote the coordinate unit vectors of our Cartesian coordinate system $O.xyz$. $\vec{e}\,[\theta]$ and $\vec{e}\,'[\theta]$ stand for $\vec{e}_x$ and $\vec{e}_y$ rotated by an angle θ around the z-axis, that is

$$\vec{e}\,[\theta] = \cos\theta\; \vec{e}_x + \sin\theta\; \vec{e}_y, \qquad \vec{e}\,'[\theta] = -\sin\theta\; \vec{e}_x + \cos\theta\; \vec{e}_y.$$

To facilitate referring to the symmetries of $\Sigma(\theta)$ we denote the identity map by $\mathcal{R}_1$, the reflection with respect to the yz-plane by $\mathcal{R}_2$, the reflection with respect to the z-axis by $\mathcal{R}_3$, and the reflection with respect to the xz-plane by $\mathcal{R}_4$. Finally we denote by H^+ the closed half-plane and denote the standard coordinates on it by $s \in [0, \infty)$ and $z \in \mathbb{R}$, so that $H^+ = \{(s, z) \in \mathbb{R}^2 : s \geq 0\}$.

In the next proposition we enumerate the properties of the Scherk surfaces which are relevant to our constructions later. Notice that these surfaces can be decomposed into a "core" which is within a finite distance from the z-axis, and four "wings" (see Definition 2.5 below). The wings depend up to Euclidean motion only on the parameter θ of the Scherk surface in consideration. The wing contained in the quadrant $\{(x, y) : x, y \geq 0\}$ is denoted by W_θ and we can describe it as the graph of a small function off a half-plane to which the wing decays exponentially away from the z-axis. Appropriate parametrizations of the wing and the asymptotic half-plane are denoted by F_θ and A_θ respectively, while f_θ denotes the function used to define the wing as a graph over the half-plane. The half-plane is parallel to the z-axis and makes an angle θ with the xz-plane; this is actually the geometric significance of the parameter θ. Finally notice that the other wings of $\Sigma(\theta)$ are images of W_θ under $\mathcal{R}_i$ $(i = 2, 3, 4)$; this reduces their description to that of W_θ.

We will need later to know that f_θ and its derivatives, as well as the rate of change of f_θ as the parameter θ changes and its derivatives, are small enough. This is ensured in 2.4.v. We will also need that the angle seen from the z-axis between the boundary line of the asymptotic half-plane of W_θ and the plane parallel to the asymptotic half-plane through the z-axis, as well as the rate of change of this angle with respect to θ, are small. This is implied by 2.4.vi. The smallness of these quantities is controlled by a small number that we call ε and is assumed to be given. It is arranged by excluding from the wings a large enough neighborhood of the z-axis, utilizing this way the asymptotic decay of the wing to ensure 2.4.iv, and being forced to observe the height b_θ (see below) from afar, to ensure the smallness of the angles above. The largeness of the excised neighborhood of the z-axis is controlled by a constant a which depends on ε and δ_θ—recall that the latter is used to ensure uniform control of the geometry of the Scherk surfaces in consideration through 2.2.

Proposition 2.4. *$\Sigma(\theta)$ is a singly periodic embedded complete minimal surface which depends smoothly on θ and has the following properties (figure 2):*

(i) $\Sigma(\theta)$ is invariant under the $\mathcal{R}_i$'s above and also under reflections with respect to the planes $\{z = n\pi\}$ $(n \in \mathbb{Z})$.

(ii) For given $\varepsilon \in (0, 10^{-3})$ there is a constant $a = a(\delta_\theta, \varepsilon) > 0$ and smooth functions $f_\theta : H^+ \to \mathbb{R}$, $A_\theta : H^+ \to E^3$, and $F_\theta : H^+ \to E^3$, such that $W_\theta := F_\theta(H^+) \subset \Sigma(\theta)$ and

$$A_\theta(s,z) = (a+s)\vec{e}\,[\theta] + z\vec{e}_z + b_\theta\,\vec{e}\,'[\theta],$$
$$F_\theta(s,z) = A_\theta(s,z) + f_\theta(s,z)\,\vec{e}\,'[\theta],$$

where $b_\theta = \sin 2\theta \log(\cot\theta)$. Moreover f_θ and F_θ depend smoothly on $\theta \in [10\,\delta_\theta, \frac{\pi}{2} - 10\,\delta_\theta]$ and (iii)-(vi) below are satisfied.

(iii) $\Sigma(\theta) \setminus \cup_{i=1}^{4} \mathcal{R}_i(W_\theta)$ *is connected and lies within distance* $a+1$ *from the* z*-axis.*

(iv) $W_\theta \subset \{(r\cos\phi, r\sin\phi, 0) : r > a,\ \phi \in [9\,\delta_\theta, \frac{\pi}{2} - 9\,\delta_\theta]\}$.

(v) $\|f_\theta : C^5(H^+, e^{-s})\| \le \varepsilon$ *and* $\|df_\theta/d\theta : C^5(H^+, e^{-s})\| \le \varepsilon$.

(vi) $|b_\theta| + |db_\theta/d\theta| < \varepsilon a$. *(Notice that the right-hand side is not small because* a *is large.)*

Proof. That $\Sigma(\theta)$ is a singly periodic embedded complete minimal surface which depends smoothly on θ follows from 2.1. (i) also follows from 2.1 by inspection. By substituting the expression for F_θ into 2.1, "solving" for f_θ, and choosing a large enough, we easily conclude (ii)-(vi). q.e.d.

Now that a precise description of the Scherk surfaces is available we give precise definitions for the wings and related concepts. Notice that the definitions below depend on having fixed the constant a above, something we assume from now on.

Definition 2.5. If R is a Euclidean motion of E^3 which fixes the z-axis we call $R(W_\theta)$ a (θ-)wing asymptotic to $R \circ A_\theta(H^+)$ and directed by $R(\vec{e}[\theta])$. We consider as standard coordinates on $R(W_\theta)$ the coordinates (s, z) defined by $(s, z) = (R \circ F_\theta)^{-1}$. We call $\mathcal{R}_i(W_\theta)$ the ith wing of $\Sigma(\theta)$. We extend the function s defined on the wings of $\Sigma(\theta)$, to a continuous function s defined on $\Sigma(\theta)$, by requiring s to vanish on the rest of $\Sigma(\theta)$. Following the notation in (1.2) we define $\Sigma_{\le 0}(\theta)$ to be the core of $\Sigma(\theta)$.

Notice that we have $\Sigma(\theta) = \Sigma_{\le 0}(\theta) \cup \bigcup_{i=1}^{4} \mathcal{R}_i(W_\theta)$, where the core and the wings, whose union makes the right-hand side, have disjoint interiors.

We proceed now to discuss the Gauss map of the Scherk surfaces. Notice that by the next proposition we can view $\Sigma(\theta)$ equipped with the (nondegenerate) metric $\nu^* g_{\mathbb{S}^2}$, as an isometric cover of

$$\mathbb{S}^2 \setminus \{(\pm\sin\theta, \pm\cos\theta, 0)\},$$

with covering map the Gauss map ν. $\Sigma(\theta)$ equipped with this pullback metric can be thought of as the union of a sequence of closed hemispheres S_n from which the four points $(\pm\sin\theta, \pm\cos\theta, 0)$ (which lie on the boundary equators of the hemispheres) have been removed. Subsequent hemispheres S_n and S_{n+1} cover opposing hemispheres of $\mathbb{S}^2$. They are joined through two opposing open arcs, which are two of the four arcs into which the equator is subdivided by the four points that we had removed. In the next junction between S_{n+1} and S_{n+2} then, the other two arcs are used. Each of the four points removed corresponds to the ∞ of one of the four wings. Preimages of small neighborhoods of such a point under the Gauss map are the complements of periodic neighborhoods of the z-axis extending a bounded distance from the z-axis.

Proposition 2.6. *The Gauss map ν of $\Sigma(\theta)$ has the following properties (see figure 3):*

(i) ν restricts to a diffeomorphism from $\Sigma(\theta) \cap \{z \in [0,\pi]\}$ onto $\mathbb{S}^2 \cap \{z \geq 0\} \setminus \{(\pm\sin\theta, \pm\cos\theta, 0)\}$.

(ii) Let E_i $(i = 1, ..., 4)$ be the arcs into which the equator $\mathbb{S}^2 \cap \{z = 0\}$ is decomposed by removing the points $(\pm\sin\theta, \pm\cos\theta, 0)$, numbered so that $(1,0,0) \in E_1$, $(0,1,0) \in E_2$, $(-1,0,0) \in E_3$, and $(0,-1,0) \in E_4$. We then have

$$\nu(\Sigma(\theta) \cap \{z = 0\}) = E_1 \cup E_3, \qquad \nu(\Sigma(\theta) \cap \{z = \pi\}) = E_2 \cup E_4.$$

(iii) $\Sigma(\theta)$ has no umbilics and $\nu^* g_{\mathbb{S}^2} = \frac{1}{2}|A|^2 g$.

Proof. The easiest way to establish this is to use the Weierstrass-Enneper representation for $\Sigma(\theta)$ [17]. q.e.d.

The above discussion and proposition motivate the definitions in 2.7. The metric h and the operator $\mathcal{L}_h$ will be useful later in understanding the behavior of the linearized operator, that is the linearization of the operator which gives the mean curvature of a graph of a given function over the minimal surface. $\mathcal{L}_h$ in particular is just the linearized operator followed by multiplication by a function—see B.1, and is often more convenient to use than the linearized operator itself.

Definition 2.7. Let $h := \nu^* g_{\mathbb{S}^2} = \frac{1}{2}|A|^2 g$ (recall 2.6.iii) and $\mathcal{L}_h := \Delta_h + 2$.

By a fundamental region of $\Sigma(\theta)$ we will mean a fundamental region for the action of the group consisting of the reflections with respect to the planes $\{z = n\pi\}$ $(n \in \mathbb{Z})$.

Clearly now the translations induce eigenfunctions in the kernel of the linearized operator on $\Sigma(\theta)$, because translating the surface leaves its mean curvature unchanged. These eigenfunctions are just $\nu \cdot \vec{e}$ for any constant vector $\vec{e}$. When $\vec{e}$ is a unit coordinate vector, $\nu \cdot \vec{e}$ is the pullback by ν of a coordinate function on $\mathbb{S}^2$. We know independently that the coordinate functions on $\mathbb{S}^2$ are the first harmonics of the sphere, that is, they are eigenfunctions for the spherical Laplacian of eigenvalue 2. Hence the $\nu \cdot \vec{e}$'s are in the kernel of $\mathcal{L}_h$, as they should be, since $\mathcal{L}_h$ and the linearized operator are related as described above.

Notice that $\nu \cdot \vec{e}_x$ is odd with respect to $\mathcal{R}_2$ and even with respect to $\mathcal{R}_4$, while $\nu \cdot \vec{e}_y$ is even and odd respectively, and $\nu \cdot \vec{e}_z$ is even with respect to both. With respect to reflections across the planes $\{z = n\pi\}$ both $\nu \cdot \vec{e}_x$ and $\nu \cdot \vec{e}_y$ are even, while $\nu \cdot \vec{e}_z$ is odd. Notice finally that $\nu \cdot \vec{e}_x$ and $\nu \cdot \vec{e}_y$ tend asymptotically on the wings—away from the z-axis—to (nonzero) constants $\pm \sin\theta$ and $\pm \cos\theta$ respectively, while $\nu \cdot \vec{e}_z$ decays exponentially fast to 0.

Understanding in detail the eigenfunctions of $\mathcal{L}_h$ whose eigenvalues are small is an integral part of the general case of this construction [16]. Contrary to the case in [13] - [14] taking $\tau \to 0$ does not decompose the surfaces in the h metric to disjoint spheres, actually changing τ in our case, that is scaling our surfaces, has no effect because h is scale-invariant. Such a decomposition would occur if we were taking $\theta \to 0$, but this is clearly forbidden by the construction. These considerations however are not important for this paper, because the symmetry which we impose simplifies considerably this and other aspects of our problem, as we can see from the following proposition:

Proposition 2.8. *Let G be the group generated by reflections across the planes $\{z = 0\}$ and $\{z = m_G\pi\}$ for some $m_G \in \mathbb{N}$. Let $\Sigma := \Sigma(\theta)$ where θ satisfies 2.2. There is an $\epsilon_\lambda = \epsilon_\lambda(m_G, \delta_\theta) > 0$ such that, the only (bounded) eigenfunctions of $\mathcal{L}_h$ on Σ/G whose eigenvalues lie in $[-\epsilon_\lambda, \epsilon_\lambda]$, are the ones in the kernel of $\mathcal{L}_h$ on Σ/G. This kernel is two-dimensional and is spanned by $\nu \cdot \vec{e}_x$ and $\nu \cdot \vec{e}_y$.*

Moreover, there are at most $C(m_G, c)$ eigenfunctions of $\mathcal{L}_h$ on Σ/G whose eigenvalues are $\leq c$, for some given $c \in \mathbb{R}$.

Proof. Clearly $\nu \cdot \vec{e}_z$ does not survive the imposed symmetry while $\nu \cdot \vec{e}_x$ and $\nu \cdot \vec{e}_y$ do. All that is needed then is to check that there are no "exceptional" eigenfunctions in the kernel. This situation has been studied before and we can refer the reader to [20, Theorem 14], where further references can also be found to finish the proof. q.e.d.

3. The desingularizing surfaces

The surfaces $\Sigma[T]$.

Recall that the four wings of a Scherk surface $\Sigma(\theta)$ are symmetrically arranged around the coordinate planes and directed by $\vec{e}\,[\theta]$ and its images under the $\mathcal{R}_i$'s. We will need perturbations of the Scherk surfaces which have wings directed by any four vectors which fail only slightly from describing a symmetric configuration. This does destroy minimality, but, as we will see later in Section 7, it relates to the way we handle the approximate kernel.

In order to systematically define these surfaces we first give the next definition (see figure 4), where we describe the collections of vectors we will be using to direct the wings of such perturbed Scherk surfaces. We also define some quantities associated to such a tetrad T: $\theta(T)$ is going to be the θ-parameter of the corresponding unperturbed Scherk surface, and the $\theta_i(T)$'s measure the failure of the vectors in T to be opposite to other vectors in T. The $\theta_i(T)$'s should be thought of as being rather small.

Definition 3.1. An acceptable tetrad of vectors, or tetrad for short, is defined to be a tetrad of vectors $T = (\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$ such that $\vec{v}_i = \vec{e}\,[\beta_i]$—recall 2.3—for some $\beta_i \in \mathbb{R}$ $(i = 1, ..., 4)$ satisfying

$$0 < \beta_2 - \beta_1 < \beta_3 - \beta_1 < \beta_4 - \beta_1 < 2\pi.$$

For such a T we also define

$$\theta(T) := \frac{3\beta_3 - \beta_1 - \beta_2 - \beta_4}{4}, \quad \theta_1(T) := \frac{\beta_3 - \beta_1 - \pi}{2},$$
$$\theta_2(T) := \frac{\beta_4 - \beta_2 - \pi}{2}.$$

Notice that in the case of a tetrad T whose ith vector is the vector directing the ith wing of $\Sigma(\theta)$ we have $\beta_1 = \theta$, $\beta_2 = \pi - \theta$, $\beta_3 = \pi + \theta$, $\beta_4 = 2\pi - \theta$, and hence $\theta(T) = \theta$ and $\theta_1(T) = \theta_2(T) = 0$. Conversely, given a tetrad T with $\theta_1(T) = \theta_2(T) = 0$, we can rotate $\Sigma(\theta(T))$ around the z-axis to get a Scherk surface whose wings are directed by the vectors in T. Rotating is not enough though for a general T, and we then need to introduce some "bending" which we describe precisely in the following definition. Notice that $Z_x(\phi)$ is designed to rotate "most" of the interiors of the first and fourth quadrants apart, while leaving the second and third quadrants pointwise fixed. (The quadrants here are taken with respect to the x and y coordinates.) $Z_y(\phi)$ pulls apart the first and second quadrants instead.

Definition 3.2. We fix once and for all a smooth family of diffeomorphisms $Z_x(\phi) : E^3 \to E^3$ parametrized by $\phi \in [-2\delta_\theta, 2\delta_\theta]$ such that:

(i) $Z_x(\phi)$ is the identity on $\{x \leq 0\}$.

(ii) $Z_x(\phi)$ is equivariant under $\mathcal{R}_4$.

(iii) On $\{(r\cos\theta', r\sin\theta', z) : r > 1, \theta' \in [9\,\delta_\theta, \frac{\pi}{2} - 9\,\delta_\theta]\}$ $Z_x(\phi)$ acts by rotation around the z-axis by ϕ.

We define also for ϕ as above $Z_y(\phi) := \mathcal{R} \circ Z_x(\phi) \circ \mathcal{R}$, where $\mathcal{R}$ is the reflection with respect to the $\{y = x\}$ plane.

We assume now we are given a T as in 3.1 which satisfies

$$\theta(T) \in [20\,\delta_\theta, \frac{\pi}{2} - 20\,\delta_\theta], \qquad \theta_1(T), \theta_2(T) \in [-2\delta_\theta, 2\delta_\theta], \tag{3.3}$$

and we proceed to describe the perturbed Scherk surface $\Sigma[T]$ whose wings are directed by the vectors of T. Notice in 3.4 that if $\theta_1(T) = \theta_2(T) = 0$, then $\Sigma[T]$ is simply $\Sigma(\theta(T))$ appropriately rotated by $\mathcal{R}$. Otherwise $Z_x(\theta_x)$ and $Z_y(\theta_y)$ are used to appropriately rotate the wings relative to each other. (That $\mathcal{Z}[T]$ is just a rotation on each of the wings follows from 3.2 and 2.4.iv.) The core of the unperturbed Scherk surface on the other hand gets genuinely deformed by $\mathcal{Z}[T]$ and its minimality is destroyed—unless $\theta_1(T) = \theta_2(T) = 0$ of course.

Definition 3.4. For a T satisfying 3.3 we define

$$\mathcal{Z}[T] := \mathcal{R} \circ Z_y(\theta_y) \circ Z_x(\theta_x),$$

where $\theta_x = -\theta_1(T) - \theta_2(T)$, $\theta_y = \theta_1(T) - \theta_2(T)$, and $\mathcal{R}$ denotes rotation around the z-axis by an angle $\frac{1}{4}(\beta_1 + \beta_2 + \beta_3 + \beta_4) - \pi$. We define a surface $\Sigma[T] := \mathcal{Z}[T](\Sigma(\theta(T)))$ and we call the image of the ith wing of $\Sigma(\theta(T))$ under $\mathcal{Z}[T]$ the ith wing of $\Sigma[T]$. Finally we define $\mathcal{R}_{T,i}$ to be the rotation by which $\mathcal{Z}[T]$ acts on the ith wing of $\Sigma[T]$.

Lemma 3.5. *$\Sigma[T]$ is a periodic embedded complete surface, and $\mathcal{Z}[T]$ induces a diffeomorphism from $\Sigma(\theta(T))$ onto $\Sigma[T]$ which is equivariant under reflections with respect to the planes $\{z = n\pi\}$ ($n \in \mathbb{Z}$). $\mathcal{R}_{T,i} \circ \mathcal{R}_i(W_\theta) \subset \Sigma[T]$ is the ith wing of $\Sigma[T]$ and is directed by $\vec{e}_i$.*

Proof. This follows easily from the definitions. To check the last statement in particular we need to verify that (see Figure 5)

$$\begin{aligned}
\vec{e}[\beta_1] &= \vec{e}[\theta(T) + \theta_x - \theta_y + (1/4)(\beta_1 + \beta_2 + \beta_3 + \beta_4) - \pi], \\
\vec{e}[\beta_2] &= \vec{e}[\pi - \theta(T) + \theta_y + (1/4)(\beta_1 + \beta_2 + \beta_3 + \beta_4) - \pi], \\
\vec{e}[\beta_3] &= \vec{e}[\pi + \theta(T) + (1/4)(\beta_1 + \beta_2 + \beta_3 + \beta_4) - \pi], \\
\vec{e}[\beta_4] &= \vec{e}[2\pi - \theta(T) - \theta_x + (1/4)(\beta_1 + \beta_2 + \beta_3 + \beta_4) - \pi],
\end{aligned}$$

which are readily obtained by substituting from the definitions. q.e.d.

The desingularizing surfaces $\Sigma[T, \underline{\varphi}, \tau]$.

At this point we have expanded the class of the Scherk surfaces so that we have surfaces whose wings are directed by the vectors in tetrads which are rather general. This allows us to "unbalance" the Scherk surfaces in a controlled way, and hence deal later with the difficulties due to the existence of approximate kernel. Even these surfaces though, $\Sigma[T]$, are not yet ready to be used as desingularizing surfaces: First of all we want to desingularize circles, but these surfaces desingularize straight lines. Second, they are at a scale much larger than the scale of the surfaces which we want desingularized, although this is clearly very easy to correct. Third, there are two reasons why we need to be able to "bend" each of the wings of the surfaces so that the complement of a "small" neighborhood of the boundary gets repositioned relative to the core of the Scherk surface (see figure 6): The first reason relates to the general philosophy of the construction and has been mentioned in the introduction already, and it is that we can use the freedom to bend the wings to adjust things so that exponential decay along the wings of various appropriate solutions to our partial differential equations can be achieved. The second reason is less important and more technical and it is to facilitate the constructions in Section 6 by allowing corrections in the final stages in fitting the wings as needed.

In this section we mostly ignore the issue of the appropriate scale—an exception is in 3.8 and 3.9 where it is convenient to keep an eye on both scales for future reference—and we resolve the other two (first and third) issues as follows: New surfaces $\Sigma[T, \underline{\varphi}, \tau]$ are constructed which depend on more parameters than before. The new parameters on top of T are a tetrad of numbers $\underline{\varphi}$ and a nonnegative number τ. When the new parameters vanish, $\Sigma[T, \underline{\varphi}, \tau]$ is simply the already defined $\Sigma[T]$. The parameter τ controls the bending of the surface introduced to make it desingularize a circle rather a straight line. Actually τ is exactly the curvature of the circle to which the axis gets bent. The ith number in $\underline{\varphi}$, φ_i, controls the bending of the ith wing, and amounts to the angle by which the wing gets bent relative to its previous position.

Before we proceed with the actual construction of the desingularizing surfaces, we have to discuss a fundamental difficulty which arises in bending the axes of the Scherk surfaces to circles: If we do this naively, for example we bend the whole surfaces instead of just the cores by using the maps $\mathcal{B}_\tau$ (see 3.6), we will get wings which decay to cones. This is unacceptable because the cones are not minimal surfaces, and hence the mean curvature created will be impossible to correct. We bypass this difficulty as follows: Recall that the wings are graphs over planes which are themselves minimal surfaces. If we want to "bend" the planes to minimal surfaces we have to bend them into catenoids rather than cones. We build then the "bent" wing as a graph over the appropriate catenoid, using the "same" function which gives the "unbent" wing as a graph over the plane. In this way the "bent" wings decay exponentially to minimal surfaces, and the mean curvature decays, also exponentially, to 0. The idea which we just described is simple, but plays a fundamental role in this paper, and so does its analogue in [16].

We start the actual construction by defining the maps which will be used to bend the cores of the surfaces $\Sigma[T]$. In the next definition notice that $\mathcal{B}_\tau$ depends smoothly on the parameter τ, fixes the xy-plane point-wise, and when $\tau \neq 0$ it wraps the z-axis isometrically around the circle $\{(\tau^{-1}+x)^2+z^2=\tau^{-2}, y=0\}$ while moving planes orthogonal to the z-axis to planes orthogonal to the same circle.

Definition 3.6. For $\tau \in \mathbb{R}$ we define $\mathcal{B}_\tau : E^3 \to E^3$ by

$$\mathcal{B}_\tau(x,y,z) = (\tau^{-1}+x)(\cos\tau z, 0, \sin\tau z) + (-\tau^{-1}, y, 0)$$

when $\tau \neq 0$, and by taking $\mathcal{B}_0$ to be the identity map.

We will need next to give a careful description of the catenoids on which we will build the bent wings. We really need to have "half-catenoids"parametrized by H^+, in a way similar to that we had the asymptotic half-planes parametrized in 2.4.ii. Moreover we need to have smooth dependence on τ which as we have already mentioned controls the amount of "bending" being done—at $\tau = 0$ the half-catenoids should smoothly deform to half-planes. This forces us to take the axes of the half-catenoids at positions depending on τ—we have seen this phenomenon already in the definition of $\mathcal{B}_\tau$. Notice also that besides

τ we need a few more parameters before the half-catenoid is uniquely specified. One way to visualize the parameters we use is as follows: "Unwrap" the catenoid to a surface invariant under translations in the direction of the z-axis by taking its preimage under $\mathcal{B}_\tau$. The x and y coordinates of the boundary line of this surface are then our r_0 and y_0 parameters, while the angle which the inward conormal makes with the xz-plane is the parameter β. Notice that in 3.9 we carefully discuss again the properties of these parametrizations of the half-catenoids, which we now give explicitly.

Definition 3.7. We define $\kappa[\tau, r_0, y_0, \beta] : H^+ \to E^3$ for given $\tau, r_0, y_0, \beta \in \mathbb{R}$ by

$$\begin{aligned}\kappa[\tau, r_0, y_0, \beta](s, z) =&(r_0 + \tau^{-1})(\cosh \tau s + \cos\beta \sinh \tau s)(\cos \tau z, 0, \sin \tau z) \\ &+ (-\tau^{-1}, y_0 + [(1 + \tau r_0) \sin\beta] s, 0),\end{aligned}$$

when $\tau \neq 0$, and by $\kappa[0, r_0, y_0, \beta](s, z) = (\cos\beta\, s, \sin\beta\, s, z)$ otherwise.

As we have already mentioned, the constructions carried out here, give us surfaces which are too large to fit in the scale of the surfaces we want to desingularize. The appropriate correction is to shrink them by a factor of order $1/\tau$. It is worthy of the effort to make this explicit for the parametrizations of the catenoids which we just defined and we do so in the next definition, while the exact relation with 3.7 becomes clear in 3.9.ii. The rescaling removes the dependence on τ and so our maps in 3.8 depend only on three parameters which are more or less like in 3.7, except for that we always "unwrap" by $\mathcal{B}_1$. Notice though the different limiting behavior we get as $\tau \to 0$ between the scaled and the unscaled versions. In the next definition we use a tilde to denote scaled quantities and we find it convenient—because of 3.9.ii—to scale the coordinates of H^+ as well.

Definition 3.8. We define $\tilde{\kappa}[\tilde{r}_0, \tilde{y}_0, \beta] : H^+ \to E^3$ for given $\tilde{r}_0, \tilde{y}_0, \beta \in \mathbb{R}$, by taking for $(\tilde{s}, \tilde{z}) \in H^+$

$$\begin{aligned}\tilde{\kappa}[\tilde{r}_0, \tilde{y}_0, \beta](\tilde{s}, \tilde{z}) =&(1 + \tilde{r}_0)(\cosh \tilde{s} + \cos\beta \sinh \tilde{s})(\cos \tilde{z}, 0, \sin \tilde{z}) \\ &+ (-1, \tilde{y}_0 + (1 + \tilde{r}_0) \sin\beta\, \tilde{s}, 0).\end{aligned}$$

Lemma 3.9. *Provided that $\tau r_0 \neq -1$ and $\tilde{r}_0 \neq -1$ the maps $\kappa = \kappa[\tau, r_0, y_0, \beta]$ and $\tilde{\kappa} = \tilde{\kappa}[\tilde{r}_0, \tilde{y}_0, \beta]$ satisfy the following:*

(i) *They are conformal minimal immersions which depend smoothly on their parameters. Moreover the pullbacks of the induced metrics by* κ *and* $\tilde\kappa$ *are* $\varrho^2(ds^2+dz^2)$ *and* $\tilde\varrho^2(d\tilde s^2+d\tilde z^2)$ *respectively where*

$$\varrho^2=(1+\tau r_0)^2(\cosh\tau s+\cos\beta\,\sinh\tau s)^2,$$
$$\tilde\varrho^2=(1+\tilde r_0)^2(\cosh\tilde s+\cos\beta\,\sinh\tilde s)^2.$$

(ii) $\tilde\kappa[\tau r_0,\tau y_0,\beta](\tau s,\tau z)\equiv\tau\,\kappa[\tau,r_0,y_0,\beta](s,z)$ *for* $\tau\neq 0$.

(iii) *Their images lie on catenoids (or planes). Moreover* $\kappa(\{s=0\})$ *is the circle of radius* $\tau^{-1}+r_0$*, centered at* $(-\tau^{-1},y_0,0)$*, and parallel to the* xz*-plane, while* $\tilde\kappa(\{\tilde s=0\})$ *is the circle of radius* $1+\tilde r_0$*, centered at* $(-1,\tilde y_0,0)$*, and also parallel to the* xz*-plane.*

(iv) $\vec e\,[\beta]$ *is the inward conormal of the image of* κ *(or* $\tilde\kappa$*) at* $\kappa(0,0)$ *(or* $\tilde\kappa(0,0)$*).*

Proof. Clearly the image lies on a catenoid (or plane), so minimality yields. (i) is obtained then by straightforward calculation. (ii-iv) follow by inspection and the definitions. q.e.d.

We concentrate our efforts now to define the "bent" wings. In order to determine a bent wing we need to know the following parameters: τ, which as we have already mentioned, amounts to the curvature of the circle to which the z-axis is "bent" by the "bending" of the surface. φ, which is the angle by which the directing vector of the wing is rotated in order to bend the wing relative to the core. θ, which determines the shape of the "unbent" wing and is the usual parameter of the Scherk surface $\Sigma(\theta)$ which contains the original wing. Finally, we need to know the position of the "unbent" wing, which is always the wing of some $\Sigma[T]$. Hence the "unbent" wing is some $R(W_\theta)$ for some Euclidean motion R which keeps the z-axis fixed. We use such an R as our fourth parameter.

Before we define the "bent" wings, we need to "bend" the half-planes to which the wings are asymptotic. This amounts to relating the parameters which we use to specify the "bent" wings to the parameters used in 3.7 to specify half-catenoids. (Recall that we intend to "bend" the asymptotic half-planes to half-catenoids.) Notice in the next definition that we have smooth dependence on the parameters and that $A[\theta,0,\mathcal{R}_1,0]$ is simply A_θ (recall 2.3 and 2.4.ii). Notice also that

$A[\theta, \varphi, R, \tau](\partial H^+)$ justifies its name because it is a line—when $\tau = 0$—or a circle—when $\tau \neq 0$—which does not depend on φ.

Definition 3.10. Given θ satisfying 2.2, $\varphi \in [-\delta_\theta, \delta_\theta]$, R a Euclidean motion fixing the z-axis, and $\tau \in [0, 1)$, we define $A[\theta, \varphi, R, \tau] : H^+ \to E^3$ as follows: If $\tau = 0$ we take

$$A[\theta, \varphi, R, 0](s, z) = R' \circ R \circ A_\theta(s, z),$$

where R' is the rotation by an angle φ around the line $R \circ A_\theta(\partial H^+)$. Otherwise we take

$$A[\theta, \varphi, R, \tau] = \kappa[\tau, r_0, y_0, \beta],$$

where the parameters r_0 and y_0 are determined by the requirement that $\kappa[\tau, r_0, y_0, \beta]$ and $\mathcal{B}_\tau \circ R \circ A_\theta$ agree on ∂H^+, and β is determined by the requirement that $\vec{e}[\beta]$ is just $R(\vec{e}[\theta])$ rotated by an angle φ around the z-axis.

We call $A[\theta, \varphi, R, \tau](\partial H^+)$ the pivot of $A[\theta, \varphi, R, \tau]$.

In the next definition we define at last the "bent" wings. As we have already described they are defined as graphs of the appropriate f_θ (recall 2.4.ii again) over the images of the maps which we have just defined. This has to be modified though somewhat, as we proceed to explain. First, in order for the "bent" wing to attach to the appropriate core, we need to transit to a different definition close to the boundary of the wing. This is the reason why $\psi[1, 0]$ appears in the formula in 3.11 below.

Second, instead of having the "bent" wings decay exponentially to the asymptotic half-catenoids, it is convenient for later use to (smoothly) truncate f_θ at an appropriate distance from the boundary of the wing. This distance has to be large enough to allow the exponential decay to make the error term introduced by the truncation negligible. On the other hand, it has to be small enough in the scale of the minimal surfaces to be desingularized, so that to avoid unnecessary complications in the discussion and construction of the initial surfaces. To achieve these conflicting aims we make our constructions dependent on a small constant $\delta_s > 0$ which will be determined later, and we carry out the truncation of f_θ at the region where s is of order δ_s/τ (see the definition of ψ_s in 3.11).

Finally, it is convenient to start to assume that τ is small enough for our purposes. We phrase this in this next definition by assuming that

$\tau \le \delta'_\tau$, where δ'_τ is a positive constant which depends only on δ_θ and is fixed so small that $A[\theta, \varphi, R, \tau]$ is ensured to be nondegenerate. The existence of such a δ'_τ is clearly implied by the definitions and 3.9. This also allows us to choose the Gauss map $\nu[\theta, \varphi, R, \tau]$ of $A[\theta, \varphi, R, \tau]$ so that it depends smoothly on the parameters and satisfies the orientation-choosing reduction $\nu[\theta, 0, R, 0] \equiv R(\vec{e}\,'[\theta])$ (recall 2.3 and 2.4.ii).

Notice in the next definition that as usual the dependence on the parameters is smooth, and that $F[\theta, 0, R, 0]$ is simply $R \circ F_\theta$ (F_θ was defined in 2.4.ii). Notice also that the restriction of $F[\theta, \varphi, R, \tau]$ to a small neighborhood of ∂H^+—to $H^+_{\le 1/3}$ for example—does not depend on φ.

Definition 3.11. With δ_s and δ'_τ as in the discussion above, we define for given θ satisfying 2.2, $\varphi \in [-\delta_\theta, \delta_\theta]$, R a Euclidean motion fixing the z-axis, and $\tau \in [0, \delta'_\tau]$, $F[\theta, \varphi, R, \tau] : H^+ \to E^3$ by

$$\begin{aligned} F[\theta, \varphi, R, \tau](s, z) =& \psi[1,0](s)\, \mathcal{B}_\tau \circ R \circ F_\theta(s, z) \\ &+ (1 - \psi[1,0](s))\, (A[\theta, \varphi, R, \tau](s, z) \\ &+ \psi_s(s)\, f_\theta(s, z)\, \nu[\theta, \varphi, R, \tau](s, z)), \end{aligned}$$

where ψ_s is defined by $\psi_s(s) = \psi[4\delta_s\tau^{-1}, 3\delta_s\tau^{-1}](s)$ if $\tau \ne 0$, and $\psi_s \equiv 1$ if $\tau = 0$, and $\nu[\theta, \varphi, R, \tau]$ is as in the discussion above.

We are now ready to describe how $\Sigma[T]$ bends to give the desingularizing surfaces (see figure 6). As usual the maps defined in 3.12 depend smoothly on their parameters, and if $\underline{\varphi}$ and τ vanish, then $\mathcal{Z}[T, \underline{\varphi}, \tau] = \mathcal{Z}[T]$. To facilitate future reference we also develop notation for the immersions of the wings and of their asymptotic half-catenoids. Notice finally that the pivots of $\mathcal{Z}[T, \underline{\varphi}, \tau]$ do not depend on $\underline{\varphi}$.

Definition 3.12. Given a tetrad T satisfying 3.3, $\underline{\varphi} = \{\varphi_i\}_{i=1}^4$ such that $|\underline{\varphi}| \le \delta_\theta$, and $\tau \in [0, \delta'_\tau]$, we define $\mathcal{Z}[T, \underline{\varphi}, \tau] : \Sigma(\theta(T)) \to E^3$, and for $i = 1, ..., 4$, $F_i[T, \underline{\varphi}, \tau] : H^+ \to E^3$ and $A_i[T, \underline{\varphi}, \tau] : H^+ \to E^3$ by taking

$$\mathcal{Z}[T, \underline{\varphi}, \tau] = \mathcal{B}_\tau \circ \mathcal{Z}[T]$$

on the core of $\Sigma(\theta(T))$, and requesting for $i = 1, ..., 4$ that:

$$\begin{aligned} \mathcal{Z}[T, \underline{\varphi}, \tau] \circ \mathcal{R}_i \circ F_{\theta(T)} = F_i[T, \underline{\varphi}, \tau] &:= F\,[\,\theta(T), \varphi_i, \mathcal{R}_{T,i} \circ \mathcal{R}_i\,, \tau\,]\,, \\ A_i[T, \underline{\varphi}, \tau] &:= A\,[\,\theta(T), \varphi_i, \mathcal{R}_{T,i} \circ \mathcal{R}_i\,, \tau\,]\,, \end{aligned}$$

which defines $\mathcal{Z}[T,\underline{\varphi},\tau]$ on the ith wing of $\Sigma(\theta(T))$, $F_i[T,\underline{\varphi},\tau]$, and $A_i[T,\underline{\varphi},\tau]$.

Finally we call the pivot of the right-hand side of the last equation the ith pivot of $\mathcal{Z}[T,\underline{\varphi},\tau]$ and we denote it by $\mathcal{C}_i[T,\underline{\varphi},\tau]$.

In the next proposition we discuss the fundamental properties of $\mathcal{Z}[T,\underline{\varphi},\tau]$, and then in Definition 3.14 we define the desingularizing surfaces and some more notation associated to them. Notice that we find convenient to truncate the desingularizing surfaces at $s = 5\delta_s/\tau$, in this way a neighborhood of their boundary when $\tau \neq 0$ is precisely catenoidal—recall that f_θ is truncated in 3.11 at a smaller value of s.

Proposition 3.13. *There is $\delta''_\tau = \delta''_\tau(\delta_\theta) \in (0, \delta'_\tau]$ such that for T and $\underline{\varphi}$ as in 3.12, and $\tau \in [0, \delta''_\tau]$, $\mathcal{Z}[T,\underline{\varphi},\tau]$ satisfies the following:*

(i) It is a smooth immersion depending smoothly on its parameters.

(ii) For each $n \in \mathbb{Z}$ it is equivariant under reflection of the domain with respect to the plane $\{z = n\pi\}$, and of the range with respect to the plane $\{z = n\pi\}$—when $\tau = 0$—or the plane through $(-\tau^{-1}, 0, 0)$, parallel to the y-axis, and forming an angle $n\tau^{-1}\pi$ with the positive x-axis—when $\tau \neq 0$.

(iii) If τ^{-1} is a positive integer then $\mathcal{Z}[T,\underline{\varphi},\tau](\Sigma_{\leq 5\delta_s/\tau}(\theta(T)))$ is an embedded surface. It contains $2\tau^{-1}$ fundamental regions under the reflections in (ii), and its boundary consists of four (round) circles which have exactly-catenoidal neighborhoods.

Proof. This follows from the various definitions by inspection.

q.e.d.

Definition 3.14. If T, $\underline{\varphi}$, and τ are as in 3.13, and $\tau^{-1} \in \mathbb{N}$, we define a smooth embedded surface with boundary by

$$\Sigma = \Sigma[T,\underline{\varphi},\tau] := \mathcal{Z}[T,\underline{\varphi},\tau](\Sigma_{\leq 5\delta_s/\tau}(\theta(T))).$$

If $\tau^{-1} \notin \mathbb{N}$ we define similarly $\Sigma[T,\underline{\varphi},\tau]$ as an immersed surface.

We also define s on $\Sigma[T,\underline{\varphi},\tau]$ to be simply the pushforward by $\mathcal{Z}[T,\underline{\varphi},\tau]$ of s on $\Sigma(\theta(T))$. The ith wing of Σ is defined to be the image under $\mathcal{Z}[T,\underline{\varphi},\tau]$ of the intersection of the ith wing of $\Sigma(\theta(T))$ with the preimage of Σ. Finally we define $\partial_i[T,\underline{\varphi},\tau]$ to be the boundary circle of Σ, which is contained in the ith wing, and we call the ith pivot of $\mathcal{Z}[T,\underline{\varphi},\tau]$, $\mathcal{C}_i[T,\underline{\varphi},\tau]$, the ith pivot of Σ as well.

By reviewing 2.4 and using the smooth dependence on τ—$\tau = 0$ included—we can summarize as follows: We have defined the surfaces $\Sigma[T, \underline{\varphi}, \tau]$ which are embedded if $\tau^{-1} \in \mathbb{N}$, or immersed in general, have a "core" $\Sigma_{\leq 0}[T, \underline{\varphi}, \tau]$ which is within a uniformly bounded distance from a circle of radius $1/\tau$ and is locally a small deformation of the core $\Sigma_{\leq 0}(\theta(T))$, and four wings which are the four connected components of the complement of the interior of the "core". $\Sigma[T, \underline{\varphi}, \tau]$ is immersed by $\mathcal{Z}[T, \underline{\varphi}, \tau]$ defined on $\Sigma(\theta(T))$. The ith wing of $\Sigma[T, \underline{\varphi}, \tau]$ is immersed by $F_i[T, \underline{\varphi}, \tau]$ restricted to $H^+_{\leq 5\delta_s/\tau}$. We can distinguish five regions on the wing: Where $s \leq 1/3$ the wing is defined in the same way as the "core" was defined and does not depend on $\underline{\varphi}$. Where $s \in [2/3, 3\delta_s/\tau]$ the wing is the graph of—essentially—$f_{\theta(T)}$ over an annulus contained in the catenoid passing through $\mathcal{C}_i[T, \underline{\varphi}, \tau]$ and $\partial_i[T, \underline{\varphi}, \tau]$. Notice that we have also a parametrization $A_i[T, \underline{\varphi}, \tau]$ of this catenoid in which these circles correspond to $\{s = 0\}$ and $\{s = 5\delta_s/\tau\}$ respectively. Where $s \in [4\delta_s/\tau, 5\delta_s/\tau]$ the wing lies on the catenoid and $F_i[T, \underline{\varphi}, \tau] = A_i[T, \underline{\varphi}, \tau]$. Finally the two remaining regions, that is where $s \in [1/3, 2/3]$ and where $s \in [3\delta_s/\tau, 4\delta_s/\tau]$, are regions of transition between their adjacent regions.

4. The mean curvature of the desingularizing surfaces

Introductory discussion.

The initial surfaces which we shall construct later in Section 6 consist of the desingularizing surfaces that we constructed in the previous section—homothetically changed—and pieces of catenoids and planes. The minimal surfaces which we construct in this paper are graphs over these initial surfaces. The functions defining these graphs satisfy a nonlinear partial differential equation. We initially concentrate on the linearization of this equation which is inhomogeneous with the inhomogeneous term involving the mean curvature of the initial surface. Since catenoids and planes are minimal surfaces, the mean curvature of the initial surfaces is supported on the desingularizing surfaces. This section is devoted to understanding the mean curvature of a desingularizing surface $\Sigma = \Sigma[T, \underline{\varphi}, \tau]$.

The mean curvature on Σ is the result of the various bendings which we had to introduce. The bending controlled by τ, which bends the axis of a Scherk surface to a circle, is really forced to us by the need to fit the desingularizing surface to the minimal surfaces that we are desingularizing. It creates mean curvature which feeds the inhomogeneous term in the linearized equation and needs to be corrected by solving the equation. Hence this part of the mean curvature has to be carefully estimated so that appropriate estimates for the solutions can be obtained.

Now as we have mentioned in the introduction, in order to solve the linearized equation and obtain appropriate estimates for the solutions, we only solve modulo functions w and $\overline{w}$, which we will define later in this section. The functions w correspond to the small eigenvalues which the operator in the linearized equation has, and allow us to make the inhomogeneous term orthogonal to the corresponding eigenfunctions. We can solve then as if there were no small eigenvalues. The functions $\overline{w}$ are accompanied by functions $\overline{u}$ on which if we acted by the linearized operator we would obtain the functions $\overline{w}$. By adding multiples of the $\overline{u}$'s to the solutions we can modify them in ways which allow us to achieve fast exponential decay along the wings.

As we discussed in the introduction and we will see in detail in Sections 7 and 8, since we are solving modulo the w's and $\overline{w}$'s, we have to be able to cancel arbitrary (small) linear combinations of them. The way we achieve this is by modifying geometrically the construction so that the mean curvature of the initial surfaces is modified by prescribed such linear combinations. As we mentioned in the introduction this is achieved by "dislocating" the pieces of the initial surfaces relative to each other.

We have already implemented these dislocations in the constructions in the previous section. This reflects in the estimate in Proposition 4.20: We see there that the mean curvature of a desingularizing surface can be decomposed into three parts: A linear combination of w's with coefficients $\theta_{i,\Sigma}$ which measure the relative bending of opposing wings—they amount to $\theta_i(T)$ modified by the further relative bending of opposing wings away from the core controlled by $\underline{\varphi}$. This part is due to the deformation of the core of the Scherk surfaces introduced in the construction of the $\Sigma[T]$'s in the beginning of the previous section. The second part is a linear combination of the $\overline{w}$'s with the φ_i's as coefficients. This part is due to the bending of the wings relative to the core. The third part is the "unwelcome" mean curvature which feeds the inhomogeneous term in the linearized equation. Most of this part is due to the fitting of the desingularizing surface to the minimal surfaces to be desingularized as we mentioned above, and its size is controlled by τ. The rest of it is due to nonlinear interactions in the creation of the previous parts and is of higher order in the parameters already mentioned.

It is interesting to reflect on the definition of the w's, $\overline{u}$'s, and $\overline{w}$'s. We more or less define them to be the linearized changes of mean curvature—for w and $\overline{w}$—or position—for $\overline{u}$—due to change of the corresponding parameters. This seems conceptually clearer than the definition of ϕ_ζ in [14, Proposition 6.7] for example—ϕ_ζ in [14] corresponds to the combination of the $\overline{w}$'s in 4.20—but it requires a certain amount of work: In our case the wings in particular are defined as graphs over catenoids, where both the catenoid and the function depend on the parameters. Understanding hence the linearized changes requires a careful use of Lemma 4.11 combined with a study of how the changes in parameters change the various quantities.

Estimates on the wings.

The main objective in this subsection is to estimate the mean curvature and its linearized change under change of the φ_i's on $\Sigma_{\geq 2}$. Before we proceed with that, we want to discuss in the next lemma the "straightening-up" of the wings, where by "straightening-up" we mean the change of the $\underline{\varphi}$ parameters towards vanishing values. Notice that we want to "straighten-up" (see Figure 7) in such a way that $\partial\Sigma$ does not move, or, to be more precise, we impose the boundary condition that the "straightened-up" new surfaces—perhaps extended beyond their boundary by using $\mathcal{Z}[T', \underline{\varphi} - \underline{\varphi}', \tau]$—pass still through $\partial\Sigma$. This condition is part of 4.1.iv. It implies the possibility that the surface which we start with satisfies $\theta_i(T) = 0$, and yet the "fully-straightened-up" surface—which in the statement of the next lemma corresponds to $\underline{\varphi}' = \underline{\varphi}$—may have $\theta_i(T') \neq 0$. This explains in particular why $\theta_{i,\Sigma}$ in 4.20 has to involve $\underline{\varphi}$.

Lemma 4.1. *There are $\delta_\phi = \delta_\phi(\delta_\theta) \in (0, \delta_\theta)$ and—recall 3.13—$\delta_\tau = \delta_\tau(\delta_\theta) \in (0, \delta''_\tau)$ such that for given a tetrad T as in 3.1, $\underline{\varphi} \in \mathbb{R}^4$, and $\tau \in (0, \delta_\tau]$, where*

$$\theta(T) \in [30\,\delta_\theta, \frac{\pi}{2} - 30\,\delta_\theta], \qquad \theta_1(T), \theta_2(T) \in [-\delta_\theta, \delta_\theta], \qquad |\underline{\varphi}| \leq \delta_\phi,$$

we have for each $\underline{\varphi}' = \{\varphi'_i\}_{i=1}^4 \in \mathbb{R}^4$ with $|\underline{\varphi}'| \leq \delta_\phi$, a T' which depends smoothly on $T, \underline{\varphi}, \tau$ and $\underline{\varphi}'$, and is characterized by the following properties:

(i) $(T', \underline{\varphi} - \underline{\varphi}', \tau)$ *satisfies the conditions in 3.13 and hence* $\mathcal{Z}[T', \underline{\varphi} - \underline{\varphi}', \tau]$ *satisfies 3.13.i-iii.*

(ii) $T' = T$ *when* $\underline{\varphi}' = 0$.

(iii) $T' = \{\vec{e}\,[\beta_i']\}_{i=1}^4$ *where each* β_i' *depends smoothly on* $\underline{\varphi}'$ *and*

$$\left| \frac{\partial \beta_i'}{\partial \varphi_j'} - \delta_{ij} \right| \le C\tau.$$

(iv) There is a smooth function $f_{\underline{\varphi}'}$ *on* $\Sigma[T, \underline{\varphi}, \tau]$ *which depends smoothly on* $T, \underline{\varphi}, \tau$ *and* $\underline{\varphi}'$, *satisfies* $f_{\underline{\varphi}'} \equiv 0$ *on* $\partial\Sigma[T, \underline{\varphi}, \tau]$, *and its graph over* $\Sigma[T, \underline{\varphi}, \tau]$ *is contained in the image of* $\mathcal{Z}[T', \underline{\varphi} - \underline{\varphi}', \tau]$ *(which contains* $\Sigma[T', \underline{\varphi} - \underline{\varphi}', \tau]$*).*

Proof. Fix for a moment a T, $\underline{\varphi}$, and τ satisfying the assumptions and suppose that $T = \{\vec{e}\,[\beta_i]\}_{i=1}^4$. By inspecting the definition and using 3.9 and 2.4.v we see that given $\underline{\beta}' = \{\beta_i'\}_{i=1}^4$ with $|\beta_i - \beta_i'| \le 2\delta_\phi$ and $\underline{\varphi}'$ as in the lemma, there is a function f on $\Sigma[T, \underline{\varphi}, \tau]$ which depends smoothly on $\underline{\beta}'$, $\underline{\varphi}'$, and $(T, \underline{\varphi}, \tau)$, and whose graph over $\Sigma[T, \underline{\varphi}, \tau]$ is contained in the image of $\mathcal{Z}[T', \underline{\varphi} - \underline{\varphi}', \tau]$ where $T' = \{\vec{e}\,[\beta_i']\}_{i=1}^4$. By the definitions f is a constant f_i on the boundary circle of $\Sigma[T, \underline{\varphi}, \tau]$ contained in the ith wing of $\Sigma[T, \underline{\varphi}, \tau]$. Moreover by using 3.9 it is easy to see that the matrix $(\partial f_j / \partial \beta_i')$ has an inverse of norm $\le C\tau$ and that the matrix $(\partial f_i / \partial \beta_j' + \partial f_i / \partial \varphi_j')$ has norm $\le C$. By applying the implicit function theorem then we can finish the proof. q.e.d.

As we have mentioned we need to study the variations along families of surfaces—the wings—which are defined as graphs of varying functions over varying surfaces—the catenoids. It is helpful to introduce some notation now to facilitate reference to the geometric invariants of the catenoids and of the wings of $\Sigma = \Sigma[T, \underline{\varphi}, \tau]$. Recall from 3.12 that the immersion of the half-catenoid on which the ith wing of Σ is defined as a graph is $A_j = A_j[T, \underline{\varphi}, \tau]$, while the immersion of the wing itself is $F_j = F_j[T, \underline{\varphi}, \tau]$. The map $F_j \circ A_j^{-1}$ maps points of the catenoid to the corresponding points of the graph above them.

Notation 4.2. Let Ω_j be the component of $\Sigma_{\ge 1}$ contained in the jth wing. Abusing the language we adopt the convention of using the same symbols for the functions and tensor fields on Ω_j, as well as their pullbacks by $F_j \circ A_j^{-1}$, to $\Omega_j' := A_j \circ F_j^{-1}(\Omega_j)$, and vice versa. To avoid confusion we use symbols without subscripts for the geometric invariants of Ω'_j contained in the asymptotic catenoid, while the geometric invariants of $\Omega_j \subset \Sigma$ are distinguished by the use of a subscript Σ, with the exception of the mean curvature of Σ which is denoted by H.

By abuse of notation also we define ϱ on Ω_j as the pushforward by A_j of the $\varrho > 0$ defined in 3.9.i—recall that A_j is just some $\kappa = \kappa[\tau, r_0, y_0, \beta]$. We have then $g = \varrho^2\ (A_j)_*(ds^2 + dz^2)$. We denote by Γ the difference of the connections induced by $(A_j)_*(ds^2 + dz^2)$ and g.

Finally, with $\underline{\varphi}'$ as in 4.1 and a fixed $i \in \{1, ..., 4\}$, we use a dot $\dot{}$ to denote the differentiation $\partial/\partial\varphi'_i|_{\underline{\varphi}'=0}$.

In the rest of this section we will be tacitly assuming that all surfaces $\Sigma[T, \underline{\varphi}, \tau]$ which we consider have $T, \underline{\varphi}, \tau$ satisfying the hypotheses in 4.1. In the next lemma we estimate the geometric quantities and their variations on the catenoids over which the wings are defined as graphs. In the following corollary we estimate some geometric invariants of the wings of Σ. We will use the lemma and its corollary repeatedly later in this section to understand the various quantities on the wings.

Lemma 4.3. *$|(\theta(T'))^{\cdot}| \leq C$ and the following are valid on $\Sigma_{\geq 1}[T, \underline{\varphi}, \tau]$:*

$$\begin{aligned}
&\|\varrho^2 - 1 : C^k(g, 1+s)\| \leq C\tau, \qquad && \|A : C^k(g)\| \leq C\tau,\\
&\|\dot{\varrho} : C^k(g)\| \leq C\tau, && \|\dot{A} : C^k(g)\| \leq C\tau^2,\\
&\|\dot{\kappa} : C^k(g)\| \leq C, && \|\nu : C^k(g)\| \leq C,
\end{aligned}$$

$$\begin{aligned}
&\|\Gamma : C^k(g)\| \leq C\tau,\\
&\|\dot{\Gamma} : C^k(g)\| \leq C\tau^2,\\
&\|\dot{\nu} : C^k(g)\| \leq C\tau,
\end{aligned}$$

where ϱ^2, g, A, ν, Γ, κ, and $\dot{}$, are as in 4.2, and all the constants C depend only on k.

Proof. The first estimate follows by 4.1.iii and the various definitions. Let $\tilde{\kappa} = \tilde{\kappa}[\tau r_0, \tau y_0, \beta]$ where A_j is $\kappa = \kappa[\tau, r_0, y_0, \beta]$. We identify $\tilde{s} = \tau s$ and $\tilde{z} = \tau z$—recall 3.9.ii—and by 3.9.i we have then $\varrho^2 = (1 + \tilde{r}_0)^2(\cosh\tilde{s} + \cos\beta\,\sinh\tilde{s})^2$. We also have by 4.1 and its proof that

$$|\dot{\tilde{r}}_0| \leq C\tau, \qquad |\dot{\beta}| \leq C\tau.$$

The estimates follow then by assuming without loss of generality δ_s small enough, by using 3.9, and finally scaling back to the right scale. q.e.d.

Corollary 4.4. *The following estimates are valid on* $\Sigma_{\geq 1}[T, \underline{\varphi}, \tau]$ *where* $\ell = 5\delta_s/\tau$*:*

$$\begin{aligned} \|\varrho^2 - 1 : C^3(g)\| &\leq C\delta_s, \\ \|g_\Sigma - g : C^3(g, e^{-s})\| &\leq C\varepsilon, \\ \||A|^2_\Sigma : C^3(g, e^{-s} + \ell^{-2})\| &\leq C\varepsilon + C\delta_s^2. \end{aligned}$$

In particular, g, g_Σ, *and* $F_{j*}(ds^2 + dz^2)$ *are all uniformly equivalent on the components of* $\Sigma_{\geq 1}[T, \underline{\varphi}, \tau]$ *by assuming without loss of generality* δ_s *and* ε *small enough.*

Proof. The first two estimates follow from 2.4.v and 4.3, and they immediately imply the equivalence of the metrics by the definitions. For the last estimate we have

$$\||A|^2_\Sigma : C^3(g, e^{-s} + \ell^{-2})\| \leq \||A|^2_\Sigma - |A|^2 : C^3(g, e^{-s})\| + \ell^2 \||A|^2 : C^3(g)\|,$$

and the proof is concluded by estimating the first summand by $C\varepsilon$ by 2.4.v, and the second by $C\delta_s^2$ by 4.3. q.e.d.

We proceed now to estimate the mean curvature and its variations under changing $\underline{\varphi}$. Since the wings decay exponentially to catenoids, we expect that the mean curvature and its variations decay also exponentially like e^{-s}. Because of the truncation of f_θ however due to ψ_s—recall 3.11—the mean curvature and its variations are larger than expected by the decay, by a factor of τ in the region where ψ_s is not locally constant, and where $s \geq 3\delta_s/\tau$. This is accommodated by reducing the rate of decay in the estimates to γ, which will be determined later, and should be thought of as a constant slightly smaller than 1. From now on we will be assuming tacitly that τ is small enough in terms of γ.

Lemma 4.5. *For* $\gamma \in (0, 1)$ *we have*

$$\|H : C^2(\Sigma_{\geq 1}[T, \underline{\varphi}, \tau], e^{-\gamma s})\| \leq C\tau,$$

where H *is the mean curvature of* $\Sigma[T, \underline{\varphi}, \tau]$.

Proof. By 2.4.v, the minimality of κ, and by assuming τ small enough in terms of γ, we can ensure that the estimate is valid where

$s \geq 3\delta_s/\tau$ because then in that region $e^{-s} < e^{-\gamma s}\tau$. We can concentrate thus our attention to the region where $s \leq 3\delta_s/\tau$ and where the stronger estimate is valid with 1 instead of γ as we prove now.

In this region we have $\psi_s \equiv 1$. Since A_θ and F_θ are minimal, and the former's second fundamental form vanishes, we can apply Lemma B.1 to conclude that

$$\Delta f_\theta = -2Q, \tag{1}$$

where Q is the Q_f of Lemma B.1 with $X = A_\theta$ and $f = \psi_s\, f_{\theta(T)}$, and Δ is the Laplacian of the flat metric induced by A_θ. Similarly then since κ is minimal by 3.9 we have

$$H = \frac{1}{2}(\Delta_g + |A|^2) f_\theta + Q',$$

where Q' is the Q_f of Lemma B.1 corresponding to $X = \kappa$ and $f = \psi_s\, f_{\theta(T)}$, and Δ_g is the Laplacian induced by g. By (1) we obtain

$$H = \frac{1}{2}|A|^2 f_\theta + Q' - \varrho^{-2} Q. \tag{2}$$

Using 2.4.v, 4.3, and B.1 we conclude the proof. q.e.d.

Lemma 4.6. $\|\dot{H} : C^1(\Sigma_{\geq 1}[T, \underline{\varphi}, \tau], e^{-\gamma s})\| \leq C\tau.$

Proof. We have

$$\dot{H} = \frac{1}{2}(\varrho^{-2})^{\cdot}\Delta f_\theta + \frac{1}{2}(|A|^2)^{\cdot} f_\theta + \frac{1}{2}(\theta(T'))^{\cdot}|A|^2\frac{\partial f_\theta}{\partial \theta} + \overline{Q},$$

where $\overline{Q} = \dot{Q}' - \varrho^{-2}\dot{Q}$, and Q', Q, and Δ are as in the proof of 4.5. Using 2.4.v, 4.3, and B.1 we conclude the proof. q.e.d.

The functions $\overline{u}_i$ and $\overline{w}_i$.

In this subsection we define and estimate the functions $\overline{u}_i$ and $\overline{w}_i$ which are related with the "bending" of the wings—controlled by $\underline{\varphi}$—we have introduced. The properties of these functions are presented in Lemma 4.17. Notice that $\overline{w}_i$ is supported close to the core by 4.17.ii. The main use of the $\overline{u}_i$'s is that they allow us to modify solutions to the appropriate Dirichlet problem on $\Sigma = \Sigma[T, \underline{\varphi}, \tau]$ so that we have exponential decay with respect to s in the spirit of A.3. This is based on 4.17.iv and is done at the expense of introducing $\overline{w}_i$'s in the inhomogeneous term of the linear equation. As we have mentioned already, we can cancel these unwelcome $\overline{w}_i$'s by choosing appropriately $\underline{\varphi}$ and appealing to 4.20—see the proof of 8.2.

Our strategy is to define $\overline{u}_i$ to be more or less the change in the immersion due to the straightening-up of the wing in the spirit of 4.1, while $\overline{w}_i$ corresponds to the change in the mean curvature. It is more practical for future use to employ the linearized changes and concentrate on the normal variation of position. Finally, if we did not modify slightly these definitions, we would obtain nonvanishing $\overline{w}_i$'s on the wings, which would make their future use awkward. Fortunately the nonvanishing quantities on $\Sigma_{\geq 1}$ are small, and so we can carry out a correction by solving the appropriate linear equation. We start by defining—Definition 4.7—and estimating—Lemma 4.8—the components of the variation field corresponding to changing φ_i. We then define the uncorrected version of $\overline{u}_i$ in 4.9, and the appropriate linearized operator in 4.10.

Definition 4.7. Let Y be the variation vector field on $\Sigma_{\geq 1}[T, \underline{\varphi}, \tau]$ due to changing φ'_i, that is on the component contained in the jth wing

$$Y = (F_j[T', \underline{\varphi} - \underline{\varphi}', \tau])^{\cdot},$$

and let $Y_{\|} := Y - (Y \cdot \nu_\Sigma)\nu_\Sigma$ be the its tangential component of Y.

Lemma 4.8. *$\|Y_{\|} : C^1(\Sigma_{\geq 1}[T, \underline{\varphi}, \tau])\| \leq C$ and*

$$\|Y \cdot \nu_\Sigma : C^1(\Sigma_{\geq 1}[T, \underline{\varphi}, \tau])\| \leq C.$$

Proof. By 4.2 we have on the component contained in the jth wing

$$Y = (\kappa + \psi_s f_{\theta(T)}\nu)^{\cdot} = \dot{\kappa} + \psi_s\, (\theta(T'))^{\cdot} \frac{\partial f_\theta}{\partial \theta}\nu + \psi_s\, f_{\theta(T)}\, \dot{\nu},$$

and the result follows by using 4.3 and—to estimate ν_Σ—2.4.v. q.e.d.

Definition 4.9. We define functions $\overline{u}'_i$ $(i = 1, ..., 4)$ on $\Sigma[T, \underline{\varphi}, \tau]$ by

$$\overline{u}'_i := \left.\frac{\partial}{\partial \varphi'_i}\right|_{\underline{\varphi}'=0} f_{\underline{\varphi}'},$$

where $f_{\underline{\varphi}'}$ is as in 4.1.

Definition 4.10. Given any smooth surface S in E^3 we define $\mathcal{L}_S := \frac{1}{2}(\Delta_S + |A|_S^2)$, where g_S, $|A|_S^2$, and Δ_S denote the first fundamental form, the square length of the second fundamental form, and the induced Laplacian on S.

The next lemma decomposes the variation of the mean curvature which we have already estimated in 4.6, to the part due to the tangential variation of position which we want to discard and do so in the following corollary, and to the rest which is the part which we really need to control.

Lemma 4.11. *On* $\Sigma_{\geq 1}[T, \underline{\varphi}, \tau]$ *we have* $\overline{u}_i' = Y \cdot \nu_\Sigma$ *and* $\dot{H} = Y_{||}(H) + \mathcal{L}_\Sigma \overline{u}_i'$.

Proof. By 4.1.iv we can factor the immersion of the component of $\Sigma_{\geq 1}[T', \underline{\varphi} - \underline{\varphi}', \tau]$ contained in the jth wing, $F_j[T', \underline{\varphi} - \underline{\varphi}', \tau]$, as the composition of a local diffeomorphism of the domain—contained in H^+—followed by $X + f_{\underline{\varphi}'}\nu_\Sigma$, where X and ν_Σ are the immersion and Gauss map of the component above when $\underline{\varphi}' = 0$. By applying $\dot{}$ to this and referring to 4.9 we conclude then that $Y = Y_{||} + \overline{u}_i'\nu_\Sigma$. The proof is completed thus by appealing to B.2. q.e.d.

Corollary 4.12. $\|\mathcal{L}_\Sigma \overline{u}_i' : C^1(\Sigma_{\geq 1}[T, \underline{\varphi}, \tau], e^{-\gamma s})\| \leq C\tau$.

Proof. Follows from 4.11 by referring to 4.6, 4.5, and 4.8. q.e.d.

We proceed now to correct $\overline{u}_i'$ to $\overline{u}_i$ so that $\mathcal{L}_\Sigma \overline{u}_i \equiv 0$ on $\Sigma_{\geq 2}$. To achieve this we apply Proposition A.3 to solve a Dirichlet problem as follows: Consider the cylinder $\Omega = [1, 5\delta_s/\tau] \times \mathbb{R}/G'$ where G' is the group generated by $(s, z) \to (s, z + 2\pi)$. Recall that $F_j = F_j[T, \underline{\varphi}, \tau]$ is the immersion of the jth wing of $\Sigma = \Sigma[T, \underline{\varphi}, \tau]$. Because of the symmetries we can consistently define

$$\chi := \varrho^{-2} F_j^*(g_\Sigma), \qquad d := \varrho^2 \, |A|_\Sigma^2 \circ F_j. \tag{4.13}$$

By the usual slight abuse of notation we have then

$$\underline{\mathcal{L}} = 2\varrho^2 \mathcal{L}_\Sigma, \tag{4.14}$$

where $\underline{\mathcal{L}}$ is as in A.2. By the next lemma we thus can apply A.3 and hence obtain the desired $\overline{u}_i$ and $\overline{w}_i$ in the next definition.

Lemma 4.15. *By assuming without loss of generality* ε *and* δ_s *small enough, we can ensure that* $N(\underline{\mathcal{L}})$ *is small enough as required by the hypotheses in A.3.*

Proof. Since $\chi - g_0 = \varrho^{-2} F_j^*(g_\Sigma - g)$ and d is as in 4.13 this follows easily by 4.4. q.e.d.

Definition 4.16. We fix now once and for all an $\alpha \in (0,1)$, and we apply A.3 to obtain $v_i = \underline{\mathcal{R}}(0, E)$, where $E = -2\varrho^2 \mathcal{L}_\Sigma \overline{u}_i' = -\underline{\mathcal{L}}\overline{u}_i'$, on each component of $\Sigma_{\geq 1}[T, \underline{\varphi}, \tau]$. We define then $\overline{u}_i$ and $\overline{w}_i$ on $\Sigma[T, \underline{\varphi}, \tau]$ by

$$\overline{u}_i := \overline{u}_i' + (\psi[1,2] \circ s) v_i, \qquad \overline{w}_i := \mathcal{L}_\Sigma \overline{u}_i.$$

Lemma 4.17. *The functions $\overline{u}_i$ and $\overline{w}_i$ ($i = 1, ..., 4$) defined as in 4.16 on $\Sigma = \Sigma[T, \underline{\varphi}, \tau]$ satisfy the following:*

(i) They depend continuously on $(T, \underline{\varphi}, \tau)$.

(ii) $\overline{w}_i$ is supported on $\Sigma_{\leq 2}$, $\|\overline{w}_i : C^{0,\alpha}(\Sigma)\| \leq C$, and $\|\overline{w}_i : C^{0,\alpha}(\Sigma_{\geq 1})\| \leq C\tau$.

(iii) $\|\overline{u}_i : C^{2,\alpha}(\Sigma)\| \leq C$.

(iv) $\mathcal{L}_\Sigma \overline{u}_i = 0$ on $\Sigma_{\geq 2}[T, \underline{\varphi}, \tau]$, $\overline{u}_i = 0$ on $\partial\Sigma$, and $|\overline{u}_i - a\delta_{ij}| \leq C\varepsilon a$ on the component of $\partial\Sigma_{\leq 2}$ contained in the jth wing.

Proof. From 3.13, 2.4.v, and 4.1 it follows that $\|\overline{u}_i' : C^3(\Sigma_{\leq 2})\| \leq C$. By 4.8 and 4.11 we have $\|\overline{u}_i' : C^3(\Sigma_{\geq 2})\| \leq C$. Thus using 4.12 and the estimates provided by A.3 we can conclude (ii) and (iii). (i) follows from reviewing the construction. Finally from reviewing the construction and using 2.4.vi, 4.1.iii, 3.7, and 2.4.v we obtain (iv) and finish the proof. q.e.d.

The functions w_i.

In this subsection we define and study the functions w_i which play the role of a "substitute kernel" in the sense of [13]. The mean curvature introduced by the "bending" which changes $\Sigma(\theta(T))$ to $\Sigma[T]$ is close to a linear combination of the w_i's and can be chosen at will by choosing $\theta_i(T)$ (see 4.20). This can be used to cancel any unwelcome combination of w_i's which is introduced when we solve the linear equation modulo the w_i's.

Definition 4.18. Let H_ϕ be the mean curvature on the surface $Z_x[\phi](\Sigma(\theta))$ and let $w_x : \Sigma(\theta) \to \mathbb{R}$ be defined by

$$w_x := \left.\frac{d}{d\phi}\right|_{\phi=0} H_\phi \circ Z_x[\phi].$$

Similarly define w_y using Z_y instead of Z_x. We also denote by w_x and w_y the pushforwards on $\Sigma[T, \underline{\varphi}, \tau]$ by $\mathcal{Z}[T, \underline{\varphi}, \tau]$ of these functions. We define then functions w_1 and w_2 on $\Sigma[T, \underline{\varphi}, \tau]$ by

$$w_1 := -w_x + w_y, \qquad w_2 := -w_x - w_y.$$

Lemma 4.19. *The functions w_i $(i = 1, 2)$ defined as in 4.18 on $\Sigma = \Sigma[T, \underline{\varphi}, \tau]$ satisfy the following:*

(i) They depend continuously on $(T, \underline{\varphi}, \tau)$.

(ii) w_i is supported on $\Sigma_{\leq 0}[T, \underline{\varphi}, \tau]$ and $\|w_i : C^{0,\alpha}(\Sigma)\| \leq C$.

(iii) Let $P : \mathbb{R}^2 \to V$, where V is the span of the pushforwards by $\mathcal{Z}[T, \underline{\varphi}, \tau]$ of $\nu \cdot \vec{e}_x$ and $\nu \cdot \vec{e}_y$ on $\Sigma(\theta(T))$, be the linear map which assigns to (η_1, η_2) the orthogonal projection in

$$L^2(\Sigma[T, \underline{\varphi}, \tau], |A|^2_\Sigma g_\Sigma/2)$$

of $(\eta_1 w_1 + \eta_2 w_2)/|A|^2_\Sigma$—which is well defined by (ii)—into V. P is then invertible and $\|P^{-1}\| \leq C$.

Proof. (i) follows easily by reviewing the construction. (ii) follows from the smooth dependence on parameters in 3.13 and the definitions. (iii) amounts to checking that the 2×2 matrix

$$\left(\frac{d}{d\phi} \int H_\phi \circ Z_a[\phi] \, \nu \cdot \vec{e}_b \right),$$

where a and b are either x or y, has a bounded by C inverse. Because of the symmetries the matrix is diagonal. The diagonal entries can be easily calculated by using the balancing formula [27], [19], or a linearized version of it [15], and the proof is complete. q.e.d.

The decomposition of the mean curvature.

We are finally ready to decompose the mean curvature to a part due to the deformation we employed to introduce nonvanishing $\theta_i(T)$'s and which part we can prescribe, a part due to the "bending" of the wings and which we can also prescribe, and finally the rest which we can only estimate as in the next proposition.

Proposition 4.20. *For $(T, \underline{\varphi}, \tau)$ as in Lemma 4.1 the mean curvature H on $\Sigma = \Sigma[T, \underline{\varphi}, \tau]$ satisfies*

$$\|H - \sum_{i=1}^{2} \theta_{i,\Sigma} w_i - \sum_{i=1}^{4} \varphi_i \overline{w}_i : C^{0,\alpha}(\Sigma, g_\Sigma, e^{-\gamma s})\| \leq C\,(\tau + |\underline{\theta}_\Sigma|^2 + |\underline{\varphi}|^2),$$

where $\underline{\theta}_\Sigma = (\theta_{1,\Sigma}, \theta_{1,\Sigma}) = (\theta_1(T) + \varphi_3 - \varphi_1, \theta_2(T) + \varphi_4 - \varphi_2)$.

Proof. On $\Sigma_{\leq 1}$, where $\overline{w}_i = \mathcal{L}_\Sigma \overline{u}'_i$, this follows from the smooth dependence on the parameters in 3.13, 4.1.iii, and B.1. On $\Sigma_{\geq 1}$ it follows from 4.17.ii, 4.19.ii, and 4.5. q.e.d.

5. The initial configurations

A description of the initial configurations.

To describe the construction of our surfaces we introduce a new Cartesian coordinate system $O.x_1x_2x_3$ in E^3. "Horizontal" and "vertical" will be used instead of "on a plane parallel to the x_1x_2-plane" and "parallel to the x_3-axis" respectively. In this section all objects defined are taken to be rotationally invariant with respect to the x_3-axis. Qualifications referring to "height" like above, below, and various others, refer to the x_3 coordinate.

Also by "catenoids" we mean "catenoids (with axis the x_3-axis in accordance with our convention above) or (horizontal) planes". The size of a catenoid is defined to be its distance from its axis (hence 0 for a plane). The "position parameter" or "position" for short of a catenoid is defined to be the x_3-coordinate of its "waist", that is of the circle on the catenoid closest to its axis, when the size is nonzero, and the x_3 coordinate of any point on the plane when the catenoid has size 0 and is therefore a plane.

A catenoidal end is defined to be the intersection of a non-planar catenoid with a halfspace of horizontal boundary, or the complement of an open disc in a plane. A catenoidal disc or annulus is defined to be a (rotationally invariant) topological closed disc or closed annulus lying on a catenoid.

Here, as well as later, we will need to have acceptable tetrads as in 3.1, but associated to the new coordinate system $O.x_1x_2x_3$. To this end we just replace the x, y, and z coordinates in Definition 3.1 (and 2.3), with the coordinates x_1, x_3, and x_2 respectively. The configurations we define here, will be used to guide us in the construction of initial surfaces in the next section. Notice that the circles in $\underline{\mathcal{C}}$ will be replaced with scaled down desingularizing surfaces.

In the next definition δ is a small positive constant which is used to provide uniform control to the geometry of the configuration. Notice that by 5.1.iii the constant δ_θ which we have been using will be defined from now on in terms of δ.

Definition 5.1. An initial configuration $\mathcal{I}$ controlled by $\delta \in (0, 10^{-3})$ consists of two finite sets $\underline{\mathcal{C}} = \{\mathcal{C}_k\}_{k=1}^{N_\mathcal{C}}$ and $\underline{\mathcal{A}} = \{\mathcal{A}_j\}_{j=1}^{N_\mathcal{A}}$ such that the following hold:

(i) Each $\mathcal{C}_k$ is a circle whose radius is in $[\delta, \delta^{-1}]$.

(ii) Each $\mathcal{A}_j$ is a catenoidal disc, annulus, or end, such that its boundary circles are contained in $\underline{\mathcal{C}}$. Moreover if $\mathcal{A}_i$ is not a disc its distance from the x_3-axis is at least δ.

(iii) Each $\mathcal{C}_k \in \underline{\mathcal{C}}$ is a boundary circle of exactly four members $\mathcal{A}_{J(i,k)}$ $(i = 1, ..., 4)$ of $\underline{\mathcal{A}}$, where J is a map from $\{1, ..., 4\} \times \{1, ..., N_\mathcal{C}\}$ to $\{1, ..., N_\mathcal{A}\}$. Moreover there exists a tetrad $T(\mathcal{C}_k)$ such that its ith vector is inward tangent to $\mathcal{A}_{J(i,k)}$ at $\mathcal{C}_k \cap \{(x_1, 0, x_3) : x_1 > 0\}$. $T(\mathcal{C}_k)$ is required to satisfy the conditions that T is required to satisfy in 4.1 with $\delta_\theta = 5\delta$.

(iv) Each $\mathcal{A}_j$ which is compact—not an end—has the property that the Dirichlet problem for $\mathcal{L}_{\mathcal{A}_j}$—recall 4.10—has no eigenvalues in $[-\delta, \delta]$ corresponding to rotationally invariant eigenfunctions.

The above definition describes essentially the situation where we have a number of catenoids intersecting along circles which we have identified, but where we allow the following modification: The regions of a catenoid bordering such a circle do not need to have opposing conormals anymore, that is we allow some change of the catenoid as we cross each of these circles. This amounts to "unbalancing" in the following sense: Notice that the (surface tension) force exerted on a circle's length element by the four catenoidal pieces adjoined to it vanishes if and only if the four pieces belong to two intersecting catenoids, which is equivalent to balancing in the sense of 5.2, where we also define simple quantities to keep track of this unbalancing.

Definition 5.2. Let $\mathcal{V}_\theta := (\mathbb{R}^2)^{N_\mathcal{C}}$. If $\mathcal{I}$ is as in 5.1 we define $\underline{\vartheta}(\mathcal{I}) = \{\underline{\theta}_k = \{\theta_{i,k}\}_{i=1}^2\}_{k=1}^{N_\mathcal{C}} \in \mathcal{V}_\theta$ by

$$\theta_{i,k} = \theta_i(T(\mathcal{C}_k)).$$

If $\underline{\vartheta}(\mathcal{I}) = 0$ we call $\mathcal{I}$ balanced.

When we want to construct embedded minimal surfaces we will need our initial configurations to satisfy a number of further conditions which we present in the next definition. Conditions 5.3.i and 5.3.ii are obviously needed. Condition 5.3.iii will be used to ensure that the ends, whose size we can not really control during the construction later, do not get perturbed to intersect each other. It may be though, that in certain cases violating this last condition, the ends get perturbed in such a way during the construction, that they still avoid each other, giving rise this way to an embedded minimal surface. Further study would be needed to settle this issue.

Definition 5.3. If $\mathcal{I}$ is as in 5.1 we call it δ-embedded if it satisfies the following:

(i) The distance between any two circles in $\underline{\mathcal{C}}$ is at least 5δ.

(ii) The interiors of the elements of $\underline{\mathcal{A}}$ are pairwise disjoint. Moreover the distance between any two sets

$$\mathcal{A}_i \setminus \{p \in E^3 : \operatorname{dist}(p, \partial\mathcal{A}_i) < \delta\}$$

is at least δ^2.

(iii) The sizes of any two catenoidal ends in $\underline{\mathcal{A}}$ differ by at least δ, so in particular there is at most one planar end.

The construction of our minimal surfaces can not be based on a single initial configuration because we do not know a priori what is the exact amount of "unbalancing" we will need. Instead we need to start with families of initial configurations which are perturbations of each other. These families are built around a distinguished configuration $\mathcal{I}$ which is balanced. The other configurations in the family are parametrized by parameters which specify the amount of unbalancing introduced and also changes in position and size of certain ends. These last changes are introduced, so we can cancel any changes introduced during the construction to ensure that the minimal surface constructed has some of its ends asymptotic to chosen ends of $\mathcal{I}$—and some more agreeing in position if not size.

Definition 5.4. A configuration $\mathcal{I}$ as in 5.1 is called a $(\delta, \underline{\mathcal{E}}', \underline{\mathcal{E}}'')$-flexible configuration if the following are true:

(i) $\mathcal{I}$ is controlled by δ and consists of $\underline{\mathcal{C}} = \{\mathcal{C}_k\}_{k=1}^{N_\mathcal{C}}$ and $\underline{\mathcal{A}} = \{\mathcal{A}_j\}_{j=1}^{N_\mathcal{A}}$.

(ii) $\underline{\mathcal{E}}' = \{\mathcal{E}_i\}_{i=1}^{N'_\mathcal{E}} \subset \underline{\mathcal{A}}$ and $\underline{\mathcal{E}}'' = \{\mathcal{E}_i\}_{i=1}^{N''_\mathcal{E}} \subset \underline{\mathcal{A}}$ are two disjoint subsets of ends.

(iii) $\mathcal{I}$ is a balanced initial configuration as in 5.2. Moreover it is equipped with a family of initial configurations $\mathcal{I}[\underline{\vartheta}, \underline{b}]$, each of which controlled by δ and consisting of $\underline{\mathcal{C}}[\underline{\vartheta}, \underline{b}] = \{\mathcal{C}_k[\underline{\vartheta}, \underline{b}]\}_{k=1}^{N_\mathcal{C}}$ and $\underline{\mathcal{A}}[\underline{\vartheta}, \underline{b}] = \{\mathcal{A}_j[\underline{\vartheta}, \underline{b}]\}_{j=1}^{N_\mathcal{A}}$, and such that the following are also true:

(iv) $\underline{\vartheta}$ takes values from $\{\underline{\vartheta} \in \mathcal{V}_\theta : |\underline{\vartheta}| \le \delta^2\}$.

(v) Let $\mathcal{V}_b := \mathbb{R}^{2N'_\mathcal{E}+N''_\mathcal{E}}$. $\underline{b} = (a'_1, ..., a'_{N'_\mathcal{E}}, b'_1, ..., b'_{N'_\mathcal{E}}, b''_1, ..., b''_{N''_\mathcal{E}})$ takes values from $\{\underline{b} \in \mathcal{V}_b : |\underline{b}| \le \delta^2\}$.

(vi) $\mathcal{I} = \mathcal{I}[0,0]$ and the maps J—recall 5.1.iii—are independent of $(\underline{\vartheta}, \underline{b})$.

(vii) Each $\mathcal{C}_k[\underline{\vartheta}, \underline{b}]$, and the position and size of the catenoid on which each $\mathcal{A}_j[\underline{\vartheta}, \underline{b}]$ lies, depend smoothly on the parameters $(\underline{\vartheta}, \underline{b})$.

(viii) $\underline{\vartheta} = \underline{\vartheta}(\mathcal{I}[\underline{\vartheta}, \underline{b}])$.

(ix) For each $\mathcal{E}_i \in \underline{\mathcal{E}}'$ the position and size of the catenoid on which $\mathcal{E}_i[\underline{\vartheta}, \underline{b}]$ lies differ from those for $\mathcal{E}_i = \mathcal{E}_i[0,0]$ by a'_i and b'_i respectively.

(x) Similarly for each $\mathcal{E}_i \in \underline{\mathcal{E}}''$ the position of the catenoid on which $\mathcal{E}_i[\underline{\vartheta}, \underline{b}]$ lies differs from the corresponding for $\mathcal{E}_i = \mathcal{E}_i[0,0]$ by b''_i.

Finally the next definition determines the families which we will be using for constructing embedded minimal surfaces.

Definition 5.5. If $\mathcal{I}$ is as in 5.4 and each $\mathcal{I}[\underline{\vartheta}, \underline{b}]$ is δ-embedded as in 5.3, then we call $\mathcal{I}$ an embedded $(\delta, \underline{\mathcal{E}}', \underline{\mathcal{E}}'')$-flexible configuration.

Constructions of initial configurations.

In this subsection we construct various initial configurations on which we will apply our main theorem to obtain the desired minimal surfaces. For a balanced initial configuration $\bigcup_{j=1}^{N_{\mathcal{A}}} \mathcal{A}_j$ has clearly to be the union of a finite number of catenoids. Reversing this we will construct initial configurations from a given collection of catenoids. We start with a definition.

Definition 5.6. Given a finite set of non-planar catenoids $\underline{\mathcal{K}} = \{\mathcal{K}_i\}_{i=1}^{N_{\mathcal{K}}}$ and a finite set of planes $\underline{\mathcal{P}} = \{\mathcal{P}_i\}_{i=1}^{N_{\mathcal{P}}}$ we say that $(\underline{\mathcal{K}}, \underline{\mathcal{P}})$ is in general position if the following conditions are satisfied:

(i) Any two of the planes or catenoids intersect transversely along circles which do not intersect any other plane or catenoid. Let $\underline{\mathcal{C}} = \{\mathcal{C}_k\}_{k=1}^{N_{\mathcal{C}}}$ be the collection of these circles.

(ii) For each catenoid in $\underline{\mathcal{K}}$ or plane in $\underline{\mathcal{P}}$ consider the connected components of the complement of $\bigcup_{k=1}^{N_{\mathcal{C}}} \mathcal{C}_k$. Let $\underline{\mathcal{A}} = \{\mathcal{A}_j\}_{j=1}^{N_{\mathcal{A}}}$ be the collection of the closures of all these components. We request that for each catenoidal annulus $\mathcal{A} \in \underline{\mathcal{A}}$ the kernel of $\mathcal{L}_{\mathcal{A}}$ on $\mathcal{A}$ with vanishing boundary data contains no rotationally invariant functions.

If the above conditions are satisfied, we call $(\underline{\mathcal{C}}, \underline{\mathcal{A}})$ the induced configuration by $(\underline{\mathcal{K}}, \underline{\mathcal{P}})$. We also define $\underline{\mathcal{E}}' \subset \underline{\mathcal{A}}$ to be the set of the top ends of each catenoid in $\underline{\mathcal{K}}$, and $\underline{\mathcal{E}}'' \subset \underline{\mathcal{A}}$ to be the set of the ends of each plane in $\underline{\mathcal{P}}$.

Lemma 5.7. *Given sets of catenoids and planes $(\underline{\mathcal{K}}, \underline{\mathcal{P}})$ in general position as in 5.6, there is a $\delta > 0$ and a $(\delta, \underline{\mathcal{E}}', \underline{\mathcal{E}}'')$-flexible configuration $\mathcal{I}$ consisting of $\underline{\mathcal{C}}$ and $\underline{\mathcal{A}}$, where $\underline{\mathcal{C}}$, $\underline{\mathcal{A}}$, $\underline{\mathcal{E}}'$ and $\underline{\mathcal{E}}''$, are as in 5.6. Moreover, by reducing δ if necessary, we can ensure that $\mathcal{I}$ is an embedded $(\delta, \underline{\mathcal{E}}', \underline{\mathcal{E}}'')$-flexible configuration, provided that there are no two catenoids of the same size in $\underline{\mathcal{K}}$ and there is at most one plane in $\underline{\mathcal{P}}$.*

Proof. It is easy to see that if we choose δ small enough, then the conditions in 5.6 guarantee that $\underline{\mathcal{C}}$ and $\underline{\mathcal{E}}$ define an initial configuration $\mathcal{I}$ as in 5.1 controlled by δ. Next we have to define $\mathcal{I}[\underline{\vartheta}, \underline{b}]$ as in 5.4. We define first instead of $\mathcal{I}[\underline{\vartheta}, \underline{b}]$ $\mathcal{I}(\underline{\vartheta}, \underline{b})$ for which we require the same conditions as for $\mathcal{I}[\underline{\vartheta}, \underline{b}]$ except for the following modification for 5.4.x: For an end $\mathcal{E}'' \in \underline{\mathcal{E}}''$, consider the planar disc $\mathcal{A} \in \underline{\mathcal{A}}$, which is contained in the same plane in $\underline{\mathcal{P}}$ as $\mathcal{E}''$. Instead of comparing then the positions of $\mathcal{E}''[\underline{\vartheta}, \underline{b}]$ and $\mathcal{E}''$, we compare the positions of $\mathcal{A}[\underline{\vartheta}, \underline{b}]$ and $\mathcal{A}$ instead.

Before we proceed to define $\mathcal{I}(\underline{\vartheta}, \underline{b})$ we renumber if necessary $\underline{\mathcal{C}} = \{\mathcal{C}_i\}_{i=1}^{N_\mathcal{C}}$ so that for each $\mathcal{C}_i$ either $\mathcal{C}_i$ lies above $\mathcal{C}_{i+1}$, or $\mathcal{C}_i$ and $\mathcal{C}_{i+1}$ are at the same height, and the radius of $\mathcal{C}_{i+1}$ is larger than the radius of $\mathcal{C}_i$ (see for example figure 1). We proceed now to define $\mathcal{I}(\underline{\vartheta}, \underline{b})$. First of all the current variant of 5.4.x specifies the catenoids on which those $\mathcal{A}_j$'s which correspond either to top ends of catenoids in $\underline{\mathcal{K}}$ or to innermost discs contained in planes in $\underline{\mathcal{P}}$ lie. Using 5.4.viii we specify then the circles in $\underline{\mathcal{C}}(\underline{\vartheta}, \underline{b})$, and the catenoids on which the remaining elements of $\underline{\mathcal{A}}(\underline{\vartheta}, \underline{b})$ lie, inductively as follows: At the nth inductive step we specify the position of $\mathcal{C}_n(\underline{\vartheta}, \underline{b})$, and the catenoids on which the $\mathcal{A}_{J(i,n)}(\underline{\vartheta}, \underline{b})$'s lie (recall 5.1.iii). This is possible because by our way of numbering the $\mathcal{C}_n$'s we already know the catenoids on which exactly two of the $\mathcal{A}_{J(i,n)}(\underline{\vartheta}, \underline{b})$'s lie.

It is clear now that if $\underline{\vartheta}$ and $\underline{b}$ are small enough, then $\mathcal{I}(\underline{\vartheta}, \underline{b})$ is well defined. Thus we can find $\underline{b}'$ such that $\mathcal{I}(\underline{\vartheta}, \underline{b}) = \mathcal{I}[\underline{\vartheta}, \underline{b}']$. The map $(\underline{\vartheta}, \underline{b}) \to (\underline{\vartheta}, \underline{b}')$ is clearly smooth and maps $(0, \underline{b})$ to $(0, \underline{b})$ by the geometry of the construction. Using the inverse function theorem then we can invert it and define $\mathcal{I}[\underline{\vartheta}, \underline{b}]$ for small enough $\underline{\vartheta}$ and $\underline{b}$. Finally checking that the remaining conditions in 5.1 and 5.4 are satisfied is straightforward provided that we choose δ small enough. Moreover if there are no two catenoids of the same size in $\underline{\mathcal{K}}$, and at most one plane in $\underline{\mathcal{P}}$, then we can arrange for the conditions in 5.3 to be satisfied by all $\mathcal{I}[\underline{\vartheta}, \underline{b}]$ by choosing δ small enough. q.e.d.

The significance of the last lemma is that, as we will see in Section 8, a collection of catenoids and planes in general position can be desingularized to give us a minimal surface which will be complete, of finite total curvature, and embedded if the above extra conditions are satisfied. It is exceptional for the catenoids and planes not to be in general position, and to present this in a systematic way we give the following definition.

Definition 5.8. Let $\mathcal{M}$ be the "configuration space" of $N_\mathcal{K}$ nonplanar catenoids and $N_\mathcal{P} \leq 1$ planes. We identify $\mathcal{M}$ with $(\mathbb{R}^+ \times \mathbb{R})^{N_\mathcal{K}} \times \mathbb{R}^{N_\mathcal{P}}$, and hence we induce a topology on $\mathcal{M}$, by using the sizes and positions of the catenoids and the positions of the planes as coordinates. Let $\mathcal{M}_{(\delta)} \subset \mathcal{M}$ consist of those $\underline{x} \in \mathcal{M}$ such that there is a neighborhood of $\underline{x}$ in $\mathcal{M}$ such that if $\underline{y}$ belongs to this neighborhood, then the following is true: The arrangement of the catenoids and planes corresponding to $\underline{y}$ induces a $\mathcal{I}$ as in 5.7, which is an embedded $(\delta, \underline{\mathcal{E}}', \underline{\mathcal{E}}'')$-flexible configuration.

We give the same definitions when $N_{\mathcal{P}} > 1$ but without requiring embeddedness.

Lemma 5.9. *In all cases $\mathcal{M}_{(\delta)}$ is an open subset of $\mathcal{M}$ increasing as δ decreases, and $\cup_\delta \mathcal{M}_{(\delta)}$ is an open dense subset of $\mathcal{M}$.*

Proof. From the proof of 5.7 it is clear that all we need to check is that those arrangements of the catenoids and planes which are not in general position, or have two catenoids of the same size in the case $N_{\mathcal{P}} \leq 1$, form a set whose complement is dense. But this is easy to see because by arbitrarily small changes in the sizes and positions of the catenoids and planes we can ensure the required conditions one by one. q.e.d.

The above constructions admit modifications which we will only mention but not carry out in detail. We will discuss them somewhat more in Section 8. The required changes in the exposition are minimal and we hope that the reader will have no problem carrying them out without further help.

Remark 5.10. The above constructions of flexible configurations could be modified as follows: We could choose not to include some of the circles of intersection in $\underline{\mathcal{C}}$. This would result in minimal surfaces which have these circles as circles of self-intersections. We could also impose certain symmetries in the construction from the beginning. This would result in minimal surfaces with some extra symmetries.

6. The initial surfaces

The construction of the initial surfaces.

In this subsection we assume a flexible configuration $\mathcal{I}$ as in 5.4 given and we construct a smooth family of initial surfaces we will use later. Our constructions will be highly symmetrical. We define next the group of imposed symmetries.

Definition 6.1. We fix an $m \in \mathbb{N}$, $m > 3$, and define $\mathcal{G}$ to be the group of symmetries generated by reflections with respect to the planes passing through the x_3-axis and forming an angle $k\pi/m$ $(k \in \mathbb{Z})$ with the x_1-axis.

Notice that the number of the fundamental regions with respect to $\mathcal{G}$ in each initial surface will be then $2m$. Before we proceed with the construction, we need to determine also the number of fundamental regions which the Scherk surface desingularizing $\mathcal{C}_k$ will have per fundamental region of the initial surface. To this end we preassign an $m_k \in \mathbb{N}$ to each $\mathcal{C}_k \in \underline{\mathcal{C}}$. We also define

$$\overline{\tau} := 1/m, \qquad \tau_k := \overline{\tau}/m_k = 1/mm_k, \qquad \underline{m} := \{m_k\}_{k=1}^{Nc}. \tag{6.2}$$

τ_k will be the τ parameter of the surface desingularizing $\mathcal{C}_k$ while $\overline{\tau}$ controls the overall size of the τ parameters. From now on unless explicitly stated otherwise we will also tacitly assume the following:

(i) Our constants C depend on δ and $\underline{m}$.

(ii) m is large enough—and hence $\overline{\tau}$ small enough—in terms of δ and $\underline{m}$ so that our constructions and estimates are valid.

Now since we have fixed m and $\underline{m}$, we can proceed to the construction of the initial surfaces. These will depend on parameters: $\underline{\vartheta}$ which control the unbalancing introduced, $\underline{b}$ which control the change of position and perhaps size of chosen ends, and finally $\underline{\phi}$ which control the bending of the wings relative to the core of each desingularizing surface used. In the next definition we specify the ranges of these parameters. Recall that $\mathcal{V}_\theta$ was defined in 5.2 and $\mathcal{V}_b$ in 5.4.v.

Definition 6.3. Let $\mathcal{V}_\varphi := (\mathbb{R}^4)^{Nc}$ and $\mathcal{V} := \mathcal{V}_\theta \times \mathcal{V}_b \times \mathcal{V}_\varphi$. We also fix a $\zeta > 0$, which will be determined in the proof of 8.2, and we define $\Xi_{\mathcal{V}} := \{\xi \in \mathcal{V} : |\xi| \leq \zeta\overline{\tau}\}$.

We fix now a $\xi = (\underline{\vartheta}, \underline{b}, \underline{\phi}) \in \Xi_{\mathcal{V}}$, where $\underline{\vartheta}$ is as in 5.2, $\underline{b}$ is as in 5.4, and $\underline{\phi} = \{\underline{\varphi}_k\}_{k=1}^{Nc}$ where $\underline{\varphi}_k = (\varphi_{i,k})_{i=1}^4$, and we proceed to construct the initial surface $M = M(\xi) = M(\underline{\vartheta}, \underline{b}, \underline{\phi})$. Let us concentrate our attention to $\mathcal{I}[\underline{\vartheta}, \underline{b}]$. To make use of the desingularizing surfaces which we constructed in Section 3 where we used a different coordinate system, let a "suitable homothety" be given by

$$(x, y, z) \to (x_1, x_2, x_3) = c(x, z, y),$$

where $c \in \mathbb{R}^+$. There is a unique composition of a suitable homothety followed by a translation which maps the circle $\mathcal{B}_{\tau_k}(z\text{-axis})$ onto $\mathcal{C}_k[\underline{\vartheta}, \underline{b}]$. We call this composition $\mathcal{H}_k$ suppressing the parameters $(\underline{\vartheta}, \underline{b})$ on which it depends. We also define a tetrad $T_k(\xi) = \{\vec{e}_{i,k}\}_{i=1}^4$ as follows: If the tetrad $T(\mathcal{C}_k[\underline{\vartheta}, \underline{b}])$ has the relation to the $\beta_{i,k}$'s which T has to the β_i's in 3.1, then $T_k(\xi)$ has the same relation to the $\beta_{i,k}+\varphi_{i,k}$'s $(i=1,...,4)$. We thus define a surface $\mathcal{S}_k(\xi)$ by

$$\mathcal{S}_k(\xi) := \mathcal{H}_k(\Sigma[T_k(\xi), \underline{\tilde{\varphi}}_k(\xi), \tau_k]), \tag{6.4}$$

where $\underline{\tilde{\varphi}}_k(\xi) = \{\tilde{\varphi}_{i,k}\}_{i=1}^4$ will be determined later. Notice however that the ith pivot of $\Sigma[T_k(\xi), \underline{\tilde{\varphi}}_k(\xi), \tau_k]$ does not depend on $\underline{\tilde{\varphi}}_k(\xi)$—recall 3.12—and hence we can define independently of $\underline{\tilde{\varphi}}_k(\xi)$

$$\mathcal{C}'_{i,k}(\xi) := \mathcal{H}_k(\mathcal{C}_i[T_k(\xi), \underline{\tilde{\varphi}}_k(\xi), \tau_k]).$$

Consider now $\mathcal{A}_{J(i,k)}[\underline{\vartheta}, \underline{b}]$. If it is an annulus, then there are i', $k' \neq k$ such that $J(i,k) = J(i',k')$ and

$$\partial\mathcal{A}_{J(i,k)}[\underline{\vartheta}, \underline{b}] = \mathcal{C}_k[\underline{\vartheta}, \underline{b}] \cup \mathcal{C}_{k'}[\underline{\vartheta}, \underline{b}].$$

By 5.1.iv, for $j = J(i,k) = J(i',k')$ there exists a catenoidal annulus $\mathcal{A}'_j(\xi)$, which is a small perturbation of $\mathcal{A}_j[\underline{\vartheta}, \underline{b}]$, and is uniquely determined by the requirement

$$\partial\mathcal{A}'_j(\xi) = \mathcal{C}'_{i,k}(\xi) \cup \mathcal{C}'_{i',k'}(\xi).$$

There is clearly a unique value for $\tilde{\varphi}_{i,k}$ such that

$$\mathcal{H}_k(\partial_i[T_k(\xi), \underline{\tilde{\varphi}}_k(\xi), \tau_k]) \subset \mathcal{A}'_j(\xi),$$

that is—recall 3.14—$\mathcal{S}_k(\xi)$ has both the image of the pivot and the corresponding boundary circle lying on $\mathcal{A}'_j(\xi)$. This implies that there is a neighborhood of this boundary circle in $\mathcal{S}_k(\xi)$ which lies in $\mathcal{A}'_j(\xi)$, that is, the two surfaces would "match" each other if we were to remove an appropriate neighborhood of the boundary of $\mathcal{A}'_j(\xi)$. To this end we define

$$\begin{aligned}\mathcal{A}''_j(\xi) := \mathcal{A}'_j(\xi) \setminus \Big(&\mathcal{H}_k \circ A_i[T_k(\xi), \underline{\tilde{\varphi}}_k(\xi), \tau_k](H^+_{\leq 4\delta_s/\tau}) \\ &\bigcup \mathcal{H}_{k'} \circ A_{i'}[T_{k'}(\xi), \underline{\tilde{\varphi}}_{k'}, \tau_{k'}](H^+_{\leq 4\delta_s/\tau})\Big),\end{aligned} \tag{6.5}$$

where $\tilde{\varphi}_{i',k'}$ has been defined similarly to $\tilde{\varphi}_{i,k}$ above.

Since we have taken care of the case that $\mathcal{A}_{J(i,k)}$ is an annulus and has two boundary circles, we concentrate in the case where $\mathcal{A}_{J(i,k)}$ is a planar disc or a catenoidal end, and hence has only one boundary circle. Let $j = J(i,k)$. Then we have $\partial\mathcal{A}_j[\underline{\vartheta}, \underline{b}] = \mathcal{C}_k[\underline{\vartheta}, \underline{b}]$, define

$$\mathcal{A}'_j(\xi) := \mathcal{H}[c^+, c](\mathcal{A}_j[\underline{\vartheta}, \underline{b}]),$$

where $\mathcal{H}[c^+, c](x_1, x_2, x_3) := c^+(x_1, x_2, x_3 + c)$ with $c^+ \in \mathbb{R}^+$ and $c \in \mathbb{R}$ chosen by the requirement that

$$\mathcal{H}[c^+, c](\mathcal{C}_k[\underline{\vartheta}, \underline{b}]) = \mathcal{C}'_{i,k}(\xi).$$

As before we can define $\tilde{\varphi}_{i,k}$ by requesting

$$\mathcal{H}_k(\partial_i[T_k(\xi), \underline{\tilde{\varphi}}_k(\xi), \tau_k]) \subset \mathcal{A}'_j(\xi),$$

and we also define

$$\mathcal{A}''_j(\xi) := \mathcal{A}'_j(\xi) \setminus \mathcal{H}_k \circ A_i[T_k(\xi), \underline{\tilde{\varphi}}_k(\xi), \tau_k](H^+_{\leq 4\delta_s/\tau}). \tag{6.6}$$

At this point we have defined $\tilde{\varphi}_{i,k}$ for all (i, k) and hence all the $\underline{\tilde{\varphi}}_k(\xi)$'s. In the next definition besides defining $M(\xi)$ we also define quantities which will be useful in discussing the changes that the parameters in ξ "suffered" during the construction: $\underline{\tilde{\vartheta}}(\xi)$ describes the precise unbalancing adopted, $\underline{\tilde{b}}(\xi)$ describes the changes of positions and sizes of chosen ends from those they had in $\mathcal{I}[\underline{\vartheta}, \underline{b}]$, and $\underline{\tilde{\phi}}(\xi)$ describes the actual bending of the wings relative to the cores. We also define M', because part of M is built as a graph over M' and we will occasionally need to refer to it.

Definition 6.7. Continuing from the above exposition we define

$$M = M(\xi) = M(\underline{\vartheta}, \underline{b}, \underline{\phi}) := \bigcup_{k=1}^{N_{\mathcal{C}}} \mathcal{S}_k(\xi) \cup \bigcup_{j=1}^{N_{\mathcal{A}}} \mathcal{A}''_j(\xi),$$

$$M' = M'(\xi) = M'(\underline{\vartheta}, \underline{b}, \underline{\phi}) := \bigcup_{j=1}^{N_{\mathcal{A}}} \mathcal{A}'_j(\xi),$$

$$\mathcal{S} = \mathcal{S}(\xi) := \bigcup_{k=1}^{N_{\mathcal{C}}} \mathcal{S}_k(\xi).$$

We push forward the function s by $\mathcal{H}_k$ to each $\mathcal{S}_k$, and then extend it to a discontinuous function on the whole M by taking $s = \max_{\mathcal{S}} s$ on the rest of M. Similarly we define s on M'.

Finally we define $\underline{\tilde{\vartheta}} = \underline{\tilde{\vartheta}}(\xi) \in \mathcal{V}_\theta$, $\underline{\tilde{b}} = \underline{\tilde{b}}(\xi) \in \mathcal{V}_b$, and $\underline{\tilde{\phi}} = \underline{\tilde{\phi}}(\xi) \in \mathcal{V}_\varphi$, as follows:

(i) $\underline{\tilde{\vartheta}} = \{\underline{\tilde{\theta}}_k\}_{k=1}^{N_{\mathcal{C}}}$ where $\underline{\tilde{\theta}}_k = \{\tilde{\theta}_{i,k}\}_{i=1}^{2}$ and $\tilde{\theta}_{i,k} = \theta_i(T_k(\xi))$.

(ii) $\underline{\tilde{b}} = (\tilde{a}'_1, ..., \tilde{a}'_{N'_{\mathcal{E}}}, \tilde{b}'_1, ..., \tilde{b}'_{N'_{\mathcal{E}}}, \tilde{b}''_1, ..., \tilde{b}''_{N''_{\mathcal{E}}})$, where for each $\mathcal{A}_j = \mathcal{E}'_l \in \underline{\mathcal{E}}'$, the size and the position of the catenoid, on which $\mathcal{A}'_j(\xi)$ lies, differ, by $\tilde{a}'_l$ and $\tilde{b}'_l$ respectively, from those of the catenoid on which $\mathcal{A}_j[\underline{\vartheta}, \underline{b}]$ lies and for each $\mathcal{A}_j = \mathcal{E}''_l \in \underline{\mathcal{E}}''$, the position of the catenoid on which $\mathcal{A}'_j(\xi)$ lies, differs by $\tilde{b}''_l$ from the position of the catenoid on which $\mathcal{A}_j[\underline{\vartheta}, \underline{b}]$ lies.

(iii) $\underline{\tilde{\phi}} = \{\underline{\tilde{\varphi}}_k\}_{k=1}^{N_C}$, where $\underline{\tilde{\varphi}}_k$ has already been defined.

Recall that as we discussed in the introduction, our strategy requires that we cover the initial surfaces by neighborhoods of the standard pieces, where each neighborhood of a standard piece consists of the standard piece and the joining pieces next to it. For the standard pieces which are (cores of) Scherk surfaces we can use the $\mathcal{S}_k(\xi)$'s as the corresponding neighborhoods. In the next definition we define the neighborhoods which we will be using for the remaining standard pieces that are catenoidal ends and annuli or planar discs.

Definition 6.8. Let $\underline{a} := 8|log\overline{\tau}| = 8\log m$. Notice that for $M = M(\xi)$ as in 6.7, each component of $M_{\geq \underline{a}}$ contains exactly one $\mathcal{A}''_j(\xi)$. The component which contains $\mathcal{A}''_j(\xi)$ will be called $\mathcal{N}_j = \mathcal{N}_j(\xi)$.

In the next proposition we present the most striking properties of the initial constructed surfaces. However when more precise information is required, a review of the details of the construction will be needed. Recall that $|\,.\,|$ stands for the maximum norm, C depends only on δ and $\underline{m}$, and $\mathcal{G}$ and $\underline{a}$ were defined in 6.1 and 6.8 respectively. Also notice that when the $(\underline{\vartheta}, \underline{b})$ parameters of circles or catenoidal pieces are not specified, we are referring to those of $\mathcal{I} = \mathcal{I}[0,0]$ (recall 5.4).

Proposition 6.9. *$M = M(\xi)$ with $\xi \in \Xi_{\mathcal{V}}$ is well defined by 6.7, and has the following properties:*

(i) M is a complete (boundaryless) smooth surface which depends smoothly on ξ.

(ii) M respects the symmetries in $\mathcal{G}$.

(iii) There is a large ball B such that $M \setminus B$ is the union of catenoidal ends which are in one-to-one correspondence with the catenoidal ends in $\underline{\mathcal{A}}$ in such a way that the sizes and positions of the catenoids on which the corresponding ends lie are close. Moreover for each

end $\mathcal{E}'_j \in \underline{\mathcal{E}}'$ there is a catenoidal end $\tilde{\mathcal{E}}'_j \subset M \setminus B$ such that the sizes and positions of the catenoids on which the two ends lie differ by $a'_j + \tilde{a}'_j$ and $b'_j + \tilde{b}'_j$ respectively. Similarly for each end $\mathcal{E}''_j \in \underline{\mathcal{E}}''$ there is a catenoidal end $\tilde{\mathcal{E}}''_j \subset M \setminus B$ such that the positions of the catenoids on which the two ends lie differ by $b''_j + \tilde{b}''_j$.

(iv) For all $i = 1, 2$ and $k = 1, ..., N_{\mathcal{C}}$ we have $\tilde{\theta}_{i,k} = \theta_{i,k} + \varphi_{i+2,k} - \varphi_{i,k}$.

(v) $|\underline{\tilde{b}}| \le C\overline{\tau}$ and $|\underline{\tilde{\phi}} + \underline{\phi}| \le C\overline{\tau}$.

(vi) $M_{\ge 1}$ is a graph over $M'_{\ge 1}$ by a function f_M such that $\|f_M : C^5(M'_{\ge \underline{a}})\| \le \overline{\tau}^3$.

(vii) Consider a sequence of initial surfaces M with varying ξ's but increasing m's. Let U be a neighborhood of $\bigcup_{k=1}^{N_{\mathcal{C}}} \mathcal{C}_k$. Then the sequence of the initial surfaces tends uniformly in C^k norm to $\bigcup_{j=1}^{N_{\mathcal{A}}} \mathcal{A}_j$ on any compact subset of $E^3 \setminus U$. Similarly $\mathcal{H}_k^{-1}(M)$ tends uniformly on compact subsets of E^3 to a Scherk surface.

Proof. That M is well defined and satisfies (i) is clear from the construction and preceding discussion. (ii) follows from 3.13. (iii) follows by searching through the definitions and 6.7.ii in particular. The expression for $\underline{\tilde{\vartheta}}$ in (iv) follows by the definition of the $T_k(\xi)$'s and 6.7.i. The first estimate in (v) follows easily by the construction and the second one because the conormals at the boundary of each $\mathcal{A}'_k$ differ from the corresponding ones for $\mathcal{A}_k$ by $\le C\overline{\tau}$ since the corresponding boundaries have been moved by $\le C\overline{\tau}$. (vi) follows from the definitions and rescaling 2.4.v (f_M is clearly supported on $\mathcal{S}(\xi)$). Finally (vii) follows from the construction and 6.3. q.e.d.

In the next corollary for future reference we review the control which we have over the parameters of the desingularizing surfaces that we use.

Corollary 6.10. *The parameters of each $\Sigma[T, \underline{\varphi}, \tau] = \mathcal{H}_k^{-1}(\mathcal{S}_k(\xi))$—recall 6.4—satisfy the following:*

(i) $50\delta \le \theta(T) \le \frac{\pi}{2} - 50\delta$.

(ii) $|\theta_i(T)| \le 3\zeta\overline{\tau}$.

(iii) $|\underline{\varphi}| \le (\zeta + C)\overline{\tau}$.

(iv) $\overline{\tau}/|\underline{m}| \le \tau \le \overline{\tau}$.

Proof. Recall from 6.4 that $T = T_k(\xi)$, $\underline{\varphi} = \underline{\tilde{\varphi}}_k(\xi)$, and $\tau = \tau_k$. (i) follows then by referring to the definition of $T_k(\xi)$, 5.4.iii, 5.1.iii, 6.3, and by assuming $\overline{\tau}$ small enough in terms of δ. (ii) follows by 6.9.iv and 6.3. (iii) follows from 6.9.v and 6.3. (iv) follows from 6.2. q.e.d.

The ends of the initial surfaces.

While the desingularizing surfaces have been studied already and the geometry of the catenoidal annuli and discs in an initial surface is more or less fully controlled by 6.9.vi, in the case of the ends we will need some more information which we present in this subsection. This information is needed in solving the linearized equation in the next section. We start with a lemma which gives parametrizations for catenoidal ends in the style needed for applying Lemma A.3.

Lemma 6.11. *A catenoidal end $\mathcal{E}$ of size a and boundary circle of radius r admits a parametrization $K_{\mathcal{E}} : H^+/G' \to \mathcal{E}$ where G' is the group generated by $(s,z) \to (s, z+2\pi)$, such that the pullbacks by $K_{\mathcal{E}}$ of the first and second fundamental forms are $\hat{\varrho}^2(ds^2+dz^2)$ and $a(ds^2-dz^2)$ respectively, where $\hat{\varrho} := r\cosh s \pm \sqrt{r^2-a^2}\sinh s$ and the dubious sign is $-$ if and only if $\mathcal{E}$ contains the waist of the catenoid on which it lies.*

Proof. By using the Enneper-Weierstrass representation it is easy to check that the general form of the expressions is correct. The correctness of the coefficients can be checked then by direct calculation. q.e.d.

It turns out that depending on the size of the Gauss image of a catenoidal end, different strategies are adopted for solving the linearized equation on it. In preparation for this we give the following definitions. Notice that as we state in 6.14 later h-small or h-large refers to the size of the punctured disk which is the Gauss image of the end in consideration. Notice also that in the case of an h-large end, the boundary circle of $\tilde{\mathcal{N}}_j(\xi)$ is large enough to ensure that $f_M = 0$ on the circle.

Definition 6.12. We fix once and for all an $\epsilon_K \in (0,\delta)$ which will be determined later.

Suppose $M = M(\xi)$ and $M' = M'(\xi)$ are as in 6.7, and $\mathcal{A}_j$ is an end for some $j = 1, ..., N_{\mathcal{A}}$. We then define $\tilde{\mathcal{N}}_j(\xi) \subset \mathcal{N}_j(\xi)$ as follows: Let r be the radius of the boundary circle of $\mathcal{A}_j$ and a the size of the catenoid on which $\mathcal{A}_j$ lies. If $r > a/\epsilon_K$ and $\mathcal{A}_j$ does not contain its waist, then we define $\tilde{\mathcal{N}}_j(\xi) := \mathcal{N}_j(\xi)$—recall 6.8; otherwise we define

$\tilde{\mathcal{N}}_j(\xi)$ by demanding that it is a catenoidal end contained in $\mathcal{N}_j(\xi)$ which does not contain its waist and has a boundary circle of radius $2a/\epsilon_K$. We call $\mathcal{A}_j$ and $\mathcal{N}_j(\xi)$ h-small in the first case, or h-large in the latter case.

For such a j we also define the following: A parametrization $K_j(\xi)$ of $\tilde{\mathcal{N}}_j(\xi)$ by $K_j(\xi) = \Pi \circ K_{\tilde{\mathcal{N}}'_j}$—recall 6.11—where $\tilde{\mathcal{N}}'_j = \Pi^{-1}(\tilde{\mathcal{N}}_j(\xi)) \subset \mathcal{A}'_j(\xi)$ and $\Pi : M'_{\geq 1} \to M_{\geq 1}$ is the projection of $M'_{\geq 1}$ to the graph over it $M_{\geq 1}$—recall 6.9.vi. A map $\tilde{\nu} : \mathcal{N}_j(\xi) \to \mathbb{S}^2$ by requiring $\tilde{\nu} \circ \Pi = \nu_{M'}$ on $\Pi^{-1}(\mathcal{N}_j(\xi))$. And finally a metric h on $\mathcal{N}_j(\xi)$ by $h := |A|^2_{\mathcal{N}_j(\xi)}\, g_{\mathcal{N}_j(\xi)}/2$.

Notice that $\mathcal{N}_j(\xi) = \tilde{\mathcal{N}}_j(\xi)$ if and only if $\mathcal{N}_j(\xi)$ is an h-small end. Otherwise—by assuming ϵ_K and δ_s small enough—$\tilde{\mathcal{N}}_j(\xi)$ does not intersect $\mathcal{S}(\xi)$. The next lemma prepares us to apply A.3 on the $\tilde{\mathcal{N}}_j(\xi)$'s in the next section.

Lemma 6.13. *Let $K_j(\xi) : \Omega \to \tilde{\mathcal{N}}_j(\xi)$ be as in the above definition where $\Omega = H^+/G'$ is as in 6.11, and let $g_0 = d\hat{s}^2 + dz^2$, $\chi := \hat{\varrho}^{-2} K_j^* g_M$, and $d := \hat{\varrho}^2 |A|^2_M \circ K_j$, where we rename into $\hat{s}$ the coordinate s on Ω to avoid confusion with the s already defined on M. Then we have*

$$\begin{aligned} \|\chi - g_0 : C^3(\Omega, g_0, e^{-\hat{s}})\| &\leq \overline{\tau}, \\ \|d : C^3(\Omega, g_0, e^{-\hat{s}})\| &\leq C\epsilon_K^2. \end{aligned}$$

Proof. Notice that $\chi = g_0$ unless $\mathcal{N}_j$ is a small end in which case $\chi - g_0$ is supported where $\hat{s} \leq 4\delta_s$ and the first inequality follows easily. The second one follows from 5.1, 6.11, and 6.12. q.e.d.

For h-large ends we will use the following lemma to study the linearized equation on them.

Lemma 6.14. *An h-large end $\mathcal{N}_j(\xi)$ as in 6.12 has as image under $\tilde{\nu}$ a round, geodesic, punctured at the center, disc of radius $\geq C\epsilon_K$. Moreover $h = \tilde{\nu}^*(g_{\mathbb{S}^2})$ on $\tilde{\mathcal{N}}_j(\xi)$ and*

$$\|h - \tilde{\nu}^*(g_{\mathbb{S}^2}) : C^3(\mathcal{N}_j(\xi), h)\| \leq \overline{\tau}.$$

Proof. The first part follows from 6.12 by rescaling to the unit catenoid. This shows also that $\tilde{\nu}$ is the Gauss map on $\tilde{\mathcal{N}}_j(\xi)$. The estimate then follows by 2.4.v and the definitions. q.e.d.

7. The linearized equation

Introductory discussion.

In this section we study the solutions to (the appropriately modified) linearized equations and we produce the required estimates for them. As we have mentioned before, the strategy for solving the linearized equation is to "quasi-localize" by concentrating our attention to neighborhoods of the "standard pieces" consisting of the standard piece in question joined with the adjacent "joining pieces". In this paper we can take as standard pieces the—perturbed during the construction of the initial surfaces—cores of the Scherk surfaces augmented with a bit of "margin", and the—also modified to fit the initial surfaces—catenoidal pieces which appear in the initial configurations. As joining pieces we take the wings of the Scherk surfaces. The exact position where the line between the various pieces can be drawn is more or less arbitrary. We found it convenient to draw it so that the neighborhoods of the standard pieces mentioned above turn out to be the $\mathcal{S}_k(\xi)$'s and the $\mathcal{N}_j(\xi)$'s (defined in 6.4 and 6.8).

To solve now the inhomogeneous linear equation on the initial surfaces, according to our approach, we have to distribute the inhomogeneous term to these neighborhoods of the standard pieces, by using a suitable partition of unity subordinate to the neighborhoods. Then we solve the linear equation with the appropriate inhomogeneous term on each such neighborhood and with boundary data which can be for example Dirichlet, or whatever else may seem convenient in each case. As we have mentioned before these "quasi-local" solutions have to be patched-up to make a global one.

This creates an error though, which we may be able to correct by iterating the process, provided it is small compared to the initial inhomogeneous term. The error can actually be arranged to be small, by using the $\overline{u}$ and $\overline{w}$ functions, as we have outlined in the introduction and Section 4. Notice that we do not need such functions on all the neighborhoods of the standard pieces in consideration; we just need one per joining piece. In our case we have indeed one $\overline{u}$ and one $\overline{w}$ per wing, and they are supported on the $\mathcal{S}_k(\xi)$ containing the wing. This is convenient since the main inhomogeneous term which we are dealing with—the mean curvature of the initial surfaces—is supported on these pieces. Finally, on those neighborhoods on which we have small eigenvalues, that is the $\mathcal{S}_k(\xi)$'s in this paper, we are forced to solve modulo the w's which we have discussed before.

This approach, where we "quasi-localize" and iterate to solve the linear equation modulo the w's and $\overline{w}$'s, appeared first in embryonic form in [13]. Its purpose there was limited: It allowed us to simplify the study of the approximate kernel keeping it finite dimensional, and perhaps simplified some of the arguments and the intuition behind them. In [13] though the decay along the joining pieces was not studied carefully enough and we used no $\overline{w}$'s. Because of this, the estimates, which we could get following the approach in [13], turned out to be insufficient for the problem solved in [14], where the full method applied here was developed. The main advantage of this approach seems to be that it provides a systematic way of setting up the construction, highlighting the idiosyncratic difficulties of each problem, and suggesting through the "geometric principle" [14] - [15] a way to resolve some of them.

In the next subsection we concentrate on solving the linearized equation on the $\mathcal{S}_k(\xi)$'s, then in the following one on the $\mathcal{N}_j(\xi)$'s, and in the final subsection we put everything together applying it also to the main case where the inhomogeneous term is the mean curvature of the initial surface.

The linearized equation on the $\mathcal{S}_k(\xi)$'s.

In this subsection we state and prove Proposition 7.1, in which we solve and estimate the inhomogeneous Dirichlet problem for the linearized equation on the $\mathcal{S}_k(\xi)$'s. Actually it is better to work in the natural scale of these pieces. To this end we fix in this subsection a $\mathcal{S}_k(\xi)$, which is as in 6.4, and we concentrate our attention to the corresponding desingularizing surface $\Sigma = \Sigma[T, \underline{\varphi}, \tau] = \mathcal{H}_k^{-1}(\mathcal{S}_k(\xi))$ whose parameters we control by 6.10. The symmetry group of our construction $\mathcal{G}$ induces a group of symmetries G_τ on Σ which is generated by the reflections with respect to the xy-plane and the plane $\mathcal{B}_\tau(\{z = m_G\pi\})$, where $m_G = m_k$ really, and so it satisfies $1 \le m_G \le |\underline{m}|$. In the rest of this subsection we assume that all the functions on Σ which we consider are invariant under the action of G_τ. From now on we fix a $\gamma \in (3/4, 1)$ (recall 4.5).

Proposition 7.1. *Given $E \in C^{0,\alpha}(\Sigma)$ there are*

$$\underline{\theta}_E = \{\theta_{E,i}\}_{i=1}^2 \in \mathbb{R}^2, \quad \underline{\varphi}_E := \{\varphi_{E,i}\}_{i=1}^4 \in \mathbb{R}^4,$$

and $v_E \in C^{2,\alpha}(\Sigma)$, such that:

(i) $\underline{\theta}_E$, $\underline{\varphi}_E$, and v_E are uniquely determined by the construction below.

(ii) $\mathcal{L}_\Sigma v_E = \sum_{i=1}^{2} \theta_{E,i}\, w_i + \sum_{i=1}^{4} \varphi_{E,i}\, \overline{w}_i + E$ *on* Σ *and* $v_E = 0$ *on* $\partial\Sigma$.

(iii) $|\underline{\theta}_E| \le C\|E\|$, *where* $\|E\| := \|E : C^{0,\alpha}(\Sigma, g_\Sigma, e^{-\gamma s/m_G})\|$.

(iv) $|\underline{\varphi}_E| \le C\|E\|$.

(v) $\|v_E : C^{2,\alpha}(\Sigma, g_\Sigma, e^{-\gamma s/m_G})\| \le C\|E\|$.

The rest of this subsection is devoted to the proof of 7.1. We first reduce to the case where E is supported on $\Sigma_{\le 2}$.

Lemma 7.2. *If 7.1 is valid when E is supported on $\Sigma_{\le 2}$, then it is valid in general.*

Proof. Assume we are given an E whose support is not restricted. We argue in a similar way as in 4.13-17: We concentrate our attention to the component of $\Sigma_{\ge 1}[T, \underline{\varphi}, \tau]$ which is contained in the ith wing of Σ for some $i = 1, ..., 4$. Consider the cylinder $\Omega = [0, 5\delta_s/m_G\tau - 1] \times \mathbb{R}/G'$ where G' is the group generated by $(s, z) \to (s, z + 2\pi)$. Let $F = F_j[T, \underline{\varphi}, \tau] \circ R$ be the reparametrization of the jth wing by using the scaling R defined by $R(s, z) = (m_G s + 1, m_G z)$. As in 4.13 we can consistently define because of the symmetries

$$\chi := m_G^{-2}\varrho^{-2}F^*(g_\Sigma), \qquad d := m_G^2\varrho^2|A|_\Sigma^2 \circ F.$$

By the usual slight abuse of notation we then have

$$\underline{\mathcal{L}} = 2m_G^2\varrho^2\mathcal{L}_\Sigma,$$

where $\underline{\mathcal{L}}$ is as in A.2. As in 4.15 we can thus ensure that we can apply A.3 to obtain $v'_E = \underline{\mathcal{R}}(0, 2m_G^2\varrho^2 E)$. We call also v'_E the pushforward by F of v'_E. We repeat the same in the other components of $\Sigma_{\ge 1}$ and define on Σ

$$E' := E - \mathcal{L}_\Sigma(\psi[1, 2] \circ s\, v'_E).$$

Clearly E' is supported on $\Sigma_{\le 2}$, and we can apply 7.1 by our assumption to obtain $v_{E'}$, $\underline{\theta}_{E'}$, $\underline{\varphi}_{E'}$ as in 7.1. We then define $\underline{\theta}_E = \underline{\theta}_{E'}$, $\underline{\varphi}_E = \underline{\varphi}_{E'}$, and $v_E = v_{E'} + (\psi[1, 2] \circ s)v'_E$. The proof is hence complete by the estimates in A.3. q.e.d.

In the next definition we define h on Σ in analogy with 2.7. Our motivation is that we can then understand the low spectrum for the appropriately modified linearized equation by comparing with the situation in 2.8. Notice that in the next definition we employ a small positive constant ϵ_h to ensure that the metric h is non-degenerate. Apart from this, ϵ_h has no effect and can be ignored.

Definition 7.3. We define a metric h on Σ by $h := (\frac{1}{2}|A|^2_\Sigma + \epsilon_h) g_\Sigma$, where $\epsilon_h > 0$ depends on $\overline{\tau}$, $\underline{m}$, and δ, and will be determined in the proof of 7.4. Given $c > 0$ we define the c-approximate kernel to be the span of those eigenfunctions of the Dirichlet problem for $\mathcal{L}_h := \Delta_h + 2|A|^2_\Sigma/(|A|^2_\Sigma + 2\epsilon_h)$ on Σ/G_τ which have corresponding eigenvalues in $[-c, c]$.

It turns out that $(\Sigma/G_\tau, h)$ and $(\Sigma(\theta(T))/G, h)$—G is as in 2.8—are close to being isometric except for small—in the h metrics—regions which do not influence the lower spectrum much. This allows us to understand the lower spectrum of $(\Sigma/G_\tau, h)$ in terms of that of $(\Sigma(\theta(T))/G, h)$, and then making use of 2.8 and 4.19 we prove in the next proposition that the inhomogeneous term can be corrected by using the w's as we have outlined before.

Lemma 7.4. *There are positive constants C and c depending only on δ and $\underline{m}$ such that given $E \in L^2(\Sigma/G_\tau, h)$ there is $\underline{\theta}_E = (\theta_{E,1}, \theta_{E,2})$ such that $(E - \sum_{i=1}^2 \theta_{E,i} w_i)/(|A|^2_\Sigma/2 + \epsilon_h)$ is $L^2(\Sigma/G_\tau, h)$-orthogonal to the c-approximate kernel and*

$$|\underline{\theta}| \leq C \| E/(|A|^2_\Sigma + 2\epsilon_h) : L^2(\Sigma/G_\tau, h)\|.$$

Proof. We focus our attention first to eigenfunctions of $\mathcal{L}_h$ on $(\Sigma/G_\tau, h)$ (defined in 7.3) of low eigenvalue, say less than 10. We prove a uniform estimate for them as follows: Let f be such an eigenfunction, that is $\mathcal{L}_h f + \lambda f = 0$, where $\lambda < 10$. It is clear by the uniform control of the geometry of $\Sigma_{\leq 2}$ that

$$\| f : C^0(\Sigma_{\leq 1})\| \leq C \| f : L^2(\Sigma/G_\tau, h)\|.$$

To estimate on $\Sigma_{\geq 1}$ we define χ, Ω, and F as in the proof of 7.2. Then by writing f for $f \circ F$ as well we have on Ω $\underline{\mathcal{L}} f = 0$ where $\underline{\mathcal{L}}$ is as in A.2 with $d = m_G^2 \varrho^2 ((1 + \lambda/2)|A|^2_\Sigma \circ F + \lambda \epsilon_h)$. As in 4.15 we can ensure that we can apply A.3, and by referring to A.3.vi we conclude that

$$\| f : C^0(\Sigma)\| \leq C \| f : L^2(\Sigma/G_\tau, h)\|. \tag{1}$$

Consider now $(\Sigma(\theta(T))/G, h)$ defined as in 2.8, and $\mathcal{L}_h$ on it defined as in 2.7. It follows from standard theory that the eigenfunctions of low eigenvalue—say < 10 again—for $\mathcal{L}_h$ on $(\Sigma(\theta(T))/G, h)$ satisfy

$$\|f : C^0(\Sigma(\theta(T)))\| \leq C\|f : L^2(\Sigma(\theta(T))/G, h)\|. \tag{2}$$

We define now for a function f on $\Sigma(\theta(T))/G$ a function $\mathcal{F}_1(f)$ on Σ/G_τ as follows: Let $\rho : \mathbb{S}^2 \to \mathbb{R}$ denote the distance from $\{(\pm\sin\theta, \pm\cos\theta, 0)\}$, which by 2.6.i is the complement of the Gauss image of $\Sigma(\theta(T))$. We define a logarithmic cutoff function $\psi^2_{\mathbb{S}} : \mathbb{S}^2 \to [0,1]$ by

$$\psi^2_{\mathbb{S}}(p) = \psi[2,1](\log\rho(p)/\log\delta_h),$$

where δ_h is a small positive constant to be determined in the course of the proof. Notice that $\psi^2_{\mathbb{S}}$ vanishes at distance $\leq \delta_h^2$ from these points, and $\psi^2_{\mathbb{S}} \equiv 1$ at distance $\geq \delta_h$ from them. Then we define $\mathcal{F}_1(f) = Z_*(f\,\psi^2_{\mathbb{S}} \circ \nu)$, where $Z = \mathcal{Z}[T, \underline{\varphi}, \tau]$. In other words $\mathcal{F}_1$ is the push-forward by Z after an appropriate truncation. Similarly for f on Σ/G_τ we define $\mathcal{F}_2(f)$ on $\Sigma(\theta(T))/G$ by $\mathcal{F}_2(f) = (f \circ Z)(\psi^2_{\mathbb{S}} \circ \nu)$.

Consider the region on $\Sigma(\theta(T))/G$ where $\psi^2_{\mathbb{S}} \circ \nu \neq 1$. By 2.6.iii and the definition of $\psi^2_{\mathbb{S}}$ we can ensure that its h-area is arbitrarily small by choosing δ_h small enough. Similarly the region on Σ/G_τ where $Z_*(\psi^2_{\mathbb{S}} \circ \nu) \neq 1$, by 4.4, has arbitrarily small area if we choose δ_h and ϵ_h small enough. With δ_h chosen now, we can ensure that on the region of $\Sigma(\theta(T))/G$ where $\psi^2_{\mathbb{S}} \circ \nu \neq 0$, Z is as close to an isometry as we like, by appealing to the smooth dependence on parameters in 3.13. ($\overline{\tau}$ by our conventions can be assumed as small as needed depending on all other constants except for ϵ_h.)

The above imply that if we have eigenfunctions of unit $L^2(h)$ norm, and of eigenvalue < 10, satisfying thus (1) or (2), and we apply $\mathcal{F}_i$ (i either 1 or 2), then their inner product can be ensured to change by an arbitrarily small amount, and the $L^2(h)$ norms of their gradients to increase also by arbitrarily small amounts. This implies that the low eigenvalues for $\mathcal{L}_h$ on $(\Sigma(\theta(T))/G, h)$ are arbitrarily close to the ones for $\mathcal{L}_h$ on $(\Sigma(\theta(T))/G, h)$.

Moreover the $\mathcal{F}_i$'s carry such eigenfunctions to functions which have arbitrarily small distance in $L^2(h)$ from linear combinations of eigenfunctions of eigenvalue close to the original one. The proof of these statements is based on the variational characterization of the eigenfunctions and eigenvalues [1]. The argument uses the fact that the $\mathcal{F}_i$'s can not increase much the Rayleigh quotients, and is by induction in increasing eigenvalues. The details are technical and elaborate though, and rather unilluminating, and since the reader can find them in [13, appendix B] we omit them.

We define now $c = \epsilon_\lambda/2$ by referring to 2.8. Using 2.8, 4.19, and the statements above we thus complete the proof. q.e.d.

We are ready now to finish the proof of 7.1. Notice that the only use of the h-metric is in obtaining L^2 estimates for the solution.

Proof of Proposition 7.1. By 7.2 we can assume that E is supported on $\Sigma_{\leq 2}$. By the smooth dependence in 3.13 and by assuming $\overline{\tau}$ small enough we have uniform control on the geometry of $\Sigma_{\leq 2}$ and hence

$$\|E/(|A|^2_\Sigma + 2\epsilon_h) : L^2(\Sigma/G_\tau, h)\| \leq C\|E\|. \tag{1}$$

Now we can apply 7.4 to obtain $\underline{\theta}_E = (\theta_{E,1}, \theta_{E,2})$ as in 7.4. Moreover, consider the Dirichlet problem on Σ

$$\mathcal{L}_h v'_E = (E - \sum_{i=1}^{2} \theta_{E,i} w_i)/(|A|^2_\Sigma/2 + \epsilon_h) \tag{2}$$

with vanishing boundary Dirichlet data. The inhomogeneous term is $L^2(\Sigma/G_\tau, h)$-orthogonal to the c-approximate kernel, (iii) is satisfied by (1) and 7.4, and the $L^2(\Sigma/G_\tau, h)$ norm of the inhomogeneous term is $\leq C\|E\|$ by 4.19 and the above. Then by the implied L^2 estimate, the fact that (2) is equivalent to

$$\mathcal{L}_\Sigma v'_E = \sum_{i=1}^{2} \theta_{E,i}\, w_i + E, \tag{3}$$

the uniform control on the geometry of $\Sigma_{\leq 3}$, and the standard linear theory, we conclude that there is a unique solution v'_E that satisfies

$$\|v'_E : C^{2,\alpha}(\Sigma_{\leq 2}, g_\Sigma)\| \leq C\|E\|. \tag{4}$$

It remains to arrange the exponential decay along the wings. We define

$$v_E = v'_E + \sum_{i=1}^{4} \varphi_{E,i}\overline{u}_i,$$

where the $\varphi_{E,i}$'s are determined as follows: First of all we can set up the situation exactly analogous to the proof of 7.2 so that A.3 applies for $\underline{\mathcal{L}}$ as in that proof, and by using F we can pull-back the restrictions of v'_E and the $\overline{u}_i$'s on the component of $\Sigma_{\geq 2}$ contained in the jth wing of Σ, to functions with the same names on Ω, a cylinder as in A.1 of length $(5\delta_s/\tau - 2)/m_G$. We then define

$$a_j = \operatorname{avg}_{\partial_0} v'_E - B(v'_E, 0), \qquad a_{i,j} = \operatorname{avg}_{\partial_0} \overline{u}_i - B(\overline{u}_i, 0),$$

where ∂_0 and B are as in A.1 and A.3. By A.3.ii and the uniqueness in A.3.vi we have that $v_E = \underline{\mathcal{R}}(v_E, 0)$ if and only if

$$a_j + \sum_{i=1}^{4} \varphi_{E,i} a_{i,j} = 0.$$

Using A.3.v, the above, and 4.17.iv, we can solve the above system of equations ($j = 1, ..., 4$) to obtain unique $\varphi_{E,i}$'s with appropriate estimates. Hence we can finish the proof by the above, 4.17, and A.3.iv.

q.e.d.

The linearized equation on the $\mathcal{N}_j(\xi)$'s.

In this subsection we fix an $\mathcal{N} = \mathcal{N}_j(\xi)$ and find a solution to $\mathcal{L}_{\mathcal{N}} v = E$ on $\mathcal{N}$, where $\mathcal{L}_{\mathcal{N}}$ is the linearized operator $\mathcal{L}_S$ of B.1 for $S = \mathcal{N}$. This is a simpler undertaking than the one in the previous subsection because of the absence of the approximate kernels, w's, $\overline{u}$'s, and $\overline{w}$'s. There is no decay required, and the geometry is simple and in uniform control, except that we have to distinguish between compact $\mathcal{N}$'s and various kinds of ends. We start by defining appropriate norms for v and E. Notice that on the ends appropriate decay has to be imposed. Sometimes we need to exclude the function from the strict control of these norms on $\mathcal{S}(\xi)$, and for this reason we define the primed versions of these norms.

Definition 7.5. For $v \in C^{r,\alpha}(\mathcal{N})$ $(r = 0, 2)$ we define $\|v : \mathcal{N}\|_r$ as follows:

(i) If $\mathcal{N}$ is a disc or an annulus, then

$$\|v : \mathcal{N}\|_r := \|v : C^{r,\alpha}(\mathcal{N}, g_{\mathcal{N}})\|.$$

(ii) If $\mathcal{N}$ is an h-small end, then

$$\|v : \mathcal{N}\|_r := \|\hat{\varrho}^{2-r} v : C^{r,\alpha}(\mathcal{N}, g_0, e^{-\gamma \hat{s}})\|,$$

where $\hat{s}$ and g_0 are as in 6.13.

(iii) If $\mathcal{N}_j$ is an h-large end, let $\tilde{\mathcal{N}} = \tilde{\mathcal{N}}_j(\xi)$ and $\hat{s}$ and g_0 be as before. We then define

$$\|v : \mathcal{N}\|_r := \|v : C^{r,\alpha}(\mathcal{N} \setminus \tilde{\mathcal{N}}, g_{\mathcal{N}})\| + \|\hat{\varrho}^{2-r} v : C^{r,\alpha}(\tilde{\mathcal{N}}, g_0, e^{-\gamma \hat{s}})\|.$$

Finally we define $\|v : \mathcal{N}\|'_r$ by replacing $\mathcal{N}$ with $\mathcal{N} \setminus \mathcal{S}(\xi)$ in the above definitions.

Lemma 7.6. *If $\mathcal{N}$ is as above, then there is a unique $v_E \in C^{2,\alpha}_{loc}(\mathcal{N}_j)$ by its construction below, such that $\mathcal{L}_{\mathcal{N}} v_E = E$ on $\mathcal{N}$, and*

$$\|v_E : \mathcal{N}\|_2 \leq C \|E : \mathcal{N}\|_0.$$

Proof. Suppose first that $\mathcal{N}$ is a disc or an annulus. Then by 5.1.iv and the fact that $\mathcal{N}$ is a small perturbation of $\mathcal{A}_j$ the Dirichlet problem for $\mathcal{L}_{\mathcal{N}}$ has no small eigenvalues. Thus there is a unique solution v_E to $\mathcal{L}_{\mathcal{N}} v_E = E$ on $\mathcal{N}$, $v_E = 0$ on $\partial\mathcal{N}$, and it satisfies the required estimate by the standard linear theory and the uniform control which we have on the geometry.

If $\mathcal{N}$ is a h-small end as in 6.12, we can apply A.4 on the setting provided by 6.13 in a way entirely analogous to the use of A.3 in the proof of 7.2, where without loss of generality we assume ϵ_K small enough. We define thus $v_E = \underline{\mathcal{R}}(2\hat{\varrho}^2 E)$, and all conditions are satisfied by A.4.

Suppose finally $\mathcal{N}$ is a large end. By applying A.4 as we just did on $\tilde{\mathcal{N}}_j(\xi)$, and arguing as in 7.2, we can assume that E is supported on a region where h—recall 6.12—and $g_{\mathcal{N}}$ are uniformly equivalent. We can rewrite our equation as $(\Delta_h + 2)v_E = 2E/|A|^2_{\mathcal{N}}$. We identify, via $\tilde{\nu}$, $\mathcal{N}$ with its image. Then by 6.14, $\Delta_h + 2$ is a small perturbation of $\Delta + 2$ where Δ is the standard Laplacian on $\mathbb{S}^2$.

In order to solve for v_E we need to specify also the boundary conditions. To ensure the absence of small eigenvalues for $\Delta + 2$ and hence $\Delta_h + 2$ as well, we impose boundary conditions as follows: Let $\mathcal{A}_j \in \underline{\mathcal{A}} = \underline{\mathcal{A}}[0,0]$ be the corresponding end to $\mathcal{N} = \mathcal{N}_j(\xi)$. If its Gauss image is close to being a hemisphere, then we impose Neumann boundary conditions, otherwise we impose Dirichlet boundary conditions. In both cases we get a solution v_E which is smooth on the closure of $\tilde{\nu}(\mathcal{N})$. By adding to it a multiple of $\nu_{\mathcal{N}} \cdot \vec{e}_3$, that is of the coordinate on $\mathbb{S}^2$ which has an extremum at the puncture, we can ensure that v_E vanishes at the puncture and still $\mathcal{L}_{\mathcal{N}} v_E = E$ on $\mathcal{N}$. This vanishing by 6.11 translates in our language to decay like $e^{-\hat{s}}$. The estimates thus follow by the standard theory. q.e.d.

The linearized equation globally on M.

We combine now the results of Lemmas 7.1 and 7.6 to obtain global solutions on the initial surfaces. We fix first an initial surface $M = M(\xi)$. We first need to define appropriate norms so that we can state our estimates later. We basically combine the norms that we have already defined on the various pieces. Notice though that when the $\mathcal{H}_k$'s are used to relate the $\mathcal{S}_k(\xi)$'s with the desingularizing surfaces, they induce a change of scale. To keep things consistent, our functions have to undergo a corresponding change of scale as well. This is the reason for the factor τ_k^{1-r} in 7.7.i and the factor $1/\tau_k$ in 7.9.

The factors b_r^{-1} in 7.7.ii now are weights reflecting that the decay functions have suffered along the wings of the desingularizing surfaces—notice that we are using the primed version of the norms so that we do not interfere with the situation on $\mathcal{S}(\xi)$—and whose decay is of order $e^{-5\gamma\delta_s/m_k\tau_k} = e^{-5\gamma\delta_s/\overline{\tau}}$. The extra factor in b_2 is to accommodate for some powers of $\overline{\tau}$ which we loose during the process of solving the linearized equation globally on M, and is insignificant compared to the exponential factor. To be consistent we have to modify also the decay in 7.7.i so that it does not exceed the one allowed in 7.7.ii; this is the reason the b_r's appear in 7.7.i as well.

Definition 7.7. Given $v \in C^{r,\alpha}_{loc}(M)$ $(r = 0, 2)$ we define $\|v\|_r$ to be the maximum of the following quantities, where $b_0 = e^{-5\gamma\delta_s/\overline{\tau}}$ and $b_2 = b_0/\overline{\tau}^{10}$:

(i) For each $k = 1, ..., N_{\mathcal{C}}$ we have—recall 6.4 and 6.2—

$$\Sigma = \Sigma[T_k(\xi), \underline{\tilde{\varphi}}_k(\xi), \tau_k] = \mathcal{H}_k^{-1}(\mathcal{S}_k(\xi)).$$

Consider the quantity

$$\tau_k^{1-r}\|v \circ \mathcal{H}_k : C^{r,\alpha}(\Sigma, g_\Sigma, \max(e^{-\gamma s/m_k}, b_r))\|.$$

(ii) For each $j = 1, ..., N_{\mathcal{A}}$ consider the quantity $b_r^{-1}\|v : \mathcal{N}_j(\xi)\|'_r$.

Notice that the imposed decay along the ends by these norms—see 7.5–is rather slow. This is because in this part we have not used the imposed symmetry. If we did and the proof of 7.6 made use of it, we could have much faster decay on the ends as stated in 7.8.

Remark 7.8. The rate of decay at the ends could be accelerated by replacing $e^{-\gamma\hat{s}}$ in 7.5 with $e^{-\gamma\hat{s}/m}$, and our theorems would be still valid.

Since on the Σ's we solve the linearized equation modulo the w's and the $\overline{w}$'s, we have to transplant them to M by using the $\mathcal{H}_k$'s and solve on M modulo these transplanted versions. In the next definition we make this systematization to facilitate future reference.

Definition 7.9. Let $\mathcal{V}' := \mathcal{V}_\theta \times \mathcal{V}_\varphi$. We define a linear map $\Theta : \mathcal{V}' \to C^\infty(M)$ by

$$\Theta(\underline{\vartheta}', \underline{\phi}') = \sum_{k=1}^{N_{\mathcal{C}}} \frac{1}{\tau_k} \mathcal{H}_{k*} \left(\sum_{i=1}^{2} \theta'_{i,k} w_i + \sum_{i=1}^{4} \varphi'_{i,k} \overline{w}_i \right),$$

where $\underline{\vartheta}' = \{\underline{\theta}'_k = \{\theta'_{i,k}\}_{i=1}^{2}\}_{k=1}^{N_{\mathcal{C}}} \in \mathcal{V}_\theta$, $\underline{\phi}' = \{\underline{\varphi}'_k = \{\varphi'_{i,k}\}_{i=1}^{4}\}_{k=1}^{N_{\mathcal{C}}} \in \mathcal{V}_\varphi$, and $\mathcal{H}_{k*}$ denotes pushforward by $\mathcal{H}_k$ of a function on $\Sigma[T_k(\xi), \tilde{\underline{\varphi}}_k(\xi), \tau_k]$ to a function on $\mathcal{S}_k(\xi)$ extended to vanish on the rest of M.

We are ready finally to state the main result of the section, Proposition 7.10, where the linearized equation is solved globally on M. As we have discussed in more detail earlier, we partition the inhomogeneous term to the $\mathcal{S}_k(\xi)$'s and the $\mathcal{N}_j(\xi)$'s, use 7.1 and 7.6 to solve and estimate on these pieces, patch up the solutions which introduces some error, and finally iterate to correct the error. A more careful examination of the proof shows that we effectively first solve on the $\mathcal{S}_k(\xi)$'s, and then cut off this solution and include the error in the inhomogeneous term on the $\mathcal{N}_j(\xi)$'s; here we use the exponential decay along the wings to ensure that the error is small even under the demanding norm which we have on the $\mathcal{N}_j(\xi)$'s. We then solve on the $\mathcal{N}_j(\xi)$'s and cut off close to their boundary. Finally we iterate. Notice that while we cut off the solutions and estimate we may loose some powers of $\overline{\tau}$, but this is accommodated either by the exponential factors or the definition of b_2.

Proposition 7.10. *Given $E \in C^{0,\alpha}(M)$ with finite $\|E\|_0$ there are $(\underline{\vartheta}_E, \underline{\phi}_E) \in \mathcal{V}'$ and $v_E \in C^{2,\alpha}(M)$, uniquely determined by the construction below, such that*

$$\mathcal{L}_M v_E = E + \Theta(\underline{\vartheta}_E, \underline{\phi}_E)$$
$$|\underline{\vartheta}| \le C\|E\|_0, \quad |\underline{\phi}| \le C\|E\|_0, \quad \|v_E\|_2 \le C\|E\|_0.$$

Proof. We take $E_0 := E$ and proceed inductively where given E_{n-1} we define E_n, v_n, $\underline{\vartheta}_n$, and $\underline{\phi}_n$, as follows: First we partition E_{n-1}. We define a cut-off function ψ on M by $\psi := \psi[5\delta_s\tau_k^{-1}, 5\delta_s\tau_k^{-1} - 1] \circ s$

on each $\mathcal{S}_k(\xi)$ for $k = 1, ..., N_{\mathcal{C}}$, and by $\psi \equiv 0$ on the rest of M. For $k = 1, ..., N_{\mathcal{C}}$ we apply 7.1 with $\Sigma = \Sigma[T_k(\xi), \underline{\tilde{\varphi}}_k(\xi), \tau_k] = \mathcal{H}_k^{-1}(\mathcal{S}_k(\xi))$ and $E = \tau_k(\psi E_{n-1}) \circ \mathcal{H}_k$ to obtain v_E, $\underline{\theta}_E$, and $\underline{\varphi}_E$. We rename $\underline{\theta}_E$ and $\underline{\varphi}_E$ to $\underline{\theta}_{n,k}$ and $\underline{\varphi}_{n,k}$ respectively, and define $v = \tau_k \mathcal{H}_{k*}(v_E)$, where $\mathcal{H}_{k*}$ is as in 7.9. By carrying this out for all k we obtain v defined on $\mathcal{S}(\xi)$, $\underline{\vartheta}_n := \{\underline{\theta}_{n,k}\}_{k=1}^{N_{\mathcal{C}}}$, and $\underline{\phi}_n := \{\underline{\varphi}_{n,k}\}_{k=1}^{N_{\mathcal{C}}}$, satisfying on $\mathcal{S}$

$$\mathcal{L}_M v = \psi E_{n-1} + \Theta(\underline{\vartheta}_n, \underline{\phi}_n). \tag{1}$$

For $j = 1, ..., N_{\mathcal{A}}$ we apply now 7.6 with $\mathcal{N} = \mathcal{N}_j(\xi)$ and $E = (1-\psi^2)E_{n-1} - [\mathcal{L}_M, \psi]v$, where $[\mathcal{L}_M, \psi]f := \mathcal{L}_M(\psi f) - \psi \mathcal{L}_M f$, to obtain a function v_E which we rename v'. By carrying this out for all j we have v' defined on $M_{\geq \underline{a}}$—recall 6.8—where it satisfies

$$\mathcal{L}_M v' = (1-\psi^2)E_{n-1} - [\mathcal{L}_M, \psi]v. \tag{2}$$

We proceed to patch-up v and v' to a function on M. To cut-off v' we introduce $\psi' := \psi[\underline{a}, \underline{a}+1] \circ s$ defined on M, and define $v_n = \psi v + \psi' v'$. Since $\psi' \equiv 1$ on the supports of $1 - \psi^2$ and $[\mathcal{L}_M, \psi]$, we conclude from (1) and (2) that on M

$$\mathcal{L}_M v_n = E_{n-1} + [\mathcal{L}_M, \psi']v' + \Theta(\underline{\vartheta}_n, \underline{\phi}_n). \tag{3}$$

We define $E_n = -[\mathcal{L}_M, \psi']v'$. By the estimates in 7.1 and 7.6 and the various definitions we obtain

$$\|E_n\|_0 \leq e^{-\delta_s/\overline{\tau}} \|E_{n-1}\|_0.$$

By taking $v_E := \sum_{n=1}^{\infty} v_n$, $\underline{\vartheta}_E := \sum_{n=1}^{\infty} \underline{\vartheta}_n$, and $\underline{\phi}_E := \sum_{n=1}^{\infty} \underline{\phi}_n$, and using the available estimates from 7.1 and 7.6, we finish the proof. q.e.d.

We can apply now 7.10 to the most important case where the inhomogeneous term is the mean curvature of M.

Corollary 7.11. *There are $v_H \in C^{2,\alpha}(M)$ and $(\underline{\vartheta}_H, \underline{\phi}_H) \in \mathcal{V}'$ such that*

$$\mathcal{L}_M v_H = H + \Theta(\underline{\vartheta}_H, \underline{\phi}_H),$$
$$|\underline{\vartheta}_H - \underline{\vartheta}| \leq C\overline{\tau}, \quad |\underline{\phi}_H + \underline{\phi}| \leq C\overline{\tau}, \quad \|v_H\|_2 \leq C\overline{\tau},$$

where $M = M(\xi) = M(\underline{\vartheta}, \underline{b}, \underline{\phi})$.

Proof. Recall that $M = M(\xi)$ with ξ as in 6.3 and the subsequent discussion. By 6.9.iv, 6.10, and 4.20 we have

$$\|H + \Theta(\underline{\vartheta} + \underline{\vartheta}', \underline{\tilde{\phi}})\|_0 \le C\overline{\tau}, \tag{1}$$

where $\underline{\tilde{\phi}}$ is as in 6.7 and $\underline{\vartheta}' = \{\underline{\theta}'_k = \{\theta'_{i,k}\}_{i=1}^2\}_{k=1}^{N_{\mathcal{C}}}$, with $\theta'_{i,k} = \tilde{\varphi}_{i+2,k} - \tilde{\varphi}_{i,k} + \varphi_{i+2,k} - \varphi_{i,k}$. Applying the proposition with $E = H + \Theta(\underline{\vartheta} + \underline{\vartheta}', \underline{\tilde{\phi}})$ we obtain v_E, $\underline{\vartheta}_E$, and $\underline{\phi}_E$. By defining $\underline{\vartheta}_H = \underline{\vartheta} + \underline{\vartheta}' + \underline{\vartheta}_E$ and $\underline{\phi}_H = \underline{\tilde{\phi}} + \underline{\phi}_E$ we thus complete the proof because the required estimates follow by (1), 7.10, and 6.9.v. q.e.d.

8. The main results

The nonlinear terms.

At this stage we have all the information needed to state and prove our theorem except for estimating the nonlinear part of the mean curvature of a graph over the initial surfaces. Therefore we fix an initial surface $M = M(\xi)$ and state the following proposition where the nonlinearity is estimated appropriately. Notice that in the proof the specific form of nonlinearity is only used when estimating over the $\tilde{\mathcal{N}}$'s, and even there the information we need is minimal.

Proposition 8.1. *Given $v \in C^{2,\alpha}(M)$ with $\|v\|_2$ smaller than a suitable constant, we have that the graph M_v of v over M is a smooth immersion and moreover*

$$\|H_v - H - \mathcal{L}_M v\|_0 \le C\|v\|_2^2,$$

where H and H_v are the mean curvature of M and M_v pulled back to M respectively.

Proof. Suppose first that $\Sigma = \mathcal{H}^{-1}(\mathcal{S}_k(\xi))$ for some $k = 1, ..., N_{\mathcal{C}}$. By appealing to 3.13 where the parameters take values on a compact set, we can ensure that $\|A : C^3(\Sigma_{\le 2}, g_\Sigma)\| \le C$. Then we have $\|A : C^3(\Sigma, g_\Sigma)\| \le C$ by using on $\Sigma_{\ge 2}$ 2.4.v and 4.3, and can apply B.1 to assert that if $\|f : C^{2,\alpha}(\Sigma, g_\Sigma, e^{-\gamma s/m_k} + b_2)\|$ is small enough, then the graph of f over Σ is an immersed surface and

$$\begin{aligned} &\|H_f - H - \mathcal{L}_\Sigma f : C^{0,\alpha}(\Sigma, g_\Sigma, e^{-\gamma s/m_k})\| \\ &\qquad \le C\|f : C^{2,\alpha}(\Sigma, g_\Sigma, e^{-\gamma s/m_k} + b_2)\|^2. \end{aligned} \tag{1}$$

Suppose now that $\mathcal{A}_j$ for some $j = 1, ..., N_{\mathcal{A}}$ is either a disc or annulus, or an h-large end. Then we take $\mathcal{A}$ to be $\mathcal{N}_j(\xi)\backslash\mathcal{S}(\xi)$ in the first case, or $\mathcal{N}_j(\xi)\backslash(\tilde{\mathcal{N}}_j(\xi)\cup\mathcal{S}(\xi))$ in the latter. In both cases we have uniform control of the geometry of $\mathcal{A}$ by 5.1, and hence $\|A : C^3(\mathcal{A}, g_{\mathcal{A}})\| \leq C$. We thus apply B.1 to conclude that if $\|v : C^{2,\alpha}(\mathcal{A}, g_M)\|$ is small enough, then M_v is an immersed surface above $\mathcal{A}$ and

$$\|H_v - H - \mathcal{L}_M v : C^{0,\alpha}(\mathcal{A}, g_M)\| \leq C\|v : C^{2,\alpha}(\mathcal{A}, g_M)\|^2. \tag{2}$$

Suppose finally that $\mathcal{A}_j$ is an end. Let $\tilde{\mathcal{N}} = \tilde{\mathcal{N}}_j(\xi)$. We would like to use the metric g_0 instead of g_M in the expression for the nonlinearity (recall 6.11 and 6.13). We can do this and replace contractions and covariant derivatives with respect to g_M in the expression for Q in B.1, with the corresponding ones with respect to g_0, provided that we replace A with $\varrho^{-2}A$, ∇A with $\varrho^{-3}(\nabla_0 A + \Gamma * A)$ where ∇_0 is the connection induced by g_0 and Γ the difference of the connections, and so on. Applying then B.1 we easily conclude that if $\|v : C^{2,\alpha}(\tilde{\mathcal{N}}, g_0, e^{-\gamma\hat{s}})\|$ is small enough, then M_v is an immersion over $\tilde{\mathcal{N}}$ and

$$\begin{aligned} &\|\hat{\varrho}^2(H_v - H - \mathcal{L}_M v) : C^{0,\alpha}(\tilde{\mathcal{N}}, g_0, e^{-\gamma\hat{s}})\| \\ &\qquad \leq C\|v : C^{2,\alpha}(\tilde{\mathcal{N}}, g_0, e^{-\gamma\hat{s}})\|^2. \end{aligned} \tag{3}$$

The $\mathcal{S}_k(\xi)$'s, $\mathcal{A}$'s, and $\tilde{\mathcal{N}}$'s above cover M, hence by (1), (2), (3) and Definitions 7.7 and 7.5 the statement follows. q.e.d.

The main theorem.

We combine now the results of the previous section with the estimates above to prove the main theorem of this paper.

Theorem 8.2. *Given a $(\delta, \underline{\mathcal{E}}', \underline{\mathcal{E}}'')$-flexible configuration $\mathcal{I}$ as in 5.4 and $\underline{m}$ as in 6.2, there is $\tilde{m} \in \mathbb{N}$ depending only on δ and $\underline{m}$ such that the following are true for any $m \in \mathbb{N}$ such that $m > \tilde{m}$: There is $\zeta > 0$ which depends only on δ and $\underline{m}$, and $\xi_0 \in \Xi_{\mathcal{V}}$ with $\Xi_{\mathcal{V}}$ as in 6.3, such that there is a smooth function v on the initial surface $M(\xi_0)$—defined in Section 6—such that $\|v\|_2 \leq C(\delta, \underline{m})/m$—recall 7.7—and the graph $\mathbf{M}_m$ by v over $M(\xi_0)$ has the following properties:*

(i) $\mathbf{M}_m$ is a complete (boundaryless) minimal smooth surface of finite total curvature.

(ii) If $\mathcal{I}$ is an embedded flexible configuration as in 5.5, then $\mathbf{M}_m$ is a properly embedded surface.

(iii) *The planes through the x_3-axis forming an angle $k\pi/m$ ($k \in \mathbb{Z}$) with the x_1-axis are planes of symmetry of $\mathbf{M}_m$.*

(iv) *There is a large ball B such that $\mathbf{M}_m \setminus B$ is the union of annular ends in one-to-one correspondence with the ends in $\underline{\mathcal{A}}$ in such a way that the following are true: The ends contained in $\mathbf{M}_m \setminus B$ which correspond to an end in $\underline{\mathcal{E}}'$ decay exponentially at infinity to the corresponding end of $\mathcal{I} = \mathcal{I}_{0,0}$. The ones that correspond to an end in $\underline{\mathcal{E}}''$ decay exponentially at ∞ to a catenoidal end of the same position as the the corresponding end of $\mathcal{I} = \mathcal{I}_{0,0}$, but of perhaps different size, the difference being at most $C(\delta, \underline{m})/m$. Finally the remaining ends decay to catenoidal ends whose size and position differ from the corresponding ends of $\mathcal{I} = \mathcal{I}_{0,0}$ by at most $C(\delta, \underline{m})/m$.*

(v) *Let U be a neighborhood of $\cup_{k=1}^{N_{\mathcal{C}}} \mathcal{C}_k$. Then as $m \to \infty$ $\mathbf{M}_m$ tends uniformly in C^k norm to $\cup_{j=1}^{N_{\mathcal{A}}} \mathcal{A}_j$ on any compact subset of $E^3 \setminus U$. Similarly $\mathcal{H}_k^{-1}(\mathbf{M}_m)$ tends uniformly on compact subsets of E^3 to the Scherk surface in such a way, that a fundamental domain under the symmetries in (iii) above tends to the union of exactly m_j fundamental domains of the Scherk surface.*

Proof. Given a function f on $M(0)$ we define $\|f : \mathcal{X}\|$ to be the maximum of the following, where $\alpha' \in (0, \alpha)$ is some fixed constant:

(1) $\|f : C^{2,\alpha'}(\tilde{\mathcal{N}}_j(0), g_0)\|$ for each end $\mathcal{N}_j(0) \subset M(0)$.

(2) $\|f : C^{2,\alpha'}(M(0) \setminus \bigcup \tilde{\mathcal{N}}_j(0), g_{M(0)})\|$.

We define then the Banach space

$$\mathcal{X} = \{f \in C^{2,\alpha'}_{loc}(M(0)) : \|f : \mathcal{X}\| < \infty\}.$$

It is easy to construct (but omit the details) a family of smooth diffeomorphisms $D_\xi : M(0) \to M(\xi)$ for $\xi \in \Xi_{\mathcal{V}}$, which depend continuously on ξ and satisfy the following: For every $f \in C^{2,\alpha}_{loc}(M(0))$ and $f' \in C^{2,\alpha}_{loc}(M(\xi))$ we have

$$\|f \circ D_\xi^{-1}\|_2 \le C\|f\|_2, \qquad \|f' \circ D_\xi\|_2 \le C\|f'\|_2. \tag{1}$$

Let $\Xi := \{(\xi, u) \in \mathcal{V} \times \mathcal{X} : |\xi| \le \zeta\overline{\tau}, \|u\|_2 \le \zeta\overline{\tau}\}$. Then we define a map $\mathcal{J} : \Xi \to \mathcal{V} \times \mathcal{X}$ as follows: Suppose we are given $(\xi, u) \in \Xi$ where

$\xi = (\underline{\vartheta}, \underline{b}, \underline{\phi})$. Let $v = (u \circ D_\xi^{-1}) - v_H$ where v_H is defined on $M = M(\xi)$ as in 7.11. By 7.11 and (1) we have

$$\|v\|_2 \leq C\,(\zeta + 1)\overline{\tau}. \tag{2}$$

By applying 8.1 we conclude that M_v is well defined and

$$\|H_v - H - \mathcal{L}_M v\|_0 \leq C(\zeta + 1)^2 \overline{\tau}^2. \tag{3}$$

Using 7.10 with $E = H_v - H - \mathcal{L}_M v$ we obtain v_E and $(\underline{\vartheta}_E, \underline{\phi}_E)$ as in 7.10. Combining the equations and estimates of 7.11, 7.10, and (3), thus yields:

$$H_v = \mathcal{L}_M(v_E + (u \circ D_\xi^{-1})) - \Theta(\underline{\vartheta}_H + \underline{\vartheta}_E, \underline{\phi}_H + \underline{\phi}_E), \tag{4}$$

$$|\underline{\vartheta} - \underline{\vartheta}_H - \underline{\vartheta}_E| \leq C\overline{\tau} + C(\zeta + 1)^2 \overline{\tau}^2,$$

$$|\underline{\phi} + \underline{\phi}_H + \underline{\phi}_E| \leq C\overline{\tau} + C(\zeta + 1)^2 \overline{\tau}^2, \tag{5}$$

$$\|v_E\|_2 \leq C(\zeta + 1)^2 \overline{\tau}^2. \tag{6}$$

We define finally

$$\mathcal{J}(\xi, u) = ((\underline{\vartheta} - \underline{\vartheta}_H - \underline{\vartheta}_E, -\tilde{\underline{b}}, \underline{\phi} + \underline{\phi}_H + \underline{\phi}_E), -v_E \circ D_\xi). \tag{7}$$

(5), (6) with (1) and 6.9.v for $\tilde{\underline{b}}$ imply that by choosing ζ large enough, and assuming $\overline{\tau}$ small enough, we can ensure $\mathcal{J}(\Xi) \subset \Xi$. Since $\alpha' < \alpha$, and because of the imposed exponential decay by the $\|\,.\,\|_2$ norm at the ends, Ξ is compact in $\mathcal{V} \times \mathcal{X}$, and is also clearly convex. Finally it is easy to check by reviewing the constructions that $\mathcal{J}$ is a continuous map. We can thus apply the Schauder fixed point theorem [5, Theorem 11.1] to assert the existence of a fixed point of $\mathcal{J}$. By (4) and (7) we find that the corresponding M_v is a minimal surface. From the standard regularity theory smoothness follows. The rest of the theorem can also be proved by using 6.9 and the smallness of v. q.e.d.

The construction of new minimal surfaces.

We apply now the main theorem to the initial configurations which we constructed in Section 5. We try to present the next statement in a way as self-contained as possible for the reader who does not wish to get involved with earlier technical definitions.

Corollary 8.3. *Given a sequence of natural numbers $\underline{m} = \{m_j\}$, there is an increasing sequence of open subsets $\mathcal{M}_m$ of the configuration* *space of $N_{\mathcal{K}}$ catenoids and $N_{\mathcal{P}} \leq 1$ planes, $\mathcal{M}$, and a decreasing sequence $\varepsilon_m \in \mathbb{R}^+$, such that $\cup \mathcal{M}_m$ is dense in $\mathcal{M}$ and $\varepsilon_m \to 0$, and for each $\underline{x} \in \mathcal{M}_m$ there is a minimal surface M which satisfies the following:*

(i) *M is a complete embedded minimal surface of finite total curvature, genus $m \sum m_j + C(\underline{x})$, and $2N_{\mathcal{K}} + N_{\mathcal{P}}$ ends.*

(ii) *The planes through the x_3-axis forming an angle $k\pi/m$ ($k \in \mathbb{Z}$) with the x_1-axis are planes of symmetry of M.*

(iii) *Let U be the open tubular neighborhood of the x_3-axis of radius ε_m^{-1}, X the intersections of $\underline{x}$ (that is points belonging to more than one catenoid or plane), and Z the uni haveon of the planes and catenoid in $\underline{x}$. Then there is an open tubular neighborhood Y of X, contained in the one of radius $2\epsilon_m$ and containing the one of radius ε_m, such that the following are true: $Y \subset U$. $U \cap (M \setminus Y)$ is a graph over $U \cap (Z \setminus Y)$ by a function f such that $|f| \leq \varepsilon_m$. $M \setminus U$ is a union of ends each of which is a graph over a catenoidal end by a function f such that $|f| \leq \varepsilon_m$ and moreover decays to it. Moreover these catenoidal ends can be put in one-to-one correspondence with the ends in $Z \setminus U$ in such a way that the positions and sizes of corresponding ends differ by ε_m at most. This difference vanishes for the top ends of each catenoid $\underline{x}$ and the position of the planar end (if there is one).*

(iv) *Each component Y_j of Y is diffeomorphic to a tubular neighborhood of the axis of periodicity of a Scherk surface quotiented out by a period in such a way that $Y_j \cap M$ maps to the intersection of the tubular neighborhood with the Scherk surface. Assuming a systematic numbering of the components, $Y_j \cap M$ has $2mm_j$ fundamental regions. If we rescale and reposition $Y_j \cap M$ suitably, and take $m \to \infty$, we get a Scherk surface in the limit.*

Proof. This follows easily from 8.3 and 5.9. q.e.d.

Notice that since we can fix the position and size of the catenoid of an end for each catenoid involved, and the position of an end for each plane involved, we have $N = 2N_{\mathcal{K}} + N_{\mathcal{P}}$ free continuous parameters in this construction, two of which are redundant because of homotheties and translations. We do not prove here the continuous dependence of these families on these parameters to keep technicalities to the minimum.

If we are interested in minimal surfaces which are not embedded, we can allow more than one plane and also choose not to desingularize some of the circles of intersection (recall 5.10). More interesting are constructions of embedded surfaces where extra symmetries are imposed. There are then essentially only two cases: Impose reflectional symmetry with respect to a horizontal plane which is not a plane contained in $\underline{\mathcal{P}}$. Or include in $\underline{\mathcal{P}}$ a plane of symmetry of the configuration of catenoids and planes, but in this case the minimal surface can not be symmetric with respect to the plane, it can be however symmetric with respect to a system of straight lines on the plane. The second case is of interest because it includes the first surfaces constructed by Hoffman and Meeks. In both cases we can only fix the sizes of the catenoids involved which serve as the continuous parameters of the construction. We leave the details to the reader.

Appendix A. Estimates on long cylinders

In this appendix we estimate and solve a Dirichlet problem which comes up repeatedly in the paper. We first define the domain of the Dirichlet problem. Recall that H^+ and its standard coordinates were defined in 2.3.

Definition A.1. We define (Ω, g_0) to be the cylinder $\Omega = H^+_{\leq \ell}/G'$, where G' is the group generated by $(s,z) \to (s, z+2\pi)$, and $\ell \in (\overline{1}0, \infty)$ is called the length of the cylinder, equipped with the standard metric $g_0 := ds^2 + dz^2$. We have $\partial\Omega = \partial_0 \cup \partial_\ell$ where ∂_0 and ∂_ℓ are the boundary circles $\{s = 0\}$ and $\{s = \ell\}$ respectively.

We are interested in solving the Dirichlet problem on the cylinder that we just defined for linear operators $\underline{\mathcal{L}}$, which are small perturbations of the standard Laplacian Δ, in the sense that the quantity $N(\underline{\mathcal{L}})$ defined in A.2 is small. Notice that we allow two different kinds of perturbations: One decays exponentially along Ω, and one does not; the second one has to be small in terms of the cylinder's length. In the applications we will need both because the $|A|^2$ on the wings consists of a part coming from the catenoid, which does not decay but is small, and a part due to taking the graph over the catenoid, which does decay but is not as small.

Definition A.2. Let $\underline{\mathcal{L}}$ denote an operator on Ω of the form

$$\underline{\mathcal{L}} = \Delta_\chi + d,$$

where χ is a C^2 Riemannian metric, and d a C^1 function on Ω. We then for $\underline{c} > 0$ define a given constant:

$$N(\underline{\mathcal{L}}) := \|\chi - g_0 : C^2(\Omega, g_0, e^{-\underline{c}s})\| + \|d : C^1(\Omega, g_0, e^{-\underline{c}s} + \ell^{-2})\|.$$

We want to solve the Dirichlet problem now for $\underline{\mathcal{L}}$ on Ω, with data an inhomogeneous term E, and boundary data vanishing on ∂_ℓ and given up to a constant on ∂_0. We need the freedom to change the data on ∂_0 by constants because we want to ensure exponential decay for the solutions assuming exponential decay for E. In the rest we assume that a Hölder exponent $\alpha \in (0,1)$ has been chosen. We work with decay rates γ which are supposed to be slightly smaller than 1.

Proposition A.3. *If $N(\underline{\mathcal{L}})$ is small enough in terms of $\underline{c}$, α, and given $\gamma \in (0,1)$ and $\epsilon > 0$ (but independently of ℓ), there is a bounded linear map*

$$\underline{\mathcal{R}} : C^{2,\alpha}(\partial_0, g_0) \times C^{0,\alpha}(\Omega, g_0, e^{-\gamma s}) \to C^{2,\alpha}(\Omega, g_0, e^{-\gamma s})$$

such that for (f, E) in $\underline{\mathcal{R}}$'s domain and $v = \underline{\mathcal{R}}(f, E)$ the following are true, where the constants C depends only on α and γ:

(i) $\underline{\mathcal{L}} v = E$ *on* Ω.

(ii) $v = f - \operatorname{avg}_{\partial_0} f + B(f, E)$ *on* ∂_0, *where* $B(f, E)$ *is a constant on* ∂_0.

(iii) $v \equiv 0$ *on* ∂_ℓ.

(iv) $\|v : C^{2,\alpha}(\Omega, g_0, e^{-\gamma s})\|$
$$\leq C\|f - \operatorname{avg}_{\partial_0} f : C^{2,\alpha}(\partial_0, g_0)\| + C\|E : C^{0,\alpha}(\Omega, g_0, e^{-\gamma s})\|.$$

(v) $|B(f, E)| \leq \epsilon\|f - \operatorname{avg}_{\partial_0} f : C^0(\partial_0)\| + C\|E : C^{0,\alpha}(\Omega, g_0, e^{-\gamma s})\|$.

(vi) If $v' \in C^2(\Omega)$ satisfies $\underline{\mathcal{L}} v' = E$ on Ω, and $v' = v$ on $\partial\Omega$, then $v' = v$ on Ω. Moreover, if E vanishes, then

$$\|v : C^0(\Omega)\| \leq 2\|v : C^0(\partial_0)\|.$$

Proof. If $\underline{\mathcal{L}} = \Delta$, the standard Laplacian with respect to the flat metric g_0, then the proposition is valid with a vanishing ϵ in (v) by the standard theory. We then allow a general $\underline{\mathcal{L}}$ but restrict to vanishing f first. The operator $\underline{\mathcal{L}} - \Delta$ thus has a small operator norm as an operator from $C^{2,\alpha}(\Omega, g_0, e^{-\gamma s})$ to $C^{0,\alpha}(\Omega, g_0, e^{-\gamma s})$, and hence the statement follows in this case except for (vi). For vanishing E and nonvanishing f we can use the previous case to correct the v which we would have for the Laplacian.

It remains to prove (vi). By using the smallness of $N(\underline{\mathcal{L}})$ we can ensure that the smallest eigenvalue of $\underline{\mathcal{L}}$ on Ω with vanishing Dirichlet conditions is at least $1/2\ell^2$. This follows by considering an arbitrary smooth function on Ω which vanishes on $\partial\Omega$, Fourier expanding on the meridian circles, and estimating in terms of the L^2 norm of the gradient (see [14, Lemma 2.26] for a similar argument). The uniqueness part follows then from the positivity of the smallest eigenvalue, and the desired estimate reduces to the case where $v = f \equiv 1$ on ∂_0, because otherwise we can produce a subdomain of Ω with a vanishing eigenvalue.

Now if $\underline{\mathcal{L}} = \Delta$ the solution is $v = (\ell - s)/\ell$. This can be corrected to the solution for $\underline{\mathcal{L}}$ of the form $\Delta + d$ with $\|d : C^1(\Omega, \ell^{-2})\|$ appropriately small, by scaling the length of the cylinder to unit, while leaving the meridian unchanged. We thus can have the estimate for v with a constant $3/2$ for example instead of 2. By using now the earlier proven parts of the proposition we can correct this v to a v for a general $\underline{\mathcal{L}}$ while establishing the estimate at the same time. q.e.d.

We occasionally will need also the existence result in A.3 for the case $\Omega = H^+/G'$, that is when the cylinder has infinite length. Then extending Definition A.2 without change to this case except for removing the term ℓ^{-2} from the expression for $N(\underline{\mathcal{L}})$, we have the following standard result which we state here to facilitate reference.

Proposition A.4. *For $\Omega = H^+/G'$ and $\underline{\mathcal{L}}$ with $N(\underline{\mathcal{L}})$ small enough in terms of α and $\gamma \in (0,1)$, there is a bounded linear map*

$$\underline{\mathcal{R}} : C^{0,\alpha}(\Omega, g_0, e^{-\gamma s}) \to C^{2,\alpha}(\Omega, g_0, e^{-\gamma s})$$

such that for E in $\underline{\mathcal{R}}$'s domain and $v = \underline{\mathcal{R}}(E)$, we have $\underline{\mathcal{L}}v = E$ on Ω, v is constant on $\partial\Omega$, and

$$\|v : C^{2,\alpha}(\Omega, g_0, e^{-\gamma s})\| \leq C(\alpha, \gamma)\|E : C^{0,\alpha}(\Omega, g_0, e^{-\gamma s})\|.$$

Proof. This is standard for $\underline{\mathcal{L}} = \Delta$. The general case follows then by treating $\underline{\mathcal{L}}$ as a perturbation of Δ. q.e.d.

Appendix B. The mean curvature of graphs

In this appendix we discuss some general facts which we use in the paper. The notation which we adopt here is independent of the notation in the rest of the paper. Let S be an immersed surface in E^3, immersed by $X : S \to E^3$, of first and second fundamental forms g and A, Gauss map ν, and mean curvature H. For $f \in C^2(S)$ we define $X_f := X + f\nu$, and when X_f is an immersion, we denote by H_f the mean curvature of the surface $X_f(S)$ pulled back by X_f to S. We thus have the following lemma in which we decompose H_f into linear and nonlinear terms in f:

Lemma B.1. *If $|fA| < 1$, then X_f is an immersion and $H_f = H + \mathcal{L}_S f + Q_f$ where $\mathcal{L}_S = \frac{1}{2}(\Delta + |A|^2)$ and*

$$Q_f = \frac{Q_1}{\sqrt{1+Q'}} + \frac{Q_2}{1+Q'+\sqrt{1+Q'}}.$$

Here Q' is a sum of terms which are contractions of at least two Φ's, Φ stands for either ∇f or fA, and Q_1 and Q_2 are sums of terms which are contractions of a number of Φ's—possibly none—and one of the following terms, where $*$ denotes contraction—all contractions are taken with respect to g:

(i) $A * \Phi * \Phi$.

(ii) $f\nabla A * \Phi$.

(iii) $fA * \nabla^2 f$.

(iv) $\nabla^2 f * \Phi * \Phi$.

Proof. The proof amounts to a local calculation. We refer the reader to [13, appendix C] for a detailed presentation. q.e.d.

Suppose now we have a variation of $X : S \to E^3$, that is a C^2 family of immersions $X_\phi : S \to E^3$ for ϕ in a neighborhood of 0 in $\mathbb{R}$, where $X_0 = X$. Let $\dot{}$ denote $\frac{\partial}{\partial\phi}\Big|_{\phi=0}$, and let $Y = \dot{X}_\phi$ be the variation vector field of the variation. Let $Y_{||}$ be the tangential to S part of Y, that is $Y = Y_{||} + (Y \cdot \nu)\nu$ is a decomposition to tangent and normal parts. In the next lemma we give an expression for the variation of the mean curvature.

Lemma B.2. *$\dot{H} = Y_{||}(H) + \mathcal{L}_S(Y \cdot \nu)$.*

Proof. The proof is by observing first that the left-hand side is linear in the variation Y, and hence the proof reduces to the cases where the X_ϕ's differ only by diffeomorphisms of the domain, or they only differ by moving along the normal lines to the image. In the first case the formula is clearly valid because H is a geometric invariant of the image. In the second case the formula follows because the linearized change in mean curvature of a graph over a surface is as implied by B.1.

q.e.d.

FIGURE 1. Example of an arrangement of two catenoids and one plane to be desingularized. (Intersection with a plane through the axis.)

Figure 2

Figure 3

Figure 4

Figure 5

FIGURE 6. Section of $\Sigma[T, \underline{\varphi}, \tau]$ and $\Sigma[T, 0, \tau]$ showing the core and portion of the 1st wing

FIGURE 7. Section of $\Sigma[T, \underline{\varphi}, \tau]$ and $\Sigma[T', \underline{\varphi} - \underline{\varphi}', \tau]$ with $\underline{\varphi} - \underline{\varphi}'$ showing the 1st wing and the core only

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