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BLASCHKE-SANTALÓ INEQUALITIES

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Two of the most important affine isoperimetric inequalities are the Blaschke-Santaló inequality (see, e.g., Gardner [6, p. 322] or Schneider [12, p. 425]) and the classical affine isoperimetric inequality of affine differential geometry (see, e.g., Schneider [12, p. 419]). These two inequalities are closely related in that given either one of these inequalities, then by well-known methods one can be quickly deduced from the other. The aim of this article is to establish a new family of analytic inequalities and their geometric counterparts. One of the members of this family of inequalities turns out to be the Blaschke-Santaló inequality.

Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$. Let B denote the unit ball (the convex hull of S^{n-1}) in $\mathbb{R}^n$, and write ω_n for the n-dimensional volume of B. Note that,

$$\omega_n = \pi^{n/2}/\Gamma(1+\tfrac{n}{2}),$$

defines ω_n for all non–negative real n (not just the positive integers). For real $p \geq 1$, define $c_{n,p}$ by

$$c_{n,p} = \frac{\omega_{n+p}}{\omega_2 \omega_n \omega_{p-1}}.$$

For real $p \geq 1$ and continuous $f : S^{n-1} \to \mathbb{R}$, let $\|f\|_p$ denote the standard L_p-norm of f; i.e.,

$$\|f\|_p = \left\{ \int_{S^{n-1}} |f(u)|^p \, du \right\}^{1/p},$$

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where the integration is always with respect to the rotation invariant probability measure on S^{n-1}.

Theorem A. *For real $p \geq 1$ and continuous $f, g : S^{n-1} \to (0, \infty)$,*

$$\int_{S^{n-1}} \int_{S^{n-1}} |u \cdot v|^p f(u) g(v)\, du\, dv \quad \geq \quad c_{n-2,p}\, \|f\|_{\frac{n}{n+p}} \|g\|_{\frac{n}{n+p}},$$

with equality if and only if there exist a $\phi \in \mathrm{GL}(n)$ and real $c_1, c_2 > 0$ such that

$$f(u) = c_1 |\phi(u)|^{-(n+p)} \qquad \text{and} \quad g(u) = c_2 |\phi^{-t}(u)|^{-(n+p)},$$

for all $u \in S^{n-1}$.

Here $u \cdot v$ denotes the standard inner product of u and v, and ϕ^{-t} denotes the inverse of the transpose of ϕ.

For the special case $p = 2$, the inequality of Theorem A is known (see e.g. Schneider [12, p. 422]) and easily established. The analytic inequality of Theorem A will be established by first proving its geometric counterpart.

For each compact star-shaped (about the origin) subset, K, of $\mathbb{R}^n$, let the norm $\|\cdot\|_{\Gamma_p^* K}$ on $\mathbb{R}^n$ be defined by

$$\|x\|_{\Gamma_p^* K} = \left\{ \frac{1}{c_{n,p} V(K)} \int_K |x \cdot y|^p dy \right\}^{1/p}, \qquad 1 \leq p \leq \infty,$$

where $V(K)$ denotes the volume of K. For the case $p = \infty$, this definition is to be interpreted as a limit as $p \to \infty$. The unit ball of this n-dimensional L_p-space is denoted by $\Gamma_p^* K$, and called the *polar p-centroid body of K*. The (unusual) normalization above was chosen so that for the unit ball B in $\mathbb{R}^n$, we have $\Gamma_p^* B = B$.

We will prove the following centro–affine inequality involving the volumes of K and its polar p-centroid body, $\Gamma_p^* K$:

Theorem B. *If K is a star body (about the origin) in $\mathbb{R}^n$, then for $1 \leq p \leq \infty$,*

$$V(K)\, V(\Gamma_p^* K) \leq \omega_n^2,$$

with equality if and only if K is an ellipsoid centered at the origin.

If K is a centered convex body (i.e., symmetric about the origin) then $\Gamma_\infty^* K$ is just the polar, K^*, of K where

$$K^* = \{x \in \mathbb{R}^n : |x \cdot y| \leq 1, \quad \text{for all } y \in K\}.$$

In this case, the inequality of Theorem B, for $p = \infty$, reduces to:

$$V(K)\,V(K^*) \le \omega_n^2,$$

with equality if and only if K is an ellipsoid.

This is the well-known Blaschke-Santaló inequality. A very recent approach to the Blaschke-Santaló inequality and the classical affine isoperimetric inequality of affine differential geometry can be found in Andrews [2].

1. Star bodies and dual mixed volumes

For quick reference, we recall some basic properties regarding star bodies and dual mixed volumes. Some recent applications of dual mixed volumes can be found in [5] and [14]. For general reference the reader may wish to consult Gardner [6] and Schneider [12].

The radial function, $\rho_K = \rho(K, \,\cdot\,) : \mathbb{R}^n \setminus \{0\} \to [0, \infty)$, of a compact, star–shaped (about the origin) $K \subset \mathbb{R}^n$, is defined, for $x \neq 0$, by $\rho(K, x) = \max\{\, \lambda \ge 0 : \lambda x \in K \,\}$. If ρ_K is positive and continuous, call K a star body (about the origin), and write $\mathcal{S}$ for the set of star bodies (about the origin) of $\mathbb{R}^n$. Two star bodies $K, L \in \mathcal{S}$ are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

From the definition of the radial function, it follows immediately that for $K \in \mathcal{S}$ and $\phi \in \mathrm{GL}(n)$ we have $\rho(\phi K, x) = \rho(K, \phi^{-1}x)$, for all $x \neq 0$. Obviously, for the unit ball, B, in $\mathbb{R}^n$, we have $\rho(B, x) = 1/|x|$, for all $x \neq 0$. Hence, if $\phi \in \mathrm{GL}(n)$ then $\rho(\phi B, x) = 1/|\phi^{-1}x|$. From the definition of the polar body, it follows immediately that $(\phi B)^* = \phi^{-t}B$. Thus, for all $x \neq 0$, we have $\rho((\phi B)^*, x) = 1/|\phi^t x|$, where ϕ^t denotes the transpose of ϕ. We summarize this in: The bodies E and E^* are centered polar reciprocal ellipsoids, if and only if, there exists a $\phi \in \mathrm{GL}(n)$ such that

$$\rho(E, u) = 1/|\phi u| \qquad \text{and} \qquad \rho(E^*, u) = 1/|\phi^{-t}u|, \tag{1.1}$$

for all $u \in S^{n-1}$.

If K is a convex body (i.e., compact convex subset with nonempty interior) in $\mathbb{R}^n$, then its support function, $h_K = h(K, \,\cdot\,) : \mathbb{R}^n \to \mathbb{R}$, is defined for $x \in \mathbb{R}^n$ by $h(K, x) = \max\{x \cdot y : y \in K\}$. If K is a centered (i.e., symmetric about the origin) convex body, then from the definitions of support function, radial function and polar body, it follows that

$$h_{K^*} = 1/\rho_K \qquad \text{and} \qquad \rho_{K^*} = 1/h_K. \tag{1.2}$$

Fix a real $p \geq 1$. For $K, L \in \mathcal{S}$, and $\epsilon > 0$, the p-harmonic radial combination $K \hat{+} \epsilon \cdot L \in \mathcal{S}$ is defined by

$$\rho(K \hat{+} \epsilon \cdot L, \ \cdot \ )^{-p} = \rho(K, \ \cdot \ )^{-p} + \epsilon \rho(L, \ \cdot \ )^{-p}.$$

While this addition and scalar multiplication are obviously dependent on p, we have not made this explicit in our choice of notation. The dual mixed volume $\tilde{V}_{-p}(K, L)$ of $K, L \in \mathcal{S}$, can be defined by

$$\frac{n}{-p} \tilde{V}_{-p}(K, L) = \lim_{\epsilon \to 0} \frac{V(K \hat{+} \epsilon \cdot L) - V(K)}{\epsilon}.$$

The definition above and the polar coordinate formula for volume give the following integral representation of the dual mixed volume $\tilde{V}_{-p}(K, L)$ for $K, L \in \mathcal{S}$:

$$\tilde{V}_{-p}(K, L) = \omega_n \int_{S^{n-1}} \rho(K, v)^{n+p} \rho(L, v)^{-p} \, dv. \tag{1.3}$$

Recall that the integration is with respect to the rotation invariant probability measure on S^{n-1}.

Unless K and L are dilates, $[\tilde{V}_{-p}(K, L)/V(K)]^{1/p}$ is strictly increasing, in p. This is an immediate consequence of the Hölder integral inequality. From the integral representation (1.3) it is easily seen that $[\tilde{V}_{-p}(K, L)/V(K)]^{1/p}$ is bounded by $\max_{u \in S^{n-1}} \rho_K(u)/\rho_L(u)$. It will be convenient to define $\tilde{V}_{-\infty}(K, L)$ by

$$\frac{\tilde{V}_{-\infty}(K, L)}{V(K)} = \lim_{p \to \infty} \left(\frac{\tilde{V}_{-p}(K, L)}{V(K)} \right)^{1/p} = \max_{u \in S^{n-1}} \frac{\rho_K(u)}{\rho_L(u)}. \tag{1.4}$$

We will need two useful properties of dual mixed volumes. First, note that for each $K \in \mathcal{S}$,

$$\tilde{V}_{-p}(K, K) = V(K), \qquad 1 \leq p \leq \infty. \tag{1.5}$$

The integral representation (1.3) together with the Hölder integral inequality immediately give the dual mixed volume inequality: If $K, L \in \mathcal{S}$ and $1 \le p < \infty$, then

$$\frac{\tilde{V}_{-\infty}(K,L)}{V(K)} \ge \left(\frac{\tilde{V}_{-p}(K,L)}{V(K)}\right)^{1/p} \ge \left(\frac{V(K)}{V(L)}\right)^{1/n}, \tag{1.6}$$

with equality, in either inequality, if and only if K and L are dilates. This inequality will provide simple proofs of two key ingredients in the proof of Theorem B.

2. Dual mixed volumes and the operator Γ_p^*

For $K \in \mathcal{S}$ and real $p \ge 1$, the p-centroid body, $\Gamma_p K$, of K is the body whose support function is given by

$$c_{n-2,p}\, h(\Gamma_p K, x)^p = \frac{\omega_n}{V(K)} \int_{S^{n-1}} |x\cdot v|^p \rho(K,v)^{n+p} dv, \tag{2.1}$$

for all $x \in \mathbb{R}^n$.

For $p = 1$, the integral operator above is the classical cosine transform which is closely related to the spherical Radon transform (see e.g. Goodey and Weil [7]).

The Minkowski integral inequality shows that $h_{\Gamma_p K}$ is the support function of a (centered) convex body. Define $\Gamma_\infty K$ as the convex body whose support function is given by

$$h(\Gamma_\infty K, u) = \lim_{p\to\infty} h(\Gamma_p K, u) = \max_{v\in S^{n-1}} |u\cdot v| \rho(K,v), \tag{2.2}$$

for $u \in S^{n-1}$. Since the pointwise convergence $h_{\Gamma_p K} \to h_{\Gamma_\infty K}$, on S^{n-1}, is a pointwise convergence of support functions, it is in fact a uniform convergence (see, e.g., Schneider [12, p. 54]). Note that the polar of $\Gamma_p K$ is denoted by $\Gamma_p^* K$, rather than $(\Gamma_p K)^*$.

From the definition of the p-centroid body we see that for $K \in \mathcal{S}$ and $\phi \in \mathrm{GL}(n)$ we have $\Gamma_p \phi K = \phi \Gamma_p K$. Thus, if E is a centered ellipsoid, then

$$\Gamma_p E = E. \tag{2.3}$$

Now (1.2), together with definition (2.1), and the integral representation (1.3), shows that for real $p \geq 1$ and $K, L \in \mathcal{S}$,

$$c_{n-2,p}\frac{\tilde{V}_{-p}(K,\Gamma_p^* L)}{V(K)} = \frac{\omega_n^2}{V(K)V(L)}\int_{S^{n-1}}\int_{S^{n-1}} |u\cdot v|^p[\rho_K(u)\rho_L(v)]^{n+p}\,du\,dv. \tag{2.4}$$

From (1.4), (1.2) and (2.2), we see that for the case $p = \infty$ we have,

$$\begin{aligned}\frac{\tilde{V}_{-\infty}(K,\Gamma_\infty^* L)}{V(K)} &= \max_{u\in S^{n-1}}\max_{v\in S^{n-1}} |u\cdot v|\rho_K(u)\rho_L(v)\\ &= \max_{u,v\in S^{n-1}} |u\cdot v|\rho_K(u)\rho_L(v).\end{aligned}$$

According to these observations we immediately get:

Lemma 2.5. *If $K, L \in \mathcal{S}$, then*

$$\tilde{V}_{-p}(K,\Gamma_p^* L)/V(K) = \tilde{V}_{-p}(L,\Gamma_p^* K)/V(L), \qquad 1 \leq p \leq \infty. \tag{2.5}$$

Taking $L = \Gamma_p^* K$ in (2.5), and using (1.5), we are led to see that for $K \in \mathcal{S}$,

$$V(K) = \tilde{V}_{-p}(K,\Gamma_p^*\Gamma_p^* K), \qquad 1 \leq p \leq \infty, \tag{2.6}$$

where $\Gamma_p^*\Gamma_p^* K$ is used to abbreviate $\Gamma_p^*(\Gamma_p^* K)$. Identity (2.6) for $p = 1$ can be found in [9].

From identity (2.6) and the dual mixed volume inequality (1.6), we have

Proposition 2.7. *If $1 \leq p \leq \infty$ and $K \in \mathcal{S}$, then*

$$V(\Gamma_p^*\Gamma_p^* K) \geq V(K),$$

with equality if and only if K and $\Gamma_p^*\Gamma_p^* K$ are dilates.

A consequence of this proposition, that will be needed in the proof of Theorem B, states that for the special case where K is a star body whose polar p-centroid body is an ellipsoid, the inequality of Theorem B holds. This is contained in:

Lemma 2.8. *If $1 \le p \le \infty$ and $K \in \mathcal{S}$ is a star body such that $\Gamma_p^* K$ is an ellipsoid, then*

$$V(K)\, V(\Gamma_p^* K) \le \omega_n^2,$$

with equality if and only if K is a centered ellipsoid.

Proof. Since $\Gamma_p^* K$ is an ellipsoid, from (2.3) it follows that

$$\Gamma_p^*(\Gamma_p^* K) = (\Gamma_p^* K)^*.$$

This, together with the trivial observation that the product of the volumes of centered polar reciprocal ellipsoids is ω_n^2, gives

$$V(\Gamma_p^* \Gamma_p^* K) = V((\Gamma_p^* K)^*) = \omega_n^2 / V(\Gamma_p^* K).$$

Combine this with the inequality of Proposition 2.7, to get the desired inequality.

To see the necessity of the equality conditions, note that from the equality conditions of Proposition 2.7 it follows that equality, in our inequality, implies that K and $\Gamma_p^* \Gamma_p^* K$ are dilates. But $\Gamma_p^*(\Gamma_p^* K) = (\Gamma_p^* K)^*$ and $\Gamma_p^* K$ is (by hypothesis) a centered ellipsoid. Hence, equality implies that K is a centered ellipsoid. q.e.d.

Define $\mathcal{Z}_p^*$ to be the class of centered convex bodies that are the range of the operator Γ_p^* on $\mathcal{S}$; i.e.,

$$\mathcal{Z}_p^* = \{\mathcal{Z}_p^* K : K \in \mathcal{S}\}.$$

As an aside, note that the closure (in the space of convex bodies) of $\mathcal{Z}_1^*$ is the class of polar projection bodies. The polars of these bodies form the class of zonoids (see e.g. [6, p. 133] and [12, p. 182]). The class $\mathcal{Z}_2^*$ is the class of centered ellipsoids.

The following lemma shows that in order to prove the inequality of Theorem B for all star bodies, we need only prove it for the class of centered convex bodies. In fact, a much smaller class than the set of centered convex bodies will suffice. These 'class reduction' methods were used in [8], and their use here may be seen as a natural extension to the Brunn–Minkowski–Firey theory (see, e.g., [10]).

Lemma 2.9. *Suppose $1 \le p \le \infty$. If the inequality of Theorem B, with its equality conditions, holds for all bodies in $\mathcal{Z}_p^*$, then the inequality of Theorem B, with its equality conditions, holds for all bodies in $\mathcal{S}$.*

Proof. Suppose $K \in \mathcal{S}$. By definition of $\mathcal{Z}_p^*$, we have $\Gamma_p^* K \in \mathcal{Z}_p^*$. Hence, the hypothesis gives

$$V(\Gamma_p^* K)\, V(\Gamma_p^* \Gamma_p^* K) \leq \omega_n^2,$$

with equality if and only if $\Gamma_p^* K$ is a centered ellipsoid. Now, Proposition 2.7 states that

$$V(K) \leq V(\Gamma_p^* \Gamma_p^* K),$$

with equality if and only if K and $\Gamma_p^* \Gamma_p^* K$ are dilates. The desired inequality is obtained by combining these inequalities.

If there is equality in the desired inequality, then $\Gamma_p^* K$ must be a centered ellipsoid, and K and $\Gamma_p^* \Gamma_p^* K$ are dilates. Hence K must be a centered ellipsoid. q.e.d.

3. Steiner symmetrization and the operator Γ_p^*

For a set $Q \subseteq \mathbb{R}^n$, and $t \in \mathbb{R}$, let

$$Q_t = \{x \in \mathbb{R}^{n-1} : (x, t) \in Q\},$$

and write $-Q_t$ for $\{-x : x \in Q_t\}$.

Lemma 3.1. *Suppose K is a centered convex body in $\mathbb{R}^n$, and $\widetilde{K}$ is the Steiner symmetral of K with respect to the hyperplane defined by $x_n = 0$. Then, for $1 \leq p \leq \infty$,*

$$\tfrac{1}{2}(\Gamma_p^* K)_t + \tfrac{1}{2}(\Gamma_p^* K)_{-t} \subseteq (\Gamma_p^* \widetilde{K})_t,$$

for all $t \in \mathbb{R}$.

Proof. First note that it suffices to prove the inclusion for $1 \leq p < \infty$. Also, without loss of generality, we may assume $V(K) = 1/c_{n,p} = V(\widetilde{K})$.

Let K' be the image of the orthogonal projection of K onto the hyperplane $x_n = 0$. For $x \in K'$, let $c(x)$ denote the chord of K that is parallel to the x_n-axis and (whose extension) passes through x. Thus,

$$c(x) = x \times [m - \tfrac{\sigma}{2}, m + \tfrac{\sigma}{2}],$$

where σ is the length of $c(x)$, and m is the x_n-coordinate of the midpoint of $c(x)$. For $(x, s) \in \widetilde{K}$ define

$$s_1 = s + m \qquad \text{and} \qquad s_2 = -s + m.$$

Thus, (x, s_1) and (x, s_2) are points in $c(x)$ that are symmetric about the midpoint of $c(x)$. While m and σ are obviously functions of $x \in K'$, we have chosen not to make this explicit in our notation.

Recall that

$$\Gamma_p^* K = \{(x,t) \in \mathbb{R}^{n-1} \times \mathbb{R} : \|(x,t)\|_{\Gamma_p^* K} \le 1\}.$$

Suppose $y_1 \in (\Gamma_p^* K)_t$, $y_2 \in (\Gamma_p^* K)_{-t}$, and let $y = \frac{1}{2}(y_1 + y_2)$. To prove the lemma we will show that $y \in (\Gamma_p^* \widetilde{K})_t$. The conditions

$$\|(y_1, t)\|_{\Gamma_p^* K} \le 1 \qquad \text{and} \qquad \|(y_2, -t)\|_{\Gamma_p^* K} \le 1,$$

give

$$\int_{(x,s)\in K} |y_1 \cdot x + ts|^p dx ds \le 1 \quad \text{and} \quad \int_{(x,s)\in K} |y_2 \cdot x - ts|^p dx ds \le 1.$$

Our aim is to show that $\|(y,t)\|_{\Gamma_p^* \widetilde{K}} \le 1$, or equivalently that

$$\int_{(x,s)\in \widetilde{K}} |y \cdot x + ts|^p dx ds \le 1.$$

Now for $(x,s) \in \widetilde{K}$, recall that $s_1 = s + m$ and $s_2 = -s + m$, and since $s = \frac{1}{2}(s_1 - s_2)$,

$$|y\cdot x + ts|^p = |\tfrac{1}{2}(y_1 + y_2)\cdot x + \tfrac{1}{2}(s_1 - s_2)t|^p \le \tfrac{1}{2}|y_1\cdot x + ts_1|^p + \tfrac{1}{2}|y_2\cdot x - ts_2|^p.$$

Thus,

$$\begin{aligned}
&\int_{\widetilde{K}} |y\cdot x + ts|^p dx ds \\
&\quad \le \frac{1}{2}\int_{\widetilde{K}} |y_1\cdot x + ts_1|^p dx ds \; + \; \frac{1}{2}\int_{\widetilde{K}} |y_2\cdot x - ts_2|^p dx ds \\
&\quad = \frac{1}{2}\int_{K'} dx \int_{-\sigma/2}^{\sigma/2} |y_1\cdot x + ts_1|^p ds \; + \; \frac{1}{2}\int_{K'} dx \int_{-\sigma/2}^{\sigma/2} |y_2\cdot x - ts_2|^p ds.
\end{aligned}$$

Since $s_1 = s + m$ and $s_2 = -s + m$, with $(x, s_1) \in c(x) \subset K$ and $(x, s_2) \in c(x) \subset K$,

$$\begin{aligned}
\int_{K'} dx \int_{-\sigma/2}^{\sigma/2} |y_1\cdot x + ts_1|^p ds &= \int_{K'} dx \int_{m-\sigma/2}^{m+\sigma/2} |y_1\cdot x + ts_1|^p ds_1 \\
&= \int_{(x,s_1)\in K} |y_1\cdot x + ts_1|^p dx ds_1,
\end{aligned}$$

and

$$\int_{K'} dx \int_{-\sigma/2}^{\sigma/2} |y_2 \cdot x - ts_2|^p ds = \int_{K'} dx \int_{m+\sigma/2}^{m-\sigma/2} |y_2 \cdot x - ts_2|^p (-ds_2)$$
$$= \int_{(x,s_2)\in K} |y_2 \cdot x - ts_2|^p dx ds_2.$$

It follows that

$$\int_{\widetilde{K}} |y \cdot x + ts|^p dx ds \le \frac{1}{2} \int_K |y_1 \cdot x + ts_1|^p dx ds_1 + \frac{1}{2} \int_K |y_2 \cdot x - ts_2|^p dx ds_2 \le 1.$$

Thus $\|(y,t)\|_{\Gamma_p^* \widetilde{K}} \le 1$, or equivalently, $(y,t) \in \Gamma_p^* \widetilde{K}$, and hence $\frac{1}{2}(y_1 + y_2) = y \in (\Gamma_p^* \widetilde{K})_t$. q.e.d.

The Brunn-Minkowski inequality can now be used to show that Steiner symmetrization does not decrease the volume of polar p-centroid bodies.

Lemma 3.2. *Suppose K is a centered convex body in $\mathbb{R}^n$ and $1 \le p \le \infty$. If $\widetilde{K}$ is the Steiner symmetral of K with respect to the hyperplane ξ, then*

$$V(\Gamma_p^* K) \le V(\Gamma_p^* \widetilde{K}),$$

with equality if and only if every $(n-1)$-dimensional slice of $\Gamma_p^* K$, parallel to ξ is centrally symmetric.

Proof. Without loss of generality, we may assume that the hyperplane ξ is the subspace of $\mathbb{R}^n$ defined by $x_n = 0$. From Lemma 3.1,

$$\mathrm{vol}_{n-1}(\tfrac{1}{2}(\Gamma_p^* K)_t + \tfrac{1}{2}(\Gamma_p^* K)_{-t}) \le \mathrm{vol}_{n-1}((\Gamma_p^* \widetilde{K})_t),$$

for all $t \in \mathbb{R}$. Since $\Gamma_p^* K$ is centered, $-(\Gamma_p^* K)_t = (\Gamma_p^* K)_{-t}$. Hence, the Brunn-Minkowski inequality (in $\mathbb{R}^{n-1}$) shows that for each t,

$$\mathrm{vol}_{n-1}((\Gamma_p^* K)_t) \le \mathrm{vol}_{n-1}(\tfrac{1}{2}(\Gamma_p^* K)_t + \tfrac{1}{2}(\Gamma_p^* K)_{-t}),$$

with equality if and only if $(\Gamma_p^* K)_t$ is a translate of $-(\Gamma_p^* K)_t$. Hence for all $t \in \mathbb{R}$,

$$\mathrm{vol}_{n-1}((\Gamma_p^* K)_t) \le \mathrm{vol}_{n-1}((\Gamma_p^* \widetilde{K})_t),$$

with equality if and only if $(\Gamma_p^* K)_t$ is centrally symmetric.

By integrating (over all $t \in \mathbb{R}$) the quantities on both sides of the last inequality, we get

$$V(\Gamma_p^* K) \le V(\Gamma_p^* \widetilde{K}),$$

with equality if and only if every $(n-1)$-dimensional slice of $\Gamma_p^* K$, parallel to the hyperplane ξ is centrally symmetric. q.e.d.

4. Proof of the theorems

We are now in a position to quickly prove the theorems presented in the introduction.

Theorem B. *If $K \in \mathcal{S}$ and $1 \le p \le \infty$, then*

$$V(K)\, V(\Gamma_p^* K) \le \omega_n^2,$$

with equality if and only if K is a centered ellipsoid.

Proof. First recall that from Lemma 2.9 it follows that we may assume that the star body K is a centered convex body. Now Lemma 3.2, and a standard Steiner symmetrization argument, show that if B_o is a dilate of the unit ball, B, chosen so that $V(B_o) = V(K)$, then

$$V(\Gamma_p^* K) \le V(\Gamma_p^* B_o).$$

Thus,

$$V(K)\, V(\Gamma_p^* K) \le V(B_o)\, V(\Gamma_p^* B_o) = \omega_n^2.$$

To obtain the necessity of the equality conditions, suppose there is equality in the last inequality. By Lemma 3.2 this implies that every $(n-1)$-dimensional slice of $\Gamma_p^* K$ is centrally symmetric. A special case of the false center theorem of Aitchison, Petty, and Rogers [1] (see Burton-Mani [4] for an alternate proof) asserts that a convex body, all of whose $(n-1)$-dimensional slices are centrally symmetric, must be an ellipsoid. Thus we conclude that $\Gamma_p^* K$ must be an ellipsoid, and now the equality conditions of Lemma 2.8 may be invoked to show that K itself is a centered ellipsoid. q.e.d.

The only tools needed to show how Theorem A follows directly from Theorem B are dual mixed volumes.

Theorem A. *For real $p \ge 1$ and continuous $f, g : S^{n-1} \to (0, \infty)$,*

$$\int_{S^{n-1}} \int_{S^{n-1}} |u \cdot v|^p f(u) g(v)\, du\, dv \quad \ge \quad c_{n-2,p}\, \|f\|_{\frac{n}{n+p}} \|g\|_{\frac{n}{n+p}},$$

with equality if and only if there exist a $\phi \in \mathrm{SL}(n)$ and $c_1, c_2 > 0$ such that

$$f(u) = c_1 |\phi(u)|^{-(n+p)} \qquad \text{and} \quad g(u) = c_2 |\phi^{-t}(u)|^{-(n+p)},$$

for all $u \in S^{n-1}$.

Proof. Define $K \in \mathcal{S}$ and $L \in \mathcal{S}$ by

$$\rho_K^{n+p} = f \qquad \text{and} \qquad \rho_L^{n+p} = g.$$

The polar coordinate formula for volume and the definitions of $K, L \in \mathcal{S}$ give

$$\|f\|_{\frac{n}{n+p}} = [V(K)/\omega_n]^{p/n} \qquad \text{and} \qquad \|g\|_{\frac{n}{n+p}} = [V(L)/\omega_n]^{p/n}.$$

From the definition of K and L, identity (2.4), the dual mixed volume inequality (1.6), and the inequality of Theorem B, we have

$$\begin{aligned}
&\int_{S^{n-1}}\int_{S^{n-1}} |u\cdot v|^p f(u)g(v)\,du\,dv \\
&= \frac{\omega_n^2}{V(K)V(L)}\int_{S^{n-1}}\int_{S^{n-1}} |u\cdot v|^p[\rho_K(u)\rho_L(v)]^{n+p}\,du\,dv \\
&= c_{n-2,p}\,\frac{\tilde{V}_{-p}(K,\Gamma_p^* L)}{V(K)} \\
&\geq c_{n-2,p}\,[V(K)/V(\Gamma_p^* L)]^{p/n} \\
&\geq c_{n-2,p}\,[V(K)V(L)/\omega_n^2]^{p/n} \\
&= c_{n-2,p}\,\|f\|_{\frac{n}{n+p}}\|g\|_{\frac{n}{n+p}}.
\end{aligned}$$

The above equation also shows that equality in the inequality of the theorem implies that there is equality in the dual mixed volume inequality and equality in the inequality of Theorem B. Thus, equality in the inequality of the theorem implies that K and $\Gamma_p^* L$ are dilates, and L is a centered ellipsoid. But if L is a centered ellipsoid, then from (2.3) we have $\Gamma_p^* L = L^*$. Thus K and L are polar reciprocal ellipsoids which are centered at the origin. The necessity of the equality conditions now follows from (1.1). q.e.d.

The preceding results show that Theorem B can be used to quickly obtain Theorem A. However, the process can be reversed. Theorem A will quickly yield Theorem B, for all real p. If we start with Theorem A, to quickly prove Theorem B we proceed as follows:

Given $K \in \mathcal{S}$, define f and g in Theorem A by

$$f = \frac{\omega_n}{V(K)}\,\rho(K,\ \cdot\ )^{n+p}, \qquad \text{and} \qquad g = \frac{\omega_n}{V(\Gamma_p^* K)}\,\rho(\Gamma_p^* K,\ \cdot\ )^{n+p}.$$

From the polar coordinate formula for volume, we get

$$\|f\|_{\frac{n}{n+p}} = [V(K)/\omega_n]^{p/n}, \qquad \text{and} \qquad \|g\|_{\frac{n}{n+p}} = [V(\Gamma_p^* K)/\omega_n]^{p/n}.$$

Thus, by (2.3) and Theorem A, we have

$$\begin{aligned}\frac{\tilde{V}_{-p}(K, \Gamma_p^*\Gamma_p^* K)}{V(K)} &= \frac{1}{c_{n-2,p}} \int_{S^{n-1}} \int_{S^{n-1}} |u \cdot v|^p f(u) g(v)\, du\, dv \\ &\geq \|f\|_{\frac{n}{n+p}} \|g\|_{\frac{n}{n+p}} \\ &= [V(K)V(\Gamma_p^* K)/\omega_n^2]^{p/n}.\end{aligned}$$

To obtain the inequality of Theorem B, recall that

$$[\tilde{V}_{-p}(K, \Gamma_p^*\Gamma_p^* K)/V(K)] = 1,$$

by (2.6).

From the equality conditions of Theorem A, we see that equality in the inequality of Theorem B would imply the existence of a $c > 0$ and a $\phi \in \mathrm{SL}(n)$ such that $\rho(K, u)/V(K) = c/|\phi u|$, for all $u \in S^{n-1}$. This, in turn, would show that, by (1.1), the star body K must be a centered ellipsoid.

5. Open problems

Mahler (see e.g. [6, p. 339] and [12, p. 427]) conjectured that for each centered (i.e., origin-symmetric) convex body, K, in $\mathbb{R}^n$,

$$V(K)V(K^*) \geq \frac{4^n}{n!}.$$

For zonoids (and thus also their polars) this is Reisner's inequality (see e.g. [6, p. 339] and [12, p. 427]).

Bourgain and Milman [3] proved the existence of an absolute constant $c > 0$, such that for each centered convex body K,

$$V(K)V(K^*) \geq \frac{c^n}{n!}.$$

Thus, there exists an absolute constant $c > 0$ such that

$$V(K)V(K^*) \geq \omega_n^2 c^n,$$

for each centered convex body, K, in $\mathbb{R}^n$.

The following problem is of considerable interest.

Problem 5.1. For each real $p \geq 1$, is there a constant $c_p > 0$, independent of n (and perhaps even independent of p), so that for each centered convex body K in $\mathbb{R}^n$,

$$V(K)V(\Gamma_p^* K) \geq \omega_n^2 c_p^n?$$

For each $K \in \mathcal{S}$, the body $\Gamma_2 K$ is an ellipsoid called the Legendre ellipsoid of K. For the case $p = 2$, the inequality of Problem 5.1 becomes

$$V(K)\, V(\Gamma_2^* K) \geq \omega_n^2\, c_2^n.$$

This inequality is one of the (equivalent) forms of the *slicing problem* (see, e.g., [6, p. 302]): Does there exist an absolute constant $c > 0$, such that each centered convex body of unit volume in $\mathbb{R}^n$, has an $(n-1)$-dimensional slice of $(n-1)$-dimensional volume greater than c?

The body $\Gamma_1 K$, with a different normalization, is called the centroid body of K. Characterizations and inequalities for centroid bodies can be found in [9] and [13]. The Busemann-Petty centroid inequality states: If K is a convex body in $\mathbb{R}^n$, then

$$V(\Gamma_1 K) \geq V(K),$$

with equality if and only if K is a centered ellipsoid.

This inequality was conjectured by Blaschke, for centered convex bodies, and proved by Petty [11], for all bodies.

For the operator Γ_2 there is a similar inequality: If K is a convex body in $\mathbb{R}^n$, then

$$V(\Gamma_2 K) \geq V(K),$$

with equality if and only if K is a centered ellipsoid.

A quick proof of this inequality can be found in e.g. [11]. Note that this inequality is an immediate consequence of Lemma 2.8 and the fact that $\Gamma_2 K$ is an ellipsoid for every K.

It is tempting to conjecture that an inequality stronger (in view of the Blaschke-Santaló inequality) than the inequality of Theorem B holds:

Problem 5.2. Is it the case that for each real $p \geq 1$ and each convex body K in $\mathbb{R}^n$,

$$V(\Gamma_p K) \geq V(K),$$

with equality if and only if K is a centered ellipsoid?

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