New York Journal of Mathematics

New York J. Math. **23** (2017) 497–503.

Alexander polynomial obstruction of bi-orderability for rationally homologically fibered knot groups

Tetsuya Ito

Abstract. We show that if the fundamental group of the complement of a rationally homologically fibered knot in a rational homology 3-sphere is bi-orderable, then its Alexander polynomial has at least one positive real root. Our argument can be applied for a finitely generated group which is an HNN extension with certain properties.

Contents

1. Introduction

A total ordering $\leq_G$ on a group G is a *bi-ordering* if $a \leq_G b$ implies both $ga \leq_G gb$ and $ag \leq_G bg$ for all $a, b, g \in G$. A group is called *bi-orderable* if it admits a bi-ordering.

The Alexander polynomial of knots and groups provides a useful criterion for the (non) bi-orderbility. In [PR03], Perron–Rolfsen showed that for a *fibered* knot $\mathcal{K}$, the knot group $\pi_1(S^3 \setminus \mathcal{K})$ is bi-orderable if its Alexander polynomial $\Delta_{\mathcal{K}}(t)$ has only positive real roots. In [ClR12], Clay–Rolfsen proved the partial converse: if the knot group $\pi_1(S^3 \setminus \mathcal{K})$ for a *fibered* knot $\mathcal{K}$, is bi-orderable, then its Alexander polynomial $\Delta_{\mathcal{K}}(t)$ has at least one positive real root. Actually their argument can be applied for not only a fibered knot group, but also a finitely generated group with finitely generated commutator subgroup.

In [ChGW15] Chiswell–Glass–Wilson showed the same result under the assumption that the group G admits a certain two generator, one relator presentation: under certain assumptions on the presentation, they showed

Received November 15, 2016; revised March 23, 2017.

2010 *Mathematics Subject Classification.* Primary 57M05, Secondary 20F60,06F15.

Key words and phrases. Bi-orderable group, Alexander polynomial.

This work was supported by JSPS KAKENHI Grant Number 15K17540.

ISSN 1076-9803/2017

that G is bi-orderable if all roots of its Alexander polynomial are positive and real, and that if G is bi-orderable then its Alexander polynomial has at least one positive real root. In [ClDN16] Clay–Desmarais–Naylor explored Chiswell–Glass–Wilson criteria to find various nonfibered knots with bi-orderable or non-bi-orderable knot groups.

In this note we prove the following (non)-bi-orderability criterion for a rationally homologically fibered knot.

Definition 1 ([GS13])**.** A null-homologous knot $\mathcal{K}$ in a rational homology 3-sphere M is *rationally homologically fibered* if $\deg \Delta_{\mathcal{K}}(t) = 2g(\mathcal{K})$, where $g(\mathcal{K})$ denotes the genus of the knot $\mathcal{K}$.

Theorem 2. *Let $\mathcal{K}$ be a rationally homologically fibered knot in a rational homology 3-sphere M. If the Alexander polynomial $\Delta_{\mathcal{K}}(t)$ has no positive real root, then the knot group $\pi_1(M \setminus \mathcal{K})$ is not bi-orderable.*

Although not all knots are rationally homologically fibered, compared with fibered knots the class of rationally homologically fibered knots is much larger. For example, the alternating knots (in S^3) are rationally homologically fibered [Cr59, Mu58], and all knots with less than or equal to 11 crossings are rationally homologically fibered, except 11_{n34},11_{n42},11_{n45}, 11_{n67},11_{n73}, 11_{n07}, 11_{n152} (in the table KnotInfo [ChaL]).

Example 3. An alternating knot $\mathcal{K} = 11a_1$ has the Alexander polynomial $\Delta_{\mathcal{K}}(t) = 2 - 12t + 30t^2 - 39t^3 + 30t^4 - 12t^5 + 2t^6$ which has no positive real root. Thus the fundamental group of its complement is not bi-orderable. ($\mathcal{K}$ is not fibered and [ClDN16] fails to find a presentation that satisfies the assumption of Chiswell–Glass–Wilson's criterion so they could not detect the non-bi-orderability)

Our argument relies on the rationally homologically fibered condition which in particular forces the Alexander polynomial to be nontrivial. Thus it is interesting to ask whether $\pi_1(M \setminus \mathcal{K})$ is bi-orderable or not when $\Delta_{\mathcal{K}}(t) = 1$.

2. Proof of theorem

Let $X = M \setminus \mathcal{K}$ be the knot complement and $G = \pi_1(M \setminus \mathcal{K})$ be the knot group. Let $\pi : \widetilde{X} \to X$ be the infinite cyclic covering of X which corresponds to the kernel of the abelianization map $\phi : G \to \mathbb{Z} = \langle t \rangle$.

The first homology group of the infinite cyclic covering $H_1(\widetilde{X};\mathbb{Q})$ has the structure of a $\mathbb{Q}[t,t^{-1}]$ module, where t acts on $\widetilde{X}$ as a deck translation. There exist $p_1(t),\dots,p_n(t) \in \mathbb{Q}[t,t^{-1}]$ and $f \in \mathbb{Z}_{\geq 0}$ such that

$$H_1(\widetilde{X};\mathbb{Q}) \cong \mathbb{Q}[t,t^{-1}]^f \oplus \bigoplus_{i=1}^{n} \mathbb{Q}[t,t^{-1}]/(p_i(t)).$$

The Alexander polynomial $\Delta_{\mathcal{K}}(t)$ is defined by

$$\Delta_{\mathcal{K}}(t) = \begin{cases} p_1(t)p_2(t)\cdots p_n(t) & (f=0) \\ 0 & (f>0). \end{cases}$$

Thus $\Delta_{\mathcal{K}}(t) \cdot h = 0$ for every $h \in H_1(\widetilde{X};\mathbb{Q})$.

Let Σ be a minimum genus Seifert surface of $\mathcal{K}$, and let $Y = M \setminus N(\Sigma)$, where $N(\Sigma) \cong \Sigma \times (-1,1)$ denotes a regular neighborhood of Σ. Let $\iota^{\pm} : \Sigma \hookrightarrow \Sigma \times \{\pm 1\} \subset Y$ be the inclusion maps. As is well-known, the infinite cyclic covering $\widetilde{X}$ is obtained by gluing infinitely many copies $\{Y_i\}_{i \in \mathbb{Z}}$ of Y along Σ, where the i-th copy Y_i and the $(i+1)$-st copy Y_{i+1} are glued by identifying $\iota^-(\Sigma) \subset Y_i$ and $\iota^+(\Sigma) \subset Y_{i+1}$. In the rest of argument, we will always take a base point of $\widetilde{X}$ so that it lies in Y_0.

For $N \geq 0$, let $Y_{[-N,N]} = \bigcup_{i=-N}^{N} Y_i \subset \widetilde{X}$, and let $i_N : Y_0 \hookrightarrow Y_{[-N,N]}$ and $j_N : Y_{[-N,N]} \hookrightarrow \widetilde{X}$ be the inclusion maps. We denote the fundamental group $\pi_1(Y_{[-N,N]})$ and $\pi_1(\widetilde{X}) = \operatorname{Ker}\phi$ by K_N and K, respectively. Since $Y_{[-N,N]}$ is compact, K_N is finitely generated.

Since $\iota^{\pm}_* : \pi_1(\Sigma) \to \pi_1(X)$ are injective, by van-Kampen theorem it follows that both $(i_N)_* : K_0 \to K_N$ and $(j_N)_* : K_N \to K$ are injective. By these inclusion maps we will always regard K_0 as a subgroup of K_N, and K_N as a subgroup of K. For $x \in K_0$, we will often simply denote $(i_N)_*(x) \in K_N$ by the same symbol x, by abuse of notation.

Proof of Theorem 2. Assume that $\mathcal{K}$ is rationally homologically fibered, and that the Alexander polynomial $\Delta_{\mathcal{K}}(t)$ has no positive real roots.

A theorem of Dubickas [Du07] says that a one-variable polynomial $f(t) \in \mathbb{Q}[t,t^{-1}]$ has no positive real roots if and only if there is a nonzero polynomial $g(t) \in \mathbb{Q}[t,t^{-1}]$ such that all the nonzero coefficients of $g(t)f(t)$ are positive. Thus there is a nonzero polynomial $D(t)$ such that all the nonzero coefficients of $D(t)\Delta_{\mathcal{K}}(t)$ are positive. We take such $D(t)$ so that $D(t)\Delta_{\mathcal{K}}(t) = \sum_{i\geq 0} a_i t^i$ with $a_0 > 0$ and $a_i \geq 0$ $(i > 0)$.

For $x \in K = \pi_1(\widetilde{X})$, we denote by $[x] \in H_1(\widetilde{X};\mathbb{Q})$ the homology class represented by x. Then $[x] = 0$ if and only if $x^r \in [K,K]$ for some $r > 0$.

Let $s \in \pi_1(X)$ be an element represented by a meridian of the knot $\mathcal{K}$. Then $t^i[x] = [s^{-i}xs^i]$. By definition of the Alexander polynomial, for each $x \in K$

$$\begin{aligned} D(t)\Delta_{\mathcal{K}}(t)[x] = \sum_{i\geq 0} a_i t^i [x] &= \sum_{i \geq 0} [s^{-i} x^{a_i} s^i] \\ &= \left[\prod_{i\geq 0} (s^{-i} x^{a_i} s^i) \right] = 0 \in H_1(\widetilde{X};\mathbb{Q}). \end{aligned}$$

This implies that there is $r(x) > 0$ such that

$$\left(\prod_{i\geq 0}(s^{-i}x^{a_i}s^i)\right)^{r(x)} \in [K,K].$$

Moreover, since $K = \bigcup_{n\geq 0} K_n$, there is $N(x) \in \mathbb{Z}$ such that

$$\left(\prod_{i\geq 0}(s^{-i}x^{a_i}s^i)\right)^{r(x)} \in [K_{N(x)},K_{N(x)}].$$

Take a finite symmetric generating set $\mathcal{X}$ of K_0. Here symmetric we mean that $x \in \mathcal{X}$ implies $x^{-1} \in \mathcal{X}$. Let $N = \max\{N(x) \mid x \in \mathcal{X}\}$, and let r be the least common multiple of $r(x)$ for $x \in \mathcal{X}$. Then for every $x \in \mathcal{X}$ we have

$$\left(\prod_{i\geq 0}(s^{-i}x^{a_i}s^i)\right)^{r} \in [K_N,K_N]. \tag{2.1}$$

Now assume to the contrary that, G is bi-orderable. Let $<_{K_N}$ be a bi-ordering on K_N which is the restriction of a bi-ordering of G. Since K_N is finitely generated, by [ClR12, Lemma 2.4] there is a $<_{K_N}$ convex normal subgroup C of K_N such that the quotient group $A_N := K_N/C$ is a nontrivial, torsion-free abelian group. Then A_N has the bi-ordering $<_{A_N}$ coming from $<_{K_N}$; $a <_{A_N} a'$ if and only if $a = P(k)$, $a' = P(k')$ $(k,k' \in K_N)$ with $k <_{K_N} k'$, where $P : K_N \to A_N$ denotes the quotient map (see [It13, Section 2] for details on abelian, bi-ordered quotients).

Lemma 1. *Let $q = P \circ (i_N)_* : K_0 \xrightarrow{(i_N)_*} K_N \xrightarrow{P} A_N$. If both*

$$(\iota^{\pm})_* : H_1(\Sigma;\mathbb{Q}) \to H_1(Y;\mathbb{Q})$$

are surjections, then q is a surjection.

Proof. By Meyer-Vietoris sequence, the surjectivity of $(\iota^{\pm})_*$ implies the surjectivity of $(i_N)_* : H_1(Y_0;\mathbb{Q}) \to H_1(Y_{[-N,N]};\mathbb{Q})$. Thus

$$(i_N)_* : H_1(Y_0;\mathbb{Z}) \to H_1(Y_{[-N,N]};\mathbb{Z})$$

is a surjection modulo torsion elements.

On the other hand, A_N is an abelian group so the map q is written as compositions

$$\begin{aligned} K_0 = \pi_1(Y_0) &\longrightarrow H_1(Y_0;\mathbb{Z}) \\ &\xrightarrow{(i_N)_*} H_1(Y_{[-N,N]};\mathbb{Z}) = K_N/[K_N,K_N] \\ &\longrightarrow K_N/C = A_N. \end{aligned}$$

All maps are surjections modulo torsion elements and A_N is torsion-free so q is a surjection. $\square$

The following lemma clarifies a role of the rationally homologically fibered assumption (cf. [GS13, Proposition 2]).

Lemma 2. *Both* $(\iota^{\pm})_* : H_1(\Sigma;\mathbb{Q}) \to H_1(Y;\mathbb{Q})$ *are surjections if and only if* $\mathcal{K}$ *is rationally homologically fibered.*

Proof. Let g be the genus of $\mathcal{K}$. By Alexander duality,

$$\dim H_1(\Sigma;\mathbb{Q}) = \dim H_1(Y;\mathbb{Q}) = 2g.$$

This shows that $(\iota^{\pm})_*$ are surjections if and only if $(\iota^{\pm})_*$ are isomorphisms, that is, if and only if they are invertible.

By the Mayer–Vietoris sequence, $H_1(\widetilde{X};\mathbb{Q})$ is written as

$$H_1(\widetilde{X};\mathbb{Q}) = \mathbb{Q}[t,t^{-1}]/\{t(\iota^+)_*(h) = (\iota^-)_*(h) \quad \forall h \in H_1(\Sigma)\}$$

Thus $\Delta_{\mathcal{K}}(t)$ is equal to the determinant of

$$t(\iota^+)_* - (\iota^-)_* : \mathbb{Q}^{2g} = H_1(\Sigma;\mathbb{Q}) \to H_1(Y;\mathbb{Q}) \cong \mathbb{Q}^{2g}.$$

If $(\iota^{\pm})_*$ are surjective, then they are invertible so

$$\Delta_{\mathcal{K}}(t) = \det(t - (\iota^+)_*^{-1}(\iota^-)_*)\det(\iota^+).$$

Since $(\iota^+)_*^{-1}(\iota^-)_*$ is invertible, $\deg \Delta_{\mathcal{K}}(t) = 2g$.

Conversely, if $\deg \Delta_{\mathcal{K}}(t) = 2g$ then

$$\Delta_{\mathcal{K}}(0) = \det((\iota^-)_*) = \det((\iota^+)_*) \neq 0$$

so both $(\iota^{\pm})_*$ are invertible. □

Now we are ready to complete the proof of Theorem 2.

By Lemma 1 and Lemma 2, if $\mathcal{K}$ is rationally homologically fibered, then q is surjective. Since $\mathcal{X}$ is a symmetric generating set, the surjectivity of q implies that there exists $x \in \mathcal{X}$ such that $1 <_{A_N} q(x)$. By definition of the quotient ordering $<_{A_N}$, $1 <_{K_N} x$. The ordering $<_{K_N}$ is the restriction of a bi-ordering of G and $0 \le a_i$ so $1 \le_{K_N} s^{-i}x^{a_i}s^i$. Therefore $1 \le_{A_N} P(s^{-i}x^{a_i}s^i)$ for all $i \ge 0$. Since $a_0 > 0$, as a consequence we get

$$1 <_{A_N} q(x) \le_{A_N} P\left(\prod_{i\ge 0}(s^{-i}x^{a_i}s^i)\right)^r.$$

On the other hand, $[K_N, K_N] \subset C$ so (2.1) implies

$$P\left(\prod_{i\ge 0}(s^{-i}x^{a_i}s^i)\right)^r = 1 \in K_N/C = A_N.$$

This is a contradiction. □

We state and prove our main theorem for the case that the group is the fundamental group of a knot complement. However, our proof can be applied for a finitely generated group represented by a certain HNN extension.

For a finitely generated group G and a surjection $\phi : G \to \mathbb{Z} = \langle t \rangle$, $H_1(\operatorname{Ker}\phi;\mathbb{Q})$ has the structure of a finitely generated $\mathbb{Q}[t,t^{-1}]$-module. The Alexander polynomial $\Delta_G^{\phi}(t)$ (with respect to ϕ) is defined similarly, and has the same property that $\Delta_G^{\phi}(t)\cdot h = 0$ for all $h \in H_1(\operatorname{Ker}\phi;\mathbb{Q})$.

In the proof of Theorem 2, besides the assumption that the Alexander polynomial has no positive real roots, what we really needed and used can be stated in terms of the groups $\operatorname{Ker}\phi$, $\pi_1(\Sigma)$ and $\pi_1(Y)$: we used the amalgamated product decomposition

$$\operatorname{Ker}\phi = \pi_1(\widetilde{X}) = \cdots *_{\pi_1(\Sigma)} \pi_1(Y) *_{\pi_1(\Sigma)} \pi_1(Y) *_{\pi_1(\Sigma)} \pi_1(Y) *_{\pi_1(\Sigma)} \cdots \tag{2.2}$$

having the properties

$$\pi_1(Y) \text{ is finitely generated.} \tag{2.3}$$

$$\text{The inclusions } \iota_*^{\pm} : \pi_1(\Sigma) \to \pi_1(Y) \text{ induce surjections} \tag{2.4}$$
$$\iota_*^{\pm} : H_1(\pi_1(\Sigma);\mathbb{Q}) \to H_1(\pi_1(Y);\mathbb{Q}).$$

Note that we used the topological assumption that $\mathcal{K}$ is a rationally homologically fibered knot in a rational homology sphere M only at Lemma 2, which is used to show the property (2.4).

In the language of group theory, the amalgamated product decomposition (2.2) comes from an expression of $\pi_1(M \setminus \mathcal{K})$ as an HNN extension

$$\pi_1(M \setminus \mathcal{K}) = *_{\pi_1(\Sigma)}\pi_1(Y) = \langle t, \pi_1(Y) \mid t^{-1}\iota^+(s)t = \iota^-(s) \quad (\forall s \in \pi_1(\Sigma))\rangle.$$

In summary, our proof of Theorem 2 actually shows the following non-bi-orderbility criterion.

Theorem 4. *Let H be a finitely generated group and A be a group (not necessarily a finitely generated). Let $\iota^{\pm} : A \to H$ be homomoprhisms such that*

$$(\iota^{\pm})_* : H_1(A;\mathbb{Q}) \to H_1(H;\mathbb{Q})$$

are surjective. Let G be a finitely generated group given by an HNN extension

$$G = *_A H = \langle t, H \mid t^{-1}\iota^+(a)t = \iota^-(a) \quad (\forall a \in A)\rangle.$$

Let $\phi : G \to \mathbb{Z}$ is a surjection given by $\phi(t) = 1$, $\phi(h) = 0$ for all $h \in H$. If the Alexander polynomial $\Delta_G^{\phi}(t)$ has no positive real root, then G is not bi-orderable.

Acknowledgements

The author thanks for Eiko Kin for valuable comments on an earlier draft of the paper. Also, the author thank for Stefan Friedl for pointing out that the proof of Theorem 2 requires the hypothesis that K is rationally homologically fibered.

References

[ChaL] Cha, Jae Choo; Livingston, Charles. Knotinfo: Table of knot invariants. `http://www.indiana.edu/~knotinfo`.

[ChGW15] Chiswell, Ian M.; Glass, Andrew M. W.; Wilson, John S. Residual nilpotence and ordering in one-relator groups and knot groups. *Math. Proc. Cambridge Philos. Soc.* **158** (2015), no. 2, 275–288. MR3310246, arXiv:1405.0994, doi: 10.1017/S0305004114000644.

[ClDN16] Clay, Adam; Desmarais, Colin; Naylor, Patrick. Testing bi-orderability of knot groups. *Canad. Math. Bull.* **59** (2016), no. 3, 472–482. MR3563729, Zbl 1350.57005, arXiv:1410.5774.

[ClR12] Clay, Adam; Rolfsen, Dale. Ordered groups, eigenvalues, knots, surgery and L-spaces. *Math. Proc. Cambridge Philos. Soc.* **152** (2012), no. 1, 115–129. MR2860419, Zbl 1241.57014, arXiv:1004.3615, doi: 10.1017/S0305004111000557.

[Cr59] Crowell, Richard. Genus of alternating link types, *Ann. of Math.* (2) **69** (1959), 258–275. MR0099665 (20 #6103b), Zbl 0111.35803, doi: 10.2307/1970181.

[Du07] Dubickas, Artūras. On roots of polynomials with positive coefficients. *Manuscripta Math.* **123** (2007), no. 3, 353–356. MR2314089, Zbl 1172.11036, doi: 10.1007/s00229-007-0101-7.

[GS13] Goda, Hiroshi; Sakasai, Takuya. Homology cylinders and sutured manifolds for homologically fibered knots. *Tokyo J. Math* **36** (2013), no. 1, 85–111. MR3112377, Zbl 1287.57022, arXiv:0807.4034, doi: 10.3836/tjm/1374497513.

[It13] Ito, Tetsuya. A remark on the Alexander polynomial criterion for the bi-orderability of fibered 3-manifold groups. *Int. Math. Res. Not. IMRN* **2013** no. 1, 156–169. MR3041697, Zbl 1278.57028, arXiv:1012.0635, doi: 10.1093/imrn/rnr264.

[Mu58] Murasugi, Kunio. On the genus of the alternating knot. II. *J. Math. Soc. Japan* **10** (1958), 235–248. MR0099664 (20 #6103a), Zbl 0106.16701, doi: 10.2969/jmsj/01030235.

[PR03] Perron, Bernard; Rolfsen, Dale. On orderability of fibred knot groups. *Math. Proc. Cambridge Philos. Soc.* **135** (2003), no. 1, 147–153. MR1990838 (2004f:57015), Zbl 1046.57008, doi: 10.1017/S0305004103006674.

(Tetsuya Ito) Department of Mathematics, Graduate School of Science, Osaka University, 1-1 Machikaneyama Toyonaka, Osaka 560-0043, JAPAN
tetito@math.sci.osaka-u.ac.jp
http://www.math.sci.osaka-u.ac.jp/~tetito/

This paper is available via http://nyjm.albany.edu/j/2017/23-22.html.