Noncommutative bundles over the multi-pullback quantum complex projective plane

Piotr M. Hajac and Jan Rudnik

ABSTRACT. We equip the multi-pullback C*-algebra $C(S_H^5)$ of a noncommutative deformation of the 5-sphere with a free $U(1)$-action, and show that its fixed-point subalgebra is isomorphic with the C*-algebra of the multi-pullback quantum complex projective plane. Our main result is the stable nontriviality of the dual tautological line bundle associated to the action. We prove it by combining Chern–Galois theory with the Milnor connecting homomorphism in K-theory. Using the Mayer–Vietoris six-term exact sequences and the functoriality of the Künneth formula, we also compute the K-groups of $C(S_H^5)$.

CONTENTS

Introduction.

This paper is a part of a bigger project devoted to the K-theory of multi-pullback noncommutative deformations of free actions on spheres defining complex and real projective spaces. The lowest-dimensional complex case is worked out in [BaHMS05, HMS06b] with the help of index theory. Herein we focus on the triple-pullback quantum complex projective plane $\mathbb{C}P_{\mathcal{T}}^2$ [HKaZ12] and its quantum 5-sphere S_H^5. Upgrading from pullback C*-algebras of [BaHMS05, HMS06b] to triple-pullback C*-algebras requires a significant change of methods. In particular, we have to take care of the cocycle condition, as explained in Section 1.2.3, to compute the K-groups of $C(S_H^5)$ and $C(\mathbb{C}P_{\mathcal{T}}^2)$ in Section 3 and [Ru12] respectively.

The main theorem of the paper is:

Theorem 2.4 *The section module $C(S_H^5)_u$ of the dual tautological line bundle over $\mathbb{C}P_{\mathcal{T}}^2$ is* not stably free *as a left $C(\mathbb{C}P_{\mathcal{T}}^2)$-module.*

The result is derived by comparing two idempotents: one coming from Chern–Galois theory applied to the $U(1)$-action on $C(S_H^5)$, and the other one obtained by applying a formula (11) for the Milnor connecting homomorphism in a K-theory exact sequence. It is the same strategy that was used to determine nontrivial generators of the K_0-group of Heegaard quantum lens spaces [HRZ13].

To explain a wider background and make the paper self-contained, we begin with a review of basic building blocks that are subsequently assembled into new results. Concerning notation, we use the unadorned tensor product $\otimes$ to denote the minimal (spatial) tensor product of C*-algebras and $\otimes_{\text{alg}}$ to denote the algebraic tensor product.

1. Preliminaries

1.1. From the Toeplitz algebra to quantum projective spaces.

1.1.1. Toeplitz algebra. There are different ways to introduce the Toeplitz algebra $\mathcal{T}$. Herein we define it as the universal C*-algebra generated by one isometry s, i.e., an element satisfying the relation $s^*s = 1$. (Throughout the paper s will always mean the generating isometry of $\mathcal{T}$.) Likewise, u will always mean the unitary element generating the C*-algebra $C(S^1)$ of all continuous complex-valued functions on the unit circle

$$S^1 := \{x \in \mathbb{C} \mid |x| = 1\}.$$

By mapping s to u, we obtain the well-known short exact sequence of C*-algebras [CoL I, CoL II]:

(1) $0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \overset{\sigma}{\longrightarrow} C(S^1) \longrightarrow 0.$

We consider the Toeplitz algebra as the C*-algebra of continuous functions on a *quantum disc*. To justify this point of view, we take the family of

universal C*-algebras generated by x satisfying $x^*x - qxx^* = 1-q$, $\|x\| = 1$, $q \in [0,1]$ [KL93]. For $q \neq 1$, the norm condition is implied by the relation, and can be omitted. For $q = 1$, it yields precisely the C*-algebra $C(D)$ of all continuous complex-valued functions on the unit disc $D := \{x \in \mathbb{C} \mid |x| \leq 1\}$. Finally, for $q = 0$, we get the Toeplitz algebra. Thus we obtain $\mathcal{T}$ as a q-deformation of $C(D)$.

Both the Toeplitz algebra $\mathcal{T}$ and $C(S^1)$ are examples of graph C*-algebras [FLR00]. Graph C*-algebras are generated by partial isometries, which come naturally equipped with a U(1)-action given by rephasing these partial isometries by unitary complex numbers (the gauge action). A key feature of the symbol map σ is that it is equivariant with respect to the gauge actions.

1.1.2. Complex projective spaces as multi-pushouts. Recall that an element of $\mathbb{C}P^n$ is an equivalence class with respect the relation

$$(x_0, \ldots, x_n) \sim (y_0, \ldots, y_n) \iff \exists \lambda \in \mathbb{C} \setminus \{0\} : (x_0, \ldots, x_n) = \lambda(y_0, \ldots, y_n).$$

We denote the equivalence class of $(x_0, \ldots, x_n)$ by $[x_0 : \ldots : x_n]$. Such a presentation of $\mathbb{C}P^n$ comes with the canonical affine open covering:

$$(2) \quad \forall\, i \in \{0, \ldots, n\} : U_i := \{[x_0 : \ldots : x_n] \in \mathbb{C}P^n \mid x_i \neq 0\} \xrightarrow{\widetilde{\psi_i}} \mathbb{C}^n.$$

The above homeomorphisms are given by

$$(3) \qquad \widetilde{\psi_i}([x_0 : \ldots : x_n]) := \left(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i} \right).$$

To express the covering subsets in C*-algebraic terms, we choose closed rather than open coverings. To this end, we define the following closed refinement of the affine covering:

$$\forall\, i \in \{0, \ldots, n\} : V_i := \left\{ [x_0 : \ldots : x_n] \in \mathbb{C}P^n \mid |x_i| = \max\{|x_0|, \ldots, |x_n|\} \right\}.$$

Note that each V_i is homeomorphic with D^n. Here the homeomorphisms are given by appropriate restrictions of $\widetilde{\psi_i}$'s denoted by ψ_i. We use the covering $\{V_i\}_i$ to present $\mathbb{C}P^n$ as a multi-pushout. More precisely, we pick indices $0 \leq i < j \leq n$, denote by ψ_{ij} the restriction of ψ_i to $V_i \cap V_j$, and take the following commutative diagram:

(4)
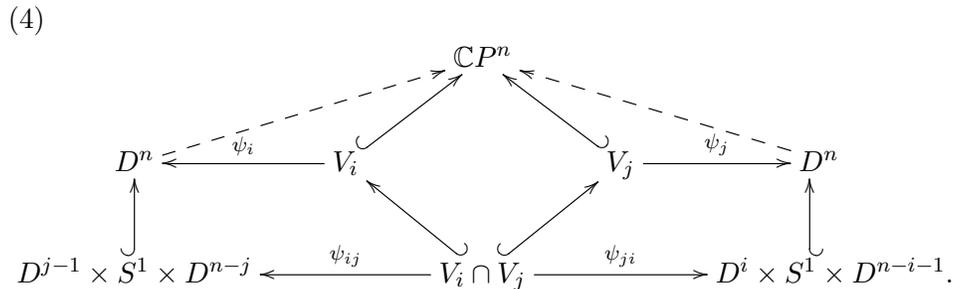
1.1.3. Multi-pullback quantum complex projective spaces. Now we combine the foregoing presentation of projective spaces with the idea that the Toeplitz algebra is the C*-algebra of functions on a quantum unit disc to construct a new type of quantum projective spaces [HKaZ12]. To define them, first we excise from diagram (4) its middle square, and dualise it to the multi-pullback diagram of unital commutative C*-algebras of functions on appropriate compact Hausdorff spaces:

(5)

$$
\begin{array}{ccc}
& C(\mathbb{C}P^n) & \\
C(D)^{\otimes n} & & C(D)^{\otimes n} \\
\downarrow \pi_j^i & \pi_i^j & \downarrow \\
C(D)^{\otimes j-1}\otimes C(S^1)\otimes C(D)^{\otimes n-j} \xleftarrow{(\psi_{ji}\circ\psi_{ij}^{-1})^*} & C(D)^{\otimes i}\otimes C(S^1)\otimes C(D)^{\otimes n-i-1}.
\end{array}
$$

This yields a multi-pullback presentation of $C(\mathbb{C}P^n)$. Then we leave $C(S^1)$ unchanged and replace $C(D)$ by $\mathcal{T}$. It turns out that the formulae for all

(6) $(\psi_{ji} \circ \psi_{ij}^{-1})^*, \quad \pi_j^i \quad \text{and} \quad \pi_i^j$

continue to make sense after these replacements, so that quantum complex projective spaces can be defined as Pedersen's *multi-pullback C*-algebras* (see [Pe99, CaM00])

(7) $B^\pi := \left\{ (b_i)_i \in \prod_{i\in J} B_i \;\middle|\; \pi_j^i(b_i) = \pi_i^j(b_j), \; \forall\, i,j \in J, \; i \neq j \right\}.$

Here each B_i is $\mathcal{T}^{\otimes n}$, $B_{ij} := \mathcal{T}^{\otimes j-1}\otimes C(S^1)\otimes \mathcal{T}^{\otimes n-j} =: B_{ji}$, $i \neq j$, and $\{\pi_j^i : B_i \to B_{ij}\}_{i,j\in J,\, i\neq j}$ is the family of C*-homomorphisms defined through commutative diagram (5) with $C(D)$ replaced by $\mathcal{T}$.

1.2. The Mayer–Vietoris six-term exact sequence. If at least one of *-homomorphisms defining a pullback of C*-algebras

(8)

$$
\begin{array}{ccc}
& A & \\
B_0 & & B_1 \\
& \pi_0 \searrow \quad \swarrow \pi_1 & \\
& B_{01} &
\end{array}
$$

is surjective (here we choose π_0), then there exists the Mayer–Vietoris six-term exact sequence (e.g., see [Bl98, Theorem 21.2.2], [BaHMS05, Section 1.3], [S84]):

$$(9) \qquad \begin{array}{ccc} K_0(A) & \longrightarrow & K_0(B_0 \oplus B_1) & \longrightarrow & K_0(B_{01}) \\ \partial_{10} \uparrow & & & & \downarrow \partial_{01} \\ K_1(B_{01}) & \longleftarrow & K_1(B_0 \oplus B_1) & \longleftarrow & K_1(A). \end{array}$$

In our applications of this exact sequence, we will need explicit formulae for connecting homomorphisms ∂_{10} and ∂_{01}.

1.2.1. Odd-to-even connecting homomorphism. Following Milnor's celebrated construction of an odd-to-even connecting homomorphism in algebraic K-theory [M71], one can derive an explicit formula for this homomorphism [R94, DHHMW12], and adapt it to unital C*-algebras (see [HRZ13, Section 0.4] for an argument of Nigel Higson).

Theorem 1.1. *Let* $U \in GL_n(B_{01})$, $(\mathrm{id} \otimes \pi_0)(c) = U^{-1}$ *and* $(\mathrm{id} \otimes \pi_0)(d) = U$. *Denote by* I_n *the identity matrix of size* n, *and put*

$$(10) \qquad p_U := \begin{pmatrix} (c(2-dc)d, 1) & (c(2-dc)(1-dc), 0) \\ ((1-dc)d, 0) & ((1-dc)^2, 0) \end{pmatrix} \in M_{2n}(A).$$

Then p_U *is an idempotent and the formula*

$$(11) \qquad \partial_{10}([U]) := [p_U] - [I_n]$$

defines an odd-to-even connecting homomorphism $\partial_{10} : K_1(B_{01}) \to K_0(A)$ *in the Mayer–Vietoris six-term exact sequence* (9).

1.2.2. Even-to-odd connecting homomorphism. Theorem 1.18 of [BaM] combined with Section 9.3.2 of [Bl98] yields:

Theorem 1.2. *Let* $p \in M_n(B_{01})$ *be a projection,* $(\mathrm{id} \otimes \pi_0)(Q_p) = p$, $Q_p^* = Q_p$, *and* I_n *be the identity matrix of size* n. *Then the formula*

$$(12) \qquad \partial_{10}([p]) := [(e^{2\pi i Q_p}, I_n)]$$

defines an even-to-odd connecting homomorphism in the Mayer–Vietoris six-term exact sequence (9).

1.2.3. Cocycle condition for multi-pullback C*-algebras. We construct algebras of functions on quantum spaces as multi-pullbacks of C^*-algebras. To make sure that this construction dually corresponds to the presentation of a quantum space as a "union of closed subspaces" (no self gluings of closed subspaces or their partial multi-pushouts; see [HZ12] for an in-depth discussion of these issues), we assume the cocycle condition. It allows us to apply the Mayer–Vietoris six-term exact sequence to multi-pullback C*-algebras by guaranteeing surjectivity of appropriate *-homomorphisms.

First we need some auxilliary definitions. Let $\{\pi_j^i : A_i \to A_{ij}\}_{i,j \in J, i \neq j}$ be a finite family of surjective C*-algebra homomorphisms. For all distinct $i, j, k \in J$, we define $A_{jk}^i := A_i / (\ker \pi_j^i + \ker \pi_k^i)$ and denote by $[\cdot]_{jk}^i : A_i \to A_{jk}^i$ the canonical surjections. (Observe that we use the fact

that the sum of closed ideals in a C*-algebra is a closed ideal [Di64, Corollary 1.8.4].) Next, we introduce the family of maps

$$(13) \qquad \pi_k^{ij} : A_{jk}^i \longrightarrow A_{ij}/\pi_j^i(\ker \pi_k^i), \qquad [b_i]_{jk}^i \longmapsto \pi_j^i(b_i) + \pi_j^i(\ker \pi_k^i),$$

for all distinct $i, j, k \in J$. Note that they are isomorphisms when all π_j^i's are surjective C*-algebra homomorphisms, as assumed herein.

We say [CaM00, in Proposition 9] that a finite family

$$\{\pi_j^i : A_i \to A_{ij}\}_{i,j \in J, i \neq j}$$

of C*-algebra surjections satisfies the *cocycle condition* if and only if, for all distinct $i, j, k \in J$:

(1) $\pi_j^i(\ker \pi_k^i) = \pi_i^j(\ker \pi_k^j)$.
(2) The isomorphisms

$$\varphi_k^{ij} := (\pi_k^{ij})^{-1} \circ \pi_k^{ji} : A_{ik}^j \to A_{jk}^i$$

satisfy $\varphi_j^{ik} = \varphi_k^{ij} \circ \varphi_i^{jk}$.

One proves ([HZ12, Theorem 1]) that a finite family

$$\{\pi_j^i : A_i \to A_{ij}\}_{i,j \in J, i \neq j}$$

of C*-algebra surjections satisfies the cocycle condition if and only if, for all $K \subsetneq J$, $k \in J \setminus K$, and $(b_i)_{i \in K} \in \bigoplus_{i \in K} A_i$ such that $\pi_j^i(b_i) = \pi_i^j(b_j)$ for all distinct $i, j \in K$, there exists $b_k \in A_k$ such that $\pi_k^i(b_i) = \pi_i^k(b_k)$ also holds for all $i \in K$. One can easily see that dually this corresponds to the statement "a quantum space is a pushout of parts, and all partial pushouts are embedded in this quantum space". This is what we usually have in mind when constructing a space from parts.

1.3. Actions of compact Hausdorff groups on unital C*-algebras.
To use the language of strong connections [H96] and facilitate some computations, we need to transform actions of compact Hausdorff groups on unital C*-algebras to coactions of their C*-algebras on unital C*-algebras. More precisely, let A be a unital C*-algebra and G a compact Hausdorff group with a group homomorphism $\alpha \colon G \ni g \mapsto \alpha_g \in \mathrm{Aut}(A)$. Then the induced coaction is

$$(14) \qquad \delta_\alpha \colon A \longrightarrow C(G, A) \cong A \otimes C(G), \quad \delta_\alpha(a)(g) := \alpha_g(a).$$

We will use the thus related action and coaction interchangeably.

Furthermore, for any compact Hausdorff group G, we can define the Hopf-algebraic structure on $C(G)$ due to its commutativity:

- the comultiplication $\Delta \colon C(G) \to C(G) \otimes C(G)$,
- the counit $\varepsilon \colon C(G) \to \mathbb{C}$,
- and the antipode $S \colon C(G) \to C(G)$,

are respectively the pullbacks of the group mulitplication, the embedding of the neutral element into G, and the inverting map $G \ni g \mapsto g^{-1} \in G$. We can also use the Heynemann–Sweedler notation (with the summation sign suppressed) for coactions and comultiplications:

$$\delta_\alpha(a) =: a_{(0)} \otimes a_{(1)},$$
$$\delta_\alpha(a)(g) = (a_{(0)} \otimes a_{(1)})(g) = a_{(0)} a_{(1)}(g),$$
$$\Delta(h) =: h_{(1)} \otimes h_{(2)},$$
$$\Delta(h)(g_1, g_2) = (h_{(1)} \otimes h_{(2)})(g_1, g_2) = h_{(1)}(g_1) h_{(2)}(g_2) = h(g_1 g_2).$$

In particular, for $G = U(1)$, the antipode is determined by $S(u) = u^{-1}$, the counit by $\varepsilon(u) = 1$, and finally the comultiplication by $\Delta(u) = u \otimes u$. The coaction of $C(U(1))$ on $\mathcal{T}$ coming from the aforementioned (Section 1.1.1) gauge action of $U(1)$ on $\mathcal{T}$ becomes

$$(15) \qquad \delta : \mathcal{T} \longrightarrow \mathcal{T} \otimes C(U(1)), \quad \delta(s) := s \otimes u.$$

1.3.1. Freeness. Following [E00], we say that an action of a compact Hausdorff group G on a unital C*-algebra A is *free* if and only if the induced coaction satisfies the following norm-density condition:

$$(16) \qquad \{(x \otimes 1)\delta_\alpha(y) \mid x, y \in A\}^{\mathrm{cls}} = A \otimes C(G).$$

Here "cls" stands for "closed linear span".

Next, let us denote by $\mathcal{O}(G)$ the dense Hopf $*$-subalgebra spanned by the matrix coefficients of finite-dimensional representations. We define the *Peter–Weyl subalgebra* of A as

$$(17) \qquad \mathcal{P}_G(A) := \left\{ a \in A \, \middle| \, \delta_\alpha(a) \in A \underset{\mathrm{alg}}{\otimes} \mathcal{O}(G) \right\}.$$

One shows that it is an $\mathcal{O}(G)$-comodule algebra which is a dense $*$-subalgebra of A (see [So11, Po95]). Moreover, the C*-algebraic freeness condition on a G-C*-algebra A is equivalent to the algebraic *principality* condition on the $\mathcal{O}(G)$-comodule algebra $\mathcal{P}_G(A)$ [BaDH]. This allows us to use crucial algebraic tools without leaving the ground of C*-algebras.

1.3.2. Strong connections and principal comodule algebras. One can prove (see [BrH09] and references therein) that a comodule algebra is principal if and only if it admits a strong connection. Therefore, we will treat the existence of a strong connection as a condition defining the principality of a comodule algebra and avoid the original definition of a principal comodule algebra. The latter is important when going beyond coactions that are algebra homomorphisms — then the existence of a strong connection is implied by principality but we do not have the reverse implication [BrH04].

Let G be a compact Hausdorff group acting on a unital C*-algebra A. A *strong connection* ℓ is a unital linear map

$$\ell : \mathcal{O}(G) \longrightarrow \mathcal{P}_G(A) \otimes_{\mathrm{alg}} \mathcal{P}_G(A)$$

satisfying:

(1) $(\mathrm{id} \otimes \delta) \circ \ell = (\ell \otimes \mathrm{id}) \circ \Delta$, $\big(((S \otimes \mathrm{id}) \circ \mathrm{flip} \circ \delta) \otimes \mathrm{id} \big) \circ \ell = (\mathrm{id} \otimes \ell) \circ \Delta$;
(2) $m \circ \ell = \varepsilon$, where $m \colon \mathcal{P}_G(A) \otimes_{\mathrm{alg}} \mathcal{P}_G(A) \to \mathcal{P}_G(A)$ is the multiplication map.

Here we abuse notation by using the same symbol for a restriction-corestriction of a map as for the map itself.

1.3.3. Associated projective modules. Let $\varrho \colon G \to GL(V)$ be a representation of a compact Hausdorff group G on a complex vector space V, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action on a unital C*-algebra A. Then the *associated module* $\mathcal{P}_G(A) \square^\varrho V$ is, by definition,

$$(18) \qquad \left\{ x \in \mathcal{P}_G(A) \underset{\mathrm{alg}}{\otimes} V \ \big| \ \forall\, g \in G : (\alpha_g \otimes \mathrm{id})(x) = \big(\mathrm{id} \otimes \varrho(g^{-1}) \big)(x) \right\}.$$

It is a left module over the fixed-point subalgebra

$$(19) \qquad\qquad A^\alpha := \{ a \in A \mid \forall\, g \in G : \alpha_g(a) = a \} =: A^G.$$

If V is finite dimensional and α is free, then $\mathcal{P}_G(A) \square^\varrho V$ is finitely generated projective [HM99]. We think of it as the section module of an *associated noncommutative vector bundle*. Furthermore, if $\dim V = 1$ and $\gamma \colon G \to GL(\mathbb{C})$ is a representation, then we obtain:

$$(20) \qquad\qquad \mathcal{P}_G(A) \overset{\gamma}{\square} \mathbb{C} = \{ a \in A \mid \delta(a) = a \otimes S(\gamma) \} =: A_{\gamma^{-1}}.$$

Modules A_γ are called *spectral subspaces*. We think of them as the section modules of associated noncommutative *line* bundles.

Now it is quite easy to apply Chern–Galois theory [BrH04, Theorem 3.1] and compute an idempotent E_γ representing the associated module A_γ using a strong connection ℓ:

$$(21) \quad A_\gamma \cong (A^\alpha)^n E^\gamma, \quad E_{ij}^\gamma := \gamma_i^R \gamma_j^L, \quad \ell(\gamma) =: \sum_{k=1}^{n} \gamma_k^L \otimes \gamma_k^R \in A_{\gamma^{-1}} \underset{\mathrm{alg}}{\otimes} A_\gamma,$$

where $\{\gamma_k^L\}_k$ is a linearly independent set.

1.3.4. Gauging coactions. Consider $A \otimes C(G)$ as a C*-algebra with the diagonal coaction

$$(22) \qquad\qquad p \otimes h \longmapsto p_{(0)} \otimes h_{(1)} \otimes p_{(1)} h_{(2)},$$

and denote by $(A \otimes C(G))_R$ the same C*-algebra, equipped instead with the coaction on the rightmost factor

$$(23) \qquad\qquad p \otimes h \longmapsto p \otimes h_{(1)} \otimes h_{(2)}.$$

Then the following map is a G-equivariant (i.e., intertwining the coactions) *gauge* isomorphism of C*-algebras:

$$(24) \qquad \widehat{\kappa} \colon (A \otimes C(G)) \longrightarrow (A \otimes C(G))_R, \quad a \otimes h \longmapsto a_{(0)} \otimes a_{(1)} h.$$

Its inverse is explicitly given by

$$(25) \qquad \widehat{\kappa}^{-1} : (A \otimes H)_R \longrightarrow (A \otimes H), \quad a \otimes h \longmapsto a_{(0)} \otimes S(a_{(1)})h.$$

2. Dual tautological line bundle

2.1. Quantum complex projective plane. We consider the case $n = 2$ of the multi-pullback deformations [HKrMZ11, Section 2] of the complex projective spaces. The C*-algebra of our quantum projective plane is given as the triple-pullback of the following diagram:

(26)

Here $\sigma_1 := \sigma \otimes \mathrm{id}$, $\sigma_2 := \mathrm{id} \otimes \sigma$, and

$$(27) \qquad C(S^1) \otimes \mathcal{T} \ni v \otimes t \xrightarrow{\Psi_{01}} S(t_{(1)}v) \otimes t_{(0)} \in C(S^1) \otimes \mathcal{T},$$

$$C(S^1) \otimes \mathcal{T} \ni v \otimes t \xrightarrow{\Psi_{02}} t_{(0)} \otimes S(t_{(1)}v) \in \mathcal{T} \otimes C(S^1),$$

$$\mathcal{T} \otimes C(S^1) \ni t \otimes v \xrightarrow{\Psi_{12}} t_{(0)} \otimes S(t_{(1)}v) \in \mathcal{T} \otimes C(S^1),$$

where $\mathcal{T} \ni t \mapsto t_{(0)} \otimes t_{(1)} \in \mathcal{T} \otimes C(S^1)$ is the coaction of (15).

2.2. Quantum complex projective plane $\mathbb{C}P_{\mathcal{T}}^2$ as quotient space $S_H^5/U(1)$. Consider the following triple-pullback diagram in which every homomorphism is given by the symbol map on the appropriate factor and identity otherwise:

(28)

Definition 2.1. *The multi-pullback C*-algebra of the family of C*-epimorphisms in* (28) *is called the C*-algebra of the* Heegaard odd quantum sphere S_H^5 *and denoted* $C(S_H^5)$.

Using the coaction (15) on $\mathcal{T}$ and the comultiplication on $C(S^1) = C(U(1))$, we define the diagonal coaction on each C*-algebra of the above diagram as in Section 1.3.4. The diagram is evidently equivariant with respect to this coaction because the symbol map is equivariant. Therefore $C(S_H^5)$ is a $U(1)$-C*-algebra. We call this $U(1)$-action on $C(S_H^5)$ *diagonal*.

In order to compute the fixed-point subalgebra for the above diagonal $U(1)$-action, we need to gauge it to an action on tensor products that acts on the rightmost $C(S^1)$-factor alone. Our goal is to show that the fixed-point subalgebra is isomorphic with $C(\mathbb{C}P_\mathcal{T}^2)$. To this end, we double the three targets of all homomorphisms in (28) to three pairs of sibling targets, so that

$$(29) \quad C(S^1) \otimes \mathcal{T} \otimes \mathcal{T} \qquad\qquad\qquad \mathcal{T} \otimes C(S^1) \otimes \mathcal{T}$$

$$C(S^1) \otimes C(S^1) \otimes \mathcal{T}$$

becomes

$$(30) \quad C(S^1) \otimes \mathcal{T} \otimes \mathcal{T} \qquad \mathcal{T} \otimes C(S^1) \otimes \mathcal{T}$$

$$C(S^1) \otimes C(S^1) \otimes \mathcal{T} \xleftarrow{\text{id}} C(S^1) \otimes C(S^1) \otimes \mathcal{T},$$

and other subdiagrams are transformed in the same fashion. Then we permute the factors in the tensor products in the top row to make $C(S^1)$ always the rightmost factor, and permute the target tensor products accordingly:

$$(31) \quad \mathcal{T} \otimes \mathcal{T} \otimes C(S^1) \qquad\qquad \mathcal{T} \otimes \mathcal{T} \otimes C(S^1)$$

$$\sigma \otimes \text{id} \otimes \text{id} \Big\downarrow \qquad\qquad\qquad \Big\downarrow \sigma \otimes \text{id} \otimes \text{id}$$

$$C(S^1) \otimes \mathcal{T} \otimes C(S^1) \xleftarrow{T_{13}} C(S^1) \otimes \mathcal{T} \otimes C(S^1).$$

Here the horizontal arrow is just the flip of the outer factors. Again, we apply analogous procedures to the other two subdiagrams. Due to the commutativity of $C(S^1)$, the thus obtained triple-pullback diagram is equivariant for the diagonal coaction, and the C*-algebra it defines is equivariantly isomorphic with the multi-pullback C*-algebra defined by diagram (28).

Now we shall gauge the diagonal action as explained in Section 1.3.4. Conjugating $T_{13} \circ (\sigma \otimes \text{id} \otimes \text{id})$ by the gauge isomorphisms (24)–(25), using the commutativity and cocommutativity of $C(S^1) = C(U(1))$, along the

lines of [HKrMZ11, Section 5.2], we get:

$$(32) \qquad \big(\widetilde{g} \circ T_{13} \circ (\sigma \otimes \mathrm{id} \otimes \mathrm{id}) \circ g^{-1}\big)(r \otimes t \otimes w)$$

$$= \big(\widetilde{g} \circ T_{13} \circ (\sigma \otimes \mathrm{id} \otimes \mathrm{id})\big)(r_{(0)} \otimes t_{(0)} \otimes S(r_{(1)}t_{(1)})w)$$

$$= (\widetilde{g} \circ T_{13})(\sigma(r)_{(1)} \otimes t_{(0)} \otimes S(\sigma(r)_{(2)}t_{(1)})w)$$

$$= \widetilde{g}(S(\sigma(r)_{(2)}t_{(1)})w \otimes t_{(0)} \otimes \sigma(r)_{(1)})$$

$$= S(\sigma(r)_{(3)}t_{(3)})w_{(1)} \otimes t_{(0)} \otimes S(\sigma(r)_{(2)}t_{(2)})w_{(2)}t_{(1)}\sigma(r)_{(1)}$$

$$= S(\sigma(r)t_{(1)})w_{(1)} \otimes t_{(0)} \otimes w_{(2)}.$$

Here g and $\widetilde{g}$ are the gauge isomorphisms on $(\mathcal{T} \otimes \mathcal{T}) \otimes C(U(1))$ and $(C(S^1) \otimes \mathcal{T}) \otimes C(U(1))$ respectively.

Much in the same way, we treat the remaining two subdiagrams of diagram (28). Summarizing, for $0 \leq i < j \leq 2$, the permuted and then gauged subdiagrams become:

(33)

$$\begin{array}{ccc}
\mathcal{T}^{\otimes 2} \otimes C(S^1) & & \mathcal{T}^{\otimes 2} \otimes C(S^1) \\
\sigma_j \downarrow & & \sigma_{i+1} \downarrow \\
\mathcal{T}^{\otimes j-1} \otimes C(S^1) \otimes \mathcal{T}^{\otimes 2-j} \otimes C(S^1) & \xleftarrow{\ \Psi_{ij}^S\ } & \mathcal{T}^{\otimes i} \otimes C(S^1) \otimes \mathcal{T}^{\otimes 1-i} \otimes C(S^1),
\end{array}$$

where

$$(34) \qquad \Psi_{01}^S(v \otimes t \otimes w) := S(vt_{(1)})w_{(1)} \otimes t_{(0)} \otimes w_{(2)},$$

$$\Psi_{02}^S(v \otimes t \otimes w) := t_{(0)} \otimes S(vt_{(1)})w_{(1)} \otimes w_{(2)},$$

$$\Psi_{12}^S(t \otimes v \otimes w) := t_{(0)} \otimes S(t_{(1)}v)w_{(1)} \otimes w_{(2)}.$$

The triple-pullback C*-algebra of the family (33) is denoted by $C(S_H^5)_R$. It is a $U(1)$-C*-algebra that is equivariantly isomorphic with $C(S_H^5)$:

$$(35) \qquad C(S_H^5) \ni \big(v^0 \otimes t^0 \otimes r^0 \ , \ t^1 \otimes v^1 \otimes r^1 \ , \ t^2 \otimes r^2 \otimes v^2\big) \longmapsto$$

$$\big(t^0{}_{(0)} \otimes r^0{}_{(0)} \otimes t^0{}_{(1)}r^0{}_{(1)}v^0 \ , \dots, \ t^2{}_{(0)} \otimes r^2{}_{(0)} \otimes t^2{}_{(1)}r^2{}_{(1)}v^2\big) \in C(S_H^5)_R.$$

This isomorphism yields an isomorhism of fixed-point subalgebras

$$(36) \qquad\qquad C(S_H^5)^{U(1)} \cong C(S_H^5)_R^{U(1)}.$$

Since the $U(1)$-action in the triple-pullback diagram defining $C(S_H^5)_R$ acts only on the rightmost factor, we conclude that $C(S_H^5)_R^{U(1)}$ is the triple-pullback C*-algebra obtained by removing all rightmost factors in (33) and taking $w = 1$ in (34). Finally, since the isomorphisms in (34) thus become the isomorphisms in (27), so that (33) becomes the defining triple-pullback diagram (26) of $C(\mathbb{C}P_\mathcal{T}^2)$, we infer that

$$(37) \qquad\qquad C(S_H^5)^{U(1)} \cong C(\mathbb{C}P_\mathcal{T}^2).$$

2.3. Strong connection for the diagonal $U(1)$-action on $C(S_H^5)$.

Theorem 2.2. *The diagonal $U(1)$-action on $C(S_H^5)$ is free.*

Proof. We prove the claim by constructing a strong connection on the Peter–Weyl comodule algebra $\mathcal{P}_{U(1)}(C(S_H^5))$ for the diagonal coaction

$$(38) \qquad \delta \colon C(S_H^5) \longrightarrow C(S_H^5) \otimes C(U(1)).$$

Let u be the generating unitary of $C(S^1)$ and let s be the generating isometry of $\mathcal{T}$. Consider the following isometries in $C(S_H^5)$:

$$
\begin{aligned}
(39) \qquad a &:= (u \otimes 1 \otimes 1, s \otimes 1 \otimes 1, s \otimes 1 \otimes 1), \\
b &:= (1 \otimes s \otimes 1, 1 \otimes u \otimes 1, 1 \otimes s \otimes 1), \\
c &:= (1 \otimes 1 \otimes s, 1 \otimes 1 \otimes s, 1 \otimes 1 \otimes u).
\end{aligned}
$$

They all commute and satisfy the equation:

$$(40) \qquad (1 - aa^*)(1 - bb^*)(1 - cc^*) = 0.$$

Now one can easily check that a strong connection

$$(41) \quad \ell \colon \mathcal{O}(U(1)) \longrightarrow \mathcal{P}_{U(1)}(C(S_H^5)) \underset{\text{alg}}{\otimes} \mathcal{P}_{U(1)}(C(S_H^5)) \subseteq C(S_H^5) \otimes C(S_H^5)$$

can be defined by the formulae:

$$
\begin{aligned}
(42) \qquad \ell(1) &= 1 \otimes 1, \quad \ell(u) = b^* \otimes b, \\
\ell(u^*) &= a \otimes a^* + b \otimes b^* + c \otimes c^* \\
&\quad - a \otimes a^* bb^* - a \otimes a^* cc^* - b \otimes b^* cc^* + a \otimes a^* bb^* cc^*.
\end{aligned}
$$

Indeed, exactly as in [HMS06a, (4.6)], we can show inductively that the formula

$$(43) \qquad \ell(x^n) := \ell(x)^{\langle 1 \rangle} \ell(x^{n-1}) \ell(x)^{\langle 2 \rangle}, \quad \ell(x) =: \ell(x)^{\langle 1 \rangle} \otimes \ell(x)^{\langle 2 \rangle}$$

(summation suppressed), has the desired properties when x is the grouplike element u or u^*. $\qquad\square$

2.4. Stable non-freeness.

Definition 2.3. *Let u be the generating unitary of $C(U(1))$ and let*

$$\delta \colon C(S_H^5) \longrightarrow C(S_H^5) \otimes C(U(1))$$

be the diagonal coaction. We call the associated module

$$C(S_H^5)_u := \{x \in C(S_H^5) \mid \delta(x) = x \otimes u\}$$

the section module of the dual tautological line bundle *over $\mathbb{C}P_{\mathcal{T}}^2$.*

It follows from the existence of a strong connection on the Peter–Weyl comodule algebra $\mathcal{P}_{U(1)}(C(S_H^5))$ that $C(S_H^5)_u$ is a finitely generated projective module over $C(S_H^5)^{U(1)} \cong C(\mathbb{C}P_{\mathcal{T}}^2)$ [HM99]. Moreover, combining (21) with (42) proves that $C(S_H^5)_u$ is isomorphic as a left $C(S_H^5)^{U(1)}$-module

with $C(S_H^5)^{U(1)}bb^*$. This allows us to prove our main result, which identifies $C(S_H^5)_u$ with a nontrivial element of $K_0(\mathbb{C}P_{\mathcal{T}}^2) = \mathbb{Z}^3$ [Ru12]:

Theorem 2.4. *The section module $C(S_H^5)_u$ of the dual tautological line bundle over $\mathbb{C}P_{\mathcal{T}}^2$ is* not stably free *as a left $C(\mathbb{C}P_{\mathcal{T}}^2)$-module.*

Proof. The gauge isomorphism (35) turns the projection bb^* representing the finitely generated projective module $C(S_H^5)_u$ to $(ss^*\otimes 1 \,,\; 1\otimes 1 \,,\; 1\otimes ss^*) \in C(\mathbb{C}P_{\mathcal{T}}^2)$. Plugging it into the iterated pullback diagram
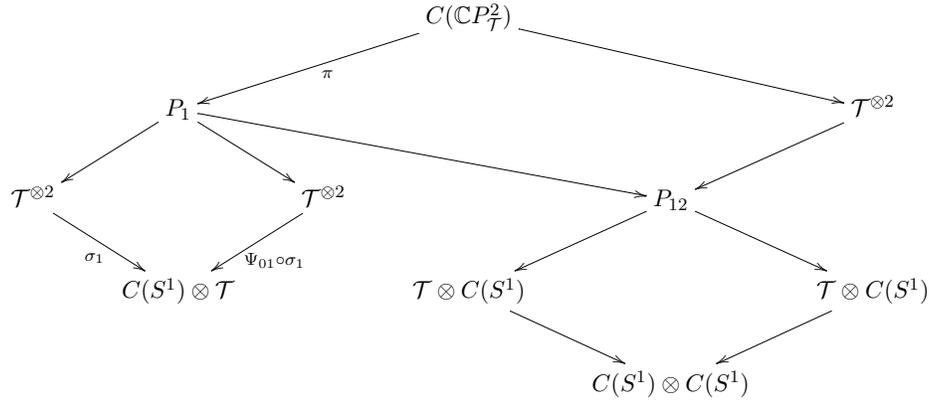
and projecting via π to P_1, we obtain $(ss^* \otimes 1 \,,\; 1 \otimes 1)$.

Furthermore, consider the Mayer–Vietoris six-term exact sequence of the pullback diagram defining P_1, and take the unitary $u \otimes 1$, whose class generates $K_1(C(S^1) \otimes \mathcal{T})$. We know from the proof of [Ru12, Theorem 2.1] that $K_0(P_1) = \mathbb{Z} \oplus \mathbb{Z}$ with one $\mathbb{Z}$ generated by $[1]$ and the other $\mathbb{Z}$ generated by $\partial_{10}([u \otimes 1])$. To compute the Milnor idempotent $p_{u\otimes 1}$ (see (10)), take a lifting of $u^{-1} \otimes 1$ to be $c := s^* \otimes 1$, and a lifting of $u \otimes 1$ to be $d := s \otimes 1$. Suppressing $\otimes 1$, we obtain

$$\partial_{10}([u \otimes 1]) = \left[\begin{pmatrix} (s^*(2 - ss^*)s, 1) & (s^*(2 - ss^*)(1 - ss^*), 0) \\ ((1 - ss^*)s, 0) & ((1 - ss^*)^2, 0) \end{pmatrix} \right] - [(1,1)]$$
$$= [(1 - ss^*, 0)].$$

Hence $[1] - \partial_{10}([u \otimes 1]) = [(ss^* \otimes 1, 1 \otimes 1)] = \pi_*[C(S_H^5)_u]$.

Finally, if $C(S_H^5)_u$ were stably free, then $\pi_*[C(S_H^5)_u] = n[1]$ for some $n \in \mathbb{N}$. This would contradict the just derived equality, so that $C(S_H^5)_u$ is not stably free. $\square$

3. K-groups of the quantum sphere S_H^5

We end this paper by showing that the K-groups of S_H^5 agree with their classical counterparts. Its C*-algebra is the triple-pullback C*-algebra (see 2.1), so that we can apply [Ru12, Corollary 1.5] to determine its K-theory.

3.1. Cocycle condition. The first step in applying [Ru12, Corollary 1.5] is verifying the cocycle condition (see Section 1.2.3).

Lemma 3.1. *The family* (28) *defining the triple-pullback C*-algebra* $C(S_H^5)$ *satisfies the cocycle condition.*

Proof. It is straightforward to check the first part of the cocycle condition. We do it only in one case as all other cases are completely analogous. For $i = 2$, $j = 1$ and $k = 0$, we obtain:

$$
(44) \qquad \pi_1^2(\ker \pi_0^2) = \pi_2^1(\ker \pi_0^1) \Leftrightarrow
$$
$$
\sigma_2(\ker \sigma_1) = \sigma_3(\ker \sigma_1) \Leftrightarrow
$$
$$
\sigma_2(\mathcal{K} \otimes \mathcal{T} \otimes C(S^1)) = \sigma_3(\mathcal{K} \otimes \mathcal{C}(S^1) \otimes \mathcal{T})) \Leftrightarrow
$$
$$
\mathcal{K} \otimes C(S^1) \otimes C(S^1) = \mathcal{K} \otimes \mathcal{C}(S^1) \otimes C(S^1).
$$

For the second part we use the following notation

$$
(45) \qquad [\cdot]_{jk}^i : B_i \to B_i/(\ker \pi_j^i + \ker \pi_k^i), \quad [\cdot]_k^{ij} : B_{ij} \to B_{ij}/\pi_j^i(\ker \pi_k^i).
$$

Again all cases are done in a similar way, so that we only check the case $i = 0$, $j = 1$, $k = 2$, i.e., we show that $\varphi_1^{02} = \varphi_2^{01} \circ \varphi_0^{12}$. For any $r \otimes t \otimes v \in \mathcal{T} \otimes \mathcal{T} \otimes C(S^1)$, the left hand side is:

$$
(46) \qquad \varphi_1^{02}\big([r \otimes t \otimes v]_{01}^2\big) = \big((\pi_1^{02})^{-1} \circ \pi_1^{20}\big)\big([r \otimes t \otimes v]_{01}^2\big)
$$
$$
= \Big[\big((\pi_2^0)^{-1} \circ \pi_0^2\big)(r \otimes t \otimes v)\Big]_{21}^0
$$
$$
= \Big[(\sigma_3^{-1} \circ \sigma_1)(r \otimes t \otimes v)\Big]_{21}^0
$$
$$
= \big[\sigma(r) \otimes t \otimes \omega(v)\big]_{21}^0,
$$

where ω is a linear splitting of σ. On the other hand, we obtain:

$$
(47) \qquad (\varphi_2^{01} \circ \varphi_0^{12})\big([r \otimes t \otimes v]_{01}^2\big) = \varphi_2^{01}\Big(\Big[\big((\pi_2^1)^{-1} \circ \pi_1^2\big)(r \otimes t \otimes v)\Big]_{20}^1\Big)
$$
$$
= \varphi_2^{01}\big(\big[r \otimes \sigma(t) \otimes \omega(v)\big]_{02}^1\big)
$$
$$
= \big((\pi_2^{01})^{-1} \circ \pi_2^{10}\big)\big(\big[r \otimes \sigma(t) \otimes \omega(v)\big]_{02}^1\big)
$$
$$
= \Big[\big((\pi_1^0)^{-1} \circ \pi_0^1\big)\big(r \otimes \sigma(t) \otimes \omega(v)\big)\Big]_{12}^0
$$
$$
= \big[\sigma(r) \otimes \omega(\sigma(t)) \otimes \omega(v)\big]_{12}^0
$$
$$
= [\sigma(r) \otimes t \otimes \omega(v)]_{12}^0.
$$

Hence the left and the right hand side agree because $[]_{jk}^i = []_{kj}^i$ for any set of distinct indices. $\qquad\square$

3.2. K-groups. We are now ready for:

Theorem 3.2. *The K-groups of the Heegaard quantum 5-sphere are:*

$$
K_0(C(S_H^5)) = \mathbb{Z} = K_1(C(S_H^5)).
$$

Proof. Lemma 3.1 allows us to apply [Ru12, Corollary 1.5] to the family of surjections in diagram (28). The first six-term exact sequence is
(48)

$$K_0(P_1) \longrightarrow K_0(\mathcal{T}^{\otimes 2} \otimes C(S^1)) \oplus K_0(\mathcal{T} \otimes C(S^1) \otimes \mathcal{T}) \dashrightarrow K_0(\mathcal{T} \otimes C(S^1)^{\otimes 2})$$

$$K_1(\mathcal{T} \otimes C(S^1)^{\otimes 2}) \dashleftarrow K_1(\mathcal{T}^{\otimes 2} \otimes C(S^1)) \oplus K_1(\mathcal{T} \otimes C(S^1) \otimes \mathcal{T}) \longleftarrow K_1(P_1).$$

with ∂_{01} on the right vertical arrow.

Here the dotted arrows are $(\mathrm{id} \otimes \sigma \otimes \mathrm{id})_* - (\mathrm{id} \otimes \mathrm{id} \otimes \sigma)_*$. With the help of the Künneth formula, the exact sequence becomes:

(49)
$$K_0(P_1) \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{(m,n)\mapsto(m-n,0)} \mathbb{Z} \oplus \mathbb{Z}$$

$$\mathbb{Z} \oplus \mathbb{Z} \xleftarrow{(m,n)\mapsto(m,-n)} \mathbb{Z} \oplus \mathbb{Z} \longleftarrow K_1(P_1).$$

Hence $K_0(P_1) = \mathbb{Z} = K_1(P_1)$.

The second diagram of [Ru12, Corollary 1.5] is
(50)

$$K_0(P_2) \longrightarrow K_0(C(S^1) \otimes \mathcal{T} \otimes C(S^1)) \oplus K_0(C(S^1)^{\otimes 2} \otimes \mathcal{T}) \dashrightarrow K_0(C(S^1)^{\otimes 3})$$

$$K_1(C(S^1)^{\otimes 3}) \dashleftarrow K_1(C(S^1) \otimes \mathcal{T} \otimes C(S^1)) \oplus K_1(C(S^1)^{\otimes 2} \otimes \mathcal{T}) \longleftarrow K_1(P_2).$$

with ∂_{10} on the left vertical arrow and ∂_{01} on the right vertical arrow.

In order to unravel this diagram, we need to take a closer look into the Künneth formula. We consider $[u] \otimes [u] \in K_1(C(S^1)) \otimes K_1(C(S^1))$ and denote its image under the Künneth isomorphism $K_1(C(S^1)) \otimes K_1(C(S^1)) \to K_0(C(S^1) \otimes C(S^1))$ by β. Using the natural leg numbering convention, we extend this notation to triple tensor products with $[1] \in K_0(\mathcal{T})$ or with $[1] \in K_0(C(S^1))$ as an appropriate factor. Next, we denote by u_i the K_1-class of a triple tensor with u as the i-th factor and $1 \in \mathcal{T}$ or $1 \in C(S^1)$ as any remaining factor. Hence $K_1(C(S^1)^{\otimes 3})$ is $\mathbb{Z}^4$ generated by u_1, u_2, u_3 and the fourth generator denoted by u_{123}. Furthermore, the above exact sequence becomes

$$K_0(P_2) \longrightarrow \mathbb{Z}[1] \oplus \mathbb{Z}\beta_{13} \oplus \mathbb{Z}[1] \oplus \mathbb{Z}\beta_{12} \dashrightarrow \mathbb{Z}[1] \oplus \mathbb{Z}\beta_{12} \oplus \mathbb{Z}\beta_{13} \oplus \mathbb{Z}\beta_{23}$$

$$\mathbb{Z}u_1 \oplus \mathbb{Z}u_2 \oplus \mathbb{Z}u_3 \oplus \mathbb{Z}u_{123} \dashleftarrow \mathbb{Z}u_1 \oplus \mathbb{Z}u_3 \oplus \mathbb{Z}u_1 \oplus \mathbb{Z}u_2 \longleftarrow K_1(P_2).$$

Now, by the functoriality of the Künneth isomorphism [Bl98, p. 232] and with the help of the diagram (56), it is easy to verify that the upper and the lower dotted maps are respectively given by

(51) $(a,b,c,d) \longmapsto (a-c,-d,b,0)$ and $(a,b,c,d) \longmapsto (a-c,-d,b,0).$

Hence, by a straightforward homological computation, we infer that

(52) $K_0(P_2) = \mathbb{Z}[1] \oplus \mathbb{Z}\,\partial_{10}(u_{123})$ and $K_1(P_2) = \mathbb{Z}[(\mathrm{u}_1,\mathrm{u}_1)] \oplus \mathbb{Z}\,\partial_{01}(\beta_{23}),$

where $u_1 := u \otimes 1 \otimes 1$.

Finally, the last diagram of [Ru12, Corollary 1.5] is

$$(53) \quad \begin{array}{ccccc} K_0(C(S_H^5)) & \longrightarrow & K_0(P_1) \oplus K_0(C(S^1) \otimes \mathcal{T}^{\otimes 2}) & \dashrightarrow & K_0(P_2) \\ \uparrow & & & & \downarrow \\ K_1(P_2) & \dashleftarrow & K_1(P_1) \oplus K_1(C(S^1) \otimes \mathcal{T}^{\otimes 2}) & \longleftarrow & K_1(C(S_H^5)). \end{array}$$

Plugging in generators into this diagram, we obtain

$$\begin{array}{ccccc} K_0(C(S_H^5)) & \longrightarrow & \mathbb{Z}[1] \oplus \mathbb{Z}[1] & \dashrightarrow & \mathbb{Z}[1] \oplus \mathbb{Z}\,\partial_{10}(u_{123}) \\ \uparrow & & & & \downarrow \\ \mathbb{Z}[(u_1, u_1)] \oplus \mathbb{Z}\,\partial_{01}(\beta_{23}) & \dashleftarrow & \mathbb{Z}\,\partial_{01}(\beta_{23}) \oplus \mathbb{Z}u_1 & \longleftarrow & K_1(C(S_H^5)). \end{array}$$

Here the upper dotted arrow is evidently given by the formula

$$(54) \qquad\qquad\qquad (a, b) \longmapsto (a - b, 0).$$

It is a bit more complicated to determine the lower dotted arrow. To this end, we denote by $b \in M_2\big(C(S^1) \otimes C(S^1)\big)$ the pullback of the Bott projection on S^2, so that $[b] = \beta$. Next, by $\bar{b} \in M_2\big(\mathcal{T} \otimes C(S^1)\big)$ we denote a self-adjoint lifting of b along $\mathrm{id}_{M_2(\mathbb{C})} \otimes (\sigma \otimes \mathrm{id})$. Then we substitute $\bar{b}$ to the formula (12) to compute both $\partial_{01}(\beta_{23}) \in K_1(P_1)$ and $\partial_{01}(\beta_{23}) \in K_1(P_2)$ at the same time. The resulting formulas will only differ in the leftmost tensor factor: for P_1 it will be $1 \in \mathcal{T}$ and for P_2 it will be $1 \in C(S^1)$. Therefore

$$(55) \qquad (\sigma_1, \sigma_1)_* \colon K_1(P_1) \ni \partial_{01}(\beta_{23}) \longmapsto \partial_{01}(\beta_{23}) \in K_1(P_2).$$

Combining this observation with the diagram

$$(56)$$

one easily checks that the desired lower dotted map is given by the formula

$$(57) \qquad\qquad\qquad (a, b) \longmapsto (-b, a).$$

Consequently, $K_0(C(S_H^5)) = \mathbb{Z} = K_0(C(S_H^5))$ as claimed. $\qquad\qquad \square$

Acknowledgements

The authors are extremely grateful to Paul Baum and Nigel Higson for pivotal K-consultations. It is a pleasure to thank Benjamin Passer for his careful proofreading of this paper.

References

[BaDH]   BAUM, PAUL F.; DE COMMER, KENNY; HAJAC, PIOTR M. Free actions of compact quantum groups on unital C*-algebras. *Documenta Mathematica*, to appear. arXiv:1304.2812.

[BaHMS05]   BAUM, PAUL F.; HAJAC, PIOTR M.; MATTHES, RAINER; SZYMAŃSKI, WOJCIECH. The K-theory of Heegaard-type quantum 3-spheres. *K-Theory* **35** (2005), no. 1–2, 159–186. MR2240219, Zbl 1111.46051, arXiv:math/0409573, doi: 10.1007/s10977-005-1550-y.

[BaM]   BAUM, PAUL F.; MEYER, R. The Baum–Connes conjecture, localisation of categories, and quantum groups. Lecture Notes on Noncommutative Geometry and Quantum Groups. *Eur. Math. Soc. Publ. House*, to appear.

[Bl98]   BLACKADAR, BRUCE. K-theory for operator algebras. Second edition. Mathematical Sciences Research Institute Publications, 5. *Cambridge University Press, Cambridge*, 1998. xx+300 pp. ISBN: 0-521-63532-2. MR1656031, Zbl 0913.46054.

[BrH04]   BRZEZIŃSKI, TOMASZ; HAJAC, PIOTR M. The Chern–Galois character. *C. R. Math. Acad. Sci. Paris* **338** (2004), no. 2, 113–116. MR2038278, Zbl 1061.16037, arXiv:math/0306436, doi: 10.1016/j.crma.2003.11.009.

[BrH09]   BRZEZIŃSKI, TOMASZ; HAJAC, PIOTR M. Galois-type extensions and equivariant projectivity. Preprint, 2009. arXiv:0901.0141.

[CaM00]   CALOW, DIRK; MATTHES, RAINER. Covering and gluing of algebras and differential algebras. *J. Geom. Phys.* **32** (2000), no. 4, 364–396. MR1734767, Zbl 0969.46051, arXiv:math/9910031, doi: 10.1016/S0393-0440(99)00035-2.

[CoL I]   COBURN, LEWIS A. The C*-algebra generated by an isometry. *Bull. Amer. Math. Soc.* **73** (1967), 722–726. MR0213906, Zbl 0153.16603, doi: 10.1090/S0002-9904-1967-11845-7.

[CoL II]   COBURN, LEWIS A. The C*-algebra generated by an isometry. II. *Trans. Amer. Math. Soc.* **137** (1969), 211–217. MR0236720, Zbl 0186.19704, doi: 10.2307/1994797.

[DHHMW12]   DĄBROWSKI, LUDWIK; HADFIELD, TOM; HAJAC, PIOTR M.; MATTHES, RAINER; WAGNER, ELMAR. Index pairings for pullbacks of C*-algebras. *Operator algebras and quantum groups*, 67–84, Banach Center Publ., 98. *Polish Acad. Sci. Inst. Math., Warsaw*, 2012. MR3059658, Zbl 1277.46037, arXiv:math/0702001, doi: 10.4064/bc98-0-4.

[Di64]   DIXMIER, JACQUES. Les C^*-algèbres et leurs représentations. Cahiers Scientifiques, Fasc. XXIX. *Gauthier-Villars & Cie, Éditeur-Imprimeur, Paris*, 1964. xi+382 pp. MR0171173, Zbl 0152.32902.

[E00]   ELLWOOD, DAVID ALEXANDRE. A new characterisation of principal actions. *J. Funct. Anal.* **173** (2000), no. 1, 49–60. MR1760277, Zbl 0960.46049, doi: 10.1006/jfan.2000.3561.

[FLR00]   FOWLER, NEAL J.; LACA, MARCELO; RAEBURN, IAIN. The C*-algebras of infinite graphs. *Proc. Amer. Math. Soc.* **128** (2000), no. 8, 2319–2327. MR1670363, Zbl 0956.46035, doi: 10.1090/S0002-9939-99-05378-2.

[H96]   HAJAC, PIOTR M. Strong connections on quantum principal bundles.
             Comm. Math. Phys, **182** (1996), no. 3, 579–617. MR1461943, Zbl
             0873.58007, arXiv:hep-th/9406129, doi: 10.1007/BF02506418.

[HKaZ12]   HAJAC, PIOTR M.; KAYGUN, ATABEY; ZIELIŃSKI, BARTOSZ. Quantum
             complex projective spaces from Toeplitz cubes. *J. Noncommutative Geom.*
             6 (2012), no. 3, 603–621. MR2956320, Zbl 1257.46040, arXiv:1008.0673,
             doi: 10.4171/JNCG/100.

[HKrMZ11]   HAJAC, PIOTR M.; KRÄHMER, ULRICH; MATTHES, RAINER; ZIELIŃSKI,
             BARTOSZ. Piecewise principal comodule algebras. *J. Noncommuta-
             tive Geom.* **5** (2011), no. 4, 591–614. MR2838527, Zbl 1258.16036,
             arXiv:0707.1344, doi: 10.4171/JNCG/88.

[HM99]   HAJAC, PIOTR M.; MAJID, SHAHN. Projective module description of the
             q-monopole. *Comm. Math. Phys.* **206** (1999), no. 2, 247–264. MR1722149,
             Zbl 0936.46052, arXiv:math/9803003, doi: 10.1007/s002200050704.

[HMS06a]   HAJAC, PIOTR M.; MATTHES, RAINER; SZYMANSKI, WOJCIECH. A
             locally trivial quantum Hopf fibration. *Algebr. Represent. Theory* **9**
             (2006), no. 2, 121–146. MR2238363, Zbl 1132.58007, arXiv:math/0112317,
             doi: 10.1007/s10468-005-3080-y.

[HMS06b]   HAJAC, PIOTR M.; MATTHES, RAINER; SZYMAŃSKI, WOJCIECH. Noncom-
             mutative index theory for mirror quantum spheres. *C. R. Math. Acad.
             Sci. Paris* **343** (2006), no. 11–12, 731–736. MR2284701, Zbl 1114.46052,
             arXiv:math/0511309, doi: 10.1016/j.crma.2006.09.021.

[HRZ13]   HAJAC, PIOTR M.; RENNIE, ADAM; ZIELIŃSKI, BARTOSZ. The *K*-
             theory of Heegaard quantum lens spaces. *J. Noncommut. Geom.* **7**
             (2013), no. 4, 1185–1216. MR3148619, Zbl 1286.58005, arXiv:1110.5897,
             doi: 10.4171/JNCG/146.

[HZ12]   HAJAC, PIOTR M.; ZIELIŃSKI, BARTOSZ. Cocycle condition for multi-
             pullbacks of algebras. *Operator algebras and quantum groups*, 239–243,
             Banach Center Publ., 98. *Polish Acad. Sci. Inst. Math., Warsaw*, 2012.
             MR3059663, Zbl 1266.81114, arXiv:1207.0087, doi: 10.4064/bc98-0-9.

[KL93]   KLIMEK, SŁAWOMIR; LESNIEWSKI, ANDRZEJ. A two-parameter quantum
             deformation of the unit disc. *J. Funct. Anal.* **115** (1993), no. 1, 1–23.
             MR1228139, Zbl 0780.58020, doi: 10.1006/jfan.1993.1078.

[M71]   MILNOR, JOHN. Introduction to algebraic *K*-theory. Annals of Mathemat-
             ics Studies, 72. *Princeton University Press, Princeton, N. J., University
             of Tokyo Press, Tokyo*, 1971. xiii+184 pp. MR0349811, Zbl 0237.18005,
             doi: 10.1515/9781400881796.

[Pe99]   PEDERSEN, GERT K. Pullback and pushout constructions in C*-algebra
             theory. *J. Funct. Anal.* **167** (1999), no. 2, 243–344. MR1716199, Zbl
             0944.46063, doi: 10.1006/jfan.1999.3456.

[Po95]   PODLEŚ, PIOTR. Symmetries of quantum spaces. Subgroups and quotient
             spaces of quantum SU(2) and SO(3) groups. *Comm. Math. Phys.* **170**
             (1995), no. 1, 1–20. MR1331688, Zbl 0853.46074, arXiv:hep-th/9402069,
             doi: 10.1007/BF02099436.

[R94]   RANICKI, ANDREW A. An explicit projection. Preprint, 1994. `http://www.
             maths.ed.ac.uk/~aar/surgery/explicit.pdf`.

[Ru12]   RUDNIK, JAN. The *K*-theory of the triple-Toeplitz deformation of the com-
             plex projective plane. *Operator algebras and quantum groups*, 303–310,
             Banach Center Publ., 98. *Polish Acad. Sci. Inst. Math., Warsaw*, 2012.
             MR3059667, Zbl 1269.46051, arXiv:1207.2066, doi: 10.4064/bc98-0-13.

[S84]   SCHOCHET, CLAUDE. Topological methods for $C*$-algebras. III. Axiomatic homology. *Pacific J. Math.* **114** (1984), no. 2, 399–445. MR0757510, Zbl 0491.46061, doi: 10.2140/pjm.1984.114.399.

[So11]   SOŁTAN, PIOTR M. On actions of compact quantum groups. *Illinois J. Math.* **55** (2011), no. 3, 953–962 (2013). MR3069291, Zbl 1272.46054, arXiv:1003.5526.

(Piotr M. Hajac) INSTYTUT MATEMATYCZNY, POLSKA AKADEMIA NAUK, UL. ŚNIADECK-ICH 8, WARSZAWA, 00-656 POLAND
`pmh@impan.pl`

(Jan Rudnik) INSTYTUT MATEMATYCZNY, POLSKA AKADEMIA NAUK, UL. ŚNIADECK-ICH 8, WARSZAWA, 00-656 POLAND
Current address: Quanticate Polska, ul. Hankiewicza 2, Warszawa, 02-103 Poland
`yarood@gmail.com`

This paper is available via `http://nyjm.albany.edu/j/2017/23-15.html`.