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Exotic group C^*-algebras in noncommutative duality

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Abstract. We show that for a locally compact group G there is a one-to-one correspondence between G-invariant weak*-closed subspaces E of the Fourier–Stieltjes algebra $B(G)$ containing $B_r(G)$ and quotients $C^*_E(G)$ of $C^*(G)$ which are intermediate between $C^*(G)$ and the reduced group algebra $C^*_r(G)$. We show that the canonical comultiplication on $C^*(G)$ descends to a coaction or a comultiplication on $C^*_E(G)$ if and only if E is an ideal or subalgebra, respectively. When α is an action of G on a C^*-algebra B, we define "E-crossed products" $B \rtimes_{\alpha,E} G$ lying between the full crossed product and the reduced one, and we conjecture that these "intermediate crossed products" satisfy an "exotic" version of crossed-product duality involving $C^*_E(G)$.

Contents

1. Introduction

It has long been known that for a locally compact group G there are many C^*-algebras between the full group C^*-algebra $C^*(G)$ and the reduced algebra $C^*_r(G)$ (see [Eym64]). However, little study has been made regarding the extent to which these intermediate algebras can be called group C^*-algebras.

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This paper is inspired by recent work of Brown and Guentner [BG], which studies such intermediate algebras for discrete groups, and [Oka], which shows that in fact there can be a continuum of such intermediate algebras. We shall consider a general locally compact group G, and show that by elementary harmonic analysis there is a one-to-one correspondence between G-invariant weak*-closed subspaces E of the Fourier–Stieltjes algebra $B(G)$ containing $B_r(G)$ and quotients $C^*_E(G)$ of $C^*(G)$ which are intermediate between $C^*(G)$ and the reduced group algebra $C^*_r(G)$.

We are primarily interested in the following results:

- E is an ideal if and only if there is a coaction

$$C^*_E(G) \to M(C^*_E(G) \otimes C^*(G)).$$

- E is a subalgebra if and only if there is a comultiplication

$$C^*_E(G) \to M(C^*_E(G) \otimes C^*_E(G)).$$

(See Propositions 3.13 and 3.16 for more precise statements.) These C^*-algebras can be used to describe various properties of G, e.g., if G is discrete and $E = \overline{B(G) \cap c_0(G)}$, then G has the Haagerup property if and only if $C^*_E(G) = C^*(G)$ (see [BG, Corollary 3.4]). Brown and Guentner also prove that (again, in the discrete case) $C^*_E(G)$ is a compact quantum group, because it carries a comultiplication, and this caught our attention since it makes a connection with noncommutative crossed-product duality.

If we have a C^*-dynamical system (B, G, α), one can form the full crossed product $B \rtimes_\alpha G$ or the reduced crossed product $B \rtimes_{\alpha,r} G$. We show in Section 6 that for E as above there is an "E-crossed product" $B \rtimes_{\alpha,E} G$, and we speculate that these "intermediate" crossed products satisfy an "exotic" version of crossed-product duality involving $C^*_E(G)$.

After a short section on preliminaries, in Section 3 we prove the above-mentioned results concerning the existence of a coaction or comultiplication on $C^*_E(G)$.

In Section 4 we briefly explore the analogue for arbitrary locally compact groups of the construction used in [BG], where for discrete groups they construct group C^*-algebras starting with ideals of $\ell^\infty(G)$.

In Section 5 we specialize (for the only time in this paper) to the discrete case, showing that a quotient $C^*_E(G)$ is a group C^*-algebra if and only if it is *topologically graded* in the sense of [Exe97].

Finally, in Section 6 we outline a possible application of our exotic group algebras to noncommutative crossed-product duality.

After this paper was circulated in preprint form, we learned that Buss and Echterhoff [BuE] have given counterexamples to Conjecture 6.12 and have proven Conjecture 6.14.

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2. Preliminaries

All ideals of C^*-algebras will be closed and two-sided. If A and B are C^*-algebras, then $A \otimes B$ will denote the minimal tensor product.

For one of our examples we will need the following elementary fact, which is surely folklore.

Lemma 2.1. *Let A be a C^*-algebra, and let I and J be ideals of A. Let $\phi : A \to A/I$ and $\psi : A \to A/J$ be the quotient maps, and define*

$$\pi = \phi \oplus \psi : A \to (A/I) \oplus (A/J).$$

Then π is surjective if and only if $A = I + J$.

Proof. First assume that π is surjective, and let $a \in A$. Choose $b \in A$ such that

$$\pi(b) = \big(\phi(a), 0\big),$$

i.e., $\phi(b) = \phi(a)$ and $\psi(b) = 0$. Then $a - b \in I$, $b \in J$, and $a = (a - b) + b$.

Conversely, assume that $A = I + J$, and let $a \in A$. Choose $b \in I$ and $c \in J$ such that $a = b + c$. Then $\psi(c) = 0$, and $\phi(c) = \phi(a)$ since $a - c \in I$. Thus

$$\pi(c) = \big(\phi(a), 0\big).$$

It follows that $\pi(A) \supset (A/I) \oplus \{0\}$, and similarly $\pi(A) \supset \{0\} \oplus (A/J)$, and hence π is onto. □

A point of notation: for a homomorphism between C^*-algebras, or for a bounded linear functional on a C^*-algebra, we use a bar to denote the unique strictly continuous extension to the multiplier algebra.

We adopt the conventions of [EKQR06] for actions and coactions of a locally compact group G on a C^*-algebra A. In particular, we use *full* coactions $\delta : A \to M(A \otimes C^*(G))$, which are nondegenerate injective homomorphisms satisfying the *coaction-nondegeneracy* property

$$\overline{\operatorname{span}}\{\delta(A)(1 \otimes C^*(G)) = A \otimes C^*(G) \tag{2.1}$$

and the *coaction identity*

$$\overline{\delta \otimes \mathrm{id}} \circ \delta = \overline{\mathrm{id} \otimes \delta_G} \circ \delta, \tag{2.2}$$

where δ_G is the canonical coaction on $C^*(G)$, determined by $\overline{\delta_G}(x) = x \otimes x$ for $x \in G$ (and where G is identified with its canonical image in $M(C^*(G))$). Recall that δ gives rise to a right $B(G)$-module structure on A^* given by

$$\omega \cdot f = \overline{\omega \otimes f} \circ \delta \quad \text{for } \omega \in A^* \text{ and } f \in B(G),$$

and also to a left $B(G)$-module structure on A given by

$$f \cdot a = \overline{\mathrm{id} \otimes f} \circ \delta(a) \quad \text{for } f \in B(G) \text{ and } a \in A,$$

and that moreover

$$(\omega \cdot f)(a) = \omega(f \cdot a) \quad \text{for all } \omega \in A^*,\ f \in B(G), \text{ and } a \in A.$$

Further recall that $1_G \cdot a = a$ for all $a \in A$, where 1_G is the constant function with value 1. In fact, suppose we have a homomorphism $\delta : A \to M(A \otimes C^*(G))$ satisfying all the conditions of a coaction except perhaps injectivity. Then δ is in fact a coaction, because injectivity follows automatically, by the following folklore trick:

Lemma 2.2. *Let $\delta : A \to M(A \otimes C^*(G))$ be a homomorphism satisfying* (2.1) *and* (2.2)*. Then for all $a \in A$ we have*

$$\overline{\mathrm{id} \otimes 1_G} \circ \delta(a) = a,$$

where $1_G \in B(G)$ is the constant function with value 1*. In particular, δ is injective and hence a coaction.*

Proof. First of all,

$$\begin{aligned} A &= \overline{\operatorname{span}}\Big\{(\mathrm{id} \otimes g)\big(\delta(a)(1 \otimes c)\big) : g \in B(G), a \in A, c \in C^*(G)\Big\} \\ &= \overline{\operatorname{span}}\big\{\overline{\mathrm{id} \otimes c \cdot g} \circ \delta(a) : g \in B(G), a \in A, c \in C^*(G)\big\} \\ &= \overline{\operatorname{span}}\big\{\overline{\mathrm{id} \otimes f} \circ \delta(a) : f \in B(G), a \in A\big\}. \end{aligned}$$

Now the following computation suffices: for all $a \in A$ and $f \in B(G)$ we have

$$\begin{aligned} &\overline{\mathrm{id} \otimes 1_G} \circ \delta\big(\overline{\mathrm{id} \otimes f} \circ \delta(a)\big) \\ &\quad = \overline{\mathrm{id} \otimes 1_G} \circ \overline{\mathrm{id} \otimes \mathrm{id} \otimes f} \circ (\delta \otimes \mathrm{id}) \circ \delta(a) \\ &\quad = \overline{\mathrm{id} \otimes 1_G \otimes f} \circ (\mathrm{id} \otimes \delta_G) \circ \delta(a) \\ &\quad = \overline{\mathrm{id} \otimes 1_G f} \circ \delta(a) \\ &\quad = \overline{\mathrm{id} \otimes f} \circ \delta(a). \end{aligned}$$

□

3. Exotic quotients of $C^*(G)$

Let G be a locally compact group,. We are interested in certain quotients $C^*_E(G)$ (see Definition 3.2 for this notation). We will always assume that ideals of C^*-algebras are closed and two-sided. Let $B(G)$ denote the Fourier–Stieltjes algebra, which we identify with the dual of $C^*(G)$. We give $B(G)$ the usual $C^*(G)$-bimodule structure: for $a, b \in C^*(G)$ and $f \in B(G)$ we define

$$\langle b, a \cdot f\rangle = \langle ba, f\rangle \quad \text{and} \quad \langle b, f \cdot a\rangle = \langle ab, f\rangle.$$

This bimodule structure extends to an $M(C^*(G))$-bimodule structure, because for $m \in M(C^*(G))$ and $f \in B(G)$ the linear functionals $a \mapsto \langle am, f\rangle$ and $a \mapsto \langle ma, f\rangle$ on $C^*(G)$ are bounded. Regarding G as canonically embedded in $M(C^*(G))$, the associated G-bimodule structure on $B(G)$ is given by

$$(x \cdot f)(y) = f(yx) \quad \text{and} \quad (f \cdot x)(y) = f(xy)$$

for $x, y \in G$ and $f \in B(G)$.

A quotient $C^*(G)/I$ is uniquely determined by the annihilator $E = I^\perp$ in $B(G)$, which is a weak*-closed subspace. We find it convenient to work in terms of E rather than I, keeping in mind that we will have $I = {}^\perp E$, the preannihilator in $C^*(G)$. First we record the following well-known property:

Lemma 3.1. *For any weak*-closed subspace E of $B(G)$, the following are equivalent:*

(1) *${}^\perp E$ is an ideal;*
(2) *E is a $C^*(G)$-subbimodule;*
(3) *E is G-invariant.*

Proof. (1)$\Leftrightarrow$(2) follows from, e.g., [Ped79, Theorem 3.10.8], and (2)$\Leftrightarrow$(3) follows by integration. $\square$

Definition 3.2. If E is a weak*-closed G-invariant subspace of $B(G)$, let $C^*_E(G)$ denote the quotient $C^*(G)/{}^\perp E$.

Note that the above definition makes sense, by Lemma 3.1.

Example 3.3. Of course we have

$$C^*(G) = C^*_{B(G)}(G).$$

Also,

$$C^*_r(G) = C^*_{B_r(G)}(G),$$

where $B_r(G)$ is the regular Fourier–Stieltjes algebra of G, because if $\lambda : C^*(G) \to C^*_r(G)$ denotes the regular representation of G then

$$(\ker \lambda)^\perp = B_r(G).$$

Recall for later use that the intersection $C_c(G) \cap B(G)$ is norm-dense in the Fourier algebra $A(G)$ (for the norm of functionals on $C^*(G)$), and is weak*-dense in $B_r(G)$ [Eym64].

Remark 3.4. If E is a weak*-closed G-invariant subspace of $B(G)$, and $q : C^*(G) \to C^*_E(G)$ is the quotient map, then the dual map

$$q^* : C^*_E(G)^* \to C^*(G)^* = B(G)$$

is an isometric isomorphism onto E, and we identify $E = C^*_E(G)^*$ and regard q^* as an inclusion map.

Inspired in part by [BG], we pause here to give another construction of the quotients $C^*_E(G)$:

(1) Start with a G-invariant, but *not necessarily weak*-closed*, subspace E of $B(G)$.
(2) Call a representation U of G on a Hilbert space H an *E-representation* if there is a dense subspace H_0 of H such that the matrix coefficients
$$x \mapsto \langle U_x \xi, \eta \rangle$$
are in E for all $\xi, \eta \in H_0$.
(3) Define a C^*-seminorm $\|\cdot\|_E$ on $C_c(G)$ by
$$\|f\|_E = \sup\{\|U(f)\| : U \text{ is an } E\text{-representation of } G\}.$$

The following lemma is presumably well-known, but we include a proof for the convenience of the reader.

Lemma 3.5. *With the above notation, let I be the ideal of $C^*(G)$ given by*

$$I = \{a \in C^*(G) : \|a\|_E = 0\}. \tag{3.1}$$

Then:

(1) $I = {}^\perp E$.
(2) *The weak*-closure $\overline{E}$ of E in $B(G)$ is G-invariant, and $C^*_{\overline{E}}(G) = C^*(G)/I$ is the Hausdorff completion of $C_c(G)$ in the seminorm $\|\cdot\|_E$.*
(3) *If E is an ideal or a subalgebra of $B(G)$, then so is $\overline{E}$.*

Proof. (1) To show that $I \subset {}^\perp E$, let $a \in I$ and $f \in E$. Since $f \in B(G)$, we can choose a representation U of G on a Hilbert space H and vectors $\xi, \eta \in H$ such that

$$f(x) = \langle U_x \xi, \eta \rangle \quad \text{for } x \in G.$$

Let K_0 be the smallest G-invariant subspace of H containing both ξ and η, and let $K = \overline{K_0}$. Then K is a closed G-invariant subspace of H, so determines a subrepresentation ρ of G. For every $\zeta, \kappa \in K_0$, the function $x \mapsto \langle U_x \zeta, \kappa \rangle$ is in E because E is G-invariant. Thus ρ is an E-representation. We have

$$\begin{aligned} |\langle a, f \rangle| &= |\langle \rho(a)\xi, \eta \rangle| \\ &\le \|\rho(a)\| \|\xi\| \|\eta\| \\ &\le \|a\|_E \|\xi\| \|\eta\| \\ &= 0. \end{aligned}$$

Thus $a \in {}^\perp E$.

For the opposite containment, suppose by way of contradiction that we can find $a \in {}^\perp E \setminus I$. Then $\|a\|_E \ne 0$, so we can also choose an E-representation U of G on a Hilbert space H such that $U(a) \ne 0$. Let H_0 be a dense subspace of H such that for all $\xi, \eta \in H_0$ the function $x \mapsto \langle U_x \xi, \eta \rangle$ is in E. By density we can choose $\xi, \eta \in H_0$ such that $\langle U(a)\xi, \eta \rangle \ne 0$. Then $g(x) = \langle U_x \xi, \eta \rangle$ defines an element $g \in E$, and we have

$$\langle a, g \rangle = \langle U(a)\xi, \eta \rangle \ne 0,$$

which is a contradiction. Therefore ${}^\perp E \subset I$, as desired.

(2) Since $I = {}^\perp E$ we have $\overline{E} = I^\perp$, which is G-invariant because I is an ideal, by Lemma 3.1. We have $I = {}^\perp \overline{E}$, so $C^*_{\overline{E}}(G) = C^*(G)/I$ by Definition 3.2. Since $C_c(G)$ is dense in $C^*(G)$, the result now follows by the definition of I in (3.1).

(3) This follows immediately from separate weak*-continuity of multiplication in $B(G)$. This is a well-known property of $B(G)$, but we include

the brief proof here for completeness: the bimodule action of $B(G)$ on the enveloping algebra $W^*(G) = B(G)^*$, given by

$$\langle a \cdot f, g\rangle = \langle a, fg\rangle = \langle f \cdot a, g\rangle \quad \text{for } a \in W^*(G), f, g \in B(G),$$

leaves $C^*(G)$ invariant, because it satisfies the submultiplicativity condition $\|a \cdot f\| \leq \|a\|\|f\|$ on norms and leaves $C_c(G) \subset C^*(G)$ invariant. Thus, if $f_i \to 0$ weak* in $B(G)$ and $g \in B(G)$, then for all $a \in C^*(G)$ we have

$$\langle a, f_i g\rangle = \langle a \cdot g, f_i\rangle \to 0. \qquad \square$$

Corollary 3.6.

(1) *A representation U of G is an E-representation if and only if, identifying U with the corresponding representation of $C^*(G)$, we have* $\ker U \supset {}^{\perp}E$.
(2) *A nondegenerate homomorphism $\tau : C^*(G) \to M(A)$, where A is a C^*-algebra, factors through a homomorphism of $C^*_E(G)$ if and only if*

$$\overline{\omega} \circ \tau \in \overline{E} \quad \textit{for all } \omega \in A^*,$$

where again $\overline{E}$ denotes the weak-closure of E.*

Proof. This follows readily from Lemma 3.5. $\square$

Remark 3.7. In light of Lemma 3.5, if we have a G-invariant subspace E of $B(G)$ that is not necessarily weak*-closed, it makes sense to, and we shall, write $C^*_E(G)$ for $C^*_{\overline{E}}(G)$. However, whenever convenient we can replace E by its weak*-closure, giving the same quotient $C^*_E(G)$.

Observation 3.8. *By Lemma* 3.5, *if E is a G-invariant subspace of $B(G)$ then:*

(1) *$C^*_E(G) = C^*(G)$ if and only if E is weak*-dense in $B(G)$.*
(2) *$C^*_E(G) = C^*_r(G)$ if and only if E is weak*-dense in $B_r(G)$.*

We record an elementary consequence of our definitions:

Lemma 3.9. *For a weak*-closed G-invariant subspace E of $B(G)$, the following are equivalent:*

(1) ${}^{\perp}E \subset \ker \lambda$.
(2) $E \supset B_r(G)$.
(3) $E \supset A(G)$.
(4) $E \supset (C_c(G) \cap B(G))$.

(5) *There is a (unique) homomorphism* $\rho : C^*_E(G) \to C^*_r(G)$ *making the diagram*

$$\begin{tikzcd} C^*(G) \arrow[dd, "\lambda"'] \arrow[dr, "q"] & \\ & C^*_E(G) \arrow[dl, dashed, "!"', "\rho"] \\ C^*_r(G) & \end{tikzcd}$$

commute.

Definition 3.10. For a weak*-closed G-invariant subspace E of $B(G)$, we say the quotient $C^*_E(G)$ is a *group* C^**-algebra of* G if the above equivalent conditions (1)–(5) are satisfied. If $B_r(G) \subsetneq E \neq B(G)$ we say the group C^*-algebra is *exotic*.

We will see in Proposition 5.1 that if G is discrete then a quotient $C^*_E(G)$ is a group C^*-algebra if and only if it is topologically graded in Exel's sense [Exe97, Definition 3.4].

We are especially interested in group C^*-algebras that carry a coaction or a comultiplication. We will need the following result, which is folklore among coaction cognoscenti:

Lemma 3.11. *If* $\delta : A \to M(A \otimes C^*(G))$ *is a coaction of* G *on a* C^**-algebra* A *and* I *is an ideal of* A*, then the following are equivalent:*

(1) *There is a coaction* $\tilde{\delta}$ *on* A/I *making the diagram*

$$\begin{tikzcd} A \arrow[r, "\delta"] \arrow[d, "q"'] & M(A \otimes C^*(G)) \arrow[d, "\overline{q \otimes \mathrm{id}}"] \\ A/I \arrow[r, "\tilde{\delta}"'] & M(A/I \otimes C^*(G)) \end{tikzcd} \tag{3.2}$$

commute (where q *is the quotient map).*

(2) $I \subset \ker \overline{q \otimes \mathrm{id}} \circ \delta$.

(3) $I^\perp$ *is a* $B(G)$*-submodule of* A^*.

Proof. This is well-known, but difficult to find in the literature, so we include the brief proof for the convenience of the reader. There exists a *homomorphism* $\tilde{\delta}$ making the diagram (3.2) commute if and only if (2) holds, and in that case $\tilde{\delta}$ will satisfy the coaction-nondegeneracy (2.1) and the coaction identity (2.2). By Lemma 2.2 this implies that $\tilde{\delta}$ is a coaction. Thus (1)⇔(2), and (2)⇔(3) follow from a routine calculation using the fact that $\{\psi \otimes f : \psi \in (A/I)^*, f \in B(G)\}$ separates the elements of

$$M(A/I \otimes C^*(G)). \qquad \square$$

Recall that the multiplication in $B(G)$ satisfies

$$\langle a, fg\rangle = \langle \delta_G(a), \overline{f \otimes g}\rangle \quad \text{for } a \in C^*(G) \text{ and } f, g \in B(G),$$

where $f \otimes g$ denotes the functional in $(C^*(G) \otimes C^*(G))^*$ determined by

$$\langle x \otimes y, \overline{f \otimes g}\rangle = f(x)g(y) \quad \text{for } x, y \in G.$$

Remark 3.12. Note that we need to explicitly state the above convention for $f \otimes g$, since we are using the minimal tensor product: if G is a group for which the canonical surjection

$$C^*(G) \otimes_{\max} C^*(G) \to C^*(G) \otimes C^*(G)$$

is noninjective[1], then

$$C^*(G) \otimes C^*(G) \neq C^*(G \times G),$$
$$(C^*(G) \otimes C^*(G))^* \neq B(G \times G),$$

because $C^*(G \times G) = C^*(G) \otimes_{\max} C^*(G)$.

Corollary 3.13. *Let E be a weak*-closed G-invariant subspace of $B(G)$, and let $q : C^*(G) \to C^*_E(G)$ be the quotient map. Then there is a coaction δ^E_G of G on $C^*_E(G)$ such that*

$$\overline{\delta^E_G}(q(x)) = q(x) \otimes x \quad \textit{for } x \in G$$

if and only if E is an ideal of $B(G)$.

Proof. Since E is the annihilator of $\ker q$, this follows immediately from Lemma 3.11. □

Recall that in Definition 3.10 we called $C^*_E(G)$ a group C^*-algebra if E is a weak*-closed G-invariant subspace of $B(G)$ containing $B_r(G)$; this latter property is automatic if E is an ideal (as long as it's nonzero):

Lemma 3.14. *Every nonzero norm-closed G-invariant ideal of $B(G)$ contains $A(G)$, and hence every nonzero weak*-closed G-invariant ideal of $B(G)$ contains $B_r(G)$.*

Proof. Let E be the ideal. It suffices to show that $E \cap A(G)$ is norm dense in $A(G)$. There exist $t \in G$ and $f \in E$ such that $f(t) \neq 0$. By [Eym64, Lemma 3.2] there exists $g \in A(G) \cap C_c(G)$ such that $g(t) \neq 0$, and then $fg \in E \cap C_c(G)$ is nonzero at t. By G-invariance of E, for all $x \in G$ there exists $f \in E$ such that $f(x) \neq 0$. Then for any $y \neq x$ we can find $g \in A(G) \cap C_c(G)$ such that $g(x) \neq 0$ and $g(y) = 0$, and so $fg \in E$ is nonzero at x and zero at y. Thus $E \cap A(G)$ is an ideal of $A(G)$ that is nowhere vanishing on G and separates points, so by [Eym64, Corollary 3.38] $E \cap A(G)$ is norm dense in $A(G)$, so we are done. □

[1] e.g., any infinite simple group with property T — see [BO08, Theorem 6.4.14 and Remark 6.4.15]

Recall that a *comultiplication* on a C^*-algebra A is a homomorphism (which we do *not* in general require to be injective) $\Delta : A \to M(A \otimes A)$ satisfying the *co-associativity* property

$$\overline{\Delta \otimes \mathrm{id}} \circ \Delta = \overline{\mathrm{id} \otimes \Delta} \circ \Delta$$

and the *nondegeneracy properties*

$$\overline{\operatorname{span}}\{\Delta(A)(1 \otimes A)\} = A \otimes A = \overline{\operatorname{span}}\{(A \otimes 1)\Delta(A)\}.$$

A C^*-algebra with a comultiplication is called a *C^*-bialgebra* (see [Kaw08] for this terminology). A comultiplication Δ on A is used to make the dual space A^* into a Banach algebra in the standard way:

$$\omega\psi := \overline{\omega \otimes \psi} \circ \Delta \quad \text{for } \omega, \psi \in A^*.$$

The following is another folklore result, proved similarly to Lemma 3.11:

Lemma 3.15. *If $\Delta : A \to M(A \otimes A)$ is a comultiplication on a C^*-algebra A and I is an ideal of A, then the following are equivalent:*

(1) *There is a comultiplication $\tilde{\Delta}$ on A/I making the diagram*

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "q"'] & M(A \otimes A) \arrow[d, "\overline{q \otimes q}"] \\ A/I \arrow[r, "\tilde{\Delta}"'] & M(A/I \otimes A/I) \end{tikzcd}$$

commute (where q is the quotient map).

(2) $I \subset \ker \overline{q \otimes q} \circ \Delta$.

(3) *$I^{\perp}$ is a subalgebra of A^*.*

We apply this to the canonical comultiplication δ_G on $C^*(G)$:

Proposition 3.16. *Let E be a weak*-closed G-invariant subspace of $B(G)$, and let $q : C^*(G) \to C^*_E(G)$ be the quotient map. Then the following are equivalent:*

(1) *There is a comultiplication Δ making the diagram*

$$\begin{tikzcd} C^*(G) \arrow[r, "\delta_G"] \arrow[d, "q"'] & M(C^*(G) \otimes C^*(G)) \arrow[d, "\overline{q \otimes q}"] \\ C^*_E(G) \arrow[r, "\Delta"'] & M(C^*_E(G) \otimes C^*_E(G)) \end{tikzcd}$$

commute.

(2) ${}^{\perp}E \subset \ker \overline{q \otimes q} \circ \delta_G$.

(3) *E is a subalgebra of $B(G)$.*

Remark 3.17. Proposition 3.16 tells us that if E is a weak*-closed G-invariant subalgebra of $B(G)$, then the group algebra $C^*_E(G)$ is a C^*-bialgebra. However, this probably does not make $C^*_E(G)$ a locally compact quantum group, since this would require an antipode. It might be difficult to investigate the general question of whether there exists *some* antipode on $C^*_E(G)$ that is compatible with the comultiplication; it seems more reasonable to ask whether the quotient map $q : C^*(G) \to C^*_E(G)$ takes the canonical antipode on $C^*(G)$ to an antipode on $C^*_E(G)$. This requires E to be closed under inverse i.e., if $f \in E$ then so is the function $f^\vee$ defined by $f^\vee(x) = f(x^{-1})$. Now, $f^\vee(x) = \overline{f^*(x)}$ where f^* is defined by $f^*(a) = \overline{f(a^*)}$ for $a \in C^*(G)$. Since $f \in E$ if and only if $f^* \in E$, we see that E is invariant under $f \mapsto f^\vee$ if and only if it is invariant under complex conjugation. In all our examples (in particular Section 4) E has this property. Note that $C^*_E(G)$ always has a Haar weight, since we can compose the canonical Haar weight on $C^*_r(G)$ with the quotient map $C^*_E(G) \to C^*_r(G)$. However, this Haar weight on $C^*_E(G)$ is faithful if and only if $E = B_r(G)$.

Remark 3.18. By Lemma 3.5, if E is a G-invariant ideal of $B(G)$ and $I = {}^\perp E$, then $\overline{E}$ is also a G-invariant ideal, so by Proposition 3.13 there is a coaction δ^E_G of G on $C^*_E(G)$ such that

$$\overline{\delta^E_G}(q(x)) = q(x) \otimes x \quad \text{for } x \in G,$$

where $q : C^*(G) \to C^*_E(G)$ is the quotient map.

Similarly, if E is a G-invariant subalgebra of $B(G)$ then $\overline{E}$ is also a G-invariant subalgebra, so by Proposition 3.16 there is a comultiplication Δ on $C^*_E(G)$ such that

$$\overline{\Delta}(q(x)) = q(x) \otimes q(x) \quad \text{for } x \in G.$$

Example 3.19. Note that if the quotient $C^*_E(G)$ is a group C^*-algebra, then the quotient map $q : C^*(G) \to C^*_E(G)$ is faithful on $C_c(G)$, and so by Lemma 3.5 $C^*_E(G)$ is the completion of $C_c(G)$ in the associated norm $\|\cdot\|_E$. However, q being faithful on $C_c(G)$ is not sufficient for $C^*_E(G)$ to be a group C^*-algebra. The simplest example of this is in [FD88, Exercise XI.38] (which we modify only slightly): let $0 \le a < b < 2\pi$, and define a surjection

$$q : C^*(\mathbb{Z}) \to C[a, b]$$

by

$$q(n)(t) = e^{int}.$$

Then the unitaries $q(n)$ are linearly independent, so q is faithful on $c_c(\mathbb{Z})$, but $q(C^*(\mathbb{Z}))$ is not a group C^*-algebra because $\ker q$ is a nontrivial ideal of $C^*(\mathbb{Z})$ and $\mathbb{Z}$ is amenable, so that $\ker \lambda = \{0\}$.

Example 3.20. The paper [EQ99] shows how to construct exotic group C^*-algebras $C^*_E(G)$ (see also [KS, Remark 9.6] for similar exotic quantum groups) with no coaction: let

$$q = \lambda \oplus 1_G,$$

where 1_G denotes the trivial 1-dimensional representation of G. The quotient $C^*_E(G)$ is a group C^*-algebra since $\ker q = \ker\lambda \cap \ker 1_G$. On the other hand, we have

$$E = (\ker q)^\perp = B_r(G) + \mathbb{C}1_G,$$

which is not an ideal of $B(G)$ unless it is all of $B(G)$, i.e., unless q is faithful; as remarked in [EQ99], this behavior would be quite bizarre, and in fact we do not know of any discrete nonamenable group with this property.

However, these quotients $C^*_E(G)$ are C^*-bialgebras, because $B_r(G)+\mathbb{C}1_G$ is a subalgebra of $B(G)$. Thus, these quotients give examples of exotic group C^*-bialgebras that are different from those in [BG, Proposition 4.4 and Remark 4.5]. It is interesting to note that these quotients of $C^*(G)$ are of a decidedly elementary variety: by Lemma 2.1 we have

$$C^*_E(G) = C^*_r(G) \oplus \mathbb{C},$$

because $C^*(G) = \ker\lambda + \ker 1_G$ since G is nonamenable. To see this latter implication, recall that if G is nonamenable then 1_G is not weakly contained in λ, so $\ker 1_G \not\supset \ker\lambda$, and hence $C^*(G) = \ker\lambda + \ker 1_G$ since $\ker 1_G$ is a maximal ideal.

Valette has a similar example in [Val84, Theorem 3.6] where he shows that if N is a closed normal subgroup of G that has property (T), then $C^*(G)$ is the direct sum of $C^*(G/N)$ and a complementary ideal.

For a different source of exotic group C^*-bialgebras, see Example 3.22.

Example 3.21. We can also find examples of group C^*-algebras with no comultiplication: modify the preceding example by taking

$$q = \lambda \oplus \gamma,$$

where γ is a nontrivial character of G (assuming that G has such characters). Then

$$(\ker q)^\perp = B_r(G) + \mathbb{C}\gamma,$$

which is not a subalgebra of $B(G)$ when G is nonamenable.

Example 3.22. Let G be a locally compact group for which the canonical surjection

$$C^*(G) \otimes_{\max} C^*(G) \to C^*(G) \otimes C^*(G) \tag{3.3}$$

is not injective. (In the second tensor product we use the minimal C^*-tensor norm as usual. See Remark 3.12.) Let I denote the kernel of this map. Since the algebraic product $B(G) \odot B(G)$ is weak*-dense in $(C^*(G) \otimes C^*(G))^*$, the annihilator $E = I^\perp$ is the weak*-closed span of functions of the form

$$(x,y) \mapsto f(x)g(y) \quad \text{for } f,g \in B(G).$$

This is clearly a subalgebra, but not an ideal, because it contains 1. Also, $E \supset B_r(G \times G)$ because the surjection (3.3) can be followed by

$$C^*(G) \otimes C^*(G) \to C^*_r(G) \otimes C^*_r(G) \cong C^*_r(G \times G).$$

Thus the canonical coaction $\delta_{G\times G}$ of $G\times G$ on $C^*(G\times G)$ descends to a comultiplication on the group C^*-algebra $C^*_E(G\times G)\cong C^*(G)\otimes C^*(G)$, but not to a coaction of $G\times G$.

4. Classical ideals

We continue to let G be an arbitrary locally compact group.

We will apply the theory of the preceding sections to group C^*-algebras $C^*_E(G)$ with E of the form

$$E=D\cap B(G),$$

where D is some familiar G-invariant set of functions on G.

Notation 4.1. If D is a G-invariant set of functions on G, we write

$$\|f\|_D=\|f\|_{D\cap B(G)},$$

and similarly $C^*_D(G)=C^*_{D\cap B(G)}(G)$.

So, for instance, we can consider $C^*_{C_c}(G)$, $C^*_{C_0(G)}(G)$, and $C^*_{L^p(G)}(G)$. In each of these cases the intersection $E=D\cap B(G)$ is a G-invariant ideal of $B(G)$, so by Remark 3.18 and Lemma 3.14 these quotients are all group C^*-algebras carrying coactions of G, and hence by Proposition 3.16 they carry comultiplications. In the case that G is discrete, $c_c(G)$, $c_0(G)$, and $\ell^p(G)$ could be regarded as classical ideals of $\ell^\infty(G)$; this is the context of Brown and Guentner's "new completions of discrete groups" [BG].

We have

$$C^*_{C_c(G)}(G)=C^*_{A(G)}(G)=C^*_r(G),$$

because $C_c(G)\cap B(G)$ is norm dense in $A(G)$, and hence weak*-dense in $B_r(G)$. However, the quotients $C^*_{C_0(G)}(G)$ and $C^*_{L^p(G)}(G)$ are more mysterious. Nevertheless, we have the following (which, for the case of discrete G, is [BG, Proposition 2.11]):

Proposition 4.2. *For all $p\le 2$ we have $C^*_{L^p(G)}(G)=C^*_r(G)$.*

Proof. Since $L^p(G)\cap B(G)$ consists of bounded functions, for $p\le 2$ we have

$$C_c(G)\cap B(G)\subset L^p(G)\cap B(G)\subset L^2(G)\cap B(G).$$

Now, if U is a representation of G having a cyclic vector ξ such that the function $x\mapsto\langle U_x\xi,\xi\rangle$ is in $L^2(G)$, then U is contained in λ (see, e.g., [Car76]), and consequently $L^2(G)\cap B(G)\subset A(G)$. Thus

$$\begin{aligned}B_r(G)&=\overline{C_c(G)\cap B(G)}^{\text{weak}^*}\\&\subset\overline{L^p(G)\cap B(G)}^{\text{weak}^*}\\&\subset\overline{L^2(G)\cap B(G)}^{\text{weak}^*}\\&\subset\overline{A(G)}^{\text{weak}^*}=B_r(G),\end{aligned}$$

and the result follows. □

Remark 4.3.

(1) The proof of Proposition 4.2 is much easier when G is discrete, because then for $\xi \in \ell^2(G)$ we have

$$\xi(x) = \langle \lambda_x \chi_{\{e\}}, \bar{\xi} \rangle,$$

so $\ell^2(G) \subset A(G)$.

(2) In general, $\overline{C_0(G) \cap B(G)}^{\text{weak}^*} \supset B_r(G)$. The containment can be proper (for perhaps the earliest result along these lines, see [Men16]). When G is discrete, this phenomenon occurs precisely when G is a-T-menable but nonamenable, by the result of [BG] mentioned in the introduction.

(3) Using the method outlined in this section, if we start with a G-invariant ideal D of $L^\infty(G)$ and put $E = \overline{D \cap B(G)}^{\text{weak}^*}$, we get many weak*-closed ideals of $B(G)$, but probably not all. For example, if we let z_F be the supremum in the universal enveloping von Neumann algebra $W^*(G) = C^*(G)^{**}$ of the support projections of finite dimensional representations of G, then it follows from [Wal75, Proposition 1, Theorem 2, Proposition 8] that $(1 - z_F) \cdot B(G)$ is an ideal of $B(G)$ and $z_F \cdot B(G) = AP(G) \cap B(G)$ is a subalgebra. It seems unlikely that for all locally compact groups G the ideal $(1 - z_F) \cdot B(G)$ arises as an intersection $D \cap B(G)$ for an ideal D of $L^\infty(G)$.

5. Graded algebras

In this short section we impose the condition that the group G is discrete. We made this a separate section for the purpose of clarity — here the assumptions on G are different from everywhere else in this paper. [Exe97, Definition 3.1] and [FD88, VIII.16.11–12] define G-graded C^*-algebras as certain quotients of Fell-bundle algebras[2]. When the fibres of the Fell bundle are 1-dimensional, each one consists of scalar multiplies of a unitary. When these unitaries can be chosen to form a representation of G, the C^*-algebra is a quotient $C^*_E(G)$.

The following can be regarded as a special case of [Exe97, Theorem 3.3]:

Proposition 5.1. *Let E be a weak*-closed G-invariant subspace of $B(G)$, and let $q : C^*(G) \to C^*_E(G)$ be the quotient map. Then the following are equivalent:*

(1) *$C^*_E(G)$ is a group C^*-algebra in the sense of Definition* 3.10.

[2][Exe97, FD88] would require the images of the fibres to be linearly independent.

(2) *There is a bounded linear functional* ω *on* $C^*_E(G)$ *such that*

$$\omega(q(x)) = \begin{cases} 1 & \text{if } x = e, \\ 0 & \text{if } x \neq e. \end{cases}$$

(3) E *contains the canonical trace* tr *on* $C^*(G)$.

(4) $E \supset B_r(G)$.

(5) *There is a (unique) homomorphism* $\rho : C^*_E(G) \to C^*_r(G)$ *making the diagram*

$$\begin{tikzcd} C^*(G) \arrow[dr, "q"] \arrow[dd, "\lambda"'] & \\ & C^*_E(G) \arrow[dl, dashed, "!"', "\rho"] \\ C^*_r(G) & \end{tikzcd}$$

commute.

Proof. Assuming (2), the composition $\omega \circ q$ coincides with tr, so tr $\in E$, and conversely if tr $\in E$ then we get a suitable ω. Thus (2) $\Leftrightarrow$ (3).

For the rest, just note that $B_r(G) = (\ker \lambda)^\perp$ is the weak*-closed G-invariant subspace generated by tr $= \chi_{\{e\}}$, and appeal to Lemma 3.9. $\square$

Remark 5.2. Condition (2) in Proposition 5.1 is precisely what Exel's [Exe97, Definition 3.4] would require to say that $C^*_E(G)$ is *topologically graded*.

6. Exotic coactions

We return to the context of an arbitrary locally compact group G.

The coactions appearing in noncommutative crossed-product duality come in a variety of flavors: *reduced* vs. *full* (see, e.g., [EKQR06, Appendix] or [HQRW11]), and, among the full ones, a spectrum with *normal* and *maximal* coactions at the extremes (see [EKQ04], for example). In this concluding section we briefly propose a new program in crossed-product duality: "exotic coactions", involving the exotic group C^*-algebras $C^*_E(G)$ in the sense of Definition 3.10. From now until Proposition 6.16 we are concerned with nonzero G-invariant weak*-closed ideals E of $B(G)$.

By Lemmas 3.9 and 3.14 the quotient $C^*_E(G) = C^*(G)/{}^\perp E$ is a group C^*-algebra. By Proposition 3.13, there is a coaction δ^E_G of G on $C^*_E(G)$ making the diagram

$$\begin{tikzcd} C^*(G) \arrow[r, "\delta_G"] \arrow[d, "q"'] & M(C^*(G) \otimes C^*(G)) \arrow[d, "\overline{q \otimes \mathrm{id}}"] \\ C^*_E(G) \arrow[r, "\delta^E_G"'] & M(C^*_E(G) \otimes C^*(G)) \end{tikzcd}$$

commute, where q is the quotient map, and by Proposition 3.16 there is a quotient comultiplication Δ on $C^*_E(G)$. Recall that we defined the *exotic* group C^*-algebras to be the ones strictly between the two extremes $C^*(G)$ and $C^*_r(G)$, corresponding to $E = B(G)$ and $E = B_r(G)$, respectively.

On one level, we could try to study coactions of Hopf C^*-algebras associated to the locally compact group G other than $C^*(G)$ and $C^*_r(G)$. But there is an inconvenient subtlety here (see Remark 3.17). However, there is a deeper level to this program, relating more directly to crossed-product duality. At the deepest level, we aim for a characterization of *all* coactions of G in terms of the quotients $C^*_E(G)$. We hasten to emphasize that at this time some of the following is speculative, and is intended merely to outline a program of study.

From now on, the unadorned term "coaction" will refer to a full coaction of G on a C^*-algebra A.

Let $\psi : (A^m, \delta^m) \to (A, \delta)$ be the maximalization of δ, so that δ^m is a maximal coaction, $\psi : A^m \to A$ is an equivariant surjection, and the crossed-product surjection

$$\psi \times G : A^m \rtimes_{\delta^m} G \to A \rtimes_\delta G$$

(for the existence of which, see [EKQR06, Lemma A.46], for example) is an isomorphism. Since δ^m is maximal, the canonical surjection

$$\Phi : A^m \rtimes_{\delta^m} G \rtimes_{\widehat{\delta^m}} G \to A^m \otimes \mathcal{K}(L^2(G))$$

is an isomorphism (this is "full-crossed-product duality"). Blurring the distinction between $A^m \rtimes_{\delta^m} G$ and the isomorphic crossed product $A \rtimes_\delta G$, and recalling that $\psi \times G : A^m \rtimes_{\delta^m} G \to A \rtimes_\delta G$ is $\widehat{\delta^m} - \widehat{\delta}$ equivariant, we can regard Φ as an isomorphism

$$A \rtimes_\delta G \rtimes_{\widehat{\delta}} G \xrightarrow[\cong]{\Phi} A^m \otimes \mathcal{K}(L^2(G)).$$

We have a surjection

$$\psi \otimes \mathrm{id} : A^m \otimes \mathcal{K}(L^2(G)) \to A \otimes \mathcal{K}(L^2(G)),$$

whose kernel is $(\ker \psi) \otimes \mathcal{K}(L^2(G))$ since $\mathcal{K}(L^2(G))$ is nuclear. Let K_δ be the inverse image under Φ of this kernel, giving an ideal of $A \rtimes_\delta G \rtimes_{\widehat{\delta}} G$ and an isomorphism Φ_δ making the diagram

$$\begin{tikzcd}
A \rtimes_\delta G \rtimes_{\widehat{\delta}} G \arrow[r, "\Phi", "\cong"'] \arrow[d, "Q"'] & A^m \otimes \mathcal{K}(L^2(G)) \arrow[d, "\psi\otimes\mathrm{id}"] \\
(A \rtimes_\delta G \rtimes_{\widehat{\delta}} G)/K_\delta \arrow[r, "\cong", "\Phi_\delta"'] & A \otimes \mathcal{K}(L^2(G))
\end{tikzcd} \tag{6.1}$$

commute, where Q is the quotient map. Adapting the techniques of [EQ02, Theorem 3.7][3], it is not hard to see that K_δ is contained in the kernel of the regular representation $\Lambda : A \rtimes_\delta G \rtimes_{\widehat{\delta}} G \to A \rtimes_\delta G \rtimes_{\widehat{\delta},r} G$.

If δ is maximal, then diagram 6.1 collapses to a single row. On the other hand, if δ is normal, then Q is the regular representation Λ and in particular

$$(A \rtimes_\delta G \rtimes_{\widehat{\delta}} G)/K_\delta = A \rtimes_\delta G \rtimes_{\widehat{\delta},r} G.$$

(In this case the isomorphism Φ_δ is "reduced-crossed-product duality".)

With the ultimate goal (which at this time remains elusive — see Conjectures 6.12 and 6.14) of achieving an "E-crossed-product duality", intermediate between full- and reduced-crossed-product dualities, below we will propose tentative definitions of "E-crossed-product duality" and "E-crossed products" $B \rtimes_{\alpha,E} G$ by actions $\alpha : G \to \operatorname{Aut} B$, and we will prove that they have the following properties:

(1) A coaction satisfies $B(G)$–crossed-product duality if and only if it is maximal.
(2) A coaction satisfies $B_r(G)$–crossed-product duality if and only if it is normal.
(3) $B \rtimes_{\alpha,B(G)} G = B \rtimes_\alpha G$.
(4) $B \rtimes_{\alpha,B_r(G)} G = B \rtimes_{\alpha,r} G$.
(5) The dual coaction $\hat{\alpha}$ on the full crossed product $B \rtimes_\alpha G$ satisfies $B(G)$-crossed-product duality.
(6) The dual coaction $\hat{\alpha}^n$ on the reduced crossed product $B \rtimes_{\alpha,r} G$ satisfies $B_r(G)$-crossed-product duality.
(7) In general, $B \rtimes_{\alpha,E} G$ is a quotient of $B \rtimes_\alpha G$ by an ideal contained in the kernel of the regular representation

$$\Lambda : B \rtimes_\alpha G \to B \rtimes_{\alpha,r} G.$$

(8) There is a dual coaction $\hat{\alpha}_E$ of G on $B \times_{\alpha,E} G$.

Definition 6.1. Define an ideal $J_{\alpha,E}$ of the crossed product $B \rtimes_\alpha G$ by

$$J_{\alpha,E} = \ker \overline{\mathrm{id} \otimes q} \circ \hat{\alpha},$$

and define the *E-crossed product* by

$$B \rtimes_{\alpha,E} G = (B \rtimes_\alpha G)/J_{\alpha,E}.$$

Note that the above properties (1)–(7) are obviously satisfied (because $\hat{\alpha}$ is maximal and $\hat{\alpha}^n$ is normal), and we now verify that (8) holds as well:

Theorem 6.2. *Let E be a nonzero weak*-closed G-invariant ideal of $B(G)$, and let $Q : B \rtimes_\alpha G \to B \rtimes_{\alpha,E} G$ be the quotient map. Then there is a*

[3]This is a convenient place to correct a slip in the last paragraph of the proof of [EQ02, Theorem 3.7]: "contains" should be replaced by "is contained in" (both times).

coaction $\hat\alpha_E$ making the diagram

$$\begin{tikzcd}
B \rtimes_\alpha G \arrow[r, "\hat\alpha"] \arrow[d, "Q"'] & M((B \rtimes_\alpha G) \otimes C^*(G)) \arrow[d, "\overline{Q\otimes \mathrm{id}}"] \\
B \rtimes_{\alpha,E} G \arrow[r, "\hat\alpha_E"'] & M((B \rtimes_{\alpha,E} G) \otimes C^*(G))
\end{tikzcd}$$

commute.

Proof. By Lemma 3.13, we must show that

$$J_{\alpha,E} \subset \ker \overline{Q \otimes \mathrm{id}} \circ \hat\alpha.$$

Let $a \in J_{\alpha,E}$, $\omega \in (B \rtimes_{\alpha,E} G)^*$, and $g \in B(G)$. Then

$$\begin{aligned}
\overline{\omega \otimes g} \circ \overline{Q \otimes \mathrm{id}} \circ \hat\alpha(a) &= \overline{Q^*\omega \otimes g} \circ \hat\alpha(a) \\
&= Q^*\omega \circ \overline{\mathrm{id} \otimes g} \circ \hat\alpha(a) \\
&= Q^*\omega(g \cdot a).
\end{aligned}$$

Now, since $Q^*\omega \in J_{\alpha,E}^\perp$, it suffices to show that $g \cdot a \in J_{\alpha,E}$. For $h \in E$ we have

$$h \cdot (g \cdot a) = (hg) \cdot a = (gh) \cdot a = g \cdot (h \cdot a) = 0,$$

because $h \cdot a = 0$ by Lemma 6.3 below. $\square$

Lemma 6.3. *With the above notation, we have:*

(1) $J_{\alpha,E} = \{a \in B \rtimes_\alpha G : E \cdot a = \{0\}\}$.
(2) $J_{\alpha,E}^\perp = \overline{\mathrm{span}}\{(B \rtimes_\alpha G)^* \cdot E\}$, *where the closure is in the weak*-topology.*

Proof. (1) For $a \in B \rtimes_\alpha G$, we have

$$\begin{aligned}
& a \in J_{\alpha,E} \\
&\Leftrightarrow \overline{\mathrm{id} \otimes q} \circ \hat\alpha(a) = 0 \\
&\Leftrightarrow \overline{\omega \otimes h} \circ \overline{\mathrm{id} \otimes q} \circ \hat\alpha(a) = 0 \\
&\qquad\qquad \text{for all } \omega \in (B \rtimes_{\alpha,E} G)^* \text{ and } h \in C_E^*(G)^* \\
&\Leftrightarrow \overline{\omega \otimes q^*h} \circ \hat\alpha(a) = 0 \\
&\qquad\qquad \text{for all } \omega \in (B \rtimes_{\alpha,E} G)^* \text{ and } h \in C_E^*(G)^* \\
&\Leftrightarrow \overline{\omega \otimes g} \circ \hat\alpha(a) = 0 \\
&\qquad\qquad \text{for all } \omega \in (B \rtimes_{\alpha,E} G)^* \text{ and } g \in E \\
&\Leftrightarrow \overline{\omega} \circ \overline{\mathrm{id} \otimes g} \circ \hat\alpha(a) = 0 \\
&\qquad\qquad \text{for all } \omega \in (B \rtimes_{\alpha,E} G)^* \text{ and } g \in E \\
&\Leftrightarrow \omega(g \cdot a) = 0 \quad \text{for all } \omega \in (B \rtimes_{\alpha,E} G)^* \text{ and } g \in E \\
&\Leftrightarrow g \cdot a = 0 \quad \text{for all } g \in E.
\end{aligned}$$

(2) If $a \in J_{\alpha,E}$, $\omega \in (B \rtimes_\alpha G)^*$, and $f \in E$,

$$(\omega \cdot f)(a) = \omega(f \cdot a) = 0,$$

so $\omega \cdot f \in J_{\alpha,E}^\perp$, and hence the left-hand side contains the right.

For the opposite containment, it suffices to show that

$$J_{\alpha,E} \supset {}^\perp\big((B \rtimes_\alpha G)^* \cdot E\big).$$

If $a \in {}^\perp((B \rtimes_\alpha G)^* \cdot E)$, then for all $\omega \in (B \rtimes_\alpha G)^*$ and $f \in E$ we have

$$0 = (\omega \cdot f)(a) = \omega(f \cdot a),$$

so $f \cdot a = 0$, and therefore $a \in J_{\alpha,E}$. $\square$

Remark 6.4. We could define a covariant representation (π, U) of the action (B, α) to be an *E-representation* if the representation U of G is an E-representation, and we could define an ideal $\tilde{J}_{\alpha,E}$ of $B \rtimes_\alpha G$ by

$$\tilde{J}_{\alpha,E} = \{a : \pi \times U(a) = 0 \text{ for every } E\text{-representation } (\pi, U)\}, \tag{6.2}$$

similarly to what is done in [BG, Definition 5.2]. It follows from Corollary 3.6 that (π, U) is an E-representation in the above sense if and only if

$$\overline{\omega} \circ U \in E \quad \text{for all } \omega \in \big(\pi \times U(B \rtimes_\alpha G)\big)^*,$$

where $i_G : C^*(G) \to M(B \rtimes_\alpha G)$ is the canonical nondegenerate homomorphism, and consequently

$$\tilde{J}_{\alpha,E}^\perp = \{\omega \in (B \rtimes_\alpha G)^* : \overline{\omega} \circ i_G \in E\}.$$

In the following lemma we show one containment that always holds between (6.2) and the ideal of Definition 6.1, after which we explain why these ideals do *not* coincide in general.

Lemma 6.5. *With the above notation, we have*

$$\tilde{J}_{\alpha,E} \subset J_{\alpha,E}.$$

Proof. If $\omega \in (B \rtimes_\alpha G)^*$ and $f \in E$, then

$$\begin{aligned}
\overline{\omega \cdot f} \circ i_G &= \overline{\omega \otimes f} \circ \overline{\hat{\alpha}} \circ i_G \\
&= \overline{\omega \otimes f} \circ \overline{i_G \otimes \mathrm{id}} \circ \delta_G \\
&= \overline{\overline{\omega} \circ i_G \otimes f} \circ \delta_G \\
&= \big(\overline{\omega} \circ i_G\big) f,
\end{aligned}$$

which is in E because $f \in E$ and E is an ideal of $B(G)$. Thus $\omega \cdot f \in \tilde{J}_{\alpha,E}^\perp$. $\square$

Example 6.6. To see that the inclusion of Lemma 6.5 can be proper, consider the extreme case $E = B_r(G)$, so that $B \rtimes_{\alpha,E} G = B \rtimes_{\alpha,r} G$. In this case $J_{\alpha,E}$ is the kernel of the regular representation $\Lambda : B \rtimes_\alpha G \to B \rtimes_{\alpha,r} G$. On the other hand, $\tilde{J}_{\alpha,E}$ comprises the elements that are killed by every representation $\pi \times U$ for which U is weakly contained in the regular representation λ of G. [QS92, Example 5.3] gives an example of an action (B, α)

having a covariant representation (π, U) for which U is weakly contained in λ but $\pi \times U$ is not weakly contained in Λ. Thus $\ker \pi \times U$ contains $\tilde{J}_{\alpha,E}$ and $J_{\alpha,E}$ has an element not contained in $\ker \pi \times U$, so $\tilde{J}_{\alpha,E}$ is properly contained in $J_{\alpha,E}$ in this case.

Definition 6.7. We say that G is *E-amenable* if there are positive definite functions h_n in E such that $h_n \to 1$ uniformly on compact sets.

Lemma 6.8. *If G is E-amenable and (A, G, α) is an action, then $J_{\alpha,E} = \{0\}$, so*

$$A \rtimes_\alpha G \cong A \rtimes_{\alpha,E} G.$$

Proof. By Lemma 6.3, we have $h_n \cdot a = 0$ for all $a \in J_{\alpha,E}$. Since $h_n \to 1$ uniformly on compact sets, it follows that $h_n \cdot a \to a$ in norm. To see this, note that since the h_n are positive definite and $h_n \to 1$, the sequence $\{h_n\}$ is bounded in $B(G)$, and certainly for $f \in C_c(G)$ we have

$$h_n \cdot \big(fa\big) = (h_n f)a \to fa$$

in norm, because the pointwise products $h_n f$ converge to f uniformly and hence in the inductive limit topology since $\operatorname{supp} f$ is compact. Therefore $J_{\alpha,E} = \{0\}$. □

Remark 6.9. In [BG, Section 5], Brown and Guentner study actions of a discrete group G on a unital abelian C^*-algebra $C(X)$, and introduce the concept of a D-amenable action, where D is a G-invariant ideal of $\ell^\infty(G)$. In particular, if G is D-amenable then every action of G is D-amenable. They show that if the action is D-amenable then $\tilde{J}_{\alpha,E} = \{0\}$, i.e.,

$$C^*_D(X \rtimes G) \cong C(X) \rtimes_\alpha G.$$

Here we have used the notation of [BG]: $C^*_D(X \rtimes G)$ denotes the quotient of the crossed product $C(X) \rtimes_\alpha G$ by the ideal $\tilde{J}_{\alpha,E}$ (although Brown and Guentner give a different, albeit equivalent, definition).

Question 6.10. With the above notation, form a weak*-closed G-invariant ideal E of $B(G)$ by taking the weak*-closure of $D \cap B(G)$. Then is the stronger statement $J_{\alpha,E} = \{0\}$ true? (One easily checks it for $E = B_r(G)$, and it is trivial for $E = B(G)$.)

Note that the techniques of [BG] rely heavily on the fact that they are using ideals of $\ell^\infty(G)$, whereas our methods require ideals of $B(G)$.

Definition 6.11. A coaction (A, δ) *satisfies E–crossed-product duality* if

$$K_\delta = J_{\widehat{\delta},E},$$

where K_δ is the ideal from (6.1) and $J_{\widehat{\delta},E}$ is the ideal associated to the dual action $\widehat{\delta}$ in Definition 6.1.

Thus (A,δ) satisfies E–crossed-product duality precisely when we have an isomorphism Φ_E making the diagram

$$\begin{tikzcd} A \rtimes_\delta G \rtimes_{\widehat{\delta}} G \arrow[r, "\Phi"] \arrow[d, "Q"'] & A \otimes \mathcal{K}(L^2(G)) \\ A \rtimes_\delta G \rtimes_{\widehat{\delta},E} G \arrow[ur, "\cong", "\Phi_E"'] & \end{tikzcd}$$

commute, where Q is the quotient map.

Conjecture 6.12. *Every coaction satisfies E–crossed-product duality for some E.*

Observation 6.13. *If E is an ideal of $B(G)$, then every group C^*-algebra $C^*_E(G)$ is an E-crossed product:*

$$C^*_E(G) = \mathbb{C} \rtimes_{\iota,E} G,$$

where ι is the trivial action of G on $\mathbb{C}$, because the kernel of the quotient map $C^*(G) \to C^*_E(G)$ is ${}^\perp E$. This generalizes the extreme cases:

(1) $C^*(G) = \mathbb{C} \rtimes_\iota G$.
(2) $C^*_r(G) = \mathbb{C} \rtimes_{\iota,r} G$.

Conjecture 6.14. *If (B,α) is an action, then the dual coaction $\hat{\alpha}_E$ on the E-crossed product $B \rtimes_{\alpha,E} G$ satisfies E–crossed-product duality.*

Remark 6.15. In particular, by Observation 6.13, Conjecture 6.14 would imply as a special case that the canonical coaction δ_G^E on the group algebra $C^*_E(G)$ satisfies E–crossed-product duality.

For our final result, we only require that E be a weak*-closed G-invariant subalgebra of $B(G)$ (but not necessarily an ideal). By Proposition 3.16, $C^*_E(G)$ carries a comultiplication Δ that is a quotient of the canonical comultiplication δ_G on $C^*(G)$.

Techniques similar to those used in the proof of Theorem 6.2, taking $g \in E$ rather than $g \in B(G)$, can be used to show:

Proposition 6.16. *Let E be a weak*-closed G-invariant subalgebra of $B(G)$, and let (B,α) be an action. Then there is a coaction Δ_α of the C^*-bialgebra $C^*_E(G)$ making the diagram*

$$\begin{tikzcd} B \rtimes_\alpha G \arrow[r, "\hat{\alpha}"] \arrow[d, "Q"'] & M((B \rtimes_\alpha G) \otimes C^*(G)) \arrow[d, "\overline{Q\otimes q}"] \\ B \rtimes_{\alpha,E} G \arrow[r, "\Delta_\alpha"'] & M((B \rtimes_{\alpha,E} G) \otimes C^*_E(G)) \end{tikzcd}$$

commute, where we use notation from Theorem 6.2.

We close with a rather vague query:

Question 6.17. What are the relationships among E-crossed products, E-coactions, and coactions of the C^*-bialgebra $C^*_E(G)$?

We hope to investigate this question, together with Conjectures 6.12 and 6.14, in future research.

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